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***Abstract*—In this work, we consider a stopping set analysis of repeat multiple-accumulate (RMA) code ensembles formed by the serial concatenation of a repetition code with multiple accumulators. The RMA codes are assumed to be iteratively decoded in a constituent code oriented fashion using maximum *a posteriori* erasure correction in the constituent codes. We give stopping set enumerators for RMA code ensembles and show that their stopping distance $h_{\min}$, defined as the size of the smallest nonempty stopping set, asymptotically grows linearly with the block length. Thus, the RMA code ensembles are *good* for the binary erasure channel. Furthermore, it is shown that, contrary to the asymptotic minimum distance $d_{\min}$, whose growth rate coefficient increases with the number of accumulate codes, the $h_{\min}$ growth rate coefficient diminishes with the number of accumulators. We also consider random puncturing and show that for sufficiently high code rates, the asymptotic $h_{\min}$ does not grow linearly with the block length, contrary to the asymptotic $d_{\min}$, whose growth rate coefficient approaches the Gilbert-Varshamov bound as the rate increases. Finally, we give iterative decoding thresholds to show the convergence properties.**

I. Introduction

The performance of a concatenated code in the error floor region for an additive white Gaussian noise channel is dominated by the distance spectrum of the code and, in particular, by its minimum distance $d_{\min}$. Turbo codes [1] and single serially concatenated codes [2] are asymptotically bad codes, in the sense that their minimum distance does not asymptotically grow linearly with the block length. Recently, it was shown that the minimum distance $d_{\min}$ of very simple serially concatenated code ensembles built from the concatenation of a repeat code with two or more accumulators increases as the number of accumulators increase. In fact, in the limit of infinitely many accumulators, the $d_{\min}$ approaches the Gilbert-Varshamov bound (GVB) [3]. Furthermore, in [4, 5], it was shown that, indeed, repeat multiple-accumulate (RMA) code ensembles are asymptotically good, in the sense that their $d_{\min}$ asymptotically grows linearly with the block length. A method for the calculation of a lower bound on the growth rate coefficient was also given in [4].

For the binary erasure channel (BEC) with iterative decoding, a similar role to that of the distance spectrum with maximum-likelihood decoding is played by the stopping set distribution. Stopping sets for iterative belief-propagation (BP) decoding of low-density parity-check (LDPC) codes were introduced in [6]. The concept of stopping sets was later extended in [7] to turbo decoding by introducing turbo stopping sets. In this work, we extend the asymptotic minimum distance analysis in [4, 5] by considering stopping sets for iterative constituent code oriented decoding using maximum *a posteriori* (MAP) erasure correction in the constituent codes of RMA codes. We derive a closed-form expression for the *subcode input-output support size enumerating function* (SIOSEF) [7] of an accumulate code and obtain stopping set enumerators for RMA code ensembles. We then analyze the asymptotic behavior of the stopping set distribution and show that RMA code ensembles exhibit an asymptotic stopping distance $h_{\min}$, defined as the size of the smallest nonempty stopping set, that grows linearly with the block length. Therefore, RMA codes are *good* for the BEC. However, contrary to the asymptotic $d_{\min}$, whose growth rate coefficient increases with the number of accumulate codes, the asymptotic $h_{\min}$ growth rate coefficient diminishes with the number of accumulators. We also consider random puncturing of RMA codes and show that for sufficiently high code rates, the asymptotic $h_{\min}$ does not grow linearly with the block length, contrary to the asymptotic $d_{\min}$, whose growth rate coefficient approaches the GVB as the rate increases [4]. Finally, we consider extrinsic information transfer (EXIT) charts [8] to analyze the convergence properties of RMA codes. We will show that repeat accumulate-accumulate (RAA) codes are good codes for the BEC, while increasing the number of accumulators decreases both the stopping distance and the convergence threshold.

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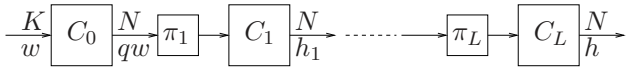

Fig. 1. Encoder structure for RMA codes.

II. Iterative Constituent Code Oriented Decoding and Stopping Sets

The encoder structure of RMA codes is depicted in Fig. 1. It is a serial concatenation of a (Kq, K) repetition code C_0 with input block length K and rate $1/q$ with $L \geq 1$ identical rate-1, memory-one, accumulators C_l, $l = 1, \ldots, L$, with

generator polynomials $g(D) = 1/(1+D)$, through interleavers $\pi_1, \ldots, \pi_L$. The overall nominal code rate (avoiding termination) is denoted by $R = K/N = 1/q$, where N denotes the output block length. Higher rates may be obtained by puncturing the output of the most inner accumulator C_L.

In this paper, we consider stopping sets for iterative constituent code oriented decoding using MAP constituent decoders (turbo decoding) over the BEC of RMA codes. With iterative constituent code oriented decoding we mean a decoding strategy that iterates between the constituent decoders. MAP decoding of the constituent codes can be implemented efficiently on a trellis representation of the constituent code [7]. Thus, the complexity of iterative constituent code oriented decoding of RMA codes is linear in the block length N when the number of constituent code activations is independent of N.

A. Stopping Sets for RMA Codes

We will now give the formal definition of a stopping set for RMA codes, adapted from the definition in [7] for turbo stopping sets. The generalization to the case with puncturing is straightforward. In the definition, we need the concept of support set of a binary vector and of a binary linear code. The support set $\chi(\mathbf{x})$ of a binary vector $\mathbf{x} = (x_1, \ldots, x_N)$ (of length N) is the set of nonzero coordinates. As an example, with $\mathbf{x} = (0, 1, 1, 0, 1)$, $\chi(\mathbf{x}) = \{2, 3, 5\}$. The support set $\chi(C)$ of a binary linear code C is the union of the support sets of each codeword in C. Finally, an interleaver will be regarded as a mapping from the set of coordinates of its input sequence to the set of coordinates of its output sequence.

Definition 1. *Let C_{RMA} denote a given RMA code with interleavers $\pi_1, \ldots, \pi_L$. A set $\mathcal{S} = \mathcal{S}(\pi_1, \ldots, \pi_L) \subseteq \{1, \ldots, N\}$ of the coordinates of the output sequence (or codeword) is a stopping set if and only if there exist $L+1$ linear subcodes $\hat{C}_l \subseteq C_l \subseteq \{0,1\}^N$, $l = 0, \ldots, L$, with support sets $\chi(\hat{C}_l)$ such that*

$$\chi(\hat{C}_L) = \mathcal{S} \text{ and } \pi_l(\chi(\hat{C}_{l-1})) = \chi(\hat{C}_l),\ l = 1, \ldots, L.$$

The size *of a stopping set $\mathcal{S}$ is its cardinality.*

In Definition 1 we used the fact that the mapping between the input support set and the output support set is an identity mapping for rate $R = 1$ encoders. Note also that Definition 1 does not exclude the empty set. Thus, the empty set is formally a stopping set of size zero. We stress the fact that the concept of stopping sets for RMA codes, as defined above, is conceptually different from the traditional concept of stopping sets used in connection with iterative BP decoding of LDPC codes, but it reduces to the traditional concept of stopping sets for Tanner graphs when the constituent codes are single parity-check codes. We can prove the following theorem.

Theorem 1. *Let C_{RMA} denote a RMA code that we use to transmit information over the BEC. The received vectors are decoded iteratively using constituent code oriented decoding using MAP erasure correction in the constituent codes until either the codeword has been recovered, or the decoder fails to progress further. Then the set of erased positions that remain when the decoder stops is equal to the unique maximum-size stopping set that is contained in the (initial) set of erased positions.*

From Theorem 1 it follows that an important parameter for code performance is the stopping distance $h_{\min}$.

III. Support Sets and Stopping Set Enumerators

Let C denote an (N, K) binary linear code. Partition all the subcodes of C of dimension d, $d = 0, \ldots, K$, into equivalence classes based on their support sets. In particular, all subcodes within a specific subcode class are required to have the same support set, but the subcodes may have different dimensions. We define the SIOSEF [7] of C as

$$A^C(W, H) = \sum_{w=0}^{K} \sum_{h=0}^{N} a^C_{w,h} W^w H^h$$

where W and H are dummy variables, and $a^C_{w,h}$ is the number of subcode classes of C of *input* support set size w and *output* support set size h. In the rest of the paper, with a slight abuse of notation, we will refer interchangeably to both $A^C(W, H)$ and $a^C_{w,h}$ as the SIOSEF of a code C.

A. Stopping Set Enumerators for RMA Code Ensembles

Benedetto *et al.* introduced in [2] the concept of *uniform interleaver* to obtain average weight enumerators for concatenated code ensembles from the weight enumerators of the constituent encoders. The same concept can be used to obtain stopping set enumerators of concatenated code ensembles by combining the SIOSEFs of the constituent encoders. Let $\bar{s}^{\mathcal{C}}_{w,h}$ be the ensemble-average *input-output stopping set size enumerating function* (IOSSEF) of the code ensemble $\mathcal{C}$ with input and output block length K and N, respectively, denoting the average number of stopping sets of input size w and output size h over $\mathcal{C}$. Also, denote by $\bar{s}^{\mathcal{C}}_{h} = \sum_{w=0}^{K} \bar{s}^{\mathcal{C}}_{w,h}$ the ensemble-average *stopping set size enumerating function* (SSEF) of the code ensemble $\mathcal{C}$, giving the average number of stopping sets of size h over $\mathcal{C}$. Using the concept of uniform interleaver [2], the IOSSEF of a RMA code ensemble $\mathcal{C}_{\mathrm{RMA}}$ can be written as [2, 7]

$$\begin{aligned}\bar{s}^{\mathcal{C}_{\mathrm{RMA}}}_{w,h} &= \sum_{h_1=0}^{N} \cdots \sum_{h_{L-1}=0}^{N} \frac{a^{C_0}_{w,qw} a^{C_1}_{qw,h_1}}{\binom{N}{qw}} \left[\prod_{l=2}^{L-1} \frac{a^{C_l}_{h_{l-1},h_l}}{\binom{N}{h_{l-1}}}\right] \frac{a^{C_L}_{h_{L-1},h}}{\binom{N}{h_{L-1}}} \\ &= \sum_{h_1=0}^{N} \cdots \sum_{h_{L-1}=0}^{N} \bar{s}^{\mathcal{C}_{\mathrm{RMA}}}_{w,h_1,\ldots,h_{L-1},h}\end{aligned} \tag{1}$$

where $\bar{s}^{\mathcal{C}_{\mathrm{RMA}}}_{w,h_1,\ldots,h_{L-1},h}$ is called the conditional support size enumerating function of $\mathcal{C}_{\mathrm{RMA}}$.

The evaluation of (1) requires the computation of the SIOSEFs of the constituent encoders, which is addressed below.

$$\frac{\binom{K}{w}\sum_{d_1=1}^{\lfloor \frac{qw}{2} \rfloor}\binom{N-h_1}{d_1}\binom{h_1-1}{d_1-1}\binom{h_1-d_1}{qw-2d_1}}{\binom{N}{qw}}\prod_{l=2}^{L-1}\frac{\sum_{d_l=1}^{\lfloor \frac{h_{l-1}}{2} \rfloor}\binom{N-h_l}{d_l}\binom{h_l-1}{d_l-1}\binom{h_l-d_l}{h_{l-1}-2d_l}}{\binom{N}{h_{l-1}}}\cdot\frac{\sum_{d_L=1}^{\lfloor \frac{h_{L-1}}{2} \rfloor}\binom{N-h}{d_L}\binom{h-1}{d_L-1}\binom{h-d_L}{h_{L-1}-2d_L}}{\binom{N}{h_{L-1}}}$$

B. SIOSEFs for Memory-One Encoders and the Repetition Code

Theorem 2. *The SIOSEF for rate-1, memory-one, convolutional encoders with generator polynomials $g(D) = 1/(1+D)$ and $g(D) = 1 + D$ that are terminated to the zero state at the end of the trellis and with input and output block length N can be given in closed form as*

$$a_{w,h}^{\frac{1}{1+D}} = a_{h,w}^{1+D} = \sum_{d=1}^{\lfloor \frac{w}{2} \rfloor}\binom{N-h}{d}\binom{h-1}{d-1}\binom{h-d}{w-2d} \qquad (2)$$

for positive input sizes w. Also, $a_{0,0}^{\frac{1}{1+D}} = a_{0,0}^{1+D} = 1$.

A detailed proof can be found in [9].

Theorem 3. *The SIOSEF for the (Kq, K) repetition code C_0 with input block length K can be given in closed form as*

$$a_{w,qw}^{C_0} = \binom{K}{w}. \qquad (3)$$

Using (2) and (3) in (1), we get the expression at the top of the page for the conditional support size enumerating function (with $w > 0$) of RMA code ensembles.

IV. Finite-Length Analysis of Stopping Distance

The ensemble-average SSEF can be used to lower bound $h_{\min}$ for finite block lengths, similarly to the case of the $d_{\min}$ [3, 10]. In particular, the following bound holds.

Lemma 1. *The probability that a code chosen randomly from the ensemble $\mathcal{C}$ with ensemble-average SSEF $\bar{s}_h^{\mathcal{C}}$ has stopping distance $h_{\min} < \hbar$ is upper-bounded as*

$$\Pr(h_{\min} < \hbar) \leq \sum_{h=1}^{\hbar-1}\bar{s}_h^{\mathcal{C}}. \qquad (4)$$

Lemma 1 can be used to obtain a probabilistic lower bound on the stopping distance of a code ensemble. In particular, if we set $\Pr(h_{\min} < \hbar) = \varepsilon$, where ε is any positive value between 0 and 1, we would expect that at least a fraction $1 - \varepsilon$ of the codes in the ensemble have a stopping distance $h_{\min}$ of at least $\hbar$. In Fig. 2, we plot the probabilistic lower bound on $h_{\min}$ from Lemma 1 for RMA codes with repeat factor $q = 4$, and $L = 2, 3$, and 4, as a function of the code length N. The bounds were obtained by setting $\varepsilon = 0.5$ in (4), i.e., at least half of the codes in the ensemble have stopping distance at least equal to the value indicated by the curves. All codes appear to have linear $h_{\min}$ growth rate. The best growth rate is obtained for $L = 2$, while increasing the number of accumulate codes decreases the growth rate. For instance, the bound on $h_{\min}$ for the RAA code ensemble indicates a stopping distance of at least 132 for block length $N = 1000$ bits. The value is reduced to 122 and to 80 for the repeat triple-accumulate (RAAA) and the repeat quadruple-accumulate (RAAAA) code ensembles, respectively.

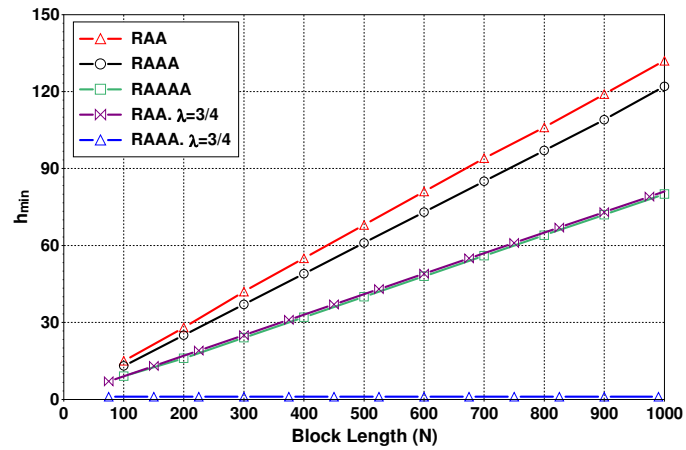

Fig. 2. Probabilistic lower bound on the stopping distance $h_{\min}$ versus block length N for RMA codes with $q = 4$ and $L = 2, 3$, and 4.

In the following, we obtain the asymptotic expression for the stopping set enumerator of RMA code ensembles, and we show that, indeed, their $h_{\min}$ asymptotically grows linearly with the block length.

V. Asymptotic Analysis of Stopping Distance

We define the asymptotic stopping set size spectral shape function as [10]

$$r_{\mathrm{s}}(\rho) = \lim_{N\longrightarrow\infty}\sup\frac{1}{N}\ln\bar{s}_{\lfloor\rho N\rfloor}^{\mathcal{C}}$$

where $\sup(\cdot)$ denotes the supremum of its argument, $\rho = \frac{h}{N}$ is the normalized stopping set size, and N is the block length. If there exists some abscissa $\rho_0 > 0$ such that $\sup_{\rho\leq\rho^*} r_{\mathrm{s}}(\rho) < 0 \quad \forall\rho^* < \rho_0$, and $r_{\mathrm{s}}(\rho) > 0$ for some $\rho > \rho_0$, then it can be shown (using Lemma 1 for example) that, with high probability, the stopping distance of most codes in the ensemble grows linearly with the block length N, with growth rate coefficient of at least ρ_0. On the other hand, if $r_{\mathrm{s}}(\rho)$ is strictly zero in the range $(0, \rho_0)$, it cannot be proved directly whether $h_{\min}$ grows linearly with the block length or not. In [5], it was shown that the spectral shape function of RMA codes, for the codeword case, exhibits this behavior, i.e., it is zero in the range $(0, \rho_0)$ and positive for some $\rho > \rho_0$, where ρ means here the normalized output weight. By combining the asymptotic spectral shapes with the use of bounding techniques, it was proved in [5, Theorem 11] that the minimum distance of RMA codes indeed grows linearly with the block length with growth rate coefficient of at least ρ_0. We remark that in the rest of the paper, with a slight abuse of language, we sometimes refer to ρ_0 as the exact value of the asymptotic growth rate coefficient. However, strictly speaking, ρ_0 is only a lower bound on it.

Now, by using Stirling's approximation for the binomial coefficient $\binom{n}{k} \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{n\mathbb{H}(k/n)}$ where $\mathbb{H}(\cdot)$ is the binary entropy function with natural logarithms, and the fact that w, $h_1, \ldots, h_{L-1}, d_1, \ldots, d_L$, and h can all be assumed to be of the same order as N [9], $\bar{s}^{\mathcal{C}_{\mathrm{RMA}}}_{w,h_1,\ldots,h_{L-1},h}$ can be written as

$$\bar{s}^{\mathcal{C}_{\mathrm{RMA}}}_{w,h_1,\ldots,h_{L-1},h} = \sum_{d_1,\ldots,d_L} \exp\left\{f(\alpha,\beta_1,\ldots,\beta_{L-1},\gamma_1,\ldots,\gamma_L,\rho)\,N + o(N)\right\}$$

when $N \longrightarrow \infty$, where $\alpha = \frac{w}{K}$ is the normalized input stopping set size, $\rho = \frac{h}{N}$ is the normalized output stopping set size, $\beta_l = \frac{h_l}{N}$ is the normalized output support set size of code C_l, $\gamma_l = \frac{d_l}{N}$, and the function $f(\cdot)$ is given by

$$\begin{aligned} f(\beta_0,\beta_1,\ldots,\beta_{L-1},\gamma_1,\ldots,\gamma_L,\rho) &= \frac{\mathbb{H}(\beta_0)}{q} - \sum_{l=1}^{L}\mathbb{H}(\beta_{l-1}) + \sum_{l=1}^{L}(1-\beta_l)\mathbb{H}\left(\frac{\gamma_l}{1-\beta_l}\right) \\ &\quad + \sum_{l=1}^{L}\beta_l\mathbb{H}\left(\frac{\gamma_l}{\beta_l}\right) + \sum_{l=1}^{L}(\beta_l-\gamma_l)\mathbb{H}\left(\frac{\beta_{l-1}-2\gamma_l}{\beta_l-\gamma_l}\right) \end{aligned} \tag{5}$$

where for conciseness we defined $\beta_0 = \alpha$ and $\beta_L = \rho$. Finally, the stopping set size spectral shape function for RMA code ensembles can be written as [4]

$$r_{\mathrm{s}}^{\mathcal{C}_{\mathrm{RMA}}}(\rho) = \sup_{\substack{0<\beta_0,\ldots,\beta_{L-1}\le 1\\ 0<\gamma_1,\ldots,\gamma_L\le 1}} f(\beta_0,\beta_1,\ldots,\beta_{L-1},\gamma_1,\ldots,\gamma_L,\rho) \tag{6}$$

To analyze the asymptotic stopping distance behavior of RMA code ensembles, we must solve the optimization problem in (5)-(6). Details can be found in [9]. The numerical evaluation of (5)-(6) is shown in Fig. 3 for RAA code ensembles with $q = 2, 3, 4, 5$, and 6. We observe that the stopping set size spectral shape function for the rate $R = 1/2$ RAA code ensemble is strictly positive, meaning that the ensemble is bad for the BEC. For $3 \le q \le 6$, the function $r_{\mathrm{s}}^{\mathcal{C}_{\mathrm{RMA}}}(\rho)$ is zero in the range $(0, \rho_0)$ and positive for some $\rho > \rho_0$. In this case, we cannot conclude directly whether $h_{\min}$ grows linearly with the block length or not. Define the function

$$\psi(u,\rho) = \sup_{\max(0,u-\rho)\le\gamma\le\min(\rho,1-\rho,u/2)} \left[-\mathbb{H}(u) + \rho\mathbb{H}\left(\frac{\gamma}{\rho}\right) + (1-\rho)\mathbb{H}\left(\frac{\gamma}{1-\rho}\right) + (\rho-\gamma)\mathbb{H}\left(\frac{u-2\gamma}{\rho-\gamma}\right)\right].$$

We can prove the following theorem (see [9] for details), extending the results in [5] to the stopping distance case.

Theorem 4. *Define* $\rho_0 = \max\{\rho^* \in [0,1/2) : r_{\mathrm{s}}^{\mathcal{C}_{\mathrm{RMA}}}(\rho) = 0\ \forall\rho \le \rho^*\}$. *Assuming that* $\lim_{u\longrightarrow 0}\frac{\psi(u,\rho)}{u} < 0\ \forall\rho < \rho_0$, *then* $\forall\rho^* > 0$

$$\lim_{N\longrightarrow\infty}\Pr\left(h_{\min} \le (\rho_0-\rho^*)N\right) = 0$$

for $L \ge 3$ *and* $q \ge 2$, *and* $L = 2$ *and* $q \ge 3$. *Thus, if* $\rho_0 > 0$ *and* $r_{\mathrm{s}}^{\mathcal{C}_{\mathrm{RMA}}}(\rho) \ge 0\ \forall\rho$, *then almost all codes in the ensemble have asymptotic stopping distance growing linearly with* N *with growth rate coefficient of at least* ρ_0.

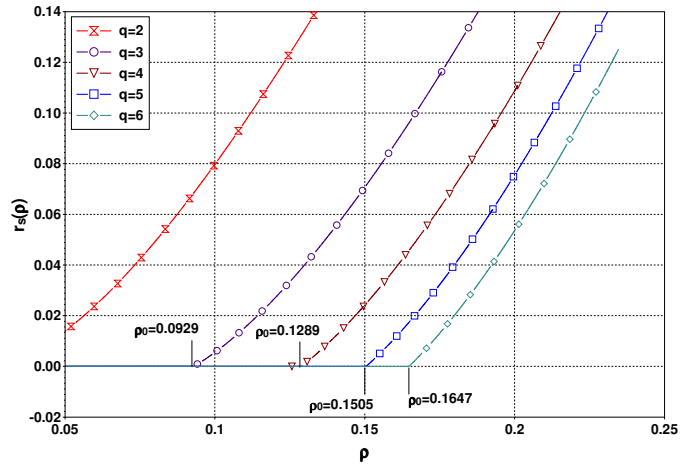

Fig. 3. Asymptotic stopping set size spectral shape function for the RAA code ensemble with $q = 2, 3, 4, 5$, and 6.

TABLE I
ASYMPTOTIC $h_{\min}$ GROWTH RATE COEFFICIENTS FOR RMA CODE ENSEMBLES.

	$q=2$	$q=3$	$q=4$	$q=5$
ρ_0 ($h_{\min}$) ($L=2$)	N/A	0.0929	0.1289	0.1505
ρ_0 ($d_{\min}$) ($L=2$) [4]	N/A	0.1323	0.1911	0.2286
ρ_0 ($h_{\min}$) ($L=3$)	0.0681	0.1037	0.1194	0.1279
ρ_0 ($d_{\min}$) ($L=3$) [4]	0.1034	0.1731	0.2143	0.2428
ρ_0 ($h_{\min}$) ($L=4$)	0.0549	0.0716	0.0784	0.0817
GVB	0.1100	0.1740	0.2145	0.2430

We remark that it can be verified that the assumption in Theorem 4 always holds for the numerical values of ρ_0 that we have found. The exact values of ρ_0 are given in Table I (except for $q = 6$). For comparison purposes we also give the asymptotic growth rate coefficient of the $d_{\min}$ computed in [4]. As expected, the asymptotic growth rate coefficient of $h_{\min}$ is smaller than for $d_{\min}$. We can now prove the following theorem (see [9] for details).

Theorem 5. *The typical* $h_{\min}$ *of RMA code ensembles for* $L \ge 3$ *and* $q \ge 2$ *grows linearly with block length.*

In Table I, we also report the asymptotic growth rate coefficient ρ_0 for RAAA and RAAAA code ensembles. Interestingly, contrary to the asymptotic $d_{\min}$ growth rate coefficient, which increases with the number of accumulators and tends to approach the GVB [4, 5], the asymptotic growth rate coefficient of $h_{\min}$ decreases with the number of accumulators concatenated in series. The results are in agreement with the finite-length analysis in the previous section.

On the other hand, the $h_{\min}$ of RA code ensembles does not grow linearly with the block length (see [9] for details).

A. RMA Codes with Random Puncturing

In this section, we consider high rate RMA code ensembles obtained by randomly puncturing the output of the most inner accumulator C_L of a RMA code ensemble. Denote by λ ($0 \le \lambda \le 1$) the puncturing permeability rate, i.e., the fraction of bits surviving after puncturing, and by $R' = R/\lambda = 1/(\lambda q)$

TABLE II
ASYMPTOTIC $h_{\min}$ GROWTH RATE COEFFICIENTS FOR PUNCTURED RMA CODE ENSEMBLES WITH $q = 4$ AND NOMINAL CODE RATE $R' = 1/(\lambda q)$.

R'	1/4	0.28	0.29	3/10	5/16	0.33	1/3	11/30	2/5	0.41	0.42	0.43
λ	1	25/28	25/29	5/6	4/5	25/33	3/4	15/22	5/8	25/41	25/42	25/43
ρ_0 $(L = 2)$	0.1289	0.1192	0.1142	0.1077	0.0977	0.0819	0.0788	0.0474	0.0188	0.0112	0.0045	N/A
ρ_0 $(L = 3)$	0.1194	0.0694	0.0528	0.0373	0.0198	0.0004	N/A	N/A	N/A	N/A	N/A	N/A
ρ_0 $(L = 4)$	0.0784	0.0112	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A

TABLE III
CONVERGENCE THRESHOLDS FOR RMA CODE ENSEMBLES.

	$q = 3$	$q = 4$	$q = 5$	$q = 6$
RAA	0.4965	0.5422	0.5719	0.5935
RAAA	0.3259	0.3531	0.3718	0.3860
RAAAA	0.1957	0.2105	0.2209	0.2290

the nominal code rate of the punctured RMA code.

The SIOSEF of a randomly punctured code $C^{\text{punct.}}$ with input support set size w, output support set size h before puncturing, and output support set size h' after puncturing is given by [4]

$$a_{w,h'}^{C^{\text{punct.}}} = \sum_{h=h'}^{N} a_{w,h}^{C} \frac{\binom{h}{h'}\binom{N-h}{\lambda N-h'}}{\binom{N}{\lambda N}}. \tag{7}$$

Using Stirling's approximation in (7) and coupling it with (5) and (6), the stopping set size spectral shape function of a punctured RMA code ensemble $r_{\text{s}}^{\mathcal{C}_{\text{RMA-punct.}}}(\rho')$, where $\rho' = \frac{h'}{\lambda N}$, can be derived.

The values of ρ_0 corresponding to $r_{\text{s}}^{\mathcal{C}_{\text{RMA-punct.}}}(\rho')$ are given in Table II for $L = 2, 3$, and 4 mother RMA code ensembles with $q = 4$ for several nominal code rates R'. Asymptotic $h_{\min}$ linear growth can be guaranteed for some rates $R' > 1/q$. However, it is interesting to note that the asymptotic linear growth rate property breaks down for heavy puncturing of the mother code ensemble. This behavior gets worse as the number of encoding stages increases. These results are also supported by the finite-length analysis. In Fig. 2, we plot the probabilistic lower bound on $h_{\min}$ from Lemma 1 for punctured $q = 4$ RMA code ensembles with $\lambda = 3/4$ and $L = 2$ and 3. The results are in agreement with the asymptotic analysis; for $L = 2$ and permeability rate $\lambda = 3/4$ linear growth rate is guaranteed. However, applying the same puncturing to the $L = 3$ RMA code ensemble breaks down the linear growth rate property. Note that these results are in contrast with the results in [4], where it was observed that the asymptotic normalized $d_{\min}$ gets closer to the GVB for higher rates with random puncturing.

VI. EXIT CHARTS ANALYSIS

In this section, we address iterative constituent code oriented decoding of RMA code ensembles on the BEC by using EXIT charts analysis [8] to estimate the convergence thresholds. Note that for the BEC the EXIT functions of the repeat and the accumulate codes can be given in closed form [11]. The convergence thresholds for RMA code ensembles for $q = 3, \ldots, 6$ are given in Table III. Note that RAAA and RAAAA code ensembles show very poor thresholds, which make them impractical. From the EXIT charts analysis and the asymptotic analysis in Section V, it arises that double serially concatenated code ensembles are good ensembles for the BEC, since they provide both high asymptotic $h_{\min}$ growth rates and good convergence behavior, while adding more encoding stages penalizes both the asymptotic $h_{\min}$ growth rate and the convergence threshold.

VII. CONCLUSION

In this paper, we considered RMA code ensembles on the BEC. By deriving a closed-form expression for the SIOSEF of the accumulate code, we obtained stopping set enumerators for RMA code ensembles. We then analyzed the asymptotic behavior of the stopping distance and showed that RMA code ensembles exhibit a stopping distance that grows linearly with the block length. Contrary to the minimum distance, whose growth rate coefficient increases with the number of accumulators, the growth rate of $h_{\min}$ diminishes with the number of accumulators. We also considered puncturing of the RMA code ensembles and showed that linear growth can be obtained for rates larger than $1/q$. However, the linear growth property breaks down for heavy puncturing.

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