# On Multipath Fading Channels at High SNR 

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#### Abstract

This work studies the capacity of multipath fading channels. A noncoherent channel model is considered, where neither the transmitter nor the receiver is cognizant of the realization of the path gains, but both are cognizant of their statistics. It is shown that if the delay spread is large in the sense that the variances of the path gains decay exponentially or slower, then capacity is bounded in the signal-to-noise ratio (SNR). For such channels, capacity does not tend to infinity as the SNR tends to infinity. In contrast, if the variances of the path gains decay faster than exponentially, then capacity is unbounded in the SNR. It is further demonstrated that if the number of paths is finite, then at high SNR capacity grows double-logarithmically with the SNR, and the capacity pre-loglog, defined as the limiting ratio of capacity to $\log \log$ SNR as SNR tends to infinity, is 1 irrespective of the number of paths.【


## 1 Introduction

We study the capacity of discrete-time multipath fading channels. In multipath fading channels, the transmitted signal propagates along a multitude of paths, and the gains and delays of these paths vary over time. In general, the path delays differ from each other, and the receiver thus observes a weighted sum of delayed replicas of the transmitted signal, where the weights are random. We shall slightly abuse nomenclature and refer to each summand in the received signal as a path, and to the corresponding weight as its path gain, even if it is in fact composed of a multitude of paths. We consider a noncoherent channel model, where transmitter and receiver are cognizant of the statistics of the path gains, but are ignorant of their realization.

Multipath fading channels arise in wireless communications, where obstacles in the surroundings reflect the transmitted signal and force it to propagate along multiple paths, and where relative movements of transmitter, receiver, and obstacles lead to time-variations of the path gains and delays. Examples of wireless communication scenarios where the receiver observes typically more than one path include radio communications (particularly if the transmitted signal is of large bandwidth as, for example, in Ultra-Wideband or in CDMA) and underwater acoustic communications.

The capacity of noncoherent multipath fading channels has been investigated extensively in the wideband regime, where the signal-to-noise ratio (SNR) is typically small. It was shown by Kennedy that, in the limit as the available bandwidth tends to infinity, the capacity of the fading channel is the same as the capacity of the additive white Gaussian noise (AWGN) channel of equal received power; see [1, Sec. 8.6] and references therein.

To the best of our knowledge, not much is known about the capacity of noncoherent multipath fading channels at high SNR. For the special case of noncoherent frequency-flat fading channels (where we only have one path), it was shown by Lapidoth \& Moser [2] that if the fading process is of finite entropy rate, then at high SNR capacity grows double-logarithmically in the SNR. This is much slower than the logarithmic growth of the AWGN capacity [3].

[^0]In this work, we study the high-SNR behavior of the capacity of noncoherent multipath fading channels (where the number of paths is typically greater than one). We demonstrate that the capacity of such channels does not merely grow slower with the SNR than the capacity of the AWGN channel, but may be even bounded in the SNR. In other words, for such channels the capacity does not necessarily tend to infinity as the SNR tends to infinity.

We derive a necessary and a sufficient condition for the capacity to be bounded in the SNR. We show that if the variances of the path gains decay exponentially or slower, then capacity is bounded in the SNR. In contrast, if the variances of the path gains decay faster than exponentially, then capacity is unbounded in the SNR. We further show that if the number of paths is finite, then at high SNR capacity increases double-logarithmically with the SNR, and the capacity pre-loglog, defined as the limiting ratio of the capacity to $\log \log$ SNR as SNR tends to infinity, is 1 irrespective of the number of paths.

The rest of this paper is organized as follows. We begin with a mathematical description of the considered channel model in Section 2. Section 3 is devoted to channel capacity. Our main results are summarized in Section 4. They follow from upper bounds and lower bounds on channel capacity, which are derived in Sections 5 and 6, respectively. Section 7 concludes the paper with a brief summary and a discussion of our results.

## 2 Channel Model

Let $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. We consider a discrete-time multipath fading channel whose channel output $Y_{k} \in \mathbb{C}$ at time $k \in \mathbb{N}$ corresponding to the time- 1 through time- $k$ channel inputs $x_{1}, \ldots, x_{k} \in \mathbb{C}$ is given by

$$
\begin{equation*}
Y_{k}=\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} x_{k-\ell}+Z_{k}, \quad k \in \mathbb{N} . \tag{1}
\end{equation*}
$$

Here $\left\{Z_{k}\right\}$ models additive noise, and $H_{k}^{(\ell)}$ denotes the time- $k$ gain of the $\ell$-th path. We assume that $\left\{Z_{k}\right\}$ is a sequence of independent and identically distributed (IID), zero-mean, variance$\sigma^{2}$, circularly-symmetric, complex Gaussian random variables. For each path $\ell \in \mathbb{N}_{0}$ (where $\mathbb{N}_{0}$ denotes the set of nonnegative integers), we assume that $\left\{H_{k}^{(\ell)}, k \in \mathbb{N}\right\}$ is a zero-mean, complex stationary process. We denote its variance and its differential entropy rate by

$$
\begin{equation*}
\alpha_{\ell} \triangleq \mathrm{E}\left[\left|H_{k}^{(\ell)}\right|^{2}\right], \quad \ell \in \mathbb{N}_{0} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{\ell} \triangleq \lim _{n \rightarrow \infty} \frac{1}{n} h\left(H_{1}^{(\ell)}, \ldots, H_{n}^{(\ell)}\right), \quad \ell \in \mathbb{N}_{0} \tag{3}
\end{equation*}
$$

We shall say that the channel has a finite number of paths, if for some finite integer $L \in \mathbb{N}_{0}$

$$
\begin{equation*}
H_{k}^{(\ell)}=0, \quad \ell>\mathrm{L}, \quad k \in \mathbb{N} \tag{4}
\end{equation*}
$$

We assume that $\alpha_{0}>0$. We further assume

$$
\begin{equation*}
\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell}<\infty \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\inf _{\ell \in \mathcal{L}} h_{\ell}>-\infty \tag{6}
\end{equation*}
$$

where the set $\mathcal{L}$ is defined as $\mathcal{L} \triangleq\left\{\ell \in \mathbb{N}_{0}: \alpha_{\ell}>0\right\}$. (When the path gains are Gaussian, then the latter condition (6) is equivalent to saying that the mean-square error in predicting the present path gain from its past is strictly positive, i.e., that the present path gain cannot be predicted perfectly from its past.) We finally assume that the processes

$$
\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\},\left\{H_{k}^{(1)}, k \in \mathbb{N}\right\}, \ldots
$$

are independent ("uncorrelated scattering"); that they are jointly independent of $\left\{Z_{k}\right\}$; and that the joint law of

$$
\left(\left\{Z_{k}\right\},\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\},\left\{H_{k}^{(1)}, k \in \mathbb{N}\right\}, \ldots\right)
$$

does not depend on the input sequence $\left\{x_{k}\right\}$. We consider a noncoherent channel model where neither transmitter nor receiver is cognizant of the realization of $\left\{H_{k}^{(\ell)}, k \in \mathbb{N}\right\}, \ell \in \mathbb{N}_{0}$, but both are aware of their law. We do not assume that the path gains are Gaussian.

## 3 Channel Capacity

Let $A_{m}^{n}$ denote the sequence $A_{m}, \ldots, A_{n}$. We define the capacity (in nats per channel use) as

$$
\begin{equation*}
C(\mathrm{SNR}) \triangleq \underline{\lim }_{n \rightarrow \infty} \frac{1}{n} \sup I\left(X_{1}^{n} ; Y_{1}^{n}\right) \tag{7}
\end{equation*}
$$

where the supremum is over all joint distributions on $X_{1}, \ldots, X_{n}$ satisfying the power constraint

$$
\begin{equation*}
\frac{1}{n} \sum_{k=1}^{n} \mathrm{E}\left[\left|X_{k}\right|^{2}\right] \leq \mathrm{P} \tag{8}
\end{equation*}
$$

and where SNR is defined as

$$
\begin{equation*}
\mathrm{SNR} \triangleq \frac{\mathrm{P}}{\sigma^{2}} \tag{9}
\end{equation*}
$$

By Fano's inequality, no rate above $C(\mathrm{SNR})$ is achievable. (See [4] for a definition of an achievable rate.) We do not claim that there is a coding theorem associated with (7), i.e., that $C(\mathrm{SNR})$ is achievable. A coding theorem will hold, for example, if the number of paths is finite, and if the processes corresponding to these paths $\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\}, \ldots,\left\{H_{k}^{(\mathrm{L})}, k \in \mathbb{N}\right\}$ are jointly ergodic, see [5, Thm. 2].

The special case of noncoherent frequency-flat fading channels (where we have only one path) was studied by Lapidoth and Moser [2]. They showed that if the fading process $\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\}$ is ergodic, then the capacity satisfies

$$
\begin{equation*}
\lim _{\mathrm{SNR} \rightarrow \infty}\{C(\mathrm{SNR})-\log \log \mathrm{SNR}\}=\log \pi+\mathrm{E}\left[\log \left|H_{1}^{(0)}\right|^{2}\right]-h_{0} \tag{10}
\end{equation*}
$$

(see [2, Thm. 4.41]), where $\log (\cdot)$ denotes the natural logarithm function. Thus, at high SNR, the capacity of noncoherent frequency-flat fading channels grows double-logarithmically with the SNR. Lapidoth and Moser concluded that communicating over noncoherent frequency-flat fading channels at high SNR is extremely power-inefficient, as one should expect to square the SNR for every additional bit per channel use ${ }^{1}$

In this paper, we show inter alia that communicating over noncoherent multipath fading channels at high SNR is not merely power-inefficient, but may be even worse: if the delay spread is large in the sense that the sequence $\left\{\alpha_{\ell}\right\}$ (which describes the variances of the path gains) decays exponentially or slower, then capacity is bounded in the SNR. For such channels, capacity does not tend to infinity as the SNR tends to infinity. The main results of this paper are presented in the following section.

## 4 Main Results

Our main results are a sufficient and a necessary condition on $\left\{\alpha_{\ell}\right\}$ for $C$ (SNR) to be bounded in SNR, as well as a characterization of the capacity pre-loglog when the number of paths is finite.

[^1]Theorem 1. Consider the above channel model. Then

$$
\begin{align*}
(i)\left(\frac{\lim _{\ell \rightarrow \infty}}{} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}>0\right) & \Longrightarrow\left(\sup _{\operatorname{SNR}>0} C(\mathrm{SNR})<\infty\right)  \tag{11}\\
(i i)\left(\lim _{\ell \rightarrow \infty} \frac{1}{\ell} \log \frac{1}{\alpha_{\ell}}=\infty\right) & \Longrightarrow\left(\sup _{\mathrm{SNR}>0} C(\mathrm{SNR})=\infty\right) \tag{12}
\end{align*}
$$

where we define $a / 0 \triangleq \infty$ for every $a>0$ and $0 / 0 \triangleq 0$.
Proof. Part (i) is proven in Section 5.1, and Part (ii) is proven in Sections 6.1 \& 6.2,
By noting that

$$
\left(\lim _{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}=0\right) \quad \Longrightarrow \quad\left(\lim _{\ell \rightarrow \infty} \frac{1}{\ell} \log \frac{1}{\alpha_{\ell}}=0\right)
$$

we obtain from Theorem the immediate corollary:
Corollary 2. Consider the above channel model. Then

$$
\begin{align*}
(i)\left(\begin{array}{l}
\left.\lim _{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}>0\right)
\end{array}\right. & \Longrightarrow\left(\sup _{\text {SNR }>0} C(\mathrm{SNR})<\infty\right)  \tag{13}\\
\text { (ii) }\left(\lim _{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}=0\right) & \Longrightarrow\left(\sup _{\text {SNR }>0} C(\mathrm{SNR})=\infty\right), \tag{14}
\end{align*}
$$

where we define $a / 0 \triangleq \infty$ for every $a>0$ and $0 / 0 \triangleq 0$.
For example, if

$$
\begin{equation*}
\alpha_{\ell}=e^{-\ell}, \quad \ell \in \mathbb{N}_{0} \tag{15}
\end{equation*}
$$

then

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}=\frac{1}{e}>0 \tag{16}
\end{equation*}
$$

and it follows from Part (i) of Corollary 2 that the capacity is bounded in the SNR. On the other hand, if

$$
\begin{equation*}
\alpha_{\ell}=\exp \left(-\ell^{\kappa}\right), \quad \ell \in \mathbb{N}_{0} \tag{17}
\end{equation*}
$$

for some $\kappa>1$, then

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}=\lim _{\ell \rightarrow \infty} \exp \left(\ell^{\kappa}-(\ell+1)^{\kappa}\right)=0 \tag{18}
\end{equation*}
$$

and it follows from Part (ii) of Corollary 2 that the capacity is unbounded in the SNR. Roughly speaking, we can say that when $\left\{\alpha_{\ell}\right\}$ decays exponentially or slower, then $C(\mathrm{SNR})$ is bounded in SNR, and when $\left\{\alpha_{\ell}\right\}$ decays faster than exponentially, then $C(\mathrm{SNR})$ is unbounded in SNR.

The condition on the left-hand side (LHS) of (14) is surely satisfied if the channel has a finite number of paths, as in this case

$$
H_{k}^{(\ell)}=0, \quad \ell>\mathrm{L}, \quad k \in \mathbb{N}
$$

which implies

$$
\alpha_{\ell}=0, \quad \ell>\mathrm{L} \quad \text { and } \quad \frac{\alpha_{\ell+1}}{\alpha_{\ell}}=\frac{0}{0} \triangleq 0, \quad \ell>\mathrm{L} .
$$

Consequently, it follows from Corollary 2 that if the number of paths is finite, then $C$ (SNR) is unbounded in SNR. However, for this case the high-SNR behavior of the capacity can be characterized more accurately: Theorem 3 ahead shows that if the number of paths is finite, then the capacity pre-loglog, defined as

$$
\begin{equation*}
\Lambda \triangleq \varlimsup_{\mathrm{SNR} \rightarrow \infty} \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}} \tag{19}
\end{equation*}
$$

is 1 irrespective of the number of paths. The pre-loglog in this case is thus the same as for frequency-flat fading.

Theorem 3. Consider the above channel model. Further assume that the number of paths is finite. Then, irrespective of the number of paths, the capacity pre-loglog is given by

$$
\begin{equation*}
\Lambda=\lim _{\mathrm{SNR} \rightarrow \infty} \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}}=1 \tag{20}
\end{equation*}
$$

Proof. See Section 5.2 for the converse and Sections $6.1 \& 6.3$ for the direct part.
When studying multipath fading channels at low or at moderate SNR, it is often assumed that the channel has a finite number of paths, even if the number of paths is in reality infinite. This assumption is commonly justified by saying that only the first ( $L+1$ ) paths are relevant, since the variances of the remaining paths are typically small and hence the influence of these paths on the capacity is marginal. As we see from Theorems 1 \& 3, this argument is not valid anymore when studying multipath fading channels at high SNR. In fact, when for example the sequence of variances $\left\{\alpha_{\ell}\right\}$ decays exponentially, then according to Part (i) of Theorem 1 the capacity is bounded in the SNR. However, if we consider only the first $(L+1)$ paths and set the other paths to zero, then it follows from Theorem 3 that, irrespective of L, the capacity increases double-logarithmically with the SNR. Thus, even though the variances of the remaining paths $\alpha_{\ell}, \ell>\mathrm{L}$ can be made arbitrarily small by choosing L sufficiently large, these paths may have a significant influence on the capacity behavior at high SNR.

The reason why paths with a small variance can affect the capacity behavior is that the capacity depends on the variance of the product between the path gains and the transmitted signal and not on the variance of the path gains only. Since at high SNR the variance of $\sum_{\ell=\mathrm{L}+1}^{\infty} H_{k}^{(\ell)} X_{k-\ell}$ might be huge even if the variance of $\sum_{\ell=\mathrm{L}+1}^{\infty} H_{k}^{(\ell)}$ is small, the relevance of a path is determined not only by its own variance but also by the power available at the transmitter. The number of paths that are needed to approximate a multipath channel typically depends on the SNR and may grow to infinity as the SNR tends to infinity.

In order to prove the above results, we derive upper and lower bounds on the capacity. Since these bounds may also be of independent interest, we summarize them in the following propositions.

Proposition 4 (Upper Bounds).
(i) Consider the above channel model. Further assume that for some $0<\rho<1$ and some $\ell_{0} \in \mathbb{N}$

$$
\alpha_{\ell_{0}}>0 \quad \text { and } \quad \frac{\alpha_{\ell+1}}{\alpha_{\ell}} \geq \rho, \quad \ell \geq \ell_{0} .
$$

Then the capacity $C(\mathrm{SNR})$ is upper bounded by

$$
\begin{equation*}
C(\mathrm{SNR}) \leq \log \frac{2 \pi^{2}}{\sqrt{\tilde{\rho}}}-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right), \quad \mathrm{SNR} \geq 0 \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{\rho}=\min \left\{\rho^{\ell_{0}-1} \frac{\alpha_{\ell_{0}}}{\max _{0 \leq \ell^{\prime}<\ell_{0}} \alpha_{\ell^{\prime}}}, \rho^{\ell_{0}}\right\} \tag{22}
\end{equation*}
$$

(ii) Consider the above channel model. Further assume that

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} \alpha_{\ell} \triangleq \alpha<\infty \tag{23}
\end{equation*}
$$

Then

$$
\begin{equation*}
\varlimsup_{\mathrm{SNR} \rightarrow \infty}\{C(\mathrm{SNR})-\log \log \mathrm{SNR}\} \leq 1+\log \pi-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right) \tag{24}
\end{equation*}
$$

Proof. Part (i) is proven in Section 5.1, and Part (ii) in Section 5.2,

For example, if $\left\{\alpha_{\ell}\right\}$ is a geometric sequence, i.e.,

$$
\alpha_{\ell}=\rho^{\ell}, \quad \ell \in \mathbb{N}_{0}
$$

for some $0<\rho<1$, and if the path gains are Gaussian and memoryless so

$$
h_{\ell}=\log \left(\pi e \alpha_{\ell}\right), \quad \ell \in \mathbb{N}_{0}
$$

then Part (i) of Proposition 4 yields

$$
\begin{equation*}
C(\mathrm{SNR}) \leq \log \frac{2 \pi}{\sqrt{\rho}}-1, \quad \mathrm{SNR} \geq 0 \tag{25}
\end{equation*}
$$

Part (ii) of Proposition 4 combines with (10) to show that the pre-loglog of a multipath fading channel can never be larger than the pre-loglog of a frequency-flat fading channel. This result is consistent with the intuition that at high SNR the multipath behavior is detrimental.

Our last result is a lower bound on the capacity. This bound is the basis for the proof of Part (ii) of Theorem 1 and for the direct part of Theorem 3.

Proposition 5 (Lower Bound). Consider the above channel model. Further assume that

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} \alpha_{\ell} \triangleq \alpha<\infty \tag{26}
\end{equation*}
$$

Let $\mathrm{L}(\mathrm{P}) \in \mathbb{N}$ be some positive integer that satisfies

$$
\begin{equation*}
\sum_{\ell=\mathrm{L}(\mathrm{P})+1}^{\infty} \alpha_{\ell} \mathrm{P} \leq \sigma^{2} \tag{27}
\end{equation*}
$$

(typically $\mathrm{L}(\mathrm{P})$ depends on P$)$, and let $\tau \in \mathbb{N}$ be some arbitrary positive integer that is allowed to depend on $\mathrm{L}(\mathrm{P})$. Then the capacity $C(\mathrm{SNR})$ is lower bounded by

$$
\begin{array}{r}
C(\mathrm{SNR}) \geq \frac{\tau}{\mathrm{L}(\mathrm{P})+\tau} \log \log \mathrm{P}^{1 / \tau}+\frac{\tau}{\mathrm{L}(\mathrm{P})+\tau}\left(\mathrm{E}\left[\log \left|H_{1}^{(0)}\right|^{2}\right]-1-2 \log \left(\sqrt{\alpha_{0}}+\sqrt{\alpha+2 \sigma^{2}}\right)\right) \\
\mathrm{P}>1 .(28)
\end{array}
$$

Proof. See Section 6.1.

## 5 Proofs of the Upper Bounds

In this section, we establish a proof of Proposition 4, which in turn will be used to prove Part (i) of Theorem 1 and the converse to Theorem 3.

Part (i) of Proposition 4 is proven in Section 5.1 and it is demonstrated that Part (i) of Theorem 1 follows immediately from this result. Section 5.2 proves Part (ii) of Proposition 4. This part provides an upper bound on the capacity pre-loglog and will be used later, together with a capacity lower bound that is derived in Section 6, to establish Theorem 3 ,

### 5.1 Bounded Capacity

We provide a proof of Part (i) of Proposition 4 by deriving an upper bound on channel capacity that holds under the assumption that for some $0<\rho<1$ and some $\ell_{0} \in \mathbb{N}_{0}$

$$
\begin{equation*}
\alpha_{\ell_{0}}>0 \quad \text { and } \quad \frac{\alpha_{\ell+1}}{\alpha_{\ell}} \geq \rho, \quad \ell \geq \ell_{0} \tag{29}
\end{equation*}
$$

As this bound is finite for $\operatorname{SNR} \geq 0$, Part (i) of Theorem 1 follows immediately from Part (i) of Proposition 4 by noting that if

$$
\varliminf_{\ell \rightarrow \infty} \frac{\alpha_{\ell+1}}{\alpha_{\ell}}>0
$$

then we can find a $0<\rho<1$ and an $\ell_{0} \in \mathbb{N}$ satisfying (29).
The proof of the desired upper bound is akin to the proof of an upper bound that was derived in [7. Sec. 6.1]. (However, [7] studies a channel whose inputs \& outputs take value in the set of real numbers rather than in $\mathbb{C}$.) It is based on (7) and on an upper bound on $\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right)$. To this end, we begin with the chain rule for mutual information [4, Thm. 2.5.2]

$$
\begin{equation*}
\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right)=\frac{1}{n} \sum_{k=1}^{\ell_{0}} I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right)+\frac{1}{n} \sum_{k=\ell_{0}+1}^{n} I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \tag{30}
\end{equation*}
$$

Each term in the first sum on the right-hand side (RHS) of (30) is upper bounded by

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) & \leq h\left(Y_{k}\right)-h\left(Y_{k} \mid Y_{1}^{k-1}, X_{1}^{n}, H_{k}^{(0)}, H_{k}^{(1)}, \ldots, H_{k}^{(k-1)}\right) \\
& \leq \log \left(\pi e\left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right]\right)\right)-\log \left(\pi e \sigma^{2}\right) \\
& \leq \log \left(1+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{SNR}\right) \tag{31}
\end{align*}
$$

where the first inequality follows because conditioning cannot increase differential entropy (4), Thm. 9.6.1]; the second inequality follows from the entropy maximizing property of Gaussian random variables [4, Thm. 9.6.5]; and the last inequality follows by upper bounding $\alpha_{\ell} \leq$ $\sup _{\ell^{\prime} \in \mathbb{N}_{0}} \alpha_{\ell^{\prime}}, \ell=0,1, \ldots, k-1$ and from the power constraint (8).

For $k=\ell_{0}+1, \ell_{0}+2, \ldots, n$, we upper bound $I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right)$ using the general upper bound for mutual information [2, Thm. 5.1]

$$
\begin{equation*}
I(X ; Y) \leq \int D(W(\cdot \mid x) \| R(\cdot)) \mathrm{Q}(x) \tag{32}
\end{equation*}
$$

where $D(\cdot \| \cdot)$ denotes relative entropy, i.e.,

$$
D\left(P_{1} \| P_{0}\right)= \begin{cases}\int_{+\infty}^{\log } \frac{\mathrm{P}_{1}}{\mathrm{P}_{0}} \mathrm{P}_{1} & \text { if } P_{1} \ll P_{0} \\ +\infty & \text { otherwise }\end{cases}
$$

$W(\cdot \mid \cdot)$ is the channel law, $Q(\cdot)$ denotes the distribution on the channel input $X$, and $R(\cdot)$ is any distribution on the output alphabet 2 Thus any choice of output distribution $R(\cdot)$ yields an upper bound on the mutual information.

For any given $Y_{1}^{k-1}=y_{1}^{k-1}$, we choose the output distribution $R(\cdot)$ to be of density

$$
\begin{equation*}
\frac{\sqrt{\beta}}{\pi^{2}\left|y_{k}\right|} \frac{1}{1+\beta\left|y_{k}\right|^{2}}, \quad y_{k} \in \mathbb{C} \tag{33}
\end{equation*}
$$

with $\beta=1 /\left(\tilde{\rho}\left|y_{k-\ell_{0}}\right|^{2}\right)$ and

$$
\begin{equation*}
\tilde{\rho}=\min \left\{\rho^{\ell_{0}-1} \frac{\alpha_{\ell_{0}}}{\max _{0 \leq \ell^{\prime}<\ell_{0}} \alpha_{\ell^{\prime}}}, \rho^{\ell_{0}}\right\} . \tag{34}
\end{equation*}
$$

(If $y_{k-\ell_{0}}=0$, then the density (33) is undefined. However, this event is of zero probability and has therefore no impact on the mutual information $I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right)$.) With this choice

$$
\begin{equation*}
0<\tilde{\rho}<1 \quad \text { and } \quad \tilde{\rho} \alpha_{\ell} \leq \alpha_{\ell+\ell_{0}}, \quad \ell \in \mathbb{N}_{0} \tag{35}
\end{equation*}
$$

Using (33) in (32), and averaging over $Y_{1}^{k-1}$, we obtain

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \leq & \frac{1}{2} \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]+\frac{1}{2} \mathrm{E}\left[\log \left(\tilde{\rho}\left|Y_{k-\ell_{0}}\right|^{2}\right)\right]+\mathrm{E}\left[\log \left(1+\frac{\left|Y_{k}\right|^{2}}{\tilde{\rho}\left|Y_{k-\ell_{0}}\right|^{2}}\right)\right] \\
& -h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right)+\log \pi^{2} \\
= & \frac{1}{2} \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\frac{1}{2} \mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right]+\mathrm{E}\left[\log \left(\tilde{\rho}\left|Y_{k-\ell_{0}}\right|^{2}+\left|Y_{k}\right|^{2}\right)\right] \\
& -h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right)+\log \frac{\pi^{2}}{\sqrt{\tilde{\rho}}} . \tag{36}
\end{align*}
$$

[^2]We bound the third and the fourth term in (36) separately. We begin with

$$
\begin{align*}
\mathrm{E}\left[\log \left(\tilde{\rho}\left|Y_{k-\ell_{0}}\right|^{2}+\left|Y_{k}\right|^{2}\right)\right] & =\mathrm{E}\left[\mathrm{E}\left[\log \left(\tilde{\rho}\left|Y_{k-\ell_{0}}\right|^{2}+\left|Y_{k}\right|^{2}\right) \mid X_{1}^{k}\right]\right] \\
& \leq \mathrm{E}\left[\log \left(\tilde{\rho} \mathrm{E}\left[\left|Y_{k-\ell_{0}}\right|^{2} \mid X_{1}^{k}\right]+\mathrm{E}\left[\left|Y_{k}\right|^{2} \mid X_{1}^{k}\right]\right)\right] \\
& =\mathrm{E}\left[\log \left((1+\tilde{\rho}) \sigma^{2}+\sum_{\ell=0}^{k-\ell_{0}-1} \tilde{\rho} \alpha_{\ell}\left|X_{k-\ell_{0}-\ell}\right|^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \\
& \leq \mathrm{E}\left[\log \left(2 \sigma^{2}+\sum_{\ell=0}^{k-\ell_{0}-1} \alpha_{\ell+\ell_{0}}\left|X_{k-\ell_{0}-\ell}\right|^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \\
& =\mathrm{E}\left[\log \left(2 \sigma^{2}+\sum_{\ell^{\prime}=\ell_{0}}^{k-1} \alpha_{\ell^{\prime}}\left|X_{k-\ell^{\prime}}\right|^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \\
& \leq \log 2+\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \tag{37}
\end{align*}
$$

where the first inequality follows by Jensen's inequality; the subsequent equality follows by evaluating the expectations; the next inequality by (35); the following equality by substituting $\ell^{\prime}=\ell+\ell_{0}$; and the last inequality follows because

$$
\sum_{\ell=\ell_{0}}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2} \leq \sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}
$$

Next we derive a lower bound on $h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right)$. Let

$$
\begin{equation*}
\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}=\left(H_{1}^{(\ell)}, H_{2}^{(\ell)}, \ldots, H_{k-1}^{(\ell)}\right), \quad \ell \in \mathbb{N}_{0} \tag{38}
\end{equation*}
$$

and let

$$
\begin{equation*}
\mathbf{H}_{1}^{k-1}=\left(\left\{H_{k^{\prime}}^{(0)}\right\}_{k^{\prime}=1}^{k-1},\left\{H_{k^{\prime}}^{(1)}\right\}_{k^{\prime}=1}^{k-1}, \ldots,\left\{H_{k^{\prime}}^{(k-1)}\right\}_{k^{\prime}=1}^{k-1}\right) \tag{39}
\end{equation*}
$$

We have

$$
\begin{align*}
h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right) & \geq h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}, \mathbf{H}_{1}^{k-1}\right) \\
& =h\left(Y_{k} \mid X_{1}^{n}, \mathbf{H}_{1}^{k-1}\right) \tag{40}
\end{align*}
$$

where the inequality follows because conditioning cannot increase differential entropy; and where the equality follows because, conditional on $\left(X_{1}^{n}, \mathbf{H}_{1}^{k-1}\right), Y_{k}$ is independent of $Y_{1}^{k-1}$. Let $\mathcal{S}_{k}$ be defined as

$$
\begin{equation*}
\mathcal{S}_{k} \triangleq\left\{\ell=0,1, \ldots, k-1:\left|x_{k-\ell}\right|^{2} \alpha_{\ell}>0\right\} \tag{41}
\end{equation*}
$$

Using the entropy power inequality [4, Thm. 16.6.3], and using that the processes

$$
\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\},\left\{H_{k}^{(1)}, k \in \mathbb{N}\right\}, \ldots
$$

are independent and jointly independent of $X_{1}^{n}$, it is shown in Appendix A that for any given $X_{1}^{n}=x_{1}^{n}$

$$
\begin{align*}
h\left(\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid\right. & \left.X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}\right) \\
\geq & \log \left(\sum_{\ell \in \mathcal{S}_{k}} e^{h\left(H_{k}^{(\ell)} X_{k-\ell} \mid X_{k-\ell}=x_{k-\ell},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)}+e^{h\left(Z_{k}\right)}\right) . \tag{42}
\end{align*}
$$

We lower bound the differential entropies on the RHS of (42) as follows. The differential entropies in the sum are lower bounded by

$$
\begin{align*}
h\left(H_{k}^{(\ell)} X_{k-\ell} \mid\right. & \left.X_{k-\ell}=x_{k-\ell},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right) \\
& =\log \left(\alpha_{\ell}\left|x_{k-\ell}\right|^{2}\right)+h\left(H_{k}^{(\ell)} \mid\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)-\log \alpha_{\ell} \\
& \geq \log \left(\alpha_{\ell}\left|x_{k-\ell}\right|^{2}\right)+\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right), \quad \ell \in \mathcal{S}_{k} \tag{43}
\end{align*}
$$

where the equality follows from the behavior of differential entropy under scaling [4, Thm. 9.6.4]; and where the inequality follows by the stationarity of the process $\left\{H_{k}^{(\ell)}, k \in \mathbb{N}\right\}$, which implies that the differential entropy

$$
h\left(H_{k}^{(\ell)} \mid\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right), \quad \ell \in \mathcal{S}_{k}
$$

cannot be smaller than the differential entropy rate $h_{\ell}$ [4, Thms. 4.2.1 \& 4.2.2], and by lower bounding $\left(h_{\ell}-\log \alpha_{\ell}\right)$ by $\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)$ (which holds for each $\ell \in \mathcal{S}_{k}$ because $\mathcal{S}_{k} \subseteq \mathcal{L}$ ). The last differential entropy on the RHS of (42) is lower bounded by

$$
\begin{equation*}
h\left(Z_{k}\right)=\log \left(\pi e \sigma^{2}\right) \geq \inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)+\log \sigma^{2} \tag{44}
\end{equation*}
$$

which follows because conditioning cannot increase differential entropy, and because Gaussian random variables maximize differential entropy:

$$
\begin{align*}
\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right) & \leq \inf _{\ell \in \mathcal{L}}\left(h\left(H_{k}^{(\ell)}\right)-\log \alpha_{\ell}\right) \\
& \leq \inf _{\ell \in \mathcal{L}}\left(\log \left(\pi e \alpha_{\ell}\right)-\log \alpha_{\ell}\right) \\
& =\log (\pi e) \tag{45}
\end{align*}
$$

Applying (43) \& (44) to (42), and averaging over $X_{1}^{n}$, yields then

$$
\begin{align*}
h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right) & \geq \mathrm{E}\left[\log \left(\sum_{\ell \in \mathcal{S}_{k}} \alpha_{\ell}\left|X_{k-\ell}\right|^{2} e^{\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)}+\sigma^{2} e^{\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)}\right)\right] \\
& =\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right]+\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right) . \tag{46}
\end{align*}
$$

Returning to the analysis of (36), we obtain from (37) and (46)

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \leq & \frac{1}{2} \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\frac{1}{2} \mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right]+\log 2+\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \\
& -\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right]-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)+\log \frac{\pi^{2}}{\sqrt{\tilde{\rho}}} \\
= & \frac{1}{2} \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\frac{1}{2} \mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right]+\mathrm{K} \tag{47}
\end{align*}
$$

where K is defined as

$$
\begin{equation*}
\mathrm{K} \triangleq \log \frac{2 \pi^{2}}{\sqrt{\tilde{\rho}}}-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right) \tag{48}
\end{equation*}
$$

Applying (47) and (31) to (30), we have

$$
\begin{align*}
& \frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right) \\
& \quad \leq \frac{1}{n} \sum_{k=1}^{\ell_{0}} \log \left(1+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{SNR}\right)+\frac{1}{n} \sum_{k=\ell_{0}+1}^{n}\left(\frac{1}{2} \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\frac{1}{2} \mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right]+\mathrm{K}\right) \\
& \quad=\frac{\ell_{0}}{n} \log \left(1+\sup _{\ell \in \mathbb{N}} \alpha_{\ell} n \mathrm{SNR}\right)+\frac{n-\ell_{0}}{n} \mathrm{~K}+\frac{1}{2 n} \sum_{k=\ell_{0}+1}^{n}\left(\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right]\right) . \tag{49}
\end{align*}
$$

To show that the RHS of (49) is bounded in the SNR, we use that, for any sequences $\left\{a_{k}\right\}$ and $\left\{b_{k}\right\}$,

$$
\begin{equation*}
\sum_{k=\ell_{0}+1}^{n}\left(a_{k}-b_{k}\right)=\sum_{k=n-\ell_{0}+1}^{n}\left(a_{k}-b_{k-n+2 \ell_{0}}\right)+\sum_{k=\ell_{0}+1}^{n-\ell_{0}}\left(a_{k}-b_{k+\ell_{0}}\right) \tag{50}
\end{equation*}
$$

Defining

$$
\begin{equation*}
a_{k} \triangleq \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right] \tag{51}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{k} \triangleq \mathrm{E}\left[\log \left|Y_{k-\ell_{0}}\right|^{2}\right] \tag{52}
\end{equation*}
$$

we have for the first sum on the RHS of (50)

$$
\begin{align*}
\sum_{k=n-\ell_{0}+1}^{n}\left(a_{k}-b_{k-n+2 \ell_{0}}\right) & =\sum_{k=n-\ell_{0}+1}^{n}\left(\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\mathrm{E}\left[\log \left|Y_{k-n+\ell_{0}}\right|^{2}\right]\right) \\
& \leq \sum_{k=n-\ell_{0}+1}^{n}\left(\log \mathrm{E}\left[\left|Y_{k}\right|^{2}\right]-\mathrm{E}\left[\log \left|Y_{k-n+\ell_{0}}\right|^{2}\right]\right) \\
& \leq \sum_{k=n-\ell_{0}+1}^{n}\left(\log \left(\sigma^{2}+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{P}\right)-\mathrm{E}\left[\log \left|Y_{k-n+\ell_{0}}\right|^{2}\right]\right) \\
& \leq \sum_{k=n-\ell_{0}+1}^{n}\left(\log \left(\sigma^{2}+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{P}\right)-\mathrm{E}\left[\log \left|Z_{k-n+\ell_{0}}\right|^{2}\right]\right) \\
& =\ell_{0} \log \left(1+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{SNR}\right)+\ell_{0} \gamma \tag{53}
\end{align*}
$$

where $\gamma \approx 0.577$ denotes Euler's constant. Here the first inequality follows by Jensen's inequality; the following inequality follows by upper bounding

$$
\mathrm{E}\left[\left|Y_{k}\right|^{2}\right]=\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] \leq \sigma^{2}+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{P}
$$

the subsequent inequality follows by noting that, conditional on $\sum_{\ell=0}^{k-n+\ell_{0}-1} H_{k-n+\ell_{0}}^{(\ell)} X_{k-n+\ell_{0}-\ell}$, we have that $\left|Y_{k-n+\ell_{0}}\right|^{2}$ is stochastically larger than $\left|Z_{k-n+\ell_{0}}\right|^{2}$, so

$$
\begin{aligned}
\mathrm{E}\left[\log \left|Y_{k-n+\ell_{0}}\right|^{2} \mid\right. & \left.\sum_{\ell=0}^{k-n+\ell_{0}-1} H_{k-n+\ell_{0}}^{(\ell)} X_{k-n+\ell_{0}-\ell}\right] \\
& \geq \mathrm{E}\left[\left.\log \left|Z_{k-n+\ell_{0}}\right|^{2}\right|_{\ell=0} ^{k-n+\ell_{0}-1} H_{k-n+\ell_{0}}^{(\ell)} X_{k-n+\ell_{0}-\ell}\right]
\end{aligned}
$$

from which we obtain the lower bound $\mathrm{E}\left[\log \left|Y_{k-n+\ell_{0}}\right|^{2}\right] \geq \mathrm{E}\left[\log \left|Z_{k-n+\ell_{0}}\right|^{2}\right]$ upon averaging over $\sum_{\ell=0}^{k-n+\ell_{0}-1} H_{k-n+\ell_{0}}^{(\ell)} X_{k-n+\ell_{0}-\ell}$ (see [2, Sec. VI-B] and in particular [2, Lemma 6.2 b )]); and the last equality follows by evaluating the expected logarithm of an exponentially distributed random variable of mean $\sigma^{2}$, i.e., $\mathrm{E}\left[\log \left|Z_{k-n+\ell_{0}}\right|^{2}\right]=\log \sigma^{2}-\gamma$.

For the second sum on the RHS of (50) we have

$$
\begin{equation*}
\sum_{k=\ell_{0}+1}^{n-\ell_{0}}\left(a_{k}-b_{k+\ell_{0}}\right)=\sum_{k=\ell_{0}+1}^{n-\ell_{0}}\left(\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]\right)=0 \tag{54}
\end{equation*}
$$

Thus applying (50)-(54) to (49) yields

$$
\begin{equation*}
\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right) \leq \frac{2 \ell_{0}}{n} \log \left(1+\sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell} n \mathrm{SNR}\right)+\frac{n-\ell_{0}}{n} \mathrm{~K}+\frac{\ell_{0}}{n} \gamma \tag{55}
\end{equation*}
$$

which tends to

$$
\mathrm{K}=\log \frac{2 \pi^{2}}{\sqrt{\tilde{\rho}}}-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\log \alpha_{\ell}\right)
$$

as $n$ tends to infinity. This proves Part (i) of Proposition 4.

### 5.2 Unbounded Capacity

We prove Part (ii) of Proposition 4 by deriving an upper bound on capacity that holds under the assumption (26), namely,

$$
\sum_{\ell=0}^{\infty} \alpha_{\ell}<\infty
$$

From this upper bound follows that

$$
\begin{equation*}
\varlimsup_{\mathrm{SNR} \rightarrow \infty}\{C(\mathrm{SNR})-\log \log \mathrm{SNR}\}<\infty \tag{56}
\end{equation*}
$$

which in turn shows that the capacity pre-loglog is upper bounded by

$$
\begin{equation*}
\Lambda \triangleq \varlimsup_{\mathrm{SNR} \rightarrow \infty} \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}} \leq 1 \tag{57}
\end{equation*}
$$

This yields the converse to Theorem 3.
As in Section 5.11, the desired upper bound follows by (7) and by deriving an upper bound on $\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right)$. To this end, we begin with the chain rule for mutual information

$$
\begin{equation*}
I\left(X_{1}^{n} ; Y_{1}^{n}\right)=\sum_{k=1}^{n} I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \tag{58}
\end{equation*}
$$

and upper bound each summand on the RHS of (58) using [2, Eq. (27)]

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \leq & \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right)+\xi\left(1+\log \mathrm{E}\left[\left|Y_{k}\right|^{2}\right]-\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]\right) \\
& +\log \Gamma(\xi)-\xi \log \xi+\log \pi \\
= & (1-\xi) \mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right]-h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right)+\xi\left(1+\log \mathrm{E}\left[\left|Y_{k}\right|^{2}\right]\right) \\
& +\log \Gamma(\xi)-\xi \log \xi+\log \pi \tag{59}
\end{align*}
$$

for any $\xi>0$. Here $\Gamma(\cdot)$ denotes the Gamma function.
We evaluate the terms on the RHS of (59) individually. We upper bound the first term using Jensen's inequality

$$
\begin{align*}
\mathrm{E}\left[\log \left|Y_{k}\right|^{2}\right] & =\mathrm{E}\left[\mathrm{E}\left[\log \left|Y_{k}\right|^{2} \mid X_{1}^{k}\right]\right] \\
& \leq \mathrm{E}\left[\log \mathrm{E}\left[\left|Y_{k}\right|^{2} \mid X_{1}^{k}\right]\right] \\
& =\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \tag{60}
\end{align*}
$$

The second term was already evaluated in (46)

$$
\begin{equation*}
h\left(Y_{k} \mid X_{1}^{n}, Y_{1}^{k-1}\right) \geq \mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right]+\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right) \tag{61}
\end{equation*}
$$

and the next term is readily evaluated as

$$
\begin{equation*}
\log \mathrm{E}\left[\left|Y_{k}\right|^{2}\right]=\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right]\right) \tag{62}
\end{equation*}
$$

Our choice of $\xi$ will satisfy $\xi<1$ (see (64) ahead). We therefore obtain, upon substituting
(60) -(62) in (59),

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \leq & (1-\xi) \mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right]-\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \\
& -\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right)+\xi\left(1+\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right]\right)\right) \\
& +\log ^{2} \Gamma(\xi)-\xi \log \xi+\log \pi \\
= & -\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right) \\
& +\xi\left(1+\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right]\right)-\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right]\right) \\
& +\log \Gamma(\xi)-\xi \log \xi+\log \pi \\
\leq & -\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right)+\log \Gamma(\xi)-\xi \log \xi+\log \pi \\
& +\xi\left(1+\log \left(1+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] / \sigma^{2}\right)\right) \tag{63}
\end{align*}
$$

where the last inequality follows by lower bounding $\mathrm{E}\left[\log \left(\sigma^{2}+\sum_{\ell=0}^{k-1} \alpha_{\ell}\left|X_{k-\ell}\right|^{2}\right)\right] \geq \log \sigma^{2}$.
We choose

$$
\begin{equation*}
\xi=\frac{1}{1+\log (1+\alpha \mathrm{SNR})} \tag{64}
\end{equation*}
$$

(where $\alpha$ was defined in (26)). Defining

$$
\begin{equation*}
\Psi(\mathrm{SNR}) \triangleq\left[\log \Gamma(\xi)-\log \frac{1}{\xi}-\left.\xi \log \xi\right|_{\xi=(1+\log (1+\alpha \mathrm{SNR}))^{-1}}\right. \tag{65}
\end{equation*}
$$

we obtain

$$
\begin{gather*}
I\left(X_{1}^{n} ; Y_{k} \mid Y_{1}^{k-1}\right) \leq-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right)+\log (1+\log (1+\alpha \mathrm{SNR}))+\Psi(\mathrm{SNR})+\log \pi \\
 \tag{66}\\
+\frac{1+\log \left(1+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] / \sigma^{2}\right)}{1+\log (1+\alpha \mathrm{SNR})} .
\end{gather*}
$$

Using (66) in (58) yields then

$$
\begin{align*}
\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right) \leq & -\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right)+\log (1+\log (1+\alpha \mathrm{SNR}))+\Psi(\mathrm{SNR})+\log \pi \\
& +\frac{1+\frac{1}{n} \sum_{k=1}^{n} \log \left(1+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] / \sigma^{2}\right)}{1+\log (1+\alpha \mathrm{SNR})} \tag{67}
\end{align*}
$$

By Jensen's inequality we have

$$
\begin{align*}
\frac{1}{n} \sum_{k=1}^{n} \log \left(1+\sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] / \sigma^{2}\right) & \leq \log \left(1+\frac{1}{n} \sum_{k=1}^{n} \sum_{\ell=0}^{k-1} \alpha_{\ell} \mathrm{E}\left[\left|X_{k-\ell}\right|^{2}\right] / \sigma^{2}\right) \\
& \leq \log (1+\alpha \mathrm{SNR}), \tag{68}
\end{align*}
$$

where the last inequality follows by rewriting the double sum as

$$
\frac{1}{n} \sum_{k=1}^{n} \mathrm{E}\left[\left|X_{k}\right|^{2}\right] / \sigma^{2} \sum_{\ell=0}^{n-k} \alpha_{\ell}
$$

and by upper bounding then $\sum_{\ell=0}^{k-n} \alpha_{\ell} \leq \alpha$ and using the power constraint (8).

Combining (68) and (67) with (7), we obtain the upper bound

$$
\begin{equation*}
C(\mathrm{SNR}) \leq-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right)+\log (1+\log (1+\alpha \mathrm{SNR}))+\Psi(\mathrm{SNR})+\log \pi+1 . \tag{69}
\end{equation*}
$$

It follows by [2, Eq. (337)] that

$$
\begin{equation*}
\lim _{\mathrm{SNR} \rightarrow \infty} \Psi(\mathrm{SNR})=\lim _{\xi \downarrow 0}\left\{\log \Gamma(\xi)-\log \frac{1}{\xi}-\xi \log \xi\right\}=0 . \tag{70}
\end{equation*}
$$

Noting that

$$
\lim _{\mathrm{SNR} \rightarrow \infty}\{\log (1+\log (1+\alpha \mathrm{SNR}))-\log \log \mathrm{SNR}\}=0,
$$

we obtain from (69) and (70) the desired result

$$
\begin{equation*}
\varlimsup_{\mathrm{SNR} \rightarrow \infty}\{C(\mathrm{SNR})-\log \log \mathrm{SNR}\} \leq 1+\log \pi-\inf _{\ell \in \mathcal{L}}\left(h_{\ell}-\alpha_{\ell}\right) . \tag{71}
\end{equation*}
$$

## 6 Proofs of the Achievability Results

In Section 6.1] we derive the lower bound on channel capacity that is presented in Proposition 5 This lower bound will be used in Sections $6.2 \& 6.3$ to prove Part (ii) of Theorem 1 and to prove the direct part of Theorem 3 respectively.

### 6.1 Lower Bound

To derive the desired lower bound on capacity, we evaluate $\frac{1}{n} I\left(X_{1}^{n} ; Y_{1}^{n}\right)$ for the following distribution on the inputs $\left\{X_{k}\right\}$.

Let $\mathrm{L}(\mathrm{P})$ be such that

$$
\begin{equation*}
\sum_{\ell=\mathrm{L}(\mathrm{P})+1}^{\infty} \alpha_{\ell} \mathrm{P} \leq \sigma^{2} . \tag{72}
\end{equation*}
$$

To shorten notation, we shall write in the following L instead of $\mathrm{L}(\mathrm{P})$. Let $\tau \in \mathbb{N}$ be some positive integer that possibly depends on L , and let $\mathbf{X}_{b}=\left(X_{b(\mathrm{~L}+\tau)+1}, \ldots, X_{(b+1)(\mathrm{L}+\tau)}\right)$. We choose $\left\{\mathbf{X}_{b}\right\}$ to be IID with

$$
\mathbf{X}_{b}=(\underbrace{0, \ldots, 0}_{\mathrm{L}}, \tilde{X}_{b \tau+1}, \ldots, \tilde{X}_{(b+1) \tau}),
$$

where $\tilde{X}_{b \tau+1}, \ldots, \tilde{X}_{(b+1) \tau}$ is a sequence of independent, zero-mean, circularly-symmetric, complex random variables with $\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}$ being uniformly distributed over the interval $\left[\log \mathrm{P}^{(\nu-1) / \tau}, \log \mathrm{P}^{\nu / \tau}\right]$, i.e., for each $\nu=1, \ldots, \tau$

$$
\log \left|\tilde{X}_{b \tau+\nu}\right|^{2} \sim \mathcal{U}\left(\left[\log \mathrm{P}^{(\nu-1) / \tau}, \log \mathrm{P}^{\nu / \tau}\right]\right) .
$$

(Here and throughout this proof we assume that $\mathrm{P}>1$.)
Let $\kappa \triangleq\left\lfloor\frac{n}{\lfloor+\tau}\right\rfloor$ (where $\lfloor a\rfloor$ denotes the largest integer that is less than or equal to $a$ ), and let $\mathbf{Y}_{b}$ denote the vector $\left(Y_{b(L+\tau)+1}, \ldots, Y_{(b+1)(\mathrm{L}+\tau)}\right)$. By the chain rule for mutual information we have

$$
\begin{align*}
I\left(X_{1}^{n} ; Y_{1}^{n}\right) & \geq I\left(\mathbf{X}_{0}^{\kappa-1} ; \mathbf{Y}_{0}^{\kappa-1}\right) \\
& =\sum_{b=0}^{\kappa-1} I\left(\mathbf{X}_{b} ; \mathbf{Y}_{0}^{\kappa-1} \mid \mathbf{X}_{0}^{b-1}\right) \\
& \geq \sum_{b=0}^{\kappa-1} I\left(\mathbf{X}_{b} ; \mathbf{Y}_{b}\right), \tag{73}
\end{align*}
$$

where the first inequality follows by restricting the number of observables; and where the last inequality follows by restricting the number of observables and by noting that $\left\{\mathbf{X}_{b}\right\}$ is IID.

We continue by lower bounding each summand on the RHS of (73). We use again the chain rule and that reducing observations cannot increase mutual information to obtain

$$
\begin{align*}
I\left(\mathbf{X}_{b} ; \mathbf{Y}_{b}\right) & =\sum_{\nu=1}^{\tau} I\left(\tilde{X}_{b \tau+\nu} ; \mathbf{Y}_{b} \mid \tilde{X}_{b \tau+1}^{b \tau+\nu-1}\right) \\
& \geq \sum_{\nu=1}^{\tau} I\left(\tilde{X}_{b \tau+\nu} ; Y_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu} \mid \tilde{X}_{b \tau+1}^{b \tau+\nu-1}\right) \\
& \geq \sum_{\nu=1}^{\tau} I\left(\tilde{X}_{b \tau+\nu} ; Y_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}\right) \tag{74}
\end{align*}
$$

where we have additionally used in the last inequality that $\tilde{X}_{b \tau+1}, \ldots, \tilde{X}_{(b+1) \tau}$ are independent.
Defining

$$
\begin{equation*}
W_{b \tau+\nu} \triangleq \sum_{\ell=1}^{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-1} H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(\ell)} X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}+Z_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu} \tag{75}
\end{equation*}
$$

each summand on the RHS of (74) can be written as

$$
\begin{equation*}
I\left(\tilde{X}_{b \tau+\nu} ; Y_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}\right)=I\left(\tilde{X}_{b \tau+\nu} ; H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(0)} \tilde{X}_{b \tau+\nu}+W_{b \tau+\nu}\right) \tag{76}
\end{equation*}
$$

A lower bound on (76) follows from the following lemma.
Lemma 6. Let the random variables $X, H$, and $W$ have finite second moments. Assume that both $X$ and $H$ are of finite differential entropy. Finally, assume that $X$ is independent of $H$; that $X$ is independent of $W$; and that $X \multimap-H \multimap-W$ forms a Markov chain. Then

$$
\begin{equation*}
I(X ; H X+W) \geq h(X)-\mathrm{E}\left[\log |X|^{2}\right]+\mathrm{E}\left[\log |H|^{2}\right]-\mathrm{E}\left[\log \left(\pi e\left(\sigma_{H}+\frac{\sigma_{W}}{|X|}\right)^{2}\right)\right] \tag{77}
\end{equation*}
$$

where $\sigma_{H}^{2} \geq 0$ and $\sigma_{H}^{2}>0$ denote the variances of $W$ and $H$. (Note that the assumptions that $X$ and $H$ have finite second moments and are of finite differential entropy guarantee that $\mathrm{E}\left[\log |X|^{2}\right]$ and $\mathrm{E}\left[\log |H|^{2}\right]$ are finite, see [2, Lemma 6.7e].)
Proof. See [10, Lemma 4].
It can be easily verified that for the channel model given in Section 2 and for the above coding scheme the lemma's conditions are satisfied. We therefore obtain from Lemma 6

$$
\begin{align*}
I\left(\tilde{X}_{b \tau+\nu} ; H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(0)} \tilde{X}_{b \tau+\nu}+W_{b \tau+\nu}\right) \geq & h\left(\tilde{X}_{b \tau+\nu}\right)-\mathrm{E}\left[\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}\right]+\mathrm{E}\left[\log \left|H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(0)}\right|^{2}\right] \\
& -\mathrm{E}\left[\log \left(\pi e\left(\sqrt{\alpha_{0}}+\frac{\sqrt{\mathrm{E}\left[\left|W_{b \tau+\nu}\right|^{2}\right]}}{\left|\tilde{X}_{b \tau+\nu}\right|}\right)^{2}\right)\right] \tag{78}
\end{align*}
$$

Using that the differential entropy of a circularly-symmetric random variable is given by (see 2, Eqs. (320) \& (316)])

$$
\begin{equation*}
h\left(\tilde{X}_{b \tau+\nu}\right)=\mathrm{E}\left[\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}\right]+h\left(\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}\right)+\log \pi \tag{79}
\end{equation*}
$$

and evaluating $h\left(\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}\right)$ for our choice of $\tilde{X}_{b \tau+\nu}$, yields for the first two terms on the RHS of (78)

$$
\begin{equation*}
h\left(\tilde{X}_{b \tau+\nu}\right)-\mathrm{E}\left[\log \left|\tilde{X}_{b \tau+\nu}\right|^{2}\right]=\log \log \mathrm{P}^{1 / \tau}+\log \pi \tag{80}
\end{equation*}
$$

We next upper bound

$$
\begin{align*}
\frac{\mathrm{E}\left[\left|W_{b \tau+\nu}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}}= & \sum_{\ell=1}^{\mathrm{L}} \alpha_{\ell} \frac{\mathrm{E}\left[\left|X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}}+\sum_{\ell=\mathrm{L}+1}^{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-1} \alpha_{\ell} \frac{\mathrm{E}\left[\left|X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}} \\
& +\frac{\sigma^{2}}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}} \tag{81}
\end{align*}
$$

To this end, we note that for our choice of $\left\{X_{k}\right\}$ and by the assumption that $\mathrm{P}>1$, we have

$$
\begin{gather*}
\mathrm{E}\left[\left|X_{\ell}\right|^{2}\right] \leq \mathrm{P}, \quad \ell \in \mathbb{N}  \tag{82}\\
\mathrm{E}\left[\left|X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}\right|^{2}\right] \leq \mathrm{P}^{(\nu-\ell) / \tau}, \quad \ell=1, \ldots, \mathrm{~L} \tag{83}
\end{gather*}
$$

and

$$
\begin{equation*}
\left|\tilde{X}_{b \tau+\nu}\right|^{2} \geq \mathrm{P}^{(\nu-1) / \tau} \geq 1 \tag{84}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
\frac{\mathrm{E}\left[\left|X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}} \leq \frac{\mathrm{P}^{(\nu-\ell) / \tau}}{\mathrm{P}^{(\nu-1) / \tau}} \leq 1, \quad \ell=1, \ldots, \mathrm{~L} \tag{85}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\mathrm{E}\left[\left|X_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-\ell}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}} \leq \mathrm{P}, \quad \ell=\mathrm{L}+1, \ldots, b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-1 \tag{86}
\end{equation*}
$$

Applying (84)-(86) to (81) yields

$$
\begin{align*}
\frac{\mathrm{E}\left[\left|W_{b \tau+\nu}\right|^{2}\right]}{\left|\tilde{X}_{b \tau+\nu}\right|^{2}} & \leq \sum_{\ell=1}^{\mathrm{L}} \alpha_{\ell}+\sum_{\ell=\mathrm{L}+1}^{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu-1} \alpha_{\ell} \mathrm{P}+\sigma^{2} \\
& \leq \alpha+\sum_{\ell=\mathrm{L}+1}^{\infty} \alpha_{\ell} \mathrm{P}+\sigma^{2} \\
& \leq \alpha+2 \sigma^{2} \tag{87}
\end{align*}
$$

with $\alpha$ being defined in (26). Here the second inequality follows because $\alpha_{\ell}, \ell \in \mathbb{N}_{0}$ and P are nonnegative, and the last inequality follows from (72).

By combining (78) with (80) \& (87), and by noting that by the stationarity of $\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\}$

$$
\mathrm{E}\left[\log \left|H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(0)}\right|^{2}\right]=\mathrm{E}\left[\log \left|H_{1}^{(0)}\right|^{2}\right]
$$

we obtain the lower bound

$$
\begin{align*}
I\left(\tilde{X}_{b \tau+\nu} ; H_{b(\mathrm{~L}+\tau)+\mathrm{L}+\nu}^{(0)} \tilde{X}_{b \tau+\nu}+W_{b \tau+\nu}\right) \geq & \log \log \mathrm{P}^{1 / \tau}+\mathrm{E}\left[\log \left|H_{1}^{(0)}\right|^{2}\right]-1 \\
& -2 \log \left(\sqrt{\alpha_{0}}+\sqrt{\alpha+2 \sigma^{2}}\right) \tag{88}
\end{align*}
$$

Note that the RHS of (88) neither depends on $\nu$ nor on $b$. We therefore have from (88), (74), and (73)

$$
\begin{equation*}
I\left(X_{1}^{n} ; Y_{1}^{n}\right) \geq \kappa \tau \log \log P^{1 / \tau}+\kappa \tau \Upsilon \tag{89}
\end{equation*}
$$

where we define $\Upsilon$ as

$$
\begin{equation*}
\Upsilon \triangleq \mathrm{E}\left[\log \left|H_{1}^{(0)}\right|^{2}\right]-1-2 \log \left(\sqrt{\alpha_{0}}+\sqrt{\alpha+2 \sigma^{2}}\right) \tag{90}
\end{equation*}
$$

Dividing the RHS of (89) by $n$, and computing the limit as $n$ tends to infinity, yields the lower bound

$$
\begin{equation*}
C(\mathrm{SNR}) \geq \frac{\tau}{\mathrm{L}+\tau} \log \log \mathrm{P}^{1 / \tau}+\frac{\tau}{\mathrm{L}+\tau} \Upsilon, \quad \mathrm{P}>1 \tag{91}
\end{equation*}
$$

where we have used that $\lim _{n \rightarrow \infty} \kappa / n=1 /(\mathrm{L}+\tau)$. This proves Proposition 5 .

### 6.2 Condition for Unbounded Capacity

We use Proposition 5 to prove Part (ii) of Theorem 1. In particular, we show that if

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \frac{1}{\ell} \log \frac{1}{\alpha_{\ell}}=\infty \tag{92}
\end{equation*}
$$

then, by cleverly choosing $\mathrm{L}(\mathrm{P})$ and $\tau$, the lower bound (28), namely,

$$
C(\mathrm{SNR}) \geq \frac{\tau}{\mathrm{L}(\mathrm{P})+\tau} \log \log \mathrm{P}^{1 / \tau}+\frac{\tau}{\mathrm{L}(\mathrm{P})+\tau} \Upsilon, \quad \mathrm{P}>1
$$

(where $\Upsilon$ is defined in (90)), can be made arbitrarily large as SNR tends to infinity. To this end, we first note that (92) implies that for every $0<\varrho<1$ we can find an $\ell_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\alpha_{\ell}<\varrho^{\ell}, \quad \ell \geq \ell_{0} \tag{93}
\end{equation*}
$$

By choosing

$$
\begin{equation*}
\mathrm{L}(\mathrm{P})=\left\lceil\frac{\log \left(\mathrm{P} / \sigma^{2} \varrho /(1-\varrho)\right)}{\log (1 / \varrho)}\right\rceil \tag{94}
\end{equation*}
$$

(where $\lceil a\rceil$ denotes the smallest integer that is greater than or equal to $a$ ) and $\tau=\mathrm{L}(\mathrm{P})$, we obtain from (28) the lower bound

$$
\begin{equation*}
C(\mathrm{SNR}) \geq \frac{1}{2} \log \frac{\log \mathrm{P}}{\left\lceil\frac{\log \left(\mathrm{P} / \sigma^{2} \varrho /(1-\varrho)\right)}{\log (1 / \varrho)}\right\rceil}+\frac{1}{2} \Upsilon, \quad \mathrm{P}>1 \tag{95}
\end{equation*}
$$

Taking the limit as SNR (and hence also $\mathrm{P}=\sigma^{2} \mathrm{SNR}$ ) tends to infinity, yields

$$
\begin{equation*}
\lim _{\mathrm{SNR} \rightarrow \infty} C(\mathrm{SNR}) \geq \frac{1}{2} \log \log \frac{1}{\varrho}+\frac{1}{2} \Upsilon . \tag{96}
\end{equation*}
$$

Since this holds for every $0<\varrho<1$

$$
\begin{equation*}
\sup _{\mathrm{SNR}>0} C(\mathrm{SNR})=\infty \tag{97}
\end{equation*}
$$

It remains to show that $\left\{\alpha_{\ell}\right\}$ and our choice of $\mathrm{L}(\mathrm{P})$ (94) satisfy the conditions (26) \& (27) of Proposition 5, namely,

$$
\sum_{\ell=0}^{\infty} \alpha_{\ell}<\infty \quad \text { and } \quad \sum_{\ell=\mathrm{L}(\mathrm{P})+1}^{\infty} \alpha_{\ell} \mathrm{P} \leq \sigma^{2}
$$

It follows immediately from (5) and (93) that $\left\{\alpha_{\ell}\right\}$ satisfies the first condition (26):

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} \alpha_{\ell}=\sum_{\ell=0}^{\ell_{0}-1} \alpha_{\ell}+\sum_{\ell=\ell_{0}}^{\infty} \alpha_{\ell}<\ell_{0} \sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell}+\sum_{\ell=\ell_{0}}^{\infty} \varrho^{\ell}=\ell_{0} \sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell}+\frac{\varrho^{\ell_{0}}}{1-\varrho}<\infty \tag{98}
\end{equation*}
$$

In order to show that $L(P)$ satisfies the second condition (27), we first note that by (93)

$$
\begin{equation*}
\sum_{\ell=\ell^{\prime}+1}^{\infty} \alpha_{\ell}<\sum_{\ell=\ell^{\prime}+1}^{\infty} \varrho^{\ell}=\varrho^{\ell^{\prime}} \frac{\varrho}{1-\varrho}, \quad \ell^{\prime} \geq \ell_{0}-1 \tag{99}
\end{equation*}
$$

Since $L(P)$ tends to infinity as $P \rightarrow \infty$, it follows that $L(P)$ is greater than $\left(\ell_{0}-1\right)$ for sufficiently large P. Furthermore, (94) implies

$$
\begin{equation*}
\varrho^{\mathrm{L}(\mathrm{P})} \frac{\varrho}{1-\varrho} \mathrm{P} \leq \sigma^{2} \tag{100}
\end{equation*}
$$

We therefore obtain from (99) and (100)

$$
\begin{equation*}
\sum_{\ell=\mathrm{L}(\mathrm{P})+1}^{\infty} \alpha_{\ell} \mathrm{P}<\varrho^{\mathrm{L}(\mathrm{P})} \frac{\varrho}{1-\varrho} \mathrm{P} \leq \sigma^{2} \tag{101}
\end{equation*}
$$

thus demonstrating that $L(P)$ satisfies (27).

### 6.3 The Pre-LogLog

We use Proposition 5 to prove Theorem 3. To this end, we first note that because the number of paths is finite, we have for some $L \in \mathbb{N}_{0}$

$$
\begin{equation*}
\alpha_{\ell}=0, \quad \ell>\mathrm{L} \tag{102}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} \alpha_{\ell}=\sum_{\ell=0}^{\mathrm{L}} \alpha_{\ell} \leq(\mathrm{L}+1) \sup _{\ell \in \mathbb{N}_{0}} \alpha_{\ell}<\infty \tag{103}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{\ell=\mathrm{L}+1}^{\infty} \alpha_{\ell} \mathrm{P}=0 \leq \sigma^{2} \tag{104}
\end{equation*}
$$

Consequently, it follows from (28) of Proposition 5 that the capacity is lower bounded by

$$
\begin{equation*}
C(\mathrm{SNR}) \geq \frac{\tau}{\mathrm{L}+\tau} \log \log \mathrm{P}^{1 / \tau}+\frac{\tau}{\mathrm{L}+\tau} \Upsilon, \quad \mathrm{P}>1 \tag{105}
\end{equation*}
$$

Dividing by $\log \log$ SNR, and computing the limit as $\mathrm{SNR} \rightarrow \infty$, yields

$$
\begin{equation*}
\lim _{\mathrm{SNR} \rightarrow \infty} \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}} \geq \frac{\tau}{\mathrm{L}+\tau} \tag{106}
\end{equation*}
$$

where we have used that for any fixed $\tau$

$$
\lim _{\mathrm{SNR} \rightarrow \infty} \frac{\log \log \mathrm{P}^{1 / \tau}}{\log \log \mathrm{SNR}}=1
$$

The lower bound on the capacity pre-loglog

$$
\begin{equation*}
\Lambda \triangleq \varlimsup_{\mathrm{SNR} \rightarrow \infty} \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}} \geq \varliminf_{\mathrm{SNR} \rightarrow \infty}^{\lim } \frac{C(\mathrm{SNR})}{\log \log \mathrm{SNR}} \geq 1 \tag{107}
\end{equation*}
$$

follows then by letting $\tau$ tend to infinity. Together with the upper bound $\Lambda \leq 1$, which was derived in Section 5.2, this proves Theorem 3.

## 7 Conclusion

We studied the high-SNR behavior of the capacity of noncoherent multipath fading channels. We demonstrated that, depending on the decay rate of the sequence $\left\{\alpha_{\ell}\right\}$, capacity may be bounded or unbounded in the SNR. We further showed that if the number of paths is finite, then at high SNR capacity grows double-logarithmically with the SNR, and the capacity pre-loglog is irrespective of the number of paths. The picture that emerges is as follows:

- If the sequence of variances $\left\{\alpha_{\ell}\right\}$ decays exponentially or slower, then capacity is bounded in the SNR.
- If the sequence of variances $\left\{\alpha_{\ell}\right\}$ decays faster than exponentially, then capacity is unbounded in the SNR.
- If the number of paths is finite, then the capacity pre-loglog is equal to 1 , irrespective of the number of paths.

The conclusions that can be drawn from these results are twofold. First, multipath channels with an infinite number of paths and multipath channels with a finite number of paths have in general completely different capacity behaviors at high SNR. Indeed, at high SNR, if the number of paths is finite, then capacity grows double-logarithmically with the SNR, whereas if the number of paths is infinite, then capacity may even be bounded in the SNR. Thus,
while for low or for moderate SNR it might be reasonable to approximate a multipath channel with infinitely many paths by a multipath channel with only a finite number paths, this is not reasonable when the SNR tends to infinity. The number of paths that are needed to approximate a multipath channel typically depends on the SNR and may grow to infinity as the SNR tends to infinity.

Second, the above results indicate that the high-SNR behavior of the capacity of multipath fading channels depends critically on the assumed channel model. Thus when studying such channels at high SNR, the channel modeling is crucial, as slight changes in the channel model might lead to completely different capacity results.

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## A Appendix to Section 5.1

To prove (42), we lower bound

$$
\begin{equation*}
h\left(\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}=\mathbf{h}_{1}^{k-1}\right) \tag{108}
\end{equation*}
$$

for a given $\mathbf{h}_{1}^{k-1}$, and average then the result over $\mathbf{H}_{1}^{k-1}$. Let $\mathcal{H}_{k}$ denote the set

$$
\begin{equation*}
\mathcal{H}_{k} \triangleq\left\{H_{k}^{(\ell)}, \ell=0, \ldots, k-1: \alpha_{\ell}=0\right\} \tag{109}
\end{equation*}
$$

We have

$$
\begin{align*}
& h\left(\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}=\mathbf{h}_{1}^{k-1}\right) \\
& \quad \geq h\left(\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}=\mathbf{h}_{1}^{k-1}, \mathcal{H}_{k}\right) \\
& \quad=h\left(\sum_{\ell \in \mathcal{S}_{k}} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}=\mathbf{h}_{1}^{k-1}, \mathcal{H}_{k}\right) \\
& \quad \geq \log \left(\sum_{\ell \in \mathcal{S}_{k}} e^{h\left(H_{k}^{(\ell)} X_{k-\ell} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}=\left\{h_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)}+e^{h\left(Z_{k}\right)}\right) \tag{110}
\end{align*}
$$

where $\mathcal{S}_{k}$ is defined in (41). Here the first inequality follows because conditioning cannot increase differential entropy; the following equality follows because differential entropy is invariant under deterministic translation [4, Thm. 9.6.3], and because the terms where $x_{k-\ell}=0$ do not contribute to the sum; and the last inequality follows by the entropy power inequality [4, Thm. 16.6.3], and because the processes

$$
\left\{H_{k}^{(0)}, k \in \mathbb{N}\right\},\left\{H_{k}^{(1)}, k \in \mathbb{N}\right\}, \ldots
$$

are independent. (Note that, for a given $\mathbf{H}_{1}^{k-1}=\mathbf{h}_{1}^{k-1}$, the conditional entropies on the RHS of (110) are possibly infinite. However, by (6) this event is of zero probability and is therefore immaterial to (110) when averaged over $\mathbf{H}_{1}^{k-1}$.)

Since the processes of the path gains are independent and jointly independent of $X_{1}^{n}$, we can compute the expectation of (110) over $\mathbf{H}_{1}^{k-1}$ by averaging (110) first over $\left(H_{1}^{(0)}, \ldots, H_{k-1}^{(0)}\right)$, then averaging the result over $\left(H_{1}^{(1)}, \ldots, H_{k-1}^{(1)}\right)$, and so on. To lower bound the individual expectations, we note that the function

$$
\begin{equation*}
f(x)=\log \left(e^{x}+\zeta\right), \quad x \in \mathbb{R} \tag{111}
\end{equation*}
$$

is convex for all $\zeta>0$. Thus, by setting for each $\ell^{\prime}=0, \ldots, k-1$

$$
\begin{align*}
\zeta_{\ell^{\prime}}= & \sum_{\substack{\ell \in \mathcal{S}_{k}, \ell<\ell^{\prime}}} e^{h\left(H_{k}^{(\ell)} X_{k-\ell} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)} \\
& +\sum_{\substack{\ell \in \mathcal{S}_{k}, \ell>\ell^{\prime}}} e^{h\left(H_{k}^{(\ell)} X_{k-\ell} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}=\left\{h_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)}+e^{h\left(Z_{k}\right)}, \tag{112}
\end{align*}
$$

it follows from Jensen's inequality

$$
\begin{align*}
& \mathrm{E}_{\left\{H_{k^{\prime}}^{\left(\ell^{\prime}\right)}\right\}_{k^{\prime}=1}^{k-1}}\left[\log \left(\mathrm{I}\left\{\ell^{\prime} \in \mathcal{S}_{k}\right\} e^{h\left(H_{k}^{\left(\ell^{\prime}\right)} X_{k-\ell^{\prime}} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{\left(\ell^{\prime}\right)}\right\}_{k^{\prime}=1}^{k-1}=\left\{h_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)}+\zeta_{\ell^{\prime}}\right)\right] \\
& \quad \geq \log \left(\mathrm{I}\left\{\ell^{\prime} \in \mathcal{S}_{k}\right\} e^{h\left(H_{k}^{\left(\ell^{\prime}\right)} X_{k-\ell^{\prime}} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{\left(\ell^{\prime}\right)}\right\}_{k^{\prime}=1}^{k-1}\right)}+\zeta_{\ell^{\prime}}\right), \quad \ell^{\prime}=0, \ldots, k-1, \tag{113}
\end{align*}
$$

where $I\{\cdot\}$ denotes the indicator function, i.e.,

$$
I\{\text { statement }\}= \begin{cases}1 & \text { if statement is true }  \tag{114}\\ 0 & \text { if statement is false }\end{cases}
$$

Averaging (110) over $\mathbf{H}_{1}^{k-1}$, and employing (113) to compute this average, yields thus

$$
\begin{align*}
h\left(\sum_{\ell=0}^{k-1} H_{k}^{(\ell)} X_{k-\ell}+Z_{k} \mid\right. & \left.X_{1}^{n}=x_{1}^{n}, \mathbf{H}_{1}^{k-1}\right) \\
& \geq \log \left(\sum_{\ell \in \mathcal{S}_{k}} e^{h\left(H_{k}^{(\ell)} X_{k-\ell} \mid X_{1}^{n}=x_{1}^{n},\left\{H_{k^{\prime}}^{(\ell)}\right\}_{k^{\prime}=1}^{k-1}\right)}+e^{h\left(Z_{k}\right)}\right) . \tag{115}
\end{align*}
$$

This proves the lower bound (421).

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[^1]:    ${ }^{1}$ Note that the capacity of coherent fading channels (where the fading realization is known to the receiver) behaves logarithmically with the SNR 6]. Thus in the coherent case it suffices to double the SNR for every additional bit per channel use.

[^2]:    ${ }^{2}$ For channels with finite input and output alphabets this inequality follows by Topsøe's identity [8] see also (9) Thm. 3.4].

