# New Constructions of a Family of 2-Generator Quasi-Cyclic Two-Weight Codes and Related Codes 

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#### Abstract

Based on cyclic simplex codes, a new construction of a family of 2 -generator quasi-cyclic two-weight codes is given. New optimal binary quasi-cyclic [195, 8, 96], [210, $8,104]$ and $[240,8,120]$ codes, good QC ternary [195, 6, 126], [208, 6, 135], [221, 6, 144] codes are thus obtained. Furthermre, binary quasi-cyclic self-complementary codes are also constructed.


## I. INTRODUCTION

A code is said to be quasi-cyclic if every cyclic shift of a codeword by $p$ positions results in another codeword [1]. Therefore quasi-cyclic (QC) codes are a generalization of cyclic codes with $p=1$.

A linear code is called projective if any two of its coordinates are linearly independent, or in other words, if the minimum distance of its dual code is at least three. A code is said to be two-weight if any non-zero codeword has a weight of $\mathrm{w}_{1}$ or $\mathrm{w}_{2}$. Two-weight codes are closely related to strongly regular graphs.

In this paper, a new construction of 2generator quasi-cyclic (QC) two-weight codes is presented. Some new good QC codes are obtained, and binary selfcomplementary codes are constructed based on the 2-generator QC codes.

## II. CYCLIC CODES AND QC CODES

## A. Cyclic Hamming Codes and Simplex

 CodesA q-ary linear [ $\mathrm{n}, \mathrm{k}, \mathrm{d}$ ] code [2] is a k dimensional subspace of an $n$-dimensional vector space over $\mathrm{GF}(\mathrm{q})$, with minimum distance d between any two codewords. A code is said to be cyclic if every cyclic shift of a codeword is also a codeword. A cyclic code is described by the polynomial algebra. A cyclic [ $\mathrm{n}, \mathrm{k}, \mathrm{d}$ ] code has a unique generator polynomial $\mathrm{g}(\mathrm{x})$. It is a polynomial with degree of $n-k$. All codewords of a cyclic code are multiples of $\mathrm{g}(\mathrm{x})$ modulo $\mathrm{x}^{\mathrm{n}}-1$.

It is well known that for any integer $k$, there is a simplex $[\mathrm{n}, \mathrm{k}, \mathrm{d}]$ code with distance $d=q^{k-1}$, where $n=\left(q^{k}-1\right) /(q-1)$. It should be noted that simplex codes are equidistance codes where $\mathrm{q}^{\mathrm{k}}$ - 1 non-zero codewords have weights of $\mathrm{q}^{\mathrm{k}-1}$.

## B. Quasi-Cyclic Codes

A code is said to be quasi-cyclic (QC) if a cyclic shift of any codeword by p positions is still a codeword. Thus a cyclic code is a QC code with $p=1$. The block length n of a QC code is a multiple of $p$, or $n=m \times p$.

Circulants, or cyclic matrices, are basic components in the generator matrix for a QC code. An $m \times m$ cyclic or circulant matrix is defined as

$$
C=\left[\begin{array}{cccc}
c_{0} & c_{1} & \cdots & c_{m-1}  \tag{1}\\
c_{m-1} & c_{0} & \cdots & c_{m-2} \\
c_{m-2} & c_{m-1} & \cdots & c_{m-3} \\
\vdots & \vdots & \cdots & \vdots \\
c_{1} & c_{2} & \cdots & c_{0}
\end{array}\right]
$$

and it is uniquely specified by a polynomial formed by the elements of its first row, $c(x)$ $=c_{0}+c_{1} x+c_{2} x^{2}+\ldots+c_{m-1} x^{m-1}$, with the least significant coefficient on the left.

A 1-generator QC code has the following form of the generator matrix [3]:

$$
\mathrm{G}=\left[\begin{array}{llll}
\mathrm{G}_{0} \mathrm{G}_{1} & \mathrm{G}_{2} & \ldots & \mathrm{G}_{\mathrm{p}-1} \tag{2}
\end{array}\right]
$$

where $\mathrm{G}_{\mathrm{i}}, \mathrm{i}=0,1,2, \ldots, \mathrm{p}-1$, are circulants of order m . Let $\mathrm{g}_{0}(\mathrm{x}), \mathrm{g}_{1}(\mathrm{x}), \ldots, \mathrm{g}_{\mathrm{p}-1}(\mathrm{x})$ are the corresponding defining polynomials.

A 2-generator QC $[m \times p, k]$ codes has the generator matrix of the following form:

$$
G=\left[\begin{array}{lll}
G_{00} & G_{01} \ldots & G_{0, p-1}  \tag{3}\\
G_{10} & G_{11} \ldots & G_{1, p-1}
\end{array}\right]
$$

where $\mathrm{G}_{\mathrm{ij}}$ are circular matrices, for $\mathrm{i}=0$, and $1, j=0,1, \ldots, p-1$.

Similarly, a 3-generator QC $[m \times p, k]$ codes has the generator matrix of the following form:

$$
G=\left[\begin{array}{llll}
G_{00} & G_{01} & \ldots & G_{0, p-1}  \tag{4}\\
G_{10} & G_{11} & \ldots & G_{1, p-1} \\
G_{20} & G_{21} & \ldots & G_{2, p-1}
\end{array}\right]
$$

where $\mathrm{G}_{\mathrm{ij}}$ are circular matrices, for $\mathrm{i}=0,1$, and $2, \mathrm{j}=0,1, \ldots, \mathrm{p}-1$.

## III. CONSTRUCTIONS OF 2GENERATOR QC TWO-WEIGHT CODES

## A. Two-Weight Codes

A linear code is called projective if any two of its coordinates are linearly
independent, or in other words, if the minimum distance of its dual code is at least three. A code is said to be two-weight if any non-zero codeword has a weight of $\mathrm{w}_{1}$ or $\mathrm{w}_{2}$, where $\mathrm{w}_{1} \neq \mathrm{w}_{2}$. A two weight code is also written as the $\left[\mathrm{n}, \mathrm{k} ; \mathrm{w}_{1}, \mathrm{w}_{2}\right.$ ] code. Twoweight codes are closely related to strongly regular graphs.

In the survey paper [4], Calderbank and Kantor presented many known families of two-weight codes. Among those families, there is a family of two-weight $\left[\mathrm{n}, \mathrm{k} ; \mathrm{w}_{1}, \mathrm{w}_{2}\right.$ ] codes over GF(q) noted by SU2, that has the following parameters:

Block length $\mathrm{n}=\mathrm{i}\left(\mathrm{q}^{\mathrm{t}}-1\right) /(\mathrm{q}-1)$
Dimension $k=2 t$
Weights $\mathrm{w}_{1}=(\mathrm{i}-1) \mathrm{q}^{\mathrm{t}-1}, \mathrm{w}_{2}=\mathrm{iq} \mathrm{q}^{\mathrm{t}-1}$
where $2 \leq \mathrm{i} \leq \mathrm{q}^{\mathrm{t}}$.
In this section, 2-generator QC twoweight codes with the same parameters as SU2 are constructed from cyclic simplex codes.

## B. Binary 2-Generator QC 2-Weight Codes

Given any positive integer k . If there exist a binary cyclic Hamming $\left[2^{\mathrm{k}}-1,2^{\mathrm{k}}-\mathrm{k}-1\right.$, 3] codes, then there exist a cyclic simplex [ $2^{\mathrm{k}}$ $-1, \mathrm{k}, 2^{\mathrm{k}-1}$ ] code. Let $\mathrm{g}_{1}(\mathrm{x})$ be the generator polynomial of the simplex code, $\mathrm{C}_{1}$. A binary 2 -generator QC two-weight [ $\left.\left(2^{\mathrm{k}}-1\right) \mathrm{p}, 2 \mathrm{k}\right]$ code can be constructed with the following generator matrix:

$$
G=\left[\begin{array}{ccc}
g_{1}(x) & g_{1}(x) & g_{1}(x)  \tag{5}\\
0 & g_{1}(x) & x g_{1}(x) \ldots g_{1}(x) \\
i-2 & g_{1}(x)
\end{array}\right]
$$

where $2 \leq \mathrm{i} \leq 2^{\mathrm{k}}$, is an integer.
Based on the generator matrix structure, and property of the simplex code, it is obvious that any non-zero codeword has a weight $w_{1}=(i-1) 2^{k-1}$, or $w_{2}=i 2^{k-1}$. So the 2-generator QC codes defined by (5) are two-weight codes in the family SU2.

Example 1. $\mathrm{n}=7, \mathrm{k}=3 . \mathrm{x}^{7}-1=(\mathrm{x}+1)$ $\left(x^{3}+x+1\right)\left(x^{3}+x^{2}+1\right)$. So a cyclic simplex
$[7,3,4]$ code is defined by $g_{1}(x)=x^{4}+x^{2}+x$ +1 . With the construction, 2-generator QC two-weight $[14,6 ; 4,8],[21,6 ; 8,12]$, [28, $6 ; 12,16],[35,6 ; 16,20],[42,6 ; 20,24],[49$, $6 ; 24,28]$ and $[56,6 ; 28,32]$ codes are obtained.

Among the QC two-weight codes obtained, some codes are optimal codes, in the sense that they meet the bound [5] on the minimum distance. Table I lists these optimal binary 2-generator QC codes constructed.

Table I OPTIMAL BINARY 2GENERATOR QC [pm, 2k] CODES

| $\mathbf{p}$ | $\mathbf{m}$ | $\mathbf{k}$ | $\mathbf{d}$ | $\mathbf{w}_{\mathbf{1}}, \mathbf{w}_{\mathbf{2}}$ |
| :--- | :--- | :--- | :--- | :--- |
| 3 | 7 | 3 | 8 | 8,12 |
| 4 | 7 | 3 | 12 | 12,16 |
| 5 | 7 | 3 | 16 | 16,20 |
| 6 | 7 | 3 | 20 | 20,24 |
| 7 | 7 | 3 | 24 | 24,28 |
| 8 | 7 | 3 | 28 | 28,32 |
| 10 | 15 | 4 | 72 | 72,80 |
| 11 | 15 | 4 | 80 | 80,88 |
| 12 | 15 | 4 | 88 | 88,96 |
| 13 | 15 | 4 | 96 | 96,104 |
| 14 | 15 | 4 | 104 | 104,112 |
| 15 | 15 | 4 | 112 | 112,120 |
| 16 | 15 | 4 | 120 | 120,128 |

Among those codes, QC [195, 8, 96], [210, 8, 104] and [240, 8, 120] codes are previously unknown[6].

## C. q-ary 2-Generator QC 2-Weight Codes

For any prime power $q$, there exist a $q$ ary cyclic simplex $\left[\left(q^{k}-1\right) /(q-1), k, q^{k-l}\right]$ code, if $q-1$ and $k$ are relatively prime. Let $\mathrm{g}_{1}(\mathrm{x})$ be the generator polynomials. Let $m=$ $\left(q^{k}-1\right) /(q-1)$. In the same way as the binary 2-generator QC code construction, we can construct a q-ary 2-generator QC two-weight [ $m \times p, 2 k$ ] code with the following generator matrix:

$$
G=\left[\begin{array}{cc}
g_{1}(x) & g_{1}(x)  \tag{6}\\
0 & a_{j} x^{i} g_{1}(x)
\end{array}\right]
$$

where $0 \leq \mathrm{i}<\mathrm{m}$, is an integer, and $\mathrm{a}_{\mathrm{j}}$ is any non-zero element in $\mathrm{GF}(\mathrm{q})$.

Example 2. $\mathrm{n}=13, \mathrm{k}=3 . \mathrm{g}_{1}(\mathrm{x})=\mathrm{x}^{10}-\mathrm{x}^{9}$ $+x^{8}-x^{6}-x^{5}+x^{4}+x^{3}+x^{2}+1$ defines a cyclic simplex [13, 3, 9] code over GF(3). So 2 -generator QC two-weight [26, 6; 9, 18] and [39, 6; 18, 27] codes can be obtained by following generator matrices:

$$
\begin{aligned}
& G=\left[\begin{array}{cc}
g_{1}(x) & g_{1}(x) \\
0 & g_{1}(x)
\end{array}\right], \\
& G=\left[\begin{array}{ccc}
g_{1}(x) & g_{1}(x) & g_{1}(x) \\
0 & g_{1}(x) & -g_{1}(x)
\end{array}\right]
\end{aligned}
$$

Also 2-generater QC two-weight [195, 6, 126], [208, 6, 135], [221, 6, 144] codes over $\mathrm{GF}(3)$ are obtained, that reach the lower bound on the minimum distance [5].

## IV. CONSTRUCTIONS OF BINARY SELF-COMPLEMENTARY CODES

A binary [ $\mathrm{n}, \mathrm{k}, \mathrm{d}$ ] code is said to be selfcomplementary if it has the property that the complementary codeword ( $\mathrm{x}_{1}+1, \mathrm{x}_{2}+1, \ldots$, $x_{n}+1$ ) is also a codeword, for any codeword $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$. For a self-complementary [ $n$, $\mathrm{k}, \mathrm{d}]$ code C, Grey-Rankin bound holds [7]:

$$
\begin{equation*}
|C| \leq \frac{8 d(n-d)}{n-(n-2 d)^{2}} \tag{7}
\end{equation*}
$$

McGuire [9] has shown that the parameters $f$ a binary linear self-complementary codes meeting the Grey-Rankin bound are

$$
\begin{align*}
& {\left[2^{2 \mathrm{k}-1}-2^{\mathrm{k}-1}, 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}-2^{\mathrm{k}-1}\right]}  \tag{8}\\
& {\left[2^{\mathrm{k}-1}+2^{\mathrm{k}-1}, 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}\right]} \tag{9}
\end{align*}
$$

These self-complementary codes are closely related to quasi-symmetric designs[8, 9]. In [7], Gulliver and Harada investigated 1generator QC self-complementary [120, 9, 56], [135, 9, 64], [496, 11, 240] and [528, 11, 256] codes. In this section, 3-generator QC self-complementary codes of the parameters as given in (8) and (9) are constructed.
A. $\left[2^{2 k-1}-2^{\mathrm{k}-1}, 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}-2^{\mathrm{k}-1}\right]$ Codes

Given a cyclic simplex $\left[2^{\mathrm{k}}-1, \mathrm{k}, 2^{\mathrm{k}-1}\right]$ code, that is defined by the generator polynomial $g_{1}(x)$. Choose $i=2^{k-1}$. Then a $2-$ generator QC two-weight $\left[2^{2 \mathrm{k}-1}-2^{\mathrm{k}-1}, \quad 2 \mathrm{k}\right.$; $\left.2^{2 \mathrm{k}-2}-2^{\mathrm{k}-1}, 2^{2 \mathrm{k}-2}\right]$ code can be constructed by ( ). So, the sum of two non-zero weights is $\left(2^{2 \mathrm{k}-2}-2^{\mathrm{k}-1}\right)+2^{2 \mathrm{k}-2}=2^{2 \mathrm{k}-1}-2^{\mathrm{k}-1}$, the block length of the code. By extending one more information digit, a 3-generator QC selfcomplementary $\left[2^{2 \mathrm{k}-1}-2^{\mathrm{k}-1}, 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}-\right.$ $2^{\mathrm{k}-1}$ ] Code is obtained by the following generator matrix:
$G=\left[\begin{array}{cccc}g_{1}(x) & g_{1}(x) & g_{1}(x) & g_{1}(x) \\ 0 & g_{1}(x) & x g_{1}(x) \ldots x^{i-2} g_{1}(x) \\ 1(x) & 1(x) & 1(x) & 1(x)\end{array}\right]$
where $1(\mathrm{x})$ is a vector of all 1 's of length $2^{\mathrm{k}}-1$.
B. $\left[2^{2 \mathrm{k}-1}+2^{\mathrm{k}-1}, 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}\right]$ Codes

Given a cyclic simplex $\left[2^{\mathrm{k}}-1, \mathrm{k}, 2^{\mathrm{k}-1}\right]$ code, that is defined by the generator polynomial $g_{1}(x)$. Choose $i=2^{k-1}+1$. Then a 2-generator QC two-weight $\left[2^{2 \mathrm{k}-1}+2^{\mathrm{k}-1}-1\right.$, $\left.2 \mathrm{k} ; 2^{2 \mathrm{k}-2}, 2^{2 \mathrm{k}-2}+2^{\mathrm{k}-1}\right]$ code can be constructed by (). So, the sum of two non-zero weights is $2^{2 k-2}+\left(2^{2 k-2}+2^{k-1}\right)=2^{2 k-1}+2^{k-1}$. By extending one more information digit, and one parity check digit, a 3-generator QC selfcomplementary $\left[2^{2 \mathrm{k}-1}+2^{\mathrm{k}-1}, \quad 2 \mathrm{k}+1,2^{2 \mathrm{k}-2}\right]$ Codes is obtained by the following generator matrix:
$G=\left[\begin{array}{ccccc}g_{1}(x) & g_{1}(x) & g_{1}(x) & g_{1}(x) & 0 \\ 0 & g_{1}(x) & x g_{1}(x) \ldots x^{i-2} g_{1}(x) & 0 \\ 1(x) & 1(x) & 1(x) & 1(x) & 1\end{array}\right]$
where $1(\mathrm{x})$ is a vector of all 1 's of length $2^{\mathrm{k}}-1$.

## V. CONCLUSION

In this paper, a new construction method for a family of two-weight codes is presented. With this construction, some new optimal and good QC codes are obtained, and binary self-complementary codes are constructed by
extending the 2-generator QC two-weight codes.

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