Universal Quantile Estimation with Feedback in the Communication-Constrained Setting

Ram Rajagopal¹, Martin J. Wainwright^{1,2}

¹ Department of Electrical Engineering and Computer Science ² Department of Statistics

University of California, Berkeley

{ramr,wainwrig}@eecs.berkeley.edu

Abstract

We consider the following problem of decentralized statistical inference: given i.i.d. samples from an unknown distribution, estimate an arbitrary quantile subject to limits on the number of bits exchanged. We analyze a standard fusion-based architecture, in which each of m sensors transmits a single bit to the fusion center, which in turn is permitted to send some number k bits of feedback. Supposing that each of m sensors receives n observations, the optimal centralized protocol yields mean-squared error decaying as $\mathcal{O}(1/[nm])$. We develop and analyze the performance of various decentralized protocols in comparison to this centralized gold-standard. First, we describe a decentralized protocol based on $k = \log(m)$ bits of feedback that is strongly consistent, and achieves the same asymptotic MSE as the centralized optimum. Second, we describe and analyze a decentralized protocol based on only a single bit (k = 1) of feedback. For step sizes independent of m, it achieves an asymptotic MSE of order $\mathcal{O}[1/(n\sqrt{m})]$, whereas for step sizes decaying as $1/\sqrt{m}$, it achieves the same $\mathcal{O}(1/[nm])$ decay in MSE as the centralized optimum. Our theoretical results are complemented by simulations, illustrating the tradeoffs between these different protocols.

Keywords: Decentralized inference; communication constraints; distributed estimation; non-parametric estimation; quantiles; sensor networks; stochastic approximation.

I. INTRODUCTION

Whereas classical statistical inference is performed in a centralized manner, many modern scientific problems and engineering systems are inherently *decentralized*: data are distributed, and cannot be aggre-

Portions of this work were presented at the International Symposium on Information Theory, Seattle, WA, July 2006.

2

gated due to various forms of communication constraints. An important example of such a decentralized system is a sensor network [6]: a set of spatially-distributed sensors collect data about the environmental state (e.g., temperature, humidity or light). Typically, these networks are based on ad hoc deployments, in which the individual sensors are low-cost, and must operate under very severe power constraints (e.g., limited battery life). In statistical terms, such communication constraints imply that the individual sensors cannot transmit the raw data; rather, they must compress or quantize the data—for instance, by reducing a continuous-valued observation to a single bit—and can transmit only this compressed representation back to the fusion center.

By now, there is a rich literature in both information theory and statistical signal processing on problems of decentralized statistical inference. A number of researchers, dating back to the seminal paper of Tenney and Sandell [16], have studied the problem of hypothesis testing under communication-constraints; see the survey papers [17], [18], [4], [19], [5] and references therein for overviews of this line of work. The hypothesis-testing problem has also been studied in the information theory community, where the analysis is asymptotic and Shannon-theoretic in nature [1], [11]. A parallel line of work deals with problem of decentralized estimation. Work in signal processing typically formulates it as a quantizer design problem and considers finite sample behavior [2], [8]; in contrast, the information-theoretic approach is asymptotic in nature, based on rate-distortion theory [20], [10]. In much of the literature on decentralized statistical inference, it is assumed that the underlying distributions are known with a specified parametric form (e.g., Gaussian). More recent work has addressed non-parametric and data-driven formulations of these problems, in which the decision-maker is simply provided samples from the unknown distribution [14], [13], [9]. For instance, Nguyen et al. [14] established statistical consistency for non-parametric approaches to decentralized hypothesis testing based on reproducing kernel Hilbert spaces. Luo [13] analyzed a nonparametric formulation of decentralized mean estimation, in which a fixed but unknown parameter is corrupted by noise with bounded support but otherwise arbitrary distribution, and shown that decentralized approaches can achieve error rates that are order-optimal with respect to the centralized optimum.

This paper addresses a different problem in decentralized non-parametric inference—namely, that of estimating an arbitrary quantile of an unknown distribution. Since there exists no unbiased estimator based on a single sample, we consider the performance of a network of m sensors, each of which collects a total of n observations in a sequential manner. Our analysis treats the standard fusion-based architecture, in which each of the m sensors transmits information to the fusion center via a communication-constrained channel. More concretely, at each observation round, each sensor is allowed to transmit a single bit to the fusion center, which in turn is permitted to send some number k bits of feedback. For a decentralized

protocol with $k = \log(m)$ bits of feedback, we prove that the algorithm achieves the order-optimal rate of the best centralized method (i.e., one with access to the full collection of raw data). We also consider a protocol that permits only a single bit of feedback, and establish that it achieves the same rate. This single-bit protocol is advantageous in that, with for a fixed target mean-squared error of the quantile estimate, it yields longer sensor lifetimes than either the centralized or full feedback protocols.

The remainder of the paper is organized as follows. We begin in Section II with background on quantile estimation, and optimal rates in the centralized setting. We then describe two algorithms for solving the corresponding decentralized version, based on log(m) and 1 bit of feedback respectively, and provide an asymptotic characterization of their performance. These theoretical results are complemented with empirical simulations. Section III contains the analysis of these two algorithms. In Section IV, we consider various extensions, including the case of feedback bits ℓ varying between the two extremes, and the effect of noise on the feedforward link. We conclude in Section V with a discussion.

II. PROBLEM SET-UP AND DECENTRALIZED ALGORITHMS

In this section, we begin with some background material on (centralized) quantile estimation, before introducing our decentralized algorithms, and stating our main theoretical results.

A. Centralized Quantile Estimation

We begin with classical background on the problem of quantile estimation (see Serfling [15] for further details). Given a real-valued random variable X, let $F(x) := \mathbb{P}[X \le x]$ be its cumulative distribution function (CDF), which is non-decreasing and right-continuous. For any $0 < \alpha < 1$, the α^{th} -quantile of X is defined as $F^{-1}(\alpha) = \theta(\alpha) := \inf \{x \in \mathbb{R} \mid F(x) \ge \alpha\}$. Moreover, if F is continuous at α , then we have $\alpha = F(\theta(\alpha))$. As a particular example, for $\alpha = 0.5$, the associated quantile is simply the median.

Now suppose that for a fixed level $\alpha^* \in (0, 1)$, we wish to estimate the quantile $\theta^* = \theta(\alpha^*)$. Rather than impose a particular parameterized form on F, we work in a non-parametric setting, in which we assume only that the distribution function F is differentiable, so that X has the density function $p_X(x) = F'(x)$ (w.r.t Lebesgue measure), and moreover that $p_X(x) > 0$ for all $x \in \mathbb{R}$. In this setting, a standard estimator for θ^* is the sample quantile $\xi_N(\alpha^*) := F_N^{-1}(\alpha^*)$ where F_N denotes the empirical distribution function based on i.i.d. samples (X_1, \ldots, X_N) . Under the conditions given above, it can be shown [15] that $\xi_N(\alpha^*)$ is strongly consistent for θ^* (i.e., $\xi_N \stackrel{a.s.}{\to} \theta^*$), and moreover that asymptotic normality holds

$$\sqrt{N}(\xi_N - \theta^*) \stackrel{d}{\to} \mathcal{N}\left(0, \frac{\alpha^*(1 - \alpha^*)}{p_X^2(\theta^*)}\right),\tag{1}$$

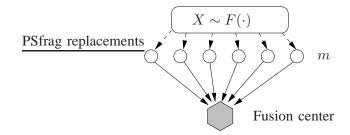


Fig. 1. Sensor network for quantile estimation with m sensors. Each sensor is permitted to transmit a 1-bit message to the fusion center; in turn, the fusion center is permitted to broadcast k bits of feedback.

so that the asymptotic MSE decreases as O(1/N), where N is the total number of samples. Although this 1/N rate is optimal, the precise form of the asymptotic variance (1) need not be in general; see Zielinski [21] for in-depth discussion of the optimal asymptotic variances that can be obtained with variants of this basic estimator under different conditions.

B. Distributed Quantile Estimation

We consider the standard network architecture illustrated in Figure 1. There are m sensors, each of which has a dedicated two-way link to a fusion center. We assume that each sensor $i \in \{1, \ldots, m\}$ collects independent samples X(i) of the random variable $X \in \mathbb{R}$ with distribution function $F(\theta) := \mathbb{P}[X \leq \theta]$. We consider a sequential version of the quantile estimation problem, in which sensor i receives measurements $X_n(i)$ at time steps $n = 0, 1, 2, \ldots$, and the fusion center forms an estimate θ_n of the quantile. The key condition—giving rise to the decentralized nature of the problem—is that communication between each sensor and the central processor is constrained, so that the sensor cannot simply relay its measurement X(i) to the central location, but rather must perform local computation, and then transmit a summary statistic to the fusion center. More concretely, we impose the following restrictions on the protocol. First, at each time step $n = 0, 1, 2, \ldots$, each sensor $i = 1, \ldots, m$ can transmit a single bit $Y_n(i)$ to the fusion center can broadcast k bits back to the sensor nodes at each time step. We analyze two distinct protocols, depending on whether $k = \log(m)$ or k = 1.

C. Protocol specification

For each protocol, all sensors are initialized with some fixed θ_0 . The algorithms are specified in terms of a constant K > 0 and step sizes $\epsilon_n > 0$ that satisfy the conditions

$$\sum_{n=0}^{\infty} \epsilon_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \epsilon_n^2 < \infty.$$
(2)

The first condition ensures infinite travel (i.e., that the sequence θ_n can reach θ^* from any starting condition), whereas the second condition (which implies that $\epsilon_n \to 0$) is required for variance reduction. A standard choice satisfying these conditions—and the one that we assume herein—is $\epsilon_n = 1/n$. With this set-up, the $\log(m)$ -bit scheme consists of the steps given in Table I. Although the most straightforward

Algorithm: Decentralized quantile estimation with log(m)-bit feedback

Given K > 0 and variable step sizes $\epsilon_n > 0$:

(a) Local decision: each sensor computes the binary decision

$$Y_{n+1}(i) \equiv Y_{n+1}(i;\theta_n) := \mathbb{I}(X_{n+1}(i) \le \theta_n),$$
(3)

and transmits it to the fusion center.

(b) Parameter update: the fusion center updates its current estimate θ_{n+1} of the quantile parameter as follows:

$$\theta_{n+1} = \theta_n + \epsilon_n K \left(\alpha^* - \frac{\sum_{i=1}^m Y_{n+1}(i)}{m} \right)$$
(4)

(c) *Feedback:* the fusion broadcasts the *m* received bits $\{Y_{n+1}(1), \ldots, Y_{n+1}(m)\}$ back to the sensors. Each sensor can then compute the updated parameter θ_{n+1} .

TABLE I: Description of the log(m)-bf algorithm.

feedback protocol is to broadcast back the *m* received bits $\{Y_{n+1}(1), \ldots, Y_{n+1}(m)\}$, as described in step (c), in fact it suffices to transmit only the $\log(m)$ bits required to perfectly describe the binomial random variable $\sum_{i=1}^{m} Y_{n+1}(i)$ in order to update θ_n . In either case, after the feedback step, each sensor knows the value of the sum $\sum_{i=1}^{m} Y_{n+1}(i)$, which (in conjunction with knowledge of m, α^* and ϵ_n) allow it to compute the updated parameter θ_{n+1} . Finally, knowledge of θ_{n+1} allows each sensor to then compute the local decision (3) in the following round.

The 1-bit feedback scheme detailed in Table II is similar, except that it requires broadcasting only a single bit (Z_{n+1}) , and involves an extra step size parameter K_m , which is specified in the statement of Theorem 2. After the feedback step of the 1-bf algorithm, each sensor has knowledge of the aggregate decision Z_{n+1} , which (in conjunction with ϵ_n and the constant β) allow it to compute the updated parameter θ_{n+1} . Knowledge of this parameter suffices to compute the local decision (5).

D. Convergence results

We now state our main results on the convergence behavior of these two distributed protocols. In all cases, we assume the step size choice $\epsilon_n = 1/n$. Given fixed $\alpha^* \in (0, 1)$, we use θ^* to denote the

Algorithm: Decentralized quantile estimation with 1-bit feedback

Given $K_m > 0$ (possibly depending on number of sensors m) and variable step sizes $\epsilon_n > 0$:

(a) Local decision: each sensor computes the binary decision

$$Y_{n+1}(i) = \mathbb{I}(X_{n+1}(i) \le \theta_n) \tag{5}$$

and transmits it to the fusion center.

(b) Aggregate decision and parameter update: The fusion center computes the aggregate decision

$$Z_{n+1} = \mathbb{I}\left(\frac{\sum_{i=1}^{m} Y_{n+1}(i)}{m} \le \alpha^*\right),\tag{6}$$

and uses it update the parameter according to

$$\theta_{n+1} = \theta_n + \epsilon_n K_m \left(Z_{n+1} - \beta \right) \tag{7}$$

where the constant β is chosen as

$$\beta = \sum_{i=0}^{\lfloor m\alpha^* \rfloor} {m \choose i} (\alpha^*)^i (1-\alpha^*)^{m-i}.$$
(8)

(c) *Feedback:* The fusion center broadcasts the aggregate decision Z_{n+1} back to the sensor nodes (one bit of feedback). Each sensor can then compute the updated parameter θ_{n+1} .

TABLE II: Description of the 1-bf algorithm.

 α^* -level quantile (i.e., such that $\mathbb{P}(X \leq \theta^*) = \alpha^*$); note that our assumption of a strictly positive density guarantees that θ^* is unique.

Theorem 1 (m-bit feedback): For any $\alpha^* \in (0,1)$, consider a random sequence $\{\theta_n\}$ generated by the m-bit feedback protocol. Then

- (a) For all initial conditions θ_0 , the sequence θ_n converges almost surely to the α^* -quantile θ^* .
- (b) Moreover, if the constant K is chosen to satisfy $p_X(\theta^*) K > \frac{1}{2}$, then

$$\sqrt{n} \left(\theta_n - \theta^*\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{K^2 \alpha^* \left(1 - \alpha^*\right)}{\left[2Kp_X(\theta^*) - 1\right]} \frac{1}{m}\right),\tag{9}$$

so that the asymptotic MSE is $O(\frac{1}{mn})$.

Remarks: After *n* steps of this decentralized protocol, a total of N = nm observations have been made, so that our discussion in Section II-A dictates (see equation (1)) that the optimal asymptotic MSE is $O(\frac{1}{nm})$. Interestingly, then, the $\log(m)$ -bit feedback decentralized protocol is order-optimal with respect to the centralized gold standard.

Before stating the analogous result for the 1-bit feedback protocol, we begin by introducing some useful notation. First, we define for any fixed $\theta \in \mathbb{R}$ the random variable

$$\bar{Y}(\theta) := \frac{1}{m} \sum_{i=1}^m Y(i;\theta) = \frac{1}{m} \sum_{i=1}^m \mathbb{I}(X(i) \le \theta).$$

Note that for each fixed θ , the distribution of $\overline{Y}(\theta)$ is binomial with parameters m and $F(\theta)$. It is convenient to define the function

$$G_m(r,y) := \sum_{i=0}^{\lfloor my \rfloor} {m \choose i} r^i (1-r)^{m-i},$$
(10)

with domain $(r, y) \in [0, 1] \times [0, 1]$. With this notation, we have

$$\mathbb{P}(\bar{Y}(\theta) \le y) = G_m(F(\theta), y).$$

Again, we fix an arbitrary $\alpha^* \in (0,1)$ and let θ^* be the associated α^* -quantile satisfying $\mathbb{P}(X \le \theta^*) = \alpha^*$.

Theorem 2 (1-bit feedback): Given a random sequence $\{\theta_n\}$ generated by the 1-bit feedback protocol, we have

(a) For any initial condition, the sequence $\theta_n \xrightarrow{a.s.} \theta^*$.

(b) Suppose that the step size K_m is chosen such that $K_m > \frac{\sqrt{2\pi\alpha^*(1-\alpha^*)}}{2p_X(\theta^*)\sqrt{m}}$, or equivalently such that

$$\gamma_m(\theta^*) := K_m \Big| \frac{\partial G_m}{\partial r}(r; \alpha^*) \Big|_{r=\alpha^*} \Big| p_X(\theta^*) > \frac{1}{2}, \tag{11}$$

then

$$\sqrt{n} \left(\theta_n - \theta^*\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{K_m^2 G_m(\alpha^*, \theta^*) \left[1 - G_m(\alpha^*, \theta^*)\right]}{2\gamma_m(\theta^*) - 1}\right)$$
(12)

(c) If we choose a *constant step size* $K_m = K$, then as $n \to \infty$, the asymptotic variance behaves as

$$\left[\frac{K^2\sqrt{2\pi\alpha^*(1-\alpha^*)}}{8Kp_X(\theta^*)\sqrt{m}-4\sqrt{2\pi\alpha^*(1-\alpha^*)}}\right],\tag{13}$$

so that the asymptotic MSE is $O\left(\frac{1}{n\sqrt{m}}\right)$. (d) If we choose a *decaying step size* $K_m = \frac{K}{\sqrt{m}}$, then

$$\frac{1}{m} \left[\frac{K^2 \sqrt{2\pi \alpha^* (1 - \alpha^*)}}{8Kp_X(\theta^*) - 4\sqrt{2\pi \alpha^* (1 - \alpha^*)}} \right],$$
(14)

so that the asymptotic MSE is $O\left(\frac{1}{nm}\right)$.

E. Comparative Analysis

It is interesting to compare the performance of each proposed decentralized algorithm to the centralized performance. Considering first the $\log(m)$ -bf scheme, suppose that we set $K = 1/p_X(\theta^*)$. Using the formula (9) from Theorem 1, we obtain that the asymptotic variance of the *m*-bf scheme with this choice of *K* is given by $\frac{\alpha^*(1-\alpha^*)}{p_X^2(\theta^*)} \frac{1}{mn}$, thus matching the asymptotics of the centralized quantile estimator (1). In fact, it can be shown that the choice $K = 1/p_X(\theta^*)$ is optimal in the sense of minimizing the asymptotic variance for our scheme, when *K* is constrained by the stability criterion in Theorem 1. In practice, however, the value $p_X(\theta^*)$ is typically not known, so that it may not be possible to implement exactly this scheme. An interesting question is whether an adaptive scheme could be used to estimate $p_X(\theta^*)$ (and hence the optimal *K* simultaneously), thereby achieving this optimal asymptotic variance. We leave this question open as an interesting direction for future work.

Turning now to the algorithm 1-bf, if we make the substitution $\bar{K} = K/\sqrt{2\pi\alpha^*(1-\alpha^*)}$ in equation (14), then we obtain the asymptotic variance

$$\frac{\pi}{2} \frac{\bar{K}^2 \,\alpha^* \left(1 - \alpha^*\right)}{\left[2\bar{K}p_X(\theta^*) - 1\right]} \,\frac{1}{m}.$$
(15)

Since the stability criterion is the same as that for *m*-bf, the optimal choice is $\bar{K} = 1/p_X(\theta^*)$. Consequently, while the (1/[mn]) rate is the same as both the centralized and decentralized *m*-bf protocols, the pre-factor for the 1-bf algorithm is $\frac{\pi}{2} \approx 1.57$ times larger than the optimized *m*-bf scheme. However, despite this loss in the pre-factor, the 1-bf protocol has substantial advantages over the *m*-bf; in particular, the network lifetime scales as O(m) compared to $O(m/\log(m))$ for the $\log(m)$ -bf scheme.

F. Simulation example

We now provide some simulation results in order to illustrate the two decentralized protocols, and the agreement between theory and practice. In particular, we consider the quantile estimation problem when the underlying distribution (which, of course, is unknown to the algorithm) is uniform on [0, 1]. In this case, we have $p_X(x) = 1$ uniformly for all $x \in [0, 1]$, so that taking the constant K = 1 ensures that the stability conditions in both Theorem 1 and 2 are satisfied. We simulate the behavior of both algorithms for $\alpha^* = 0.3$ over a range of choices for the network size m. Figure 2(a) illustrates several sample paths of m-bit feedback protocol, showing the convergence to the correct θ^* .

For comparison to our theory, we measure the empirical variance by averaging the error $\hat{e}_n = \sqrt{n}(\theta_n - \theta^*)$ over L = 20 runs. The normalization by \sqrt{n} is used to isolate the effect of increasing m, the number of nodes in the network. We estimate the variance by running algorithm for n = 2000 steps, and computing

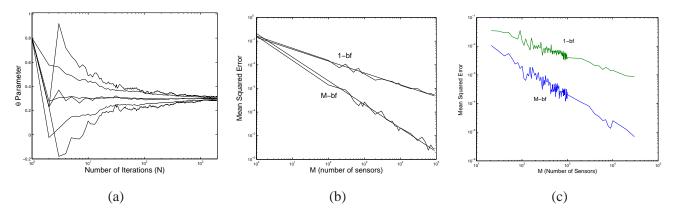


Fig. 2. Convergence of θ_n to θ^* with m = 11 nodes, and quantile level $\alpha^* = 0.3$. (b) Log-log plots of the variance against m for both algorithms (log(m)-bf and 1-bf) with constant step sizes, and comparison to the theoretically-predicted rate (solid straight lines). (c) Log-log plots of log(m)-bf with constant step size versus 1-bf algorithm with decaying step size.

the empirical variance of \hat{e}_n for time steps n = 1800 through to n = 2000. Figure 2(b) shows these empirically computed variances, and a comparison to the theoretical predictions of Theorems 1 and 2 for constant step size; note the excellent agreement between theory and practice. Panel (c) shows the comparison between the $\log(m)$ -bf algorithm, and the 1-bf algorithm with decaying $1/\sqrt{m}$ step size. Here the asymptotic MSE of both algorithms decays like 1/m for $\log m$ up to roughly 500; after this point, our fixed choice of n is insufficient to reveal the asymptotic behavior.

III. ANALYSIS

In this section, we turn to the proofs of Theorem 1 and 2, which exploit results from the stochastic approximation literature [12], [3]. In particular, both types of parameter updates (4) and (7) can be written in the general form

$$\theta_{n+1} = \theta_n + \epsilon_n H(\theta_n, Y_{n+1}), \tag{16}$$

where $Y_{n+1} = (Y_{n+1}(1), \dots, Y_{n+1}(m))$. Note that the step size choice $\epsilon_n = 1/n$ satisfies the conditions in equation (2). Moreover, the sequence (θ_n, Y_{n+1}) is Markov, since θ_n and Y_{n+1} depend on the past only via θ_{n-1} and Y_n . We begin by stating some known results from stochastic approximation, applicable to such Markov sequences, that will be used in our analysis.

For each fixed $\theta \in \mathbb{R}$, let $\mu_{\theta}(\cdot)$ denote the distribution of Y conditioned on θ . A key quantity in the analysis of stochastic approximation algorithms is the averaged function

$$h(\theta) := \int H(\theta, y) \mu_{\theta}(dy) = \mathbb{E} \left[H(\theta, Y) \mid \theta \right].$$
(17)

We assume (as is true for our cases) that this expectation exists. Now the differential equation method dictates that under suitable conditions, the asymptotic behavior of the update (16) is determined essentially by the behavior of the ODE $\frac{d\theta}{dt} = h(\theta(t))$.

Almost sure convergence: Suppose that the following attractiveness condition

$$h(\theta) \left[\theta - \theta^*\right] < 0 \quad \text{for all } \theta \neq \theta^*$$
 (18)

is satisfied. If, in addition, the variance $R(\theta) := \operatorname{Var}[H(\theta; Y) \mid \theta]$ is bounded, then we are guaranteed that $\theta_n \xrightarrow{a.s.} \theta^*$ (see §5.1 in Benveniste et al. [3]).

Asymptotic normality: In our updates, the random variables Y_n take the form $Y_n = g(X_n, \theta_n)$ where the X_n are i.i.d. random variables. Suppose that the following stability condition is satisfied:

$$\gamma(\theta^*) := -\frac{dh}{d\theta}(\theta^*) > \frac{1}{2}.$$
(19)

Then we have

$$\sqrt{n} (\theta_n - \theta^*) \xrightarrow{d} \mathcal{N}\left(0, \frac{R(\theta^*)}{2\gamma(\theta^*) - 1}\right)$$
 (20)

See §3.1.2 in Benveniste et al. [3] for further details.

A. Proof of Theorem 1

(a) The *m*-bit feedback algorithm is a special case of the general update (16), with $\epsilon_n = \frac{1}{n}$ and $H(\theta_n, Y_{n+1}) = K \left[\alpha^* - \frac{1}{m} \sum_{i=1}^m Y_{n+1}(i; \theta_n) \right]$. Computing the averaged function (17), we have

$$h(\theta) = K\mathbb{E}\left[\alpha^* - \frac{1}{m}\sum_{i=1}^m Y_{n+1}(i) \mid \theta_n\right]$$
$$= K\left(\alpha^* - F(\theta_n)\right),$$

where $F(\theta_n) = \mathbb{P}(X \leq \theta_n)$. We then observe that θ^* satisfies the attractiveness condition (18), since

$$\left[\theta - \theta^*\right] h(\theta_n) = K \left[\theta - \theta^*\right] \left[\alpha^* - F(\theta_n)\right] < 0$$

for all $\theta \neq \theta^*$, by the monotonicity of the cumulative distribution function. Finally, we compute the conditional variance of *H* as follows:

$$R(\theta_n) = K^2 \operatorname{Var} \left[\alpha^* - \frac{\sum_{i=1}^m Y_{n+1}(i)}{m} \mid \theta_n \right]$$
$$= \frac{K^2}{m} F(\theta_n) \left[1 - F(\theta_n) \right] \le \frac{K^2}{4m},$$
(21)

using the fact that H is a sum of m Bernoulli variables that are conditionally i.i.d. (given θ_n). Thus, we can conclude that $\theta_n \to \theta^*$ almost surely.

(b) Note that $\gamma(\theta^*) = -\frac{dh}{d\theta}(\theta^*) = Kp_X(\theta^*) > \frac{1}{2}$, so that the stability condition (19) holds. Applying the asymptotic normality result (20) with the variance $R(\theta^*) = \frac{K^2}{m}\alpha^*(1-\alpha^*)$ (computed from equation (21)) yields the claim.

B. Proof of Theorem 2

This argument involves additional analysis, due to the aggregate decision (6) taken by the fusion center. Since the decision Z_{n+1} is a Bernoulli random variable; we begin by computing its parameter. Each transmitted bit $Y_{n+1}(i)$ is $Ber(F(\theta_n))$, where we recall the notation $F(\theta) := \mathbb{P}(X \le \theta)$. Using the definition (10), we have the equivalences

$$\mathbb{P}(Z_{n+1}=1) = G_m(F(\theta_n), \alpha^*)$$
(22a)

$$\beta = G_m(\alpha^*, \alpha^*) = G_m(F(\theta^*), \alpha^*).$$
(22b)

We start with the following result:

Lemma 1: For fixed $x \in [0,1]$, the function $f(r) := G_m(r,x)$ is non-negative, differentiable and monotonically decreasing.

Proof: Non-negativity and differentiability are immediate. To establish monotonicity, note that $f(r) = \mathbb{P}(\sum_{i=1}^{m} Y_i \leq xm)$, where the Y_i are i.i.d. Ber(r) variates. Consider a second Ber(r') sequence Y'_i with r' > r. Then the sum $\sum_{i=1}^{m} Y'_i$ stochastically dominates $\sum_{i=1}^{m} Y_i$, so that f(r) < f(r') as required.

To establish almost sure convergence, we use a similar approach as in the previous theorem. Using the equivalences (22), we compute the function h as follows

$$h(\theta) = K_m \mathbb{E} \left[Z_{n+1} - \beta \mid \theta \right]$$
$$= K_m \left[G_m(F(\theta), \alpha^*) - G_m(F(\theta^*), \alpha^*) \right]$$

Next we establish the attractiveness condition (18). In particular, for any θ such that $F(\theta) \neq F(\theta^*)$, we calculate that $h(\theta) \ [\theta - \theta^*]$ is given by

$$K_m \Big\{ G_m(F(\theta_n), \alpha^*) - G_m(F(\theta^*), \alpha^*) \Big\} [\theta_n - \theta^*] < 0,$$

where the inequality follows from the fact that $G_m(r, x)$ is monotonically decreasing in r for each fixed $x \in [0, 1]$ (using Lemma 1), and that the function F is monotonically increasing. Finally, computing the variance $R(\theta) := \operatorname{Var} [H(\theta, Y) \mid \theta]$, we have

$$R(\theta) = K_m^2 G_m(F(\theta), \alpha^*) [1 - G_m(F(\theta), \alpha^*)] \le \frac{K_m^2}{4}$$

11

since (conditioned on θ), the decision Z_{n+1} is Bernoulli with parameter $G_m(F(\theta); \alpha^*)$. Thus, we can conclude that $\theta_n \to \theta^*$ almost surely.

(b) To show asymptotic normality, we need to verify the stability condition. By chain rule, we have $\frac{h}{d\theta}(\theta^*) = K_m \frac{\partial G_m}{\partial r}(r, \alpha^*) \Big|_{r=F(\theta)} p_X(\theta)$. From Lemma 1, we have $\frac{\partial G_m}{\partial r}(F(\theta), \alpha^*) < 0$, so that the stability condition holds as long as $\gamma_m(\theta^*) > \frac{1}{2}$ (where γ_m is defined in the statement). Thus, asymptotic normality holds.

In order to compute the asymptotic variance, we need to investigate the behavior of $R(\theta^*)$ and $\gamma(\theta^*)$ as $m \to +\infty$. First examining $R(\theta^*)$, the central limit theorem guarantees that $G_m(F(\theta^*), y) \to \Phi\left(\sqrt{m}\frac{y-\alpha^*}{\alpha^*(1-\alpha^*)}\right)$. Consequently, we have

$$R(\theta^*) = K_m^2 G_m(F(\theta^*), \alpha^*) [1 - G_m(F(\theta^*), \alpha^*)] \to \frac{K_m^2}{4}.$$

We now turn to the behavior of $\gamma(\theta^*)$. We first prove a lemma to characterize the asymptotic behavior of $G_m(r, \alpha^*)$:

Lemma 2: (a) The partial derivative of $G_m(r, x)$ with respect to r is given by:

$$\frac{\partial G_m(r,x)}{\partial r} = \frac{\mathbb{E}[X\mathbb{I}(X \le xm)] - \mathbb{E}[X]\mathbb{E}[\mathbb{I}(X \le xm)]}{r(1-r)},$$
(23)

where X is binomial with parameters (m, x), and mean $\mathbb{E}[X] = xm$.

(b) Moreover, as $m \to +\infty$, we have

$$\frac{\partial G_m(r,\alpha^*)}{\partial r}\Big|_{r=F(\theta^*)} \to -\sqrt{\frac{m}{2\pi\alpha^*(1-\alpha^*)}}.$$

Proof: (a) Computing the partial derivative, we have

$$\begin{aligned} \frac{\partial G_m(r,x)}{\partial r} &= \sum_{i=0}^{\lfloor m\alpha^* \rfloor} \binom{m}{i} \left[ir^{i-1}(1-r)^{m-i} - (m-i)r^i(1-r)^{m-i-1} \right] \\ &= \frac{1}{r(1-r)} \sum_{i=0}^{\lfloor mx \rfloor} \binom{m}{i} (i-mr)r^i(1-r)^{m-i} \\ &= \frac{1}{r(1-r)} \left(\sum_{i=0}^{\lfloor mx \rfloor} \binom{m}{i} r^i(1-r)^{m-i} - mr \sum_{i=0}^{\lfloor mx \rfloor} \binom{m}{i} r^i(1-r)^{m-i} \right) \\ &= \frac{1}{r(1-r)} \left(\mathbb{E}[X\mathbb{I}(X \le mx)] - \mathbb{E}[X]\mathbb{E}[\mathbb{I}(X \le mx)] \right), \end{aligned}$$

as claimed.

(b) We derive this limiting behavior by applying classical asymptotics to the form of $\frac{\partial G_m(r,\alpha^*)}{\partial r}$ given in

part (a). Defining $Z_m = \frac{X - \alpha^* m}{\sqrt{m}}$, the central limit theorem yields that:

$$Z_m \stackrel{d}{\to} Z \sim N(0, a)$$

$$a := \alpha^* (1 - \alpha^*)$$
(24)

Moreover, in this binomial case, we actually have $\mathbb{E}[|Z_m|] \to \mathbb{E}[|Z|] = \sqrt{\frac{2a}{\pi}}$.

First, since $\mathbb{E}[X] = \alpha^* m$ and $\mathbb{E}[\mathbb{I}(X \le \alpha^* m)] \to \frac{1}{2}$ by the CLT, we have

$$\mathbb{E}[X] \mathbb{E}[\mathbb{I}(X \le \alpha^* m)] \to \frac{\alpha^* m}{2}.$$
(25)

Let us now re-write the first term in the representation (23) of $\frac{\partial G_m(r,\alpha^*)}{\partial r}$ as

$$\mathbb{E}[X\mathbb{I}(X \le \alpha^* m)] = \alpha^* m \mathbb{E}[\mathbb{I}(X \le \alpha^* m)] + \sqrt{m} \mathbb{E}[Z_m \mathbb{I}(Z_m \le 0)]$$

$$\rightarrow \frac{\alpha^* m}{2} - \sqrt{m} \sqrt{\frac{a}{2\pi}}$$
(26)

since $\mathbb{E}[\mathbb{I}(X \leq \alpha^* m)] \to 1/2$ and

$$\mathbb{E}[Z_m \,\mathbb{I}(Z_m \le 0)] \ \to \ \mathbb{E}[Z\mathbb{I}(Z \le 0)] = \frac{1}{2}\mathbb{E}[|Z|] = \sqrt{\frac{a}{2\pi}}$$

Putting together the limits (25) and (26), we conclude that $\frac{\partial G_m(r,\alpha^*)}{\partial r}\Big|_{r=\alpha^*}$ converges to

$$\frac{1}{\alpha^*(1-\alpha^*)} \left[\left\{ \frac{\alpha^*m}{2} - \sqrt{m}\sqrt{\frac{\alpha^*(1-\alpha^*)}{2\pi}} \right\} - \frac{\alpha^*m}{2} \right] = -\sqrt{\frac{m}{2\pi\alpha^*(1-\alpha^*)}}$$

as claimed.

Returning now to the proof of the theorem, we use Lemma 2 and put the pieces together to obtain that $\frac{R(\theta^*)}{2K_m \left| \frac{\partial G_m(r,\theta^*)}{\partial r} \right|_{r=0^*} \left| p_X(\theta^*) - 1 \right|}$ converges to

$$\frac{K_m^2/4}{\frac{2K_m\sqrt{m}p_X(\theta^*)}{\sqrt{2\pi\alpha^*(1-\alpha^*)}} - 1} = \frac{1}{m} \left[\frac{K^2\sqrt{2\pi\alpha^*(1-\alpha^*)}}{8Kp_X(\theta^*) - 4\sqrt{2\pi\alpha^*(1-\alpha^*)}} \right]$$

with $K > \frac{\sqrt{2\pi\alpha^*(1-\alpha^*)}}{2p_X(\theta^*)}$ for stability, thus completing the proof of the theorem.

IV. Some extensions

In this section, we consider some extensions of the algorithms and analysis from the preceding sections, including variations in the number of feedback bits, and the effects of noise.

A. Different levels of feedback

We first consider the generalization of the preceding analysis to the case when the fusion center some number of bits between 1 and m. The basic idea is to apply a quantizer with 2ℓ levels, corresponding to $\log_2(2\ell)$ bits, on the update of the stochastic gradient algorithm. Note that the extremes $\ell = 1$ and $\ell = 2^{m-1}$ correspond to the previously studied protocols. Given 2ℓ levels, we partition the real line as

$$-\infty = s_{-\ell} < s_{-\ell+1} < \dots < s_{\ell-1} < s_{\ell} = +\infty,$$
(27)

where the remaining breakpoints $\{s_k\}$ are to be specified. With this partition fixed, we define a quantization function Q_ℓ

$$\mathcal{Q}_{\ell}(X) := r_k \quad \text{if } X \in (s_k, s_{k+1}] \text{ for } k = -\ell, \dots, \ell - 1,$$
(28)

where the 2ℓ quantized values $(r_{-\ell}, \ldots, r_{\ell-1})$ are to be chosen. In the setting of the algorithm to be proposed, the quantizer is applied to binomial random variables X with parameters (m, r). Recall the function $G_m(r, x)$, as defined in equation (10), corresponding to the probability $\mathbb{P}[X \leq mx]$. Let us define a new function $G_{m,\ell}$, corresponding to the expected value of the quantizer when applied to such a binomial variate, as follows

$$G_{m,\ell}(r,x) := \sum_{k=-\ell}^{\ell-1} r_k \left\{ G_m(r,x-s_k) - G_m(r,x-s_{k+1}) \right\}.$$
 (29)

With these definitions, the general $\log_2(2\ell)$ feedback algorithm takes the form shown in Table III.

In order to understand the choice of the offset parameter β defined in equation (33), we compute the expected value of the quantizer function, when $\theta_n = \theta^*$, as follows

$$\mathbb{E}\Big[\mathcal{Q}_{\ell}\left[\alpha^{*} - \frac{\sum_{i=1}^{m} Y_{n+1}(i)}{m}\right] \mid \theta_{n} = \theta^{*}\Big] = \sum_{k=-\ell}^{\ell-1} r_{k} \mathbb{P}\left[(\alpha^{*} - s_{k+1}) < \frac{\bar{Y}(\theta^{*})}{m} \le (\alpha^{*} - s_{k})\right]$$
$$= \sum_{k=-\ell}^{\ell-1} r_{k} \left[G_{m}(F(\theta^{*}), \alpha^{*} - s_{k}) - G_{m}(F(\theta^{*}), \alpha^{*} - s_{k+1})\right]$$
$$= G_{m,\ell}(F(\theta^{*}), \alpha^{*}).$$

The following result, analogous to Theorem 2, characterizes the behavior of this general protocol:

Theorem 3 (General feedback scheme): Given a random sequence $\{\theta_n\}$ generated by the general $\log_2(2\ell)$ bit feedback protocol, there exist choices of partition $\{s_k\}$ and quantization levels $\{r_k\}$ such that:

(a) For any initial condition, the sequence $\theta_n \xrightarrow{a.s.} \theta^*$.

Algorithm: Decentralized quantile estimation with $\log_2(2\ell)$ -bits feedback

Given $K_m > 0$ (possibly depending on number of sensors m) and variable step sizes $\epsilon_n > 0$:

(a) Local decision: each sensor computes the binary decision

$$Y_{n+1}(i) = \mathbb{I}(X_{n+1}(i) \le \theta_n) \tag{30}$$

and transmits it to the fusion center.

(b) Aggregate decision and parameter update: The fusion center computes the quantized aggregate decision variable

$$Z_{n+1} = Q_{\ell} \left[\alpha^* - \frac{\sum_{i=1}^m Y_{n+1}(i)}{m} \right],$$
(31)

and uses it update the parameter according to

$$\theta_{n+1} = \theta_n + \epsilon_n K_m \left(Z_{n+1} - \beta \right) \tag{32}$$

where the constant β is chosen as

$$\beta := G_{m,\ell}(F(\theta^*), \alpha^*). \tag{33}$$

(c) *Feedback:* The fusion center broadcasts the aggregate quantized decision Z_{n+1} back to the sensor nodes, using its $\log_2(2\ell)$ bits of feedback. The sensor nodes can then compute the updated parameter θ_{n+1} .

TABLE III: Description of the general algorithm, with $\log_2(2\ell)$ bits of feedback.

(b) There exists a choice of *decaying step size* (i.e., $K_m \simeq \frac{1}{\sqrt{m}}$) such that the asymptotic variance of the protocol is given by $\frac{\kappa(\alpha^*, Q_\ell)}{mn}$, where the constant has the form

$$\kappa(\alpha^*, \mathcal{Q}_{\ell}) := 2\pi \frac{\sum_{k=-\ell}^{\ell-1} r_k^2 \Delta G_m(s_k, s_{k+1}) - \beta^2}{\left(\sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k, s_{k+1})\right)^2},$$
(34)

with

$$\Delta G_m(s_k, s_{k+1}) = G_m(F(\theta^*), \alpha^* - s_k) - G_m(F(\theta^*), \alpha^* - s_{k+1}), \text{ and } (35a)$$

$$\Delta_m(s_k, s_{k+1}) = \exp\left(-\frac{ms_k^2}{2\alpha^*(1-\alpha^*)}\right) - \exp\left(-\frac{ms_{k+1}^2}{2\alpha^*(1-\alpha^*)}\right).$$
(35b)

We provide a formal proof of Theorem 3 in the Appendix. Figure 3(a) illustrates how the constant factor κ , as defined in equation (34) decreases as the number of levels ℓ in an uniform quantizer is increased.

In order to provide comparison with results from the previous section, let us see how the two extreme cases (1 bit and m feedback) can be obtained as special case. For the 1-bit case, the quantizer has $\ell = 1$ levels with breakpoints $s_{-1} = -\infty$, $s_0 = 0$, $s_1 = +\infty$, and quantizer outputs $r_{-1} = 0$ and $r_1 = 1$. By

making the appropriate substitutions, we obtain:

$$\kappa(\alpha^*, Q_1) = 2\pi \frac{\Delta G_m(s_0, s_1) - \beta^2}{\Delta_m(s_0, s_1)}, \qquad \beta^2 = G_{m,\ell}(F(\theta^*), \alpha^*)^2$$

$$\Delta G_m(s_0, s_1) = G_{m,\ell}(F(\theta^*), \alpha^*) \qquad \text{and} \qquad \Delta_m(s_0, s_1)) = 1.$$

By applying the central limit theorem, we conclude that

$$\Delta G_m(s_0, s_1) - \beta^2 = G_{m,\ell}(F(\theta^*), \alpha^*)(1 - G_{m,\ell}(F(\theta^*), \alpha^*)) \to 1/4,$$

as established earlier. Thus $\kappa(\alpha^*, Q_1) \to \pi/2$ as $m \to \infty$, recovering the result of Theorem 2. Similarly, the results for *m*-bf can be recovered by setting the parameters

$$r_{k-\ell} = \alpha^* - \frac{m-k}{m}, \quad \text{for} \quad k = 0, ..., m, \quad \text{and}$$
$$s_i = r_i. \tag{36}$$

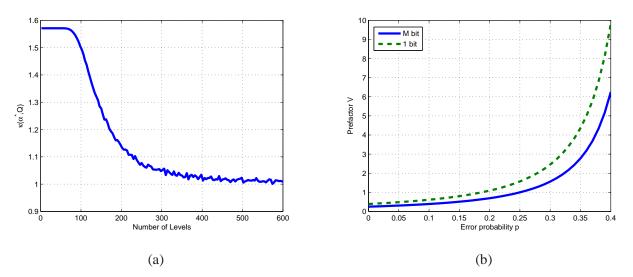


Fig. 3. (a) Plots of the asymptotic variance $\kappa(\alpha^*, Q_\ell)$ defined in equation (34) versus the number of levels ℓ in a uniform quantizer, corresponding to $\log_2(2\ell)$ bits of feedback, for a sensor network with m = 4000 nodes. The plots show the asymptotic variance rescaled by the centralized gold standard, so that it starts at $\pi/2$ for $\ell = 2$, and decreases towards 1 as ℓ is increased towards m/2. (b) Plots of the asymptotic variances $V_m(\epsilon)$ and $V_1(\epsilon)$ defined in equation (39) as the feedforward noise parameter ϵ is increased from 0 towards $\frac{1}{2}$.

B. Extensions to noisy links

We now briefly consider the effect of communication noise on our algorithms. There are two types of noise to consider: (a) *feedforward*, meaning noise in the link from sensor node to fusion center, and (b)

feedback, meaning noise in the feedback link from fusion center to the sensor nodes. Here we show that feedforward noise can be handled in a relatively straightforward way in our algorithmic framework. On the other hand, feedback noise requires a different analysis, as the different sensors may loose synchronicity in their updating procedure. Although a thorough analysis of such asynchronicity is an interesting topic for future research, we note that assuming noiseless feedback is not unreasonable, since the fusion center typically has greater transmission power.

Focusing then on the case of feedforward noise, let us assume that the link between each sensor and the fusion center acts as a binary symmetric channel (BSC) with probability $\epsilon \in [0, \frac{1}{2})$. More precisely, if a bit $x \in \{0, 1\}$ is transmitted, then the received bit y has the (conditional) distribution

$$\mathbb{P}(y \mid x) = \begin{cases} 1 - \epsilon & \text{if } x = y \\ \epsilon & \text{if } x \neq y. \end{cases}$$
(37)

With this bit-flipping noise, the updates (both equation (4) and (7)) need to be modified so as to correct for the bias introduced by the channel noise. If α^* denotes the desired quantile, then in the presence of BSC(ϵ) noise, both algorithms should be run with the modified parameter

$$\widetilde{\alpha}(\epsilon) := (1 - 2\epsilon)\alpha^* + \epsilon.$$
(38)

Note that $\tilde{\alpha}(\epsilon)$ ranges between α^* (for the noiseless case $\epsilon = 0$), to a quantity arbitrarily close to $\frac{1}{2}$, as the channel approaches the extreme of pure noise ($\epsilon = \frac{1}{2}$). The following lemma shows that for all $\epsilon < \frac{1}{2}$, this adjustment (38) suffices to correct the algorithm. Moreover, it specifies how the resulting asymptotic variance depends on the noise parameter:

Proposition 1: Suppose that each of the m feedforward links from sensor to fusion center are modeled as i.i.d. BSC channels with probability $\epsilon \in [0, \frac{1}{2})$. Then the m-bf or 1-bf algorithms, with the adjusted $\tilde{\alpha}(\epsilon)$, are strongly consistent in computing the α^* -quantile. Moreover, with appropriate step size choices, their asymptotic MSEs scale as 1/(mn) with respective pre-factors given by

$$V_m(\epsilon) := \frac{K^2 \,\widetilde{\alpha}(\epsilon) \,(1 - \widetilde{\alpha}(\epsilon))}{\left[2K(1 - 2\epsilon)p_X(\theta^*) - 1\right]} \tag{39a}$$

$$V_{1}(\epsilon) := \left[\frac{K^{2}\sqrt{2\pi\widetilde{\alpha}(\epsilon)(1-\widetilde{\alpha}(\epsilon))}}{8K(1-2\epsilon)p_{X}(\theta^{*}) - 4\sqrt{2\pi\widetilde{\alpha}(\epsilon)(1-\widetilde{\alpha}(\epsilon))}}\right].$$
(39b)

In both cases, the asymptotic MSE is minimal for $\epsilon = 0$.

Proof: If sensor node *i* transmits a bit $Y_{n+1}(i)$ at round n+1, then the fusion center receives the random variable

$$Y_{n+1}(i) = Y_{n+1}(i) \oplus W_{n+1},$$

where W_{n+1} is Bernoulli with parameter ϵ , and \oplus denotes addition modulo two. Since W_{n+1} is independent of the transmitted bit (which is Bernoulli with parameter $F(\theta_n)$), the received value $\tilde{Y}_{n+1}(i)$ is also Bernoulli, with parameter

$$\epsilon * F(\theta_n) = \epsilon (1 - F(\theta_n)) + (1 - \epsilon) F(\theta_n) = \epsilon + (1 - 2\epsilon) F(\theta_n).$$
(40)

Consequently, if we set $\tilde{\alpha}(\epsilon)$ according to equation (38), both algorithms will have their unique fixed point when $F(\theta) = \alpha^*$, so will compute the α^* -quantile of X. The claimed form of the asymptotic variances follows from by performing calculations analogous to the proofs of Theorems 1 and 2. In particular, the partial derivative with respect to θ now has a multiplicative factor $(1 - 2\epsilon)$, arising from equation (40) and the chain rule. To establish that the asymptotic variance is minimized at $\epsilon = 0$, it suffices to note that the derivative of the MSE with respect to ϵ is positive, so that it is an increasing function of ϵ .

Of course, both the algorithms will fail, as would be expected, if $\epsilon = 1/2$ corresponding to pure noise. However, as summarized in Proposition 1, as long as $\epsilon < \frac{1}{2}$, feedforward noise does not affect the asymptotic rate itself, but rather only the pre-factor in front of the 1/(mn) rate. Figure 3(b) shows how the asymptotic variances $V_m(\epsilon)$ and $V_1(\epsilon)$ behave as ϵ is increased towards $\epsilon = \frac{1}{2}$.

V. DISCUSSION

In this paper, we have proposed and analyzed different approaches to the problem of decentralized quantile estimation under communication constraints. Our analysis focused on the fusion-centric architecture, in which a set of m sensor nodes each collect an observation at each time step. After n rounds of this process, the centralized oracle would be able to estimate an arbitrary quantile with mean-squared error of the order $\mathcal{O}(1/(mn))$. In the decentralized formulation considered here, each sensor node is allowed to transmit only a single bit of information to the fusion center. We then considered a range of decentralized algorithms, indexed by the number of feedback bits that the fusion center is allowed to transmit back to the sensor nodes. In the simplest case, we showed that an log m-bit feedback algorithm achieves the same asymptotic variance $\mathcal{O}(1/(mn))$ as the centralized estimator. More interestingly, we also showed that that a 1-bit feedback scheme, with suitably designed step sizes, can also achieve the same asymptotic variance as the centralized oracle. We also showed that using intermediate amounts of feedback (between 1 and m bits) does not alter the scaling behavior, but improves the constant. Finally, we showed how our algorithm can be adapted to the case of noise in the feedforward links from sensor nodes to fusion center, and the resulting effect on the asymptotic variance.

Our analysis in the current paper has focused only on the fusion center architecture illustrated in Figure 1. A natural generalization is to consider a more general communication network, specified by an undirected graph on the sensor nodes. One possible formulation is to allow only pairs of sensor nodes connected by an edge in this communication graph to exchange a bit of information at each round. In this framework, the problem considered in this paper effectively corresponds to the complete graph, in which every node communicates with every other node at each round. This more general formulation raises interesting questions as to the effect of graph topology on the achievable rates and asymptotic variances.

Acknowledgements

We would like to thank Prof. Pravin Varaiya for some discussion that led to the initial ideas on this subject. RR was supported by the California Department of Transportation through the California PATH program. MJW was partially supported by NSF grant DMS-0605165 and an NSF CAREER award CCF-0545862.

APPENDIX

Proof of Theorem 3:

We proceed in an analogous manner to the proof of Theorem 1:

Lemma 3: For fixed $x \in [0,1]$, the function $G_{m,\ell}(r,x)$ is non-negative, differentiable and monotonically decreasing.

Proof: First notice that by definition:

$$G_{m,\ell}(r,x) = \mathbb{E}\Big[\mathcal{Q}_\ell\left[x - \frac{X}{m}\right]\Big],\tag{41}$$

where X is a Bin(r,m) random variable. Note that if $X' \sim Bin(r',m)$, with r' > r, then certainly $\mathbb{P}(X' \leq n) \leq \mathbb{P}(X \leq n)$ —meaning that X' stochastically dominates X. For any constant x, $\mathbb{P}(x - \frac{X'}{m} \leq s) \geq \mathbb{P}(x - \frac{X}{m} \leq s)$. Furthermore, by the quantizer is, by definition, a monotonically nondecreasing function. Consequently, a standard result on stochastic domination [7, §4.12] implies that $G_{m,\ell}(r,x) \geq G_{m,\ell}(r',x)$. Differentiability follows from the definition of the function.

The finiteness of the variance of the quantization step is clear by construction; more specifically, a crude upper bound is r_{ℓ}^2 . Thus, analogous to the previous theorems, Lemma 3 is used to establish almost sure convergence.

19

Now, some straightforward algebra using the results of Lemma 2 shows that the partial derivative $\frac{\partial G_{m,\ell}(r,x)}{\partial r}$ is

$$\frac{1}{r(1-r)}\sum_{k=-\ell}^{\ell-1} r_k \left\{ \mathbb{E}\left[X \mathbb{I}\left(x - s_{k+1} \le \frac{X}{m} \le x - s_k \right) \right] - \mathbb{E}[X] \mathbb{P}\left[x - s_{k+1} \le \frac{X}{m} \le x - s_k \right] \right\},$$
(42)

This will be used next. To compute the asymptotic variance, we again exploit asymptotic normality (see equation (24)) as before:

$$\mathbb{E}[X\mathbb{I}(m(\alpha^* - s_{k+1}) \le X \le m(\alpha^* - s_k))] = \mathbb{E}\left[X\mathbb{I}\left(-\sqrt{m}s_{k+1} \le \frac{X - \alpha^*m}{\sqrt{m}} \le -\sqrt{m}s_k\right)\right]$$
$$= \sqrt{m}\mathbb{E}\left[(Z + \alpha^*\sqrt{m})\mathbb{I}\left(-\sqrt{m}s_{k+1} \le Z \le -\sqrt{m}s_k\right)\right] + S$$
$$= \sqrt{m}\mathbb{E}\left[Z\mathbb{I}\left(-\sqrt{m}s_{k+1} \le Z \le -\sqrt{m}s_k\right)\right] + S$$
$$\rightarrow -\sqrt{m}\int_{\sqrt{m}s_k}^{\sqrt{m}s_{k+1}} z \frac{\exp\left(\frac{-z^2}{2a}\right)}{\sqrt{2\pi a}}dz + S$$
$$S := \mathbb{E}[X]P(m(x - s_{k+1}) \le X \le m(x - s_k))$$

Now make the definition, which corresponds to solving the integral above:

$$\Delta_m(s_k, s_{k+1}) = \left(\exp\left(-\frac{ms_k^2}{2\alpha^*(1-\alpha^*)}\right) - \exp\left(-\frac{ms_{k+1}^2}{2\alpha^*(1-\alpha^*)}\right)\right)$$

Thus, plugging into Equation 42, noticing that S cancels:

$$\frac{\partial G_{m,\ell}(r,\alpha^*)}{\partial r}\Big|_{r=F(\theta^*)} \to -\sqrt{\frac{m}{2\pi\alpha^*(1-\alpha^*)}}\sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k,s_{k+1})$$

A side note is that if one chooses $s_0 = 0$, we are guaranteed that at least one $\Delta_m(s_k, s_{k+1})$ does not go to zero in a fixed quantizer (i.e. a quantizer where the levels s_k do not depend on m). But the correction factor expression, and as a matter of fact, the optimum quantization of Gaussian, suggests that the levels s_k scale as $1/\sqrt{m}$. In this case, the factor is a constant, independent of m.

We now need to compute $R(\theta^*)$ for the quantized updated. It is also straightforward to see that this quantity is given by:

$$R(\theta^*) = K_m^2 \sum_{k=-\ell}^{\ell-1} r_k^2(G_m(F(\theta^*), \alpha^* - s_k) - G_m(F(\theta^*), \alpha^* - s_{k+1})) - \beta^2$$

Putting everything together we obtain the asymptotic variance estimate for the more general quantizer converges to:

$$\frac{R(\theta^*)}{2K_m \left| \frac{\partial G_{m,\ell}(r,\theta^*)}{\partial r} \right|_{r=\alpha^*} \left| p_X(\theta^*) - 1 \right|} \rightarrow \frac{K_m^2 \sum_{k=-\ell}^{\ell-1} r_k^2 (G_m(F(\theta^*), \alpha^* - s_k) - G_m(F(\theta^*), \alpha^* - s_{k+1})) - \beta^2}{\frac{2K_m \sqrt{m} \sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k, s_{k+1}) p_X(\theta^*)}{\sqrt{2\pi \alpha^* (1-\alpha^*)}} - 1}$$

Set a gain $K = \frac{K_m \sqrt{m} \sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k, s_{k+1})}{\sqrt{2\pi\alpha^*(1-\alpha^*)}}$ and we have the final expression for the variance:

$$2\pi \frac{\sum_{k=-\ell}^{\ell-1} r_k^2 \Delta G_m(s_k, s_{k+1}) - \beta^2}{\left(\sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k, s_{k+1})\right)^2} \left[\frac{K^2 \alpha^*(1-\alpha^*)}{2K p_X(\theta^*) - 1} \frac{1}{m}\right]$$

Where $\Delta G_m(s_k, s_{k+1}) = G_m(\alpha^*, \alpha^* - s_k) - G_m(\alpha^*, \alpha^* - s_{k+1})$. The constant $\kappa(\alpha^*, Q_\ell)$ defines the performance of the algorithm for different quantization choices:

$$\kappa(\alpha^*, \mathcal{Q}_{\ell}) = 2\pi \frac{\sum_{k=-\ell}^{\ell-1} r_k^2 \Delta G_m(s_k, s_{k+1}) - \beta^2}{\left(\sum_{k=-\ell}^{\ell-1} r_k \Delta_m(s_k, s_{k+1})\right)^2}$$

The rate with respect to m is the same, independent of quantization. It is clear from previous analysis that if the best quantizers are chosen $1 \le \kappa(\alpha^*, \mathcal{Q}_\ell) \le \frac{2\pi}{4}$. Obviously $\kappa(\alpha^*, \mathcal{Q}_\ell)$ over the class of optimal quantizers is a decreasing function of ℓ .

REFERENCES

- S. Amari and T. S. Han. Statistical inference under multiterminal rate restrictions: A differential geometric approach. *IEEE Trans. Info. Theory*, 35(2):217–227, March 1989.
- [2] E. Ayanoglu. On optimal quantization of noisy sources. IEEE Trans. Info. Theory, 36(6):1450-1452, 1990.
- [3] A. Benveniste, M. Metivier, and P. Priouret. Adaptive Algorithms and Stochastic Approximations. Springer-Verlag, New York, NY, 1990.
- [4] R. S. Blum, S. A. Kassam, and H. V. Poor. Distributed detection with multiple sensors: Part ii—advanced topics. *Proceedings of the IEEE*, 85:64–79, January 1997.
- [5] J. F. Chamberland and V. V. Veeravalli. Asymptotic results for decentralized detection in power constrained wireless sensor networks. *IEEE Journal on Selected Areas in Communication*, 22(6):1007–1015, August 2004.
- [6] C. Chong and S. P. Kumar. Sensor networks: Evolution, opportunities, and challenges. *Proceedings of the IEEE*, 91:1247–1256, 2003.
- [7] G.R. Grimmett and D.R. Stirzaker. *Probability and random processes*. Oxford Science Publications, Clarendon Press, Oxford, 1992.
- [8] J. A. Gubner. Decentralized estimation and quantization. IEEE Trans. Info. Theory, 39(4):1456–1459, 1993.

DRAFT

- J. Han, P. K. Varshney, and V. C. Vannicola. Some results on distributed nonparametric detection. In Proc. 29th Conf. on Decision and Control, pages 2698–2703, 1990.
- [10] T. S. Han and S. Amari. Statistical inference under multiterminal data compression. *IEEE Trans. Info. Theory*, 44(6):2300–2324, October 1998.
- [11] T. S. Han and K. Kobayashi. Exponential-type error probabilities for multiterminal hypothesis testing. *IEEE Trans. Info. Theory*, 35(1):2–14, January 1989.
- [12] H. J. Kushner and G. G. Yin. Stochastic Approximation Algorithms and Applications. Springer-Verlag, New York, NY, 1997.
- [13] Z.Q. Luo. Universal decentralized estimation in a bandwidth-constrained sensor network. *IEEE Trans. Info. Theory*, 51(6):2210–2219, 2005.
- [14] X. Nguyen, M. J. Wainwright, and M. I. Jordan. Nonparametric decentralized detection using kernel methods. *IEEE Trans. Signal Processing*, 53(11):4053–4066, November 2005.
- [15] R. J. Serfling. Approximation Theorems of Mathematical Statistics. Wiley Series in Probability and Statistics. Wiley, 1980.
- [16] R. R. Tenney and N. R. Jr. Sandell. Detection with distributed sensors. *IEEE Trans. Aero. Electron. Sys.*, 17:501–510, 1981.
- [17] J. N. Tsitsiklis. Decentralized detection. In Advances in Statistical Signal Processing, pages 297–344. JAI Press, 1993.
- [18] V. V. Veeravalli, T. Basar, and H. V. Poor. Decentralized sequential detection with a fusion center performing the sequential test. *IEEE Trans. Info. Theory*, 39(2):433–442, 1993.
- [19] R. Viswanathan and P. K. Varshney. Distributed detection with multiple sensors: Part i—fundamentals. *Proceedings of the IEEE*, 85:54–63, January 1997.
- [20] Z. Zhang and T. Berger. Estimation via compressed information. IEEE Trans. Info. Theory, 34(2):198-211, 1988.
- [21] R. Zielinski. Optimal quantile estimators: Small sample approach. Technical report, Inst. of Math. Pol. Academy of Sci., 2004.