New Constructions of a Family of 2-Generator Quasi-Cyclic Two-Weight Codes and Related Codes

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Abstract: Based on cyclic simplex codes, a new construction of a family of 2-generator quasi-cyclic two-weight codes is given. New optimal binary quasi-cyclic [195, 8, 96], [210, 8, 104] and [240, 8, 120] codes, good QC ternary [195, 6, 126], [208, 6, 135], [221, 6, 144] codes are thus obtained. Furthermre, binary quasi-cyclic self-complementary codes are also constructed.

I. INTRODUCTION

A code is said to be quasi-cyclic if every cyclic shift of a codeword by p positions results in another codeword [1]. Therefore quasi-cyclic (QC) codes are a generalization of cyclic codes with p = 1.

A linear code is called projective if any two of its coordinates are linearly independent, or in other words, if the minimum distance of its dual code is at least three. A code is said to be two-weight if any non-zero codeword has a weight of w_1 or w_2 . Two-weight codes are closely related to strongly regular graphs.

In this paper, a new construction of 2generator quasi-cyclic (QC) two-weight codes is presented. Some new good QC codes are obtained, and binary selfcomplementary codes are constructed based on the 2-generator QC codes.

II. CYCLIC CODES AND QC CODES

A. Cyclic Hamming Codes and Simplex Codes

A q-ary linear [n, k, d] code [2] is a kdimensional subspace of an n-dimensional vector space over GF(q), with minimum distance d between any two codewords. A code is said to be cyclic if every cyclic shift of a codeword is also a codeword. A cyclic code is described by the polynomial algebra. A cyclic [n, k, d] code has a unique generator polynomial g(x). It is a polynomial with degree of n - k. All codewords of a cyclic code are multiples of g(x) modulo $x^n - 1$.

It is well known that for any integer k, there is a simplex [n, k, d] code with distance $d = q^{k-1}$, where $n = (q^k - 1)/(q - 1)$. It should be noted that simplex codes are equidistance codes where $q^k - 1$ non-zero codewords have weights of q^{k-1} .

B. Quasi-Cyclic Codes

A code is said to be quasi-cyclic (QC) if a cyclic shift of any codeword by p positions is still a codeword. Thus a cyclic code is a QC code with p = 1. The block length n of a QC code is a multiple of p, or $n = m \times p$.

Circulants, or cyclic matrices, are basic components in the generator matrix for a QC code. An $m \times m$ cyclic or circulant matrix is defined as

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{m-1} \\ c_{m-1} & c_0 & \cdots & c_{m-2} \\ c_{m-2} & c_{m-1} & \cdots & c_{m-3} \\ \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}$$
(1)

and it is uniquely specified by a polynomial formed by the elements of its first row, $c(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{m-1} x^{m-1}$, with the least significant coefficient on the left.

A 1-generator QC code has the following form of the generator matrix [3]:

$$G = [G_0 G_1 G_2 \dots G_{p-1}]$$
(2)

where $G_{i,i} = 0, 1, 2, ..., p-1$, are circulants of order m. Let $g_0(x), g_1(x), ..., g_{p-1}(x)$ are the corresponding defining polynomials.

A 2-generator QC $[m \times p, k]$ codes has the generator matrix of the following form:

$$G = \begin{bmatrix} G_{00} & G_{01} \dots & G_{0,p-1} \\ G_{10} & G_{11} \dots & G_{1,p-1} \end{bmatrix}$$
(3)

where G_{ij} are circular matrices, for i = 0, and 1, j = 0, 1, ..., p-1.

Similarly, a 3-generator QC $[m \times p, k]$ codes has the generator matrix of the following form:

$$G = \begin{bmatrix} G_{00} & G_{01} & \dots & G_{0,p-1} \\ G_{10} & G_{11} & \dots & G_{1,p-1} \\ G_{20} & G_{21} & \dots & G_{2,p-1} \end{bmatrix}$$
(4)

where G_{ij} are circular matrices, for i = 0, 1, and 2, j = 0, 1, ..., p-1.

III. CONSTRUCTIONS OF 2-GENERATOR QC TWO-WEIGHT CODES

A. Two-Weight Codes

A linear code is called projective if any two of its coordinates are linearly independent, or in other words, if the minimum distance of its dual code is at least three. A code is said to be two-weight if any non-zero codeword has a weight of w_1 or w_2 , where $w_1 \neq w_2$. A two weight code is also written as the [n, k; w_1 , w_2] code. Two-weight codes are closely related to strongly regular graphs.

In the survey paper [4], Calderbank and Kantor presented many known families of two-weight codes. Among those families, there is a family of two-weight [n, k; w_1 , w_2] codes over GF(q) noted by SU2, that has the following parameters:

Block length $n = i(q^t - 1)/(q - 1)$ Dimension k = 2tWeights $w_1 = (i - 1) q^{t-1}$, $w_2 = iq^{t-1}$ where $2 \le i \le q^t$.

In this section, 2-generator QC twoweight codes with the same parameters as SU2 are constructed from cyclic simplex codes.

B. Binary 2-Generator QC 2-Weight Codes

Given any positive integer k. If there exist a binary cyclic Hamming $[2^k -1, 2^k - k - 1, 3]$ codes, then there exist a cyclic simplex $[2^k -1, k, 2^{k-1}]$ code. Let $g_1(x)$ be the generator polynomial of the simplex code, C_1 . A binary 2-generator QC two-weight $[(2^k -1)p, 2k]$ code can be constructed with the following generator matrix:

$$G = \begin{bmatrix} g_1(x) g_1(x) g_1(x) \dots g_1(x) \\ 0 g_1(x) x g_1(x) \dots x^{i-2} g_1(x) \end{bmatrix}$$
(5)
where $2 \le i \le 2^k$, is an integer.

Based on the generator matrix structure, and property of the simplex code, it is obvious that any non-zero codeword has a weight $w_1 = (i - 1) 2^{k-1}$, or $w_2 = i2^{k-1}$. So the 2-generator QC codes defined by (5) are two-weight codes in the family SU2.

Example 1. n = 7, k = 3. $x^7 - 1 = (x + 1)(x^3 + x + 1)(x^3 + x^2 + 1)$. So a cyclic simplex

[7, 3, 4] code is defined by $g_1(x) = x^4 + x^2 + x + 1$. With the construction, 2-generator QC two-weight [14, 6; 4, 8], [21, 6; 8, 12], [28, 6; 12, 16], [35, 6; 16, 20], [42, 6; 20, 24], [49, 6; 24, 28] and [56, 6; 28, 32] codes are obtained.

Among the QC two-weight codes obtained, some codes are optimal codes, in the sense that they meet the bound [5] on the minimum distance. Table I lists these optimal binary 2-generator QC codes constructed.

Table I OPTIMA	L BINARY 2-

р	m	k	d	w ₁ , w ₂
3	7	3	8	8, 12
4	7	3	12	12, 16
5	7	3	16	16, 20
6	7	3	20	20, 24
7	7	3	24	24, 28
8	7	3	28	28, 32
10	15	4	72	72, 80
11	15	4	80	80, 88
12	15	4	88	88, 96
13	15	4	96	96, 104
14	15	4	104	104, 112
15	15	4	112	112, 120
16	15	4	120	120, 128

Among those codes, QC [195, 8, 96], [210, 8, 104] and [240, 8, 120] codes are previously unknown[6].

C. q-ary 2-Generator QC 2-Weight Codes

For any prime power *q*, there exist a *q*ary cyclic simplex $[(q^k - 1)/(q-1), k, q^{k-1}]$ code, if *q* -1 and *k* are relatively prime. Let $g_1(x)$ be the generator polynomials. Let $m = (q^k - 1)/(q-1)$. In the same way as the binary 2-generator QC code construction, we can construct a *q*-ary 2-generator QC two-weight $[m \times p, 2k]$ code with the following generator matrix:

$$G = \begin{bmatrix} g_1(x) & g_1(x) \\ 0 & a_j x^i g_1(x) \end{bmatrix}$$
(6)

where $0 \le i < m$, is an integer, and a_j is any non-zero element in GF(q).

Example 2. n = 13, k = 3. $g_1(x) = x^{10} - x^9 + x^8 - x^6 - x^5 + x^4 + x^3 + x^2 + 1$ defines a cyclic simplex [13, 3, 9] code over GF(3). So 2-generator QC two-weight [26, 6; 9, 18] and [39, 6; 18, 27] codes can be obtained by following generator matrices:

$$G = \begin{bmatrix} g_1(x) g_1(x) \\ 0 & g_1(x) \end{bmatrix},$$

$$G = \begin{bmatrix} g_1(x) g_1(x) & g_1(x) \\ 0 & g_1(x) & -g_1(x) \end{bmatrix}$$

Also 2-generater QC two-weight [195, 6, 126], [208, 6, 135], [221, 6, 144] codes over GF(3) are obtained, that reach the lower bound on the minimum distance [5].

IV. CONSTRUCTIONS OF BINARY SELF-COMPLEMENTARY CODES

A binary [n, k, d] code is said to be selfcomplementary if it has the property that the complementary codeword $(x_1+1, x_2+1, ..., x_n+1)$ is also a codeword, for any codeword $(x_1, x_2, ..., x_n)$. For a self-complementary [n, k, d] code C, Grey-Rankin bound holds [7]:

$$|C| \le \frac{8d(n-d)}{n - (n-2d)^2}$$
 (7)

McGuire [9] has shown that the parameters f a binary linear self-complementary codes meeting the Grey-Rankin bound are

$$[2^{2k-1} - 2^{k-1}, 2k + 1, 2^{2k-2} - 2^{k-1}]$$
(8)
$$[2^{2k-1} + 2^{k-1}, 2k + 1, 2^{2k-2}]$$
(9)

These self-complementary codes are closely related to quasi-symmetric designs[8, 9]. In [7], Gulliver and Harada investigated 1-generator QC self-complementary [120, 9, 56], [135, 9, 64], [496, 11, 240] and [528, 11, 256] codes. In this section, 3-generator QC self-complementary codes of the parameters as given in (8) and (9) are constructed.

A.
$$[2^{2k-1} - 2^{k-1}, 2k + 1, 2^{2k-2} - 2^{k-1}]$$
 Codes

Given a cyclic simplex $[2^{k} - 1, k, 2^{k-1}]$ code, that is defined by the generator polynomial $g_1(x)$. Choose $i = 2^{k-1}$. Then a 2-generator QC two-weight $[2^{2k-1} - 2^{k-1}, 2k; 2^{2k-2} - 2^{k-1}, 2^{2k-2}]$ code can be constructed by (). So, the sum of two non-zero weights is $(2^{2k-2} - 2^{k-1}) + 2^{2k-2} = 2^{2k-1} - 2^{k-1}$, the block length of the code. By extending one more information digit, a 3-generator QC self-complementary $[2^{2k-1} - 2^{k-1}, 2k + 1, 2^{2k-2} - 2^{k-1}]$ Code is obtained by the following generator matrix:

$$G = \begin{bmatrix} g_1(x) & g_1(x) & g_1(x) & g_1(x) \\ 0 & g_1(x) & xg_1(x) \dots x^{i-2}g_1(x) \\ 1(x) & 1(x) & 1(x) & 1(x) \end{bmatrix}$$
(10)

where 1(x) is a vector of all 1's of length $2^k - 1$.

B.
$$[2^{2k-1} + 2^{k-1}, 2k + 1, 2^{2k-2}]$$
 Codes

Given a cyclic simplex $[2^{k} - 1, k, 2^{k-1}]$ code, that is defined by the generator polynomial $g_1(x)$. Choose $i = 2^{k-1} + 1$. Then a 2-generator QC two-weight $[2^{2k-1} + 2^{k-1} - 1, 2k; 2^{2k-2}, 2^{2k-2} + 2^{k-1}]$ code can be constructed by (). So, the sum of two non-zero weights is $2^{2k-2} + (2^{2k-2} + 2^{k-1}) = 2^{2k-1} + 2^{k-1}$. By extending one more information digit, and one parity check digit, a 3-generator QC selfcomplementary $[2^{2k-1} + 2^{k-1}, 2k + 1, 2^{2k-2}]$ Codes is obtained by the following generator matrix:

$$G = \begin{bmatrix} g_1(x) & g_1(x) & g_1(x) & g_1(x) & 0\\ 0 & g_1(x) & xg_1(x) \dots x^{i-2}g_1(x) \\ 1(x) & 1(x) & 1(x) & 1(x) & 1 \end{bmatrix} (11)$$

where 1(x) is a vector of all 1's of length $2^k - 1$.

V. CONCLUSION

In this paper, a new construction method for a family of two-weight codes is presented. With this construction, some new optimal and good QC codes are obtained, and binary self-complementary codes are constructed by extending the 2-generator QC two-weight codes.

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