Construction and Covering Properties of Constant-Dimension Codes

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Abstract

Constant-dimension codes (CDCs) have been investigated for noncoherent error correction in random network coding. The maximum cardinality of CDCs with given minimum distance and how to construct optimal CDCs are both open problems, although CDCs obtained by lifting Gabidulin codes, referred to as KK codes, are nearly optimal. In this paper, we first construct a new class of CDCs based on KK codes, referred to as augmented KK codes, whose cardinalities are greater than previously proposed CDCs. We then propose a low-complexity decoding algorithm for our augmented KK codes using that for KK codes. Our decoding algorithm corrects more errors than a bounded subspace distance decoder by taking advantage of the structure of our augmented KK codes. In the rest of the paper we investigate the covering properties of CDCs. We first derive bounds on the minimum cardinality of a CDC with a given covering radius and then determine the asymptotic behavior of this quantity. Moreover, we show that liftings of rank metric codes have the highest possible covering radius, and hence liftings of rank metric codes are **not optimal** packing CDCs. Finally, we construct good covering CDCs by permuting liftings of rank metric codes.

I. Introduction

While random network coding [1]–[3] has proved to be a powerful tool for disseminating information in networks, it is highly susceptible to errors caused by various sources. Thus, error control for random network coding is critical and has received growing attention recently. Error control schemes proposed for random network coding assume two types of transmission models: some (see, for example, [4]–[9]) depend on and take advantage of the underlying network topology or the particular linear network coding operations performed at various network nodes; others [10], [11] assume that the transmitter

and receiver have no knowledge of such channel transfer characteristics. The contrast is similar to that between coherent and noncoherent communication systems. Data transmission in noncoherent random network coding can be viewed as sending subspaces through an operator channel [10]. Error correction in noncoherent random network coding can hence be treated as a coding problem where codewords are linear subspaces and codes are subsets of the projective space of a vector space over a finite field. Similar to codes defined over complex Grassmannians for noncoherent multiple-antenna channels, codes defined in Grassmannians associated with the vector space play a significant role in error control for noncoherent random network coding; Such codes are referred to as constant-dimension codes (CDCs) [10]. In addition to the subspace metric defined in [10], an injection metric was defined for subspace codes over adversarial channels in [12].

Construction of CDCs has received growing attention in the literature recently. In [10], a Singleton bound for CDCs and a family of codes were proposed, which are nearly Singleton-bound-achieving and referred to as KK codes henceforth. A multi-step construction of CDCs was proposed in [13], and we call these codes Skachek codes; Skachek codes have larger cardinalities than KK codes in some scenarios, and reduce to KK codes otherwise. Further constructions for small parameter values were given in [14] and the Johnson bound for CDCs was derived in [15]. Although the CDCs in [15] are optimal in the sense of the Johnson bound, they exist in some special cases only. Despite these previous works, the maximum cardinality of a CDC with a given minimum distance and how to construct optimal CDCs remain open problems.

Although the packing properties of CDCs were investigated in [10], [13]–[15], the covering properties of CDCs have received little attention in the literature. Covering properties are significant for error control codes, and the covering radius is a basic geometric parameter of a code [16]. For instance, the covering radius can be viewed as a measure of performance: if the minimum distance decoding is used, then the covering radius is the maximum weight of a correctable error vector [17]; if the code is used for data compression, then the covering radius is a measure of the maximum distortion [17]. The covering radius is also crucial for code design: if the covering radius is no less than the minimum distance of a code, then there exists a supercode with the same minimum distance and greater cardinality.

This paper has two main contributions. First, we introduce a new class of CDCs, referred to as augmented KK codes. The cardinalities of our augmented KK codes are **always greater** than those of KK codes, and in **most** cases the cardinalities of our augmented KK code are greater than those of Skachek codes. Thus our augmented KK codes represent a step toward solving the open problem (construction of optimal CDCs) mentioned above. Furthermore, we propose an efficient decoding algorithm for our

augmented KK codes using the bounded subspace distance decoding algorithm in [10]. Our decoding algorithm corrects more errors than a bounded subspace distance decoder. Second, we investigate the covering properties of CDCs. We first derive some key geometric results for Grassmannians. Using these results, we derive upper and lower bounds on the minimum cardinality of a CDC with a given covering radius. Since these bounds are asymptotically tight, we also determine the asymptotic behavior of the minimum cardinality of a CDC with a given covering radius. Although liftings of rank metric codes can be used to construct packing CDCs that are optimal up to a scalar (see, for example, those in [10]), we show that all liftings of rank metric codes have the greatest covering radius possible; our result further implies that liftings of rank metric codes are **not optimal** packing CDCs. We also construct good covering CDCs by permuting liftings of rank metric codes.

The rest of the paper is organized as follows. To be self-contained, Section II reviews some necessary background. In Section III, we present our augmented KK codes and a decoding algorithm for these codes. In Section IV, we investigate the covering properties of CDCs.

II. Preliminaries

A. Subspace codes

We refer to the set of all subspaces of $\mathrm{GF}(q)^n$ with dimension r as the Grassmannian of dimension r and denote it as $E_r(q,n)$; we refer to $E(q,n) = \bigcup_{r=0}^{n} E_r(q,n)$ as the projective space. For $U, V \in E(q,n)$, both the *subspace metric* [10, (3)]

$$d_{\mathrm{s}}(U,V) \stackrel{\text{def}}{=} \dim(U+V) - \dim(U \cap V) = 2\dim(U+V) - \dim(U) - \dim(V) \tag{1}$$

and *injection metric* [12, Def. 1]

$$d_{\mathrm{i}}(U,V) \stackrel{\text{def}}{=} \frac{1}{2} d_{\mathrm{s}}(U,V) + \frac{1}{2} |\dim(U) - \dim(V)| = \dim(U+V) - \min\{\dim(U), \dim(V)\} \tag{2}$$

are metrics over $E(q,n)$. A *subspace code* is a nonempty subset of $E(q,n)$. The minimum subspace (respectively, injection) distance of a subspace code is the minimum subspace (respectively, injection) distance over all pairs of distinct codewords.

B. CDCs and rank metric codes

The Grassmannian $E_r(q,n)$ endowed with both the subspace and injection metrics forms an association scheme [10], [18]. For all $U, V \in E_r(q,n)$, $d_{\mathrm{s}}(U,V) = 2d_{\mathrm{i}}(U,V)$ and the injection distance provides a

natural distance spectrum, i.e., $0 \leq d_{\text{I}}(U,V) \leq r$. We have $|E_r(q,n)| = \begin{bmatrix} n \\ r \end{bmatrix}$, where $\begin{bmatrix} n \\ r \end{bmatrix} = \prod_{i=0}^{r-1} \frac{q^n - q^i}{q^r - q^i}$ is the Gaussian polynomial [19], which satisfies

$$q^{r(n-r)} \leq \begin{bmatrix} n \\ r \end{bmatrix} < K_q^{-1} q^{r(n-r)} \tag{3}$$

for all $0 \leq r \leq n$, where $K_q = \prod_{j=1}^{\infty}(1-q^{-j})$ [20]. We denote the number of subspaces in $E_r(q,n)$ at distance d from a given subspace as $N_{\text{C}}(d) = q^{d^2} \begin{bmatrix} r \\ d \end{bmatrix} \begin{bmatrix} n-r \\ d \end{bmatrix}$ [10], and denote a ball in $E_r(q,n)$ of radius t around a subspace U and its volume as $B_t(U)$ and $V_{\text{C}}(t) = \sum_{d=0}^{t} N_{\text{C}}(d)$, respectively.

A subset of $E_r(q,n)$ is called a constant-dimension code (CDC). A CDC is thus a subspace code whose codewords have the same dimension. We denote the **maximum** cardinality of a CDC in $E_r(q,n)$ with minimum distance d as $A_{\text{C}}(q,n,r,d)$. Constructions of CDCs and bounds on $A_{\text{C}}(q,n,r,d)$ have been given in [10], [13]–[15], [21]. In particular, $A_{\text{C}}(q,n,r,1) = \begin{bmatrix} n \\ r \end{bmatrix}$ and it is shown [13], [15] for $r \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $2 \leq d \leq r$,

$$\frac{q^{n(r-d+1)} - q^{(r+l)(r-d+1)}}{q^{r(r-d+1)} - 1} \leq A_{\text{C}}(q,n,r,d) \leq \frac{\begin{bmatrix} n \\ r-d+1 \end{bmatrix}}{\begin{bmatrix} r \\ r-d+1 \end{bmatrix}}, \tag{4}$$

where $l \equiv n \mod r$. We denote the lower bound on $A_{\text{C}}(q,n,r,d)$ in (4) as $L(q,n,r,d)$. Since the lower bound is due to the class of codes proposed by Skachek [13], we refer to these codes as Skachek codes.

CDCs are closely related to rank metric codes [22]–[24], which can be viewed as sets of matrices in $\text{GF}(q)^{m\times n}$. The rank distance between two matrices $\mathbf{C}, \mathbf{D} \in \text{GF}(q)^{m\times n}$ is defined as $d_{\text{R}}(\mathbf{C},\mathbf{D}) \stackrel{\text{def}}{=} \text{rk}(\mathbf{C}-\mathbf{D})$. The maximum cardinality of a rank metric code in $\text{GF}(q)^{m\times n}$ with minimum distance d is given by $\min\{q^{m(n-d+1)}, q^{n(m-d+1)}\}$ and codes that achieve this cardinality are referred to as MRD codes. In this paper, we shall only consider MRD codes that are either introduced independently in [22]–[24] for $n \leq m$, or their transpose codes for $n > m$. The number of matrices in $\text{GF}(q)^{m\times n}$ with rank d is denoted as $N_{\text{R}}(q,m,n,d) = \begin{bmatrix} n \\ d \end{bmatrix} \prod_{i=0}^{d-1}(q^m - q^i)$, and the volume of a ball with rank radius t in $\text{GF}(q)^{m\times n}$ as $V_{\text{R}}(q,m,n,t) = \sum_{d=0}^{t} N_{\text{R}}(q,m,n,d)$. The minimum cardinality $K_{\text{R}}(q^m,n,\rho)$ of a code in $\text{GF}(q)^{m\times n}$ with rank covering radius ρ is studied in [25], [26] and satisfies $K_{\text{R}}(q^m,n,\rho) = K_{\text{R}}(q^n,m,\rho)$ [25].

CDCs are related to rank metric codes through the lifting operation [11]. Denoting the row space of a matrix $\mathbf{M}$ as $R(\mathbf{M})$, the lifting of $\mathbf{C} \in \text{GF}(q)^{r\times(n-r)}$ is defined as $I(\mathbf{C}) = R(\mathbf{I}_r|\mathbf{C}) \in E_r(q,n)$. For all $\mathbf{C}, \mathbf{D} \in \text{GF}(q)^{r\times(n-r)}$, we have $d_{\text{I}}(I(\mathbf{C}), I(\mathbf{D})) = d_{\text{R}}(\mathbf{C},\mathbf{D})$ [11]. A KK code in $E_r(q,n)$ with minimum injection distance d is the lifting $I(\mathcal{C}) \subseteq E_r(q,n)$ of an MRD code $\mathcal{C} \subseteq \text{GF}(q)^{r\times(n-r)}$ with minimum rank distance d and cardinality $\min\{q^{(n-r)(r-d+1)}, q^{r(n-r-d+1)}\}$. An efficient bounded subspace distance decoding algorithm for KK codes was also given in [10]. Although the algorithm was presented for $r \leq \frac{n}{2}$, it can be easily generalized to all r.

III. Construction of CDCs

In this section, we construct a new class of CDCs which contain KK codes as proper subsets. Thus we call them augmented KK codes. We will show that the cardinalities of our augmented KK codes are always greater than those of KK codes, and that in most cases the cardinalities of our augmented KK code are greater than those of Skachek codes. Furthermore, we propose a low-complexity decoder for our augmented KK codes based on the bounded subspace distance decoder in [10]. Since dual CDCs preserve the distance, we assume $r \leq \frac{n}{2}$ without loss of generality.

A. Augmented KK codes

Our augmented KK code is so named because it has a layered structure and the first layer is simply a KK code. We denote a KK code in $E_r(q,n)$ with minimum injection distance d ($d \leq r$ by definition) and cardinality $q^{(n-r)(r-d+1)}$ as $\mathcal{E}^0$. For $1 \leq k \leq \lfloor \frac{r}{d} \rfloor$, we first define two MRD codes $\mathcal{C}^k$ and $\mathcal{D}^k$, and then construct $\mathcal{E}^k$ based on $\mathcal{C}^k$ and $\mathcal{D}^k$. $\mathcal{C}^k$ is an MRD code in $\mathrm{GF}(q)^{(r-kd)\times kd}$ with minimum distance d for $k \leq \lfloor \frac{r}{d} \rfloor - 1$ ($\lfloor \frac{n-r}{d} \rfloor \geq \lfloor \frac{r}{d} \rfloor$) and $\mathcal{C}^{\lfloor \frac{r}{d} \rfloor} = \{\mathbf{0}\} \subseteq \mathrm{GF}(q)^{(r-\lfloor \frac{r}{d} \rfloor d)\times \lfloor \frac{r}{d} \rfloor d}$; $\mathcal{D}^k$ is an MRD code in $\mathrm{GF}(q)^{r\times(n-r-kd)}$ with minimum distance d for $k \leq \lfloor \frac{n-r}{d} \rfloor - 1$ and $\mathcal{D}^{\lfloor \frac{n-r}{d} \rfloor} = \{\mathbf{0}\} \subseteq \mathrm{GF}(q)^{r\times\left(n-r-\lfloor \frac{n-r}{d} \rfloor d\right)}$. For $1 \leq k < \lfloor \frac{r}{d} \rfloor$, the block lengths of $\mathcal{C}^k$ and $\mathcal{D}^k$ are at least d, and hence existence of MRD codes with the parameters mentioned above is trivial. For $1 \leq k \leq \lfloor \frac{r}{d} \rfloor$, $I(\mathcal{C}^k)$ and $I(\mathcal{D}^k)$ are either trivial codes or KK codes with minimum injection distance d in $E_{r-kd}(q,r)$ and $E_r(q,n-kd)$, respectively. For $1 \leq k \leq \lfloor \frac{r}{d} \rfloor$, $\mathbf{C}_i^k \in \mathcal{C}^k$, and $\mathbf{D}_j^k \in \mathcal{D}^k$, we define $E_{i,j}^k \in E_r(q,n)$ as the row space of $\left(\begin{array}{c|c|c} \mathbf{I}_{r-kd} & \mathbf{C}_i^k & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{kd} \end{array} \begin{array}{c} \\ \\ \mathbf{D}_j^k \end{array}\right)$ and $\mathcal{E}^k = \{E_{i,j}^k\}_{i,j=0}^{|\mathcal{C}^k|-1,|\mathcal{D}^k|-1}$. Our augmented KK code is simply $\mathcal{E} = \bigcup_{k=0}^{\lfloor \frac{r}{d} \rfloor} \mathcal{E}^k$. In order to determine its minimum distance, we first establish two technical results. First, for any two matrices $\mathbf{A} \in \mathrm{GF}(q)^{a\times n}$, $\mathbf{B} \in \mathrm{GF}(q)^{b\times n}$, by (1) and (2) we can easily show that

$$d_{\mathrm{s}}(R(\mathbf{A}),R(\mathbf{B})) = 2\mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T) - \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) \geq |\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})|, \tag{5}$$

$$d_{\mathrm{I}}(R(\mathbf{A}),R(\mathbf{B})) = \mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T) - \min\{\mathrm{rk}(\mathbf{A}),\mathrm{rk}(\mathbf{B})\} \geq |\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})|. \tag{6}$$

Second, we show that truncating the generator matrices of two subspaces in $E(q,n)$ can only reduce the (subspace or injection) distance between them.

Lemma 1: Suppose $0 \leq n_1 \leq n$. Let $\mathbf{A} = (\mathbf{A}_1|\mathbf{A}_2) \in \mathrm{GF}(q)^{a\times n}$, $\mathbf{B} = (\mathbf{B}_1|\mathbf{B}_2) \in \mathrm{GF}(q)^{b\times n}$, where $\mathbf{A}_1 \in \mathrm{GF}(q)^{a\times n_1}$ and $\mathbf{B}_1 \in \mathrm{GF}(q)^{b\times n_1}$. Then for $i = 1$ and 2, $d_{\mathrm{s}}(R(\mathbf{A}_i),R(\mathbf{B}_i)) \leq d_{\mathrm{s}}(R(\mathbf{A}),R(\mathbf{B}))$ and $d_{\mathrm{I}}(R(\mathbf{A}_i),R(\mathbf{B}_i)) \leq d_{\mathrm{I}}(R(\mathbf{A}),R(\mathbf{B}))$.

Proof: It suffices to prove it for $i = 1$ and $n_1 = n-1$. We need to distinguish two cases, depending on $\mathrm{rk}(\mathbf{A}_1^T|\mathbf{B}_1^T)$. First, if $\mathrm{rk}(\mathbf{A}_1^T|\mathbf{B}_1^T) = \mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T)$, then it is easily shown that $\mathrm{rk}(\mathbf{A}_1) = \mathrm{rk}(\mathbf{A})$ and $\mathrm{rk}(\mathbf{B}_1) = \mathrm{rk}(\mathbf{B})$, and hence $d_{\mathrm{s}}(R(\mathbf{A}_1), R(\mathbf{B}_1)) = d_{\mathrm{s}}(R(\mathbf{A}), R(\mathbf{B}))$ and $d_{\mathrm{I}}(R(\mathbf{A}_1), R(\mathbf{B}_1)) = d_{\mathrm{I}}(R(\mathbf{A}), R(\mathbf{B}))$ by (5) and (6), respectively. Second, if $\mathrm{rk}(\mathbf{A}_1^T|\mathbf{B}_1^T) = \mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T)-1$, then $d_{\mathrm{s}}(R(\mathbf{A}_1), R(\mathbf{B}_1)) = 2\mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T) - 2 - \mathrm{rk}(\mathbf{A}_1) - \mathrm{rk}(\mathbf{B}_1) \leq d_{\mathrm{s}}(R(\mathbf{A}), R(\mathbf{B}))$ by (5) and $d_{\mathrm{I}}(R(\mathbf{A}_1), R(\mathbf{B}_1)) = \mathrm{rk}(\mathbf{A}^T|\mathbf{B}^T) - 1 - \min\{\mathrm{rk}(\mathbf{A}_1), \mathrm{rk}(\mathbf{B}_1)\} \leq d_{\mathrm{I}}(R(\mathbf{A}), R(\mathbf{B}))$ by (6). ■

Proposition 1: $\mathcal{E}$ has minimum injection distance d.

Proof: We show that any two codewords $E_{i,j}^k, E_{a,b}^c \in \mathcal{E}$ are at injection distance at least d using Lemma 1. When $c \neq k$, let us assume $c < k$ without loss of generality, and then $d_{\mathrm{I}}(E_{i,j}^k, E_{a,b}^c) \geq d_{\mathrm{I}}(R(\mathbf{I}_{r-kd}|\mathbf{0}), R(\mathbf{I}_{r-cd})) = (k-c)d \geq d$. When $c = k$ and $a \neq i$, then $d_{\mathrm{I}}(E_{i,j}^k, E_{a,b}^k) \geq d_{\mathrm{I}}(I(\mathbf{C}_i^k), I(\mathbf{C}_a^k)) \geq d$. When $c = k$, $a = i$, and $b \neq j$, then $d_{\mathrm{I}}(E_{i,j}^k, E_{i,b}^k) \geq d_{\mathrm{I}}(I(\mathbf{D}_j^k), I(\mathbf{D}_b^k)) \geq d$. ■

Let us first determine the cardinality of our augmented KK codes. By construction, $\mathcal{E}$ has cardinality $|\mathcal{E}| = q^{(n-r)(r-d+1)} + \sum_{k=1}^{\lfloor \frac{r}{d} \rfloor} |\mathcal{C}^k||\mathcal{D}^k|$, where $|\mathcal{C}^{\lfloor \frac{r}{d} \rfloor}| = 1$ and $|\mathcal{C}^k| = \min\{q^{(r-kd)(kd-d+1)}, q^{kd(r-kd-d+1)}\}$ for $1 \leq k \leq \lfloor \frac{r}{d} \rfloor - 1$ and $|\mathcal{D}^{\lfloor \frac{n-r}{d} \rfloor}| = 1$ and
$|\mathcal{D}^k| = \min\{q^{r(n-r-kd-d+1)}, q^{(n-r-kd)(r-d+1)}\}$ for $1 \leq k \leq \lfloor \frac{n-r}{d} \rfloor - 1$.

Let us compare the cardinality of our augmented KK codes to those of KK and Skachek codes. Note that all three codes are CDCs with minimum injection distance d in $E_r(q, n)$. First, it is easily shown that our augmented KK codes properly contain KK codes for all parameter values. This is a clear distinction from Skachek codes with cardinality $L(q, n, r, d)$, which by (4) reduce to KK codes for $3r > n$. In order to compare our codes to Skachek codes when $3r \leq n$, we first remark that (4) and (3) lead to $L(q, n, r, d) - q^{(n-r)(r-d+1)} < K_q^{-1} q^{(n-2r)(r-d+1)}$. Also, we have $|\mathcal{E}| \geq q^{(n-r)(r-d+1)} + |\mathcal{C}^1||\mathcal{D}^1| \geq q^{(n-r)(r-d+1)} + q^{(n-r-d)(r-d+1)}$. Hence $|\mathcal{E}| - q^{(n-r)(r-d+1)} > K_q q^{(r-d)(r-d+1)}(L(q, n, r, d) - q^{(n-r)(r-d+1)})$, and our augmented KK codes have a greater cardinality than Skachek codes when $d < r$. We emphasize that for CDCs of dimension r, their minimum injection distance d satisfies $d \leq r$. A Skachek code is constructed in multiple steps, and in the i-th step ($i \geq 1$), subspaces that correspond to a KK code in $E_r(q, n - ir)$ are added to the code. When $d = r$, $\mathcal{E}$ is actually the code obtained after the first step.

B. Decoding of augmented KK codes

Let $\mathbf{A} = (\mathbf{A}_0|\mathbf{A}_3) \in \mathrm{GF}(q)^{a \times n}$ be the received matrix, where $\mathbf{A}_0 \in \mathrm{GF}(q)^{a \times r}$ and $\mathbf{A}_3 \in \mathrm{GF}(q)^{a \times (n-r)}$. We propose a decoding algorithm that either produces the unique codeword in $\mathcal{E}$ closest to $R(\mathbf{A})$ in the subspace metric or returns a failure. Suppose the minimum subspace distance of our augmented KK codes is denoted as $2d$, a bounded distance decoder would find the codeword that is closest to $R(\mathbf{A})$

up to subspace distance $d-1$. Our decoding algorithm always returns the correct codeword if it is at subspace distance at most $d-1$ from the received subspace, thus correcting more errors than a bounded subspace distance decoder.

Given the layered structure of $\mathcal{E}$, our decoding algorithm for $\mathcal{E}$ is based on a decoding algorithm for $\mathcal{E}^k$, shown below in Algorithm 1, for any k. We denote the codewords in $\mathcal{E}^0$ as $E^0_{0,j}$ for $0 \leq j \leq |\mathcal{E}^0|-1$.

Algorithm 1: $\mathrm{EBDD}(k, \mathbf{A})$.

Input: k and $\mathbf{A} = (\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3) \in \mathrm{GF}(q)^{a\times n}$, $\mathbf{A}_1 \in \mathrm{GF}(q)^{a\times(r-kd)}$, $\mathbf{A}_2 \in \mathrm{GF}(q)^{a\times kd}$, $\mathbf{A}_3 \in \mathrm{GF}(q)^{a\times(n-r)}$.

Output: $(E^k_{i,j}, d_k, f_k)$.

1.1 If $k = 0$, use the decoder for $\mathcal{E}^0$ to obtain $E^0_{0,j}$, calculate $d_k = d_{\mathrm{s}}(R(\mathbf{A}), E^0_{0,j})$, and return $(E^0_{0,j}, d_k, 0)$. If the decoder returns a failure, return $(I(\mathbf{0}), d, 0)$.

1.2 Use the decoder of $I(\mathcal{C}^k)$ on $(\mathbf{A}_1|\mathbf{A}_2)$ to obtain $\mathbf{C}^k_i$. If the decoder returns a failure, set $\mathbf{C}^k_i = \mathbf{0}$, $\mathbf{D}^k_j = \mathbf{0}$ and return $(E^k_{i,j}, d, 0)$.

1.3 Use the decoder of $I(\mathcal{D}^k)$ on $(\mathbf{A}_1|\mathbf{A}_3)$ to obtain $\mathbf{D}^k_j$. If the decoder returns a failure, set $\mathbf{D}^k_j = \mathbf{0}$ and return $(E^k_{i,j}, d, 0)$.

1.4 Calculate $d_k = d_{\mathrm{s}}(R(\mathbf{A}), E^k_{i,j})$ and $f_k = 2d - \max\{d_{\mathrm{s}}(R(\mathbf{A}_1|\mathbf{A}_2), I(\mathbf{C}^k_i)), d_{\mathrm{s}}(R(\mathbf{A}_1|\mathbf{A}_3), I(\mathbf{D}^k_j))\}$ and return $(E^k_{i,j}, d_k, f_k)$.

Algorithm 1 is based on the bounded distance decoder proposed in [10]. When $k = 0$, $\mathcal{E}^0$ is simply a KK code, and the algorithm in [10] is used directly; when $k \geq 1$, given the structure of $\mathcal{E}^k$, two decoding attempts are made based on $(\mathbf{A}_1|\mathbf{A}_2)$ and $(\mathbf{A}_1|\mathbf{A}_3)$, and both are based on the decoding algorithm in [10].

We remark that Algorithm 1 always return $(E^k_{i,j}, d_k, f_k)$. If a unique nearest codeword in $\mathcal{E}^k$ at distance no more than $d-1$ from $R(\mathbf{A})$ exists, then by Lemma 1 Steps 1.2 and 1.3 succeed and Algorithm 1 returns the unique nearest codeword in $E^k_{i,j}$. However, when such unique codeword in $\mathcal{E}^k$ at distance no more than $d-1$ does not exist, the return value f_k can be used to find the unique nearest codeword because f_k is a lower bound on the distance from the received subspace to any other codeword in $\mathcal{E}^k$. Also, when $f_k = 0$, Algorithm 2 below always returns a failure. Thus, we call Algorithm 1 an enhanced bounded distance decoder.

Lemma 2: Suppose the output of $\mathrm{EBDD}(k, \mathbf{A})$ is $(E^k_{i,j}, d_k, f_k)$, then $d_{\mathrm{s}}(R(\mathbf{A}), E^k_{u,v}) \geq f_k$ for any $E^k_{u,v} \in \mathcal{E}^k$ provided $(u, v) \neq (i, j)$.

Proof: The case $f_k = 0$ is trivial, and it suffices to consider $f_k = \min\{2d - d_{\mathrm{s}}(R(\mathbf{A}_1|\mathbf{A}_2), I(\mathbf{C}^k_i)), 2d-$

$d_{\rm s}(R(\mathbf{A}_1|\mathbf{A}_3), I(\mathbf{D}_j^k))\}$. When $u \neq i$, Lemma 1 yields

$$\begin{aligned} d_{\rm s}(R(\mathbf{A}), E_{u,v}^k) &\geq d_{\rm s}(R(\mathbf{A}_1|\mathbf{A}_2), I(\mathbf{C}_u^k)) \\ &\geq d_{\rm s}(I(\mathbf{C}_i^k), I(\mathbf{C}_u^k)) - d_{\rm s}(R(\mathbf{A}_1|\mathbf{A}_2), I(\mathbf{C}_i^k)) \\ &\geq 2d - d_{\rm s}(R(\mathbf{A}_1|\mathbf{A}_2), I(\mathbf{C}_i^k)) \geq f_k. \end{aligned}$$

Similarly, when $v \neq j$, we obtain $d_{\rm s}(R(\mathbf{A}), E_{u,v}^k) \geq 2d - d_{\rm s}(R(\mathbf{A}_1|\mathbf{A}_3), I(\mathbf{D}_j^k)) \geq f_k$. ■

The algorithm for $\mathcal{E}$ thus follows.

Algorithm 2: Decoder for $\mathcal{E}$.

Input: $\mathbf{A} = (\mathbf{A}_0|\mathbf{A}_3) \in \mathrm{GF}(q)^{a\times n}$, $\mathbf{A}_0 \in \mathrm{GF}(q)^{a\times r}$, $\mathbf{A}_3 \in \mathrm{GF}(q)^{a\times(n-r)}$.

Output: Either a failure or the unique nearest codeword in $\mathcal{E}$ from $R(\mathbf{A})$.

2.1 If $\mathrm{rk}(\mathbf{A}) < r - d + 1$, return a failure.

2.2 Calculate $r - \mathrm{rk}(\mathbf{A}_0) = ld + m$ where $0 \leq l \leq \left\lfloor \frac{r}{d} \right\rfloor$ and $0 \leq m < d$.

2.3 Call $\mathrm{EBDD}(l, \mathbf{A})$ to obtain $(E_{i,j}^l, d_l, f_l)$. If $d_l \leq d-1$, return $E_{i,j}^l$.

2.4 If $m = 0$, return a failure. Otherwise, call $\mathrm{EBDD}(l+1, \mathbf{A})$ to obtain $(E_{s,t}^{l+1}, d_{l+1}, f_{l+1})$. If $d_{l+1} \leq d-1$, return $E_{s,t}^{l+1}$.

2.5 If $d_l < \min\{d+m, f_l, d_{l+1}, f_{l+1}, 2d-m\}$, return $E_{i,j}^l$. If $d_{l+1} < \min\{d+m, d_l, f_l, f_{l+1}, 2d-m\}$, return $E_{s,t}^{l+1}$.

2.6 Return a failure.

Proposition 2: If the received subspace is at subspace distance at most $d-1$ from a codeword in $\mathcal{E}$, then Algorithm 2 returns this codeword. Otherwise, Algorithm 2 returns either a failure or the unique codeword closest to the received subspace in the subspace metric.

Proof: We first show that Algorithm 2 returns the unique nearest codeword in $\mathcal{E}$ to the received subspace if it is at subspace distance at most $d-1$. For all $1 \leq k \leq \left\lfloor \frac{r}{d} \right\rfloor$ and $E_{u,v}^k \in \mathcal{E}^k$, Lemma 1 and (5) yield

$$d_{\rm s}(R(\mathbf{A}), E_{u,v}^k) \geq d_{\rm s}(R(\mathbf{A}_0), I(\mathbf{C}_u^k)) \geq |r - kd - \mathrm{rk}(\mathbf{A}_0)| = |(l-k)d + m|. \tag{7}$$

Similarly (5) yields $d_{\rm s}(R(\mathbf{A}), E_{0,v}^0) \geq ld + m$ for any v. Hence $d_{\rm s}(R(\mathbf{A}), \mathcal{E}^k) \geq d$ for $k \leq l-1$ or $k \geq l+2$. Therefore, the unique nearest codeword is either in $\mathcal{E}^l$ or $\mathcal{E}^{l+1}$ and applying Algorithm 1 for $\mathcal{E}^l$ and $\mathcal{E}^{l+1}$ always returns the nearest codeword.

We now show that when the distance from the received subspace to the code is at least d, Algorithm 2 either produces the unique nearest codeword or returns a failure. First, by (7), $d_{\rm s}(R(\mathbf{A}), \mathcal{E}^{l-1}) = d + m$ and $d_{\rm s}(R(\mathbf{A}), \mathcal{E}^{l+2}) = 2d - m$, while $d_{\rm s}(R(\mathbf{A}), \mathcal{E}^k) \geq 2d$ for $k \leq l-2$ or $k \geq l+3$. Also, by Lemma

2, $d_{\rm s}(R(\mathbf{A}), E^l_{u,v}) \geq f_l$ for all $(u,v) \neq (i,j)$ and $d_{\rm s}(R(\mathbf{A}), \mathcal{E}^{l+1}) \geq \min\{d_{l+1}, f_{l+1}\}$. Therefore, if $d_l < \min\{d+m, f_l, d_{l+1}, f_{l+1}, 2d-m\}$, then $E^l_{i,j}$ is the unique codeword closest to $R(\mathbf{A})$. Similarly, if $d_{l+1} < \min\{d+m, d_l, f_l, f_{l+1}, 2d-m\}$, then $E^{l+1}_{s,t}$ is the unique codeword closest to $R(\mathbf{A})$. ■

We note that when $\mathrm{rk}(\mathbf{A}) < r-d+1$, by (5) Steps 1.2 and 1.3 would both fail, and Algorithm 2 will return a failure. We also justify why Algorithm 2 returns a failure if $d_l \geq d$ and $m=0$ in Step 2.3. Suppose $d_l \geq d$ and $m=0$ and we apply Algorithm 1 for $\mathcal{E}^{l+1}$. Then we have $d_l \geq d+m$ and by (7) $d_{l+1} \geq |d-m| = d+m$. Therefore, neither inequality in Step 2.5 is satisfied and the decoder returns a failure.

By Proposition 2, Algorithm 2 decodes beyond the half distance. However, the decoding radius of Algorithm 2 is limited. It is easy to see that the decoding radius of Algorithm 2 is at most $d + \lfloor \frac{d}{2} \rfloor$ due to the terms $d+m$ and $2d-m$ in the inequalities in Step 2.5. We emphasize that this is just an upper bound, and its tightness is unknown. Suppose $r - \mathrm{rk}(\mathbf{A}_0) = ld+m$, when Algorithm 2 decodes beyond half distance, it is necessary that f_l and f_{l+1} be both nonzero in Step 2.5. This implies that the row space of $(\mathbf{A}_1|\mathbf{A}_2)$ is at subspace distance no more than $d-1$ from $I(\mathcal{C}^l)$ and $I(\mathcal{C}^{l+1})$ and that the row spaces of $(\mathbf{A}_1|\mathbf{A}_3)$ are at subspace distance no more than $d-1$ from $I(\mathcal{D}^l)$ and $I(\mathcal{D}^{l+1})$.

We note that the inequalities in Step 2.5 are strict in order to ensure that the output of the decoder is the **unique** nearest codeword from the received subspace. However, if one of the nearest codewords is an acceptable outcome, then equality can be included in the inequalities in Step 2.5.

Our decoding algorithm can be readily simplified in order to obtain a bounded subspace distance decoder, by removing Step 2.5. We emphasize that the general decoding algorithm has the same order of complexity as this simplified bounded subspace distance decoding algorithm.

Finally, we note that the decoding algorithms and discussions above consider the subspace metric. It is also remarkable that our decoder remains the same if the injection metric is used instead. We formalize this by the following proposition.

Proposition 3: If the received subspace is at injection distance at most $d-1$ from a codeword in $\mathcal{E}$, then Algorithm 2 returns this codeword. Otherwise, Algorithm 2 returns either a failure or the unique codeword closest to the received subspace in the injection metric.

The proof of Proposition 3 is based on the observation that a codeword in a CDC is closest to the received subspace in the subspace metric if and only if the codeword is closest to the received subspace in the injection metric by (2), and is hence omitted.

The complexity of the bounded subspace distance decoder in [10] for a KK code in $E(q,n)$ is on the order of $O(n^2)$ operations over $\mathrm{GF}(q)^{n-r}$ for $r \leq \frac{n}{2}$, which is hence the complexity of decoding $\mathcal{E}^0$.

This algorithm can be easily generalized to include the case where $r > \frac{n}{2}$, and we obtain a complexity on the order of $O(n^2)$ operations over $\mathrm{GF}(q^{\max\{r,n-r\}})$. Thus the complexity of decoding $I(\mathcal{C}^k)$ and $I(\mathcal{D}^k)$ for $k \geq 1$ is on the order of $O(r^2)$ operations over $\mathrm{GF}(q^{\max\{kd,r-kd\}})$ and $O((n-kd)^2)$ operations over $\mathrm{GF}(q^{\max\{r,n-kd-r\}})$, respectively. The complexity of the decoding algorithm for $\mathcal{E}^k$ is on the order of the maximum of these two quantities. It is easily shown that the complexity is maximized for $k = 0$, that is, our decoding algorithm has the same order of complexity as the algorithm for the KK code $\mathcal{E}^0$.

IV. Covering properties of CDCs

The packing properties of CDCs have been studied in [10], [13]–[15], [21] and an asymptotic packing rate of CDCs was defined and determined in [10]. Henceforth in this section, we focus on the covering properties of CDCs in the Grassmannian instead. We emphasize that since $d_{\mathrm{S}}(U,V) = 2d_{\mathrm{I}}(U,V)$ for all $U, V \in E_r(q,n)$, we consider only the injection distance in this section. Furthermore, since $d_{\mathrm{I}}(U,V) = d_{\mathrm{I}}(U^\perp, V^\perp)$ for all $U, V \in E_r(q,n)$, without loss of generality we assume that $r \leq \left\lfloor \frac{n}{2} \right\rfloor$ in this section.

A. Properties of balls in the Grassmannian

We first investigate the properties of balls in the Grassmannian $E_r(q,n)$, which will be instrumental in our study of covering properties of CDCs. First, we derive bounds on the volume of balls in $E_r(q,n)$.

Lemma 3: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 \leq t \leq r$, $q^{t(n-t)} \leq V_{\mathrm{C}}(t) < K_q^{-2} q^{t(n-t)}$.

Proof: First, we have $V_{\mathrm{C}}(t) \geq N_{\mathrm{C}}(t) \geq q^{t(n-t)}$ by (3). Also, $N_{\mathrm{C}}(d) < K_q^{-1} N_{\mathrm{R}}(q, n-r, r, d)$, and hence $V_{\mathrm{C}}(t) < K_q^{-1} V_{\mathrm{R}}(q, n-r, r, t) < K_q^{-2} q^{t(n-t)}$ as $V_{\mathrm{R}}(q, n-r, r, t) < K_q^{-1} q^{t(n-t)}$ [20, Lemma 9]. ■

We now determine the volume of the intersection of two **spheres** of radii u and s respectively and distance d between their centers, which is referred to as the intersection number $J_{\mathrm{C}}(u,s,d)$ of the association scheme [27]. The intersection number is an important parameter of an association scheme.

Lemma 4: For all u, s, and d between 0 and r,

$$J_{\mathrm{C}}(u,s,d) = \frac{1}{\begin{bmatrix} n \\ r \end{bmatrix} N_{\mathrm{C}}(d)} \sum_{i=0}^{r} \mu_i E_u(i) E_s(i) E_d(i),$$

where $\mu_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$ and $E_j(i)$ is a q-Eberlein polynomial [28]:

$$E_j(i) = \sum_{l=0}^{j} (-1)^{j-l} q^{li + \binom{j-l}{2}} \begin{bmatrix} r-l \\ r-j \end{bmatrix} \begin{bmatrix} r-l \\ i \end{bmatrix} \begin{bmatrix} n-r+l-i \\ l \end{bmatrix}.$$

Although Lemma 4 is obtained by a direct application of Theorems 3.5 and 3.6 in [29, Chapter II], we present it formally here since it is a fundamental geometric property of the Grassmannian and is very instrumental in our study of CDCs. We also obtain a recursion formula for $J_{\mathrm{C}}(u,s,d)$.

Lemma 5: $J_{\mathrm{C}}(u,s,d)$ satisfies the following recursion: $J_{\mathrm{C}}(0,s,d)=\delta_{s,d}$, $J_{\mathrm{C}}(u,0,d)=\delta_{u,d}$, and

$$c_{u+1}J_{\mathrm{C}}(u+1,s,d)=b_{s-1}J_{\mathrm{C}}(u,s-1,d)+(a_s-a_u)J_{\mathrm{C}}(u,s,d)+c_{s+1}J_{\mathrm{C}}(u,s+1,d)-b_{u-1}J_{\mathrm{C}}(u-1,s,d),$$

where $c_j=J_{\mathrm{C}}(1,j-1,j)=\begin{bmatrix}j\\1\end{bmatrix}^2$, $b_j=J_{\mathrm{C}}(1,j+1,j)=q^{2j+1}\begin{bmatrix}r-j\\1\end{bmatrix}\begin{bmatrix}n-r-j\\1\end{bmatrix}$, and $a_j=J_{\mathrm{C}}(1,j,j)=N_{\mathrm{C}}(1)-b_j-c_j$ for $0\le j\le r$.

The proof follows directly from [27, Lemma 4.1.7], [27, Theorem 9.3.3], and [27, Chapter 4, (1a)], and hence is omitted. Let $I_{\mathrm{C}}(u,s,d)$ denote the intersection of two **balls** in $E_r(q,n)$ with radii u and s and distance d between their centers. Since $I_{\mathrm{C}}(u,s,d)=\sum_{i=0}^{u}\sum_{j=0}^{s}J_{\mathrm{C}}(i,j,d)$, Lemma 4 also leads to an analytical expression for $I_{\mathrm{C}}(u,s,d)$. Proposition 4 below shows that $I_{\mathrm{C}}(u,s,d)$ decreases as d increases.

Proposition 4: For all u and s, $I_{\mathrm{C}}(u,s,d)$ is a non-increasing function of d.

The proof of Proposition 4 is given in Appendix A. Therefore, the minimum nonzero intersection between two balls with radii u and s in $E_r(q,n)$ is given by $I_{\mathrm{C}}(u,s,u+s)=J_{\mathrm{C}}(u,s,u+s)$ for $u+s\le r$. By Lemma 5, it is easily shown that $J_{\mathrm{C}}(u,s,u+s)=\begin{bmatrix}u+s\\u\end{bmatrix}^2$ for all u and s when $u+s\le r$.

We derive below an upper bound on the union of balls in $E_r(q,n)$ with the same radius.

Lemma 6: The volume of the union of *any* K balls in $E_r(q,n)$ with radius ρ is at most

$$\begin{aligned}B_{\mathrm{C}}(K,\rho) \quad &= KV_{\mathrm{C}}(\rho)-\sum_{a=1}^{l}[A_{\mathrm{C}}(q,n,r,r-a+1)-A_{\mathrm{C}}(q,n,r,r-a+2)]I_{\mathrm{C}}(\rho,\rho,r-a+1)\\ &\quad -[K-A_{\mathrm{C}}(q,n,r,r-l+1)]I_{\mathrm{C}}(\rho,\rho,r-l),\end{aligned}\tag{8}$$

where $l=\max\{a: K\ge A_{\mathrm{C}}(q,n,r,r-a+1)\}$.

Proof: Let $\{U_i\}_{i=0}^{K-1}$ denote the centers of K balls with radius ρ and let $\mathcal{V}_j=\{U_i\}_{i=0}^{j-1}$ for $1\le j\le K$. Without loss of generality, we assume that the centers are labeled such that $d_{\mathrm{I}}(U_j,\mathcal{V}_j)$ is non-increasing for $j\ge 1$. For $1\le a\le l$ and $A_{\mathrm{C}}(q,n,r,r-a+2)\le j<A_{\mathrm{C}}(q,n,r,r-a+1)$, we have $d_{\mathrm{I}}(U_j,\mathcal{V}_j)=d_{\mathrm{I}}(\mathcal{V}_{j+1})\le r-a+1$. By Proposition 4, U_j hence covers at most $V_{\mathrm{C}}(\rho)-I_{\mathrm{C}}(\rho,\rho,r-a+1)$ subspaces that are not previously covered by balls centered at $\mathcal{V}_j$. ■

We remark that using any upper bound on $A_{\mathrm{C}}(q,n,r,r-a+1)$ in the proof of Lemma 6 leads to a valid upper bound on $B_{\mathrm{C}}(K,\rho)$. Hence, although the value of $A_{\mathrm{C}}(q,n,r,r-a+1)$ is unknown in general, the upper bound in (4) can be used in (8) in order to obtain an upper bound on the volume of the union on balls in the Grassmannian.

B. Covering CDCs

The *covering radius* of a CDC $\mathcal{C}\subseteq E_r(q,n)$ is defined as $\rho=\max_{U\in E_r(q,n)}d_{\mathrm{I}}(U,\mathcal{C})$. We denote the minimum cardinality of a CDC in $E_r(q,n)$ with covering radius ρ as $K_{\mathrm{C}}(q,n,r,\rho)$. Since $K_{\mathrm{C}}(q,n,n-$

$r,\rho) = K_{\mathrm{C}}(q,n,r,\rho)$, we assume $r \leq \left\lfloor \frac{n}{2} \right\rfloor$. Also, $K_{\mathrm{C}}(q,n,r,0) = \begin{bmatrix} n \\ r \end{bmatrix}$ and $K_{\mathrm{C}}(q,n,r,r) = 1$, hence we assume $0 < \rho < r$ henceforth. We first derive lower bounds on $K_{\mathrm{C}}(q,n,r,\rho)$.

Lemma 7: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \geq \min\left\{K : B_{\mathrm{C}}(K,\rho) \geq \begin{bmatrix} n \\ r \end{bmatrix}\right\} \geq \frac{\begin{bmatrix} n \\ r \end{bmatrix}}{V_{\mathrm{C}}(\rho)}$.

Proof: Let $\mathcal{C}$ be a CDC with cardinality $K_{\mathrm{C}}(q,n,r,\rho)$ and covering radius ρ. Then the balls around the codewords cover the $\begin{bmatrix} n \\ r \end{bmatrix}$ subspaces in $E_r(q,n)$; however, by Lemma 6, they cannot cover more than $B_{\mathrm{C}}(|\mathcal{C}|,\rho)$ subspaces. Therefore, $B_{\mathrm{C}}(K_{\mathrm{C}}(q,n,r,\rho),\rho) \geq \begin{bmatrix} n \\ r \end{bmatrix}$ and we obtain the first inequality. Since $B_{\mathrm{C}}(K,\rho) \leq KV_{\mathrm{C}}(\rho)$ for all K, we obtain the second inequality. ■

The second lower bound in Lemma 7 is referred to as the sphere covering bound for CDCs. This bound can also be refined by considering the distance distribution of a covering code.

Proposition 5: For $0 \leq \delta \leq \rho$, let $T_\delta = \min \sum_{i=0}^{r} A_i(\delta)$, where the minimum is taken over all integer sequences $\{A_i(\delta)\}$ which satisfy $A_i(\delta) = 0$ for $0 \leq i \leq \delta - 1$, $1 \leq A_\delta(\delta) \leq N_{\mathrm{C}}(\delta)$, $0 \leq A_i(\delta) \leq N_{\mathrm{C}}(i)$ for $\delta + 1 \leq i \leq r$, and $\sum_{i=0}^{r} A_i(\delta) \sum_{s=0}^{\rho} J_{\mathrm{C}}(l,s,i) \geq N_{\mathrm{C}}(l)$ for $0 \leq l \leq r$. Then $K_{\mathrm{C}}(q,n,r,\rho) \geq \max_{0 \leq \delta \leq \rho} T_\delta$.

Proof: Let $\mathcal{C}$ be a CDC with covering radius ρ. For any $U \in E_r(q,n)$ at distance δ from $\mathcal{C}$, let $A_i(\delta)$ denote the number of codewords at distance i from U. Then $\sum_{i=0}^{r} A_i(\delta) = |\mathcal{C}|$ and we easily obtain $A_i(\delta) = 0$ for $0 \leq i \leq \delta - 1$, $1 \leq A_\delta(\delta) \leq N_{\mathrm{C}}(\delta)$, and $0 \leq A_i(\delta) \leq N_{\mathrm{C}}(i)$ for $\delta + 1 \leq i \leq r$. Also, for $0 \leq l \leq r$, all the subspaces at distance l from U are covered, hence $\sum_{i=0}^{r} A_i(\delta) \sum_{s=0}^{\rho} J_{\mathrm{C}}(l,s,i) \geq N_{\mathrm{C}}(l)$. ■

We remark that Proposition 5 is a tighter lower bound than the sphere covering bound. However, determining T_δ is computationally infeasible for large parameter values.

Another set of linear inequalities is obtained from the inner distribution $\{a_i\}$ of a covering code $\mathcal{C}$, defined as $a_i \stackrel{\text{def}}{=} \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} |\{D \in \mathcal{C} : d_{\mathrm{I}}(C,D) = i\}|$ for $0 \leq i \leq r$ [30].

Proposition 6: Let $t = \min \sum_{i=0}^{r} a_i$, where the minimum is taken over all sequences $\{a_i\}$ satisfying $a_0 = 1$, $0 \leq a_i \leq N_{\mathrm{C}}(i)$ for $1 \leq i \leq r$, $\sum_{i=0}^{r} a_i \sum_{s=0}^{\rho} J_{\mathrm{C}}(l,s,i) \geq N_{\mathrm{C}}(l)$ for $0 \leq l \leq r$, and $\sum_{i=0}^{r} a_i \frac{E_i(l)}{N_{\mathrm{C}}(i)} \geq 0$ for $0 \leq l \leq r$. Then $K_{\mathrm{C}}(q,n,r,\rho) \geq t$.

Proof: Let $\mathcal{C}$ be a CDC with covering radius ρ and inner distribution $\{a_i\}$. Proposition 5 yields $0 \leq a_i \leq N_{\mathrm{C}}(i)$ for $1 \leq i \leq r$, $\sum_{i=0}^{r} a_i \sum_{s=0}^{\rho} J_{\mathrm{C}}(l,s,i) \geq N_{\mathrm{C}}(l)$ for $0 \leq l \leq r$, while $a_0 = 1$ follows the definition of a_i. By the generalized MacWilliams inequalities [30, Theorem 3], $\sum_{i=0}^{r} a_i F_l(i) \geq 0$, where $F_l(i) = \frac{\mu_l}{N_{\mathrm{C}}(i)} E_i(l)$ are the q-numbers of the association scheme [30, (15)], which yields $\sum_{i=0}^{r} a_i \frac{E_i(l)}{N_{\mathrm{C}}(i)} \geq 0$. Since $\sum_{i=0}^{r} a_i = |\mathcal{C}|$ we obtain that $|\mathcal{C}| \geq t$. ■

Lower bounds on covering codes with the Hamming metric can be obtained through the concept of the excess of a code [31]. This concept being independent of the underlying metric, it was adapted to

the rank metric in [25]. We adapt it to the injection metric for CDCs below, thus obtaining the lower bound in Proposition 7.

Proposition 7: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \geq \frac{\begin{bmatrix} n \\ r \end{bmatrix}}{V_{\mathrm{C}}(\rho) - \frac{\epsilon}{\delta} N_{\mathrm{C}}(\rho)}$, where $\epsilon \stackrel{\text{def}}{=} \left\lceil \frac{b_\rho}{c_{\rho+1}} \right\rceil c_{\rho+1} - b_\rho$, $\delta \stackrel{\text{def}}{=} N_{\mathrm{C}}(1) - c_\rho + 2\epsilon$, and b_ρ and $c_{\rho+1}$ are defined in Lemma 5.

The proof of Proposition 7 is given in Appendix B. We now derive upper bounds on $K_{\mathrm{C}}(q,n,r,\rho)$. First, we investigate how to expand covering CDCs.

Lemma 8: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \leq K_{\mathrm{C}}(q,n-1,r,\rho-1) \leq \begin{bmatrix} n-\rho \\ r \end{bmatrix}$, and $K_{\mathrm{C}}(q,n,r,\rho) \leq K_{\mathrm{C}}(q,n,r-1,\rho-1) \leq \begin{bmatrix} n \\ r-\rho \end{bmatrix}$.

The proof of Lemma 8 is given in Appendix C. The next upper bound is a straightforward adaptation of [25, Proposition 12].

Proposition 8: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \leq \left\{ 1 - \log_{\begin{bmatrix} n \\ r \end{bmatrix}} \left(\begin{bmatrix} n \\ r \end{bmatrix} - V_{\mathrm{C}}(\rho) \right) \right\}^{-1} + 1$.

The proof of Proposition 8 is given in Appendix D. The next bound is a direct application of [16, Theorem 12.2.1].

Proposition 9: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \leq \frac{\begin{bmatrix} n \\ r \end{bmatrix}}{V_{\mathrm{C}}(\rho)} \{1 + \ln V_{\mathrm{C}}(\rho)\}$.

The bound in Proposition 9 can be refined by applying the greedy algorithm described in [32] to CDCs.

Proposition 10: Let k_0 be the cardinality of an augmented KK code with minimum distance $2\rho+1$ in $E_r(q,n)$ for $2\rho < r$ and $k_0 = 1$ for $2\rho \geq r$. Then for all $k \geq k_0$, there exists a CDC with cardinality k which covers at least $\begin{bmatrix} n \\ r \end{bmatrix} - u_k$ subspaces, where $u_{k_0} \stackrel{\text{def}}{=} \begin{bmatrix} n \\ r \end{bmatrix} - k_0 V_{\mathrm{C}}(\rho)$ and $u_{k+1} = u_k - \left\lceil \frac{u_k V_{\mathrm{C}}(\rho)}{\min\left\{ \begin{bmatrix} n \\ r \end{bmatrix} - k, B_{\mathrm{C}}(u_k,\rho) \right\}} \right\rceil$ for all $k \geq k_0$. Thus $K_{\mathrm{C}}(q,n,r,\rho) \leq \min\{k : u_k = 0\}$.

The proof of Proposition 10 is given in Appendix E.

Using the bounds derived above, we finally determine the asymptotic behavior of $K_{\mathrm{C}}(q,n,r,\rho)$. The rate of a covering CDC $\mathcal{C} \subseteq E_r(q,n)$ is defined as $\frac{\log_q |\mathcal{C}|}{\log_q |E_r(q,n)|}$. We remark that this rate is defined in a combinatorial sense: the rate describes how well a CDC covers the Grassmannian. We use the following normalized parameters: $r' = \frac{r}{n}$, $\rho' = \frac{\rho}{n}$, and the asymptotic rate $k_{\mathrm{C}}(r',\rho') = \liminf_{n\to\infty} \frac{\log_q K_{\mathrm{C}}(q,n,r,\rho)}{\log_q \begin{bmatrix} n \\ r \end{bmatrix}}$.

Proposition 11: For all $0 \leq \rho' \leq r' \leq \frac{1}{2}$, $k_{\mathrm{C}}(r',\rho') = 1 - \frac{\rho'(1-\rho')}{r'(1-r')}$.

Proof: The bounds on $V_{\mathrm{C}}(\rho)$ in Lemma 3 together with the sphere covering bound yield $K_{\mathrm{C}}(q,n,r,\rho) > K_q^2 q^{r(n-r)-\rho(n-\rho)}$. Using the bounds on the Gaussian polynomial in Section II-B, we obtain $k_{\mathrm{C}}(r',\rho') \geq 1 - \frac{\rho'(1-\rho')}{r'(1-r')}$. Also, Proposition 9 leads to $K_{\mathrm{C}}(q,n,r,\rho) < K_q^{-1} q^{r(n-r)-\rho(n-\rho)} [1 + \ln(K_q^{-2}) + \rho(n-\rho)\ln q]$, which asymptotically becomes $k_{\mathrm{C}}(r',\rho') \leq 1 - \frac{\rho'(1-\rho')}{r'(1-r')}$. ∎

The proof of Proposition 11 indicates that $K_{\mathrm{C}}(q,n,r,\rho)$ is on the order of $q^{r(n-r)-\rho(n-\rho)}$.

We finish this section by studying the covering properties of liftings of rank metric codes. We first prove that they have maximum covering radius.

Lemma 9: Let $I(\mathcal{C}) \subseteq E_r(q,n)$ be the lifting of a rank metric code in $\mathrm{GF}(q)^{r\times(n-r)}$. Then $I(\mathcal{C})$ has covering radius r.

Proof: Let $D \in E_r(q,n)$ be generated by $(\mathbf{0}|\mathbf{D}_1)$, where $\mathbf{D}_1 \in \mathrm{GF}(q)^{r\times(n-r)}$ has rank r. Then, for any codeword $I(\mathbf{C})$ generated by $(\mathbf{I}_r|\mathbf{C})$, it is easily seen that $d_{\mathrm{I}}(D, I(\mathbf{C})) = d_{\mathrm{I}}(R(\mathbf{0}), R(\mathbf{I}_r)) = r$ by Lemma 1. ■

Lemma 9 is significant for the design of CDCs. It is shown in [10] that liftings of rank metric codes can be used to construct nearly optimal packing CDCs. However, Lemma 9 indicates that for any lifting of a rank metric code, there exists a subspace at distance r from the code. Hence, adding this subspace to the code leads to a supercode with higher cardinality and the same minimum distance since $d \leq r$. Thus an optimal CDC cannot be designed from a lifting of a rank metric code.

Although liftings of rank metric codes have poor covering properties, below we construct a class of covering CDCs by using permuted liftings of rank metric covering codes. We thus relate the minimum cardinality of a covering CDC to that of a covering code with the rank metric. For all n and r, we denote the set of subsets of $\{0,1,\ldots,n-1\}$ with cardinality r as S_n^r. For all $J \in S_n^r$ and all $\mathbf{C} \in \mathrm{GF}(q)^{r\times(n-r)}$, let $I(J,\mathbf{C}) = R(\pi(\mathbf{I}_r|\mathbf{C})) \in E_r(q,n)$, where π is the permutation of $\{0,1,\ldots,n-1\}$ satisfying $J = \{\pi(0),\pi(1),\ldots,\pi(r-1)\}$, $\pi(0) < \pi(1) < \ldots < \pi(r-1)$, and $\pi(r) < \pi(r+1) < \ldots < \pi(n-1)$. We remark that π is uniquely determined by J. It is easily shown that $d_{\mathrm{I}}(I(J,\mathbf{C}), I(J,\mathbf{D})) = d_{\mathrm{R}}(\mathbf{C},\mathbf{D})$ for all $J \in S_n^r$ and all $\mathbf{C},\mathbf{D} \in \mathrm{GF}(q)^{r\times(n-r)}$.

Proposition 12: For all q, n, $r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $0 < \rho < r$, $K_{\mathrm{C}}(q,n,r,\rho) \leq \binom{n}{r} K_{\mathrm{R}}(q^{n-r},r,\rho)$.

Proof: Let $\mathcal{C} \subseteq \mathrm{GF}(q)^{r\times(n-r)}$ have rank covering radius ρ and cardinality $K_{\mathrm{R}}(q^{n-r},r,\rho)$. We show below that $L(\mathcal{C}) = \{I(J,\mathbf{C}) : J \in S_n^r, \mathbf{C} \in \mathcal{C}\}$ is a CDC with covering radius ρ. Any $U \in E_r(q,n)$ can be expressed as $I(J,\mathbf{V})$ for some $J \in S_n^r$ and some $\mathbf{V} \in \mathrm{GF}(q)^{r\times(n-r)}$. Also, by definition, there exists $\mathbf{C} \in \mathcal{C}$ such that $d_{\mathrm{R}}(\mathbf{C},\mathbf{V}) \leq \rho$ and hence $d_{\mathrm{I}}(U, I(J,\mathbf{C})) = d_{\mathrm{R}}(\mathbf{C},\mathbf{V}) \leq \rho$. Thus $L(\mathcal{C})$ has covering radius ρ and cardinality $\leq \binom{n}{r} K_{\mathrm{R}}(q^{n-r},r,\rho)$. ■

It is shown in [25] that for $r \leq n-r$, $K_{\mathrm{R}}(q^{n-r},r,\rho)$ is on the order of $q^{r(n-r)-\rho(n-\rho)}$, which is also the order of $K_{\mathrm{C}}(q,n,r,\rho)$. The bound in Proposition 12 is relatively tighter for large q since $\binom{n}{r}$ is independent of q.

APPENDIX

A. Proof of Proposition 4

Before proving Proposition 4, we introduce some useful notations. For $0 \leq d \leq r$, we denote $U_d = R(\mathbf{I}_r|\mathbf{P}_d) \in E_r(q,n)$, where $\mathbf{P}_d = \left(\begin{array}{c|c} \mathbf{I}_d & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \in \mathrm{GF}(q)^{r\times(n-r)}$, hence $d_{\mathrm{I}}(U_0, U_d) = d$ for all $0 \leq d \leq r$. We also denote the set of all generator matrices of all subspaces in $B_u(U_0) \cap B_s(U_d)$ as $F(u,s,d)$, hence $|F(u,s,d)| = I_{\mathrm{C}}(u,s,d)\prod_{i=0}^{r-1}(q^r - q^i)$.

Lemma 10: Let $\mathbf{X} = (\mathbf{A}|\mathbf{B}) \in \mathrm{GF}(q)^{r\times n}$, where $\mathbf{A}$ and $\mathbf{B}$ have r and $n-r$ columns, respectively. Furthermore, we denote $\mathbf{A} = (\mathbf{A}_1|\mathbf{a}|\mathbf{A}_2)$ and $\mathbf{B} = (\mathbf{B}_1|\mathbf{b}|\mathbf{B}_2)$, where $\mathbf{a}$ and $\mathbf{b}$ are the d-th columns of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then $\mathbf{X} \in F(u,s,d)$ if and only if $\mathrm{rk}(\mathbf{X}) = r$, $\mathrm{rk}(\mathbf{B}) \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{b} - \mathbf{a}|\mathbf{B}_2) \leq s$.

Proof: First, $\mathbf{X}$ is the generator matrix of some $V \in E_r(q,n)$ if and only if $\mathrm{rk}(\mathbf{X}) = r$. Also, it is easily shown that $d_{\mathrm{I}}(V, U_0) = \mathrm{rk}(\mathbf{B})$ and $d_{\mathrm{I}}(V, U_d) = \mathrm{rk}(\mathbf{B} - \mathbf{A}\mathbf{P}_d) = \mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{b} - \mathbf{a}|\mathbf{B}_2)$. Therefore, $\mathbf{X} \in F(u,s,d)$ if and only if $\mathrm{rk}(\mathbf{X}) = r$, $\mathrm{rk}(\mathbf{B}) \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{b} - \mathbf{a}|\mathbf{B}_2) \leq s$. ■

We now give the proof of Proposition 4.

Proof: It suffices to show that $I_{\mathrm{C}}(u,s,d) \leq I_{\mathrm{C}}(u,s,d-1)$ for any $d \geq 1$. We do so by first defining a mapping ϕ from $F(u,s,d)$ to $F(u,s,d-1)$ and then proving it is injective. Let $\mathbf{X} \in F(u,s,d)$, then by Lemma 10, $\mathrm{rk}(\mathbf{X}) = r$, $\mathrm{rk}(\mathbf{B}) \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{b} - \mathbf{a}|\mathbf{B}_2) \leq s$. Since the mapping ϕ only modifies $\mathbf{b}$, we shall denote $\phi(\mathbf{X}) = \mathbf{Y} = (\mathbf{A}|\mathbf{B}_1|\mathbf{c}|\mathbf{B}_2)$. We hence have to show that $\mathrm{rk}(\mathbf{Y}) = r$, $\mathrm{rk}(\mathbf{B}_1|\mathbf{c}|\mathbf{B}_2) \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{c}|\mathbf{B}_2) \leq s$. We need to distinguish three cases.

- Case I: $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2) \leq s-1$. In this case, $\mathbf{c} = \mathbf{b}$. Note that $\mathrm{rk}(\mathbf{Y}) = r$, $\mathrm{rk}(\mathbf{B}) \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{c}|\mathbf{B}_2) \leq \mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2) + 1 \leq s$.
- Case II: $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2) = s$ and $\mathrm{rk}(\mathbf{B}_1|\mathbf{B}_2) \leq u-1$. In this case, $\mathbf{c} = \mathbf{b} - \mathbf{a}$. Note that $\mathrm{rk}(\mathbf{Y}) = r$, $\mathrm{rk}(\mathbf{B}_1|\mathbf{c}|\mathbf{B}_2) \leq \mathrm{rk}(\mathbf{B}) + 1 \leq u$, and $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{c}|\mathbf{B}_2) = \mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{b} - \mathbf{a}|\mathbf{B}_2) = s$.
- Case III: $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2) = s$ and $\mathrm{rk}(\mathbf{B}_1|\mathbf{B}_2) = u$. We denote the column space of a matrix $\mathbf{D}$ as $C(\mathbf{D})$. We have $\mathbf{b} - \mathbf{a} \in C(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2)$ and $\mathbf{b} \in C(\mathbf{B}_1|\mathbf{B}_2)$. Hence $\mathbf{a} \in C(\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_1 - \mathbf{A}_1)$. Denoting $C(\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_1 - \mathbf{A}_1) = C(\mathbf{B}_1|\mathbf{B}_2) \oplus S$, where S is a fixed subspace of $C(\mathbf{B}_1 - \mathbf{A}_1)$, $\mathbf{a}$ can be uniquely expressed as $\mathbf{a} = \mathbf{r} + \mathbf{s}$, where $\mathbf{r} \in C(\mathbf{B}_1|\mathbf{B}_2)$ and $\mathbf{s} \in S$. In this case, $\mathbf{c} = \mathbf{b} - \mathbf{r}$. Since $\mathbf{b} \in C(\mathbf{B}_1|\mathbf{B}_2)$, $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathbf{A}|\mathbf{B}_1|\mathbf{B}_2) = r = \mathrm{rk}(\mathbf{Y})$. Also, since $\mathbf{c} \in C(\mathbf{B}_1|\mathbf{B}_2)$, $\mathrm{rk}(\mathbf{B}_1|\mathbf{c}|\mathbf{B}_2) = \mathrm{rk}(\mathbf{B}_1|\mathbf{B}_2) = u$. Finally, $\mathbf{c} = \mathbf{b} - \mathbf{a} + \mathbf{s} \in C(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{B}_2)$, therefore $\mathrm{rk}(\mathbf{B}_1 - \mathbf{A}_1|\mathbf{c}|\mathbf{B}_2) = s$.

It is easy to show that ϕ is injective. Therefore, $|F(u,s,d)| \leq |F(u,s,d-1)|$ and $I_{\mathrm{C}}(u,s,d) \leq I_{\mathrm{C}}(u,s,d-1)$. ■

B. Proof of Proposition 7

We adapt below the notations in [31], [33] to the injection metric for CDCs. For all $V \subseteq E_r(q,n)$ and a CDC $\mathcal{C} \subseteq E_r(q,n)$ with covering radius ρ, the excess on V by $\mathcal{C}$ is defined to be $E_{\mathcal{C}}(V) \stackrel{\text{def}}{=} \sum_{C\in\mathcal{C}} |B_\rho(C)\cap V| - |V|$. Hence if $\{W_i\}$ is a family of disjoint subsets of $E_r(q,n)$, then $E_{\mathcal{C}}\left(\bigcup_i W_i\right) = \sum_i E_{\mathcal{C}}(W_i)$. We define $\mathcal{Z} \stackrel{\text{def}}{=} \{Z \in E_r(q,n) : E_{\mathcal{C}}(\{Z\}) \geq 1\}$, i.e., $\mathcal{Z}$ is the set of subspaces covered by at least two codewords in $\mathcal{C}$. It follows that $|\mathcal{Z}| \leq E_{\mathcal{C}}(\mathcal{Z}) = E_{\mathcal{C}}(E_r(q,n)) = |\mathcal{C}|V_{\rm C}(\rho) - \begin{bmatrix} n \\ r \end{bmatrix}$.

Before proving Proposition 7, we need the following adaptation of [31, Lemma 8]. Let $\mathcal{C}$ be a code in $E_r(q,n)$ with covering radius ρ. We define $\mathcal{A} \stackrel{\text{def}}{=} \{U \in E_r(q,n) : d_{\rm I}(U,\mathcal{C}) = \rho\}$.

Lemma 11: For $U \in \mathcal{A}\backslash\mathcal{Z}$ and $0 < \rho < r$, we have $E_{\mathcal{C}}(B_1(U)) \geq \epsilon$.

Proof: Since $U \notin \mathcal{Z}$, there is a unique $C_0 \in \mathcal{C}$ such that $d_{\rm I}(U,C_0) = \rho$. We have $|B_\rho(C_0)\cap B_1(U)| = I_{\rm C}(\rho,1,\rho) = J_{\rm C}(\rho,0,\rho) + J_{\rm C}(\rho,1,\rho) + J_{\rm C}(\rho-1,1,\rho) = 1 + a_\rho + c_\rho$. For any codeword $C_1 \in \mathcal{C}$ satisfying $d_{\rm I}(U,C_1) = \rho+1$, by Lemma 5 we have $|B_\rho(C_1) \cap B_1(U)| = J_{\rm C}(\rho,1,\rho+1) = c_{\rho+1}$. Finally, for all other codewords $C_2 \in \mathcal{C}$ at distance $> \rho+1$ from U, we have $|B_\rho(C_2) \cap B_1(U)| = 0$. Denoting $N \stackrel{\text{def}}{=} |\{C_1 \in \mathcal{C} : d_{\rm I}(U,C_1) = \rho+1\}|$, we obtain

$$\begin{aligned} E_{\mathcal{C}}(B_1(U)) &= \sum_{C\in\mathcal{C}} |B_\rho(C)\cap B_1(U)| - |B_1(U)| \\ &= 1 + a_\rho + c_\rho + N c_{\rho+1} - N_{\rm C}(1) - 1 = -b_\rho + N c_{\rho+1} \\ &\equiv -b_\rho \mod c_{\rho+1}. \end{aligned}$$

The proof is completed by realizing that $-b_\rho < 0$, while $E_{\mathcal{C}}(B_1(U))$ is a non-negative integer. ■

We now establish a key lemma.

Lemma 12: If $Z \in \mathcal{Z}$ and $0 < \rho < r$, then $|\mathcal{A} \cap B_1(Z)| \leq V_{\rm C}(1) - c_\rho$.

Proof: By definition of ρ, there exists $C \in \mathcal{C}$ such that $d_{\rm I}(Z,C) \leq \rho$. By Proposition 4, $|B_1(Z) \cap B_{\rho-1}(C)| \geq c_\rho$, with equality achieved for $d_{\rm I}(Z,C) = \rho$. A subspace at distance $\leq \rho-1$ from any codeword does not belong to $\mathcal{A}$. Therefore, $B_1(Z) \cap B_{\rho-1}(C) \subseteq B_1(Z)\backslash\mathcal{A}$, and hence $|\mathcal{A}\cap B_1(Z)| = |B_1(Z)| - |B_1(Z)\backslash\mathcal{A}| \leq V_{\rm C}(1) - |B_1(Z)\cap B_{\rho-1}(C)|$. ■

We now give a proof of Proposition 7.

Proof: For a code $\mathcal{C}$ with covering radius ρ and $\epsilon \geq 1$,

$$\begin{aligned} \gamma &\stackrel{\text{def}}{=} \epsilon\left\{\begin{bmatrix} n \\ r \end{bmatrix} - |\mathcal{C}|V_{\rm C}(\rho-1)\right\} - (\epsilon-1)\left\{|\mathcal{C}|V_{\rm C}(\rho) - \begin{bmatrix} n \\ r \end{bmatrix}\right\} \quad (9) \\ &\leq \epsilon|\mathcal{A}| - (\epsilon-1)|\mathcal{Z}| \quad (10) \\ &\leq \epsilon|\mathcal{A}| - (\epsilon-1)|\mathcal{A}\cap\mathcal{Z}| = \epsilon|\mathcal{A}\backslash\mathcal{Z}| + |\mathcal{A}\cap\mathcal{Z}|, \end{aligned}$$

where (10) follows from $|\mathcal{Z}| \leq |\mathcal{C}|V_{\rm C}(\rho) - \begin{bmatrix} n \\ r \end{bmatrix}$.

$$\begin{aligned} \gamma &\leq \sum_{A \in \mathcal{A} \setminus \mathcal{Z}} E_{\mathcal{C}}(B_1(A)) + \sum_{A \in \mathcal{A} \cap \mathcal{Z}} E_{\mathcal{C}}(B_1(A)) \\ &= \sum_{A \in \mathcal{A}} E_{\mathcal{C}}(B_1(A)), \end{aligned} \tag{11}$$

where (11) follows from Lemma 11 and $|\mathcal{A} \cap \mathcal{Z}| \leq E_{\mathcal{C}}(\mathcal{A} \cap \mathcal{Z})$.

$$\begin{aligned} \gamma &\leq \sum_{A \in \mathcal{A}} \sum_{U \in B_1(A) \cap \mathcal{Z}} E_{\mathcal{C}}(\{U\}) \\ &= \sum_{U \in \mathcal{Z}} \sum_{A \in B_1(U) \cap \mathcal{A}} E_{\mathcal{C}}(\{U\}) = \sum_{U \in \mathcal{Z}} |\mathcal{A} \cap B_1(U)| E_{\mathcal{C}}(\{U\}), \end{aligned} \tag{12}$$

where (12) follows the fact that the second summation is over disjoint sets $\{U\}$. By Lemma 12, we obtain

$$\begin{aligned} \gamma &\leq \sum_{U \in \mathcal{Z}} (V_{\rm C}(1) - c_\rho)\, E_{\mathcal{C}}(\{U\}) \\ &= (V_{\rm C}(1) - c_\rho)\, E_{\mathcal{C}}(\mathcal{Z}) \\ &= (V_{\rm C}(1) - c_\rho) \left\{ |\mathcal{C}|V_{\rm C}(\rho) - \begin{bmatrix} n \\ r \end{bmatrix} \right\}. \end{aligned} \tag{13}$$

Combining (13) and (9), we obtain the bound in Proposition 7. ■

C. Proof of Lemma 8

Proof: Let $\mathcal{C}$ be a code in $E_r(q, n-1)$ with covering radius $\rho-1$ and cardinality $K_{\rm C}(q, n-1, r, \rho-1)$. Define the code $\mathcal{C}_1 \subseteq E_r(q, n)$ as $\mathcal{C}_1 = \{R(\mathbf{C}|\mathbf{0}) : R(\mathbf{C}) \in \mathcal{C}\}$. For any $U_1 \in E_r(q, n)$ with generator matrix $\mathbf{U}_1 = (\mathbf{U}|\mathbf{u})$, where $\mathbf{U} \in \mathrm{GF}(q)^{r \times n-1}$ and $\mathbf{u} \in \mathrm{GF}(q)^{r \times 1}$, we prove that there exists $C_1 \in \mathcal{C}_1$ generated by $\mathbf{C}_1 = (\mathbf{C}|\mathbf{0})$ such that $d_{\rm I}(C_1, U_1) \leq \rho$. We remark that $\mathrm{rk}(\mathbf{U})$ is equal to either r or $r-1$. First, if $\mathrm{rk}(\mathbf{U}) = r$, then there exists $C \in \mathcal{C}$ such that $\mathrm{rk}(\mathbf{C}^T|\mathbf{U}^T) \leq r + \rho - 1$. Second, if $\mathrm{rk}(\mathbf{U}) = r-1$, then let $\mathbf{U}_0$ be $r-1$ linearly independent rows of $\mathbf{U}$. For any $\mathbf{v} \in \mathrm{GF}(q)^{n-1}$, $\mathbf{v} \notin R(\mathbf{U}_0)$, there exists $C \in \mathcal{C}$ such that $r + \rho - 1 \geq \mathrm{rk}(\mathbf{C}^T|\mathbf{U}_0^T|\mathbf{v}^T) \geq \mathrm{rk}(\mathbf{C}^T|\mathbf{U}_0^T) = \mathrm{rk}(\mathbf{C}^T|\mathbf{U}^T)$. Hence $\mathrm{rk}(\mathbf{C}_1^T|\mathbf{U}_1^T) \leq r + \rho$ and $d_{\rm I}(C_1, U_1) \leq \rho$. Thus $\mathcal{C}_1$ has covering radius at most ρ and hence $K_{\rm C}(q, n, r, \rho) \leq K_{\rm C}(q, n-1, r, \rho-1)$, which applied ρ times yields $K_{\rm C}(q, n, r, \rho) \leq K_{\rm C}(q, n-\rho, r, 0) = \begin{bmatrix} n-\rho \\ r \end{bmatrix}$.

Similarly, let $\mathcal{D}$ be a code in $E_{r-1}(q, n)$ with covering radius $\rho-1$ and cardinality $K_{\rm C}(q, n, r-1, \rho-1)$. Define the code $\mathcal{D}_1 = \{R((\mathbf{D}^T|\mathbf{d}^T)^T) : R(\mathbf{D}) \subseteq \mathcal{D}\} \in E_r(q, n)$, where $\mathbf{d} \in \mathrm{GF}(q)^n$ is chosen at random

such that $\mathrm{rk}(\mathbf{D}^T|\mathbf{d}^T) = r$. We remark that $|\mathcal{D}_1| \leq |\mathcal{D}|$. For any $V_1 \in E_r(q,n)$ with generator matrix $\mathbf{V}_1 = (\mathbf{V}^T|\mathbf{v}^T)^T$, there exists $D_1 \in \mathcal{D}_1$ with generator matrix $\mathbf{D}_1 = (\mathbf{D}^T|\mathbf{d}^T)^T$ with $\mathrm{rk}(\mathbf{D}^T|\mathbf{V}^T) \leq r+\rho-2$. Thus $\mathrm{rk}(\mathbf{D}_1^T|\mathbf{V}_1^T) \leq r+\rho$ and $\mathcal{D}_1$ has covering radius at most ρ. Thus $K_{\mathrm{C}}(q,n,r,\rho) \leq |\mathcal{D}_1| \leq K_{\mathrm{C}}(q,n,r-1,\rho-1)$ which applied ρ times yields $K_{\mathrm{C}}(q,n,r,\rho) \leq K_{\mathrm{C}}(q,n,r-\rho,0) = \begin{bmatrix} n \\ r-\rho \end{bmatrix}$. ■

D. Proof of Proposition 8

Proof: Denoting the set of all codes of cardinality K in $E_r(q,n)$ as S_K, we have $|S_K| = \binom{Q}{K}$, where $Q \stackrel{\text{def}}{=} \begin{bmatrix} n \\ r \end{bmatrix}$. For any code $\mathcal{C} \in K$ we denote the number of subspaces in $E_r(q,n)$ at distance $> \rho$ from $\mathcal{C}$ as $P(\mathcal{C})$. The average value of $P(\mathcal{C})$ for all codes $\mathcal{C} \in S_K$ is given by

$$\begin{aligned}
\frac{1}{|S_K|}\sum_{\mathcal{C}\in S_K} P(\mathcal{C}) &= \frac{1}{|S_K|}\sum_{\mathcal{C}\in S_K}\sum_{\substack{U\in E_r(q,n)\\ d_{\mathrm{I}}(U,\mathcal{C})>\rho}} 1 \\
&= \frac{1}{|S_K|}\sum_{U\in E_r(q,n)}\sum_{\substack{\mathcal{C}\in S_K\\ d_{\mathrm{I}}(U,\mathcal{C})>\rho}} 1 \\
&= \frac{1}{|S_K|}\sum_{U\in E_r(q,n)}\binom{Q-V_{\mathrm{C}}(\rho)}{K} \qquad (14)\\
&= \frac{Q}{|S_K|}\binom{Q-V_{\mathrm{C}}(\rho)}{K}.
\end{aligned}$$

Eq. (14) comes from the fact that there are $\binom{Q-V_{\mathrm{C}}(\rho)}{K}$ codes with cardinality K that do not cover U. For all K, there exists a code $\mathcal{C}' \in S_K$ for which $P(\mathcal{C}')$ is no more than the average, i.e., $P(\mathcal{C}') \leq Q\binom{Q}{K}^{-1}\binom{Q-V_{\mathrm{C}}(\rho)}{K} \leq Q\left(1-Q^{-1}V_{\mathrm{C}}(\rho)\right)^K$. For $K = \left\lfloor -\frac{1}{\log_Q(1-Q^{-1}V_{\mathrm{C}}(\rho))}\right\rfloor + 1$, $P(\mathcal{C}') \leq Q\left(1-Q^{-1}V_{\mathrm{C}}(\rho)\right)^K < 1$ and $\mathcal{C}'$ has covering radius at most ρ. ■

E. Proof of Proposition 10

Proof: The proof is by induction on k. First, an augmented KK code is a code with cardinality k_0 and minimum distance $2\rho+1$ for $2\rho < r$, which hence leaves u_{k_0} subspaces uncovered; for $2\rho \geq r$, a single codeword covers $V_{\mathrm{C}}(\rho)$ subspaces. Second, suppose there exists a code with cardinality k which leaves **exactly** v_k ($v_k \leq u_k$) subspaces uncovered, and denote the set of uncovered subspaces as V_k. Let G be the graph where the vertex set is $E_r(q,n)$ and two vertices are adjacent if and only if their distance is at most ρ. Let $\mathbf{A}$ be the adjacency matrix of G and $\mathbf{A}_k$ be the v_k columns of $\mathbf{A}$ corresponding to V_k. There are $v_k V_{\mathrm{C}}(\rho)$ ones in $\mathbf{A}_k$, distributed across $|N(V_k)|$ rows, where $N(V_k)$ is the neighborhood [34] of V_k. By construction, $N(V_k)$ does not contain any codeword, hence $|N(V_k)| \leq \begin{bmatrix} n \\ r \end{bmatrix} - k$. Also, by Lemma 6, $|N(V_k)| \leq B_{\mathrm{C}}(v_k,\rho) \leq B_{\mathrm{C}}(u_k,\rho)$. Thus $|N(V_k)| \leq \min\left\{\begin{bmatrix} n \\ r \end{bmatrix} - k, B_{\mathrm{C}}(u_k,\rho)\right\}$ and there exists

a row with at least $\left\lceil \frac{v_k V_{\mathrm{C}}(\rho)}{\min\left\{\begin{bmatrix} n \\ r \end{bmatrix} - k, B_{\mathrm{C}}(u_k,\rho)\right\}} \right\rceil$ ones in $\mathbf{A}_k$. Adding the subspace corresponding to this row to the code, we obtain a code with cardinality $k+1$ which leaves at most $v_k - \left\lceil \frac{v_k V_{\mathrm{C}}(\rho)}{\min\left\{\begin{bmatrix} n \\ r \end{bmatrix} - k, B_{\mathrm{C}}(u_k,\rho)\right\}} \right\rceil \leq u_{k+1}$ subspaces uncovered. ■

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