# A Fundamental Characterization of Stability in Broadcast Queueing Systems

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Abstract—Stability with respect to a given scheduling policy has become an important issue for wireless communication systems, but it is hard to prove in particular scenarios. In this paper two simple conditions for stability in broadcast channels are derived, which are easy to check. Heuristically, the conditions imply that if the queue length in the system becomes large, the rate allocation is always the solution of a weighted sum rate maximization problem. Furthermore, the change of the weight factors between two time slots becomes smaller and the weight factors of the users, whose queues are bounded while the other queues expand, tend to zero. Then it is shown that for any mean arrival rate vector inside the ergodic achievable rate region the system is stable in the strong sense when the given scheduling policy complies with the conditions. In this case the policy is so-called throughput-optimal. Subsequently, some results on the necessity of the presented conditions are provided. Finally, in several application examples it is shown that the results in the paper provide a convenient way to verify the throughput-optimal policies.

## I. INTRODUCTION

In wireless communication systems input data packets arrive randomly at the transmitter and queue up in a buffer awaiting the transmission. The transmission rate for each user is determined by a scheduling policy which acts according to the system state. One major challenge for the policy design is to improve the system throughput by taking advantage of the channel variations. However, policies which consider only the channel state will cause unfairness among the users due to the randomness in the data traffic. Even more severely, some users may suffer from long queueing delay or even buffer overflow.

In order to avoid this situation, the scheduling policy can take the queue states into account. An essential requirement then is that the queues are kept finite for all users and do not blow up over time, such that the queueing system is stable. The stability of the queueing system is determined by the arrival traffic, the transmission capacity and the applied scheduling policy. A scheduling policy is called *throughput-optimal* if it keeps the system stable for any set of mean arrival rates that lies in the ergodic achievable rate region. Throughputoptimality is a desirable feature of scheduling policies, since the system can offer a maximal possible traffic load and keeps all queues stable at the same time. There already exist a number of throughput-optimal scheduling policies in previous work, e.g. the Maximum Weight Matching (MWM) policy [1]–[4], the exponential rule [5], the Queue Proportional Scheduling (QPS) [6], the Idle State Prediction Scheduling (ISPS) [7]. Throughput-optimality of these policies is proven using techniques which are adapted to the particular policy and in fact the proofs can be rather involved [1], [5]–[7]. Eryilmaz et al. provided some sufficient conditions under which the scheduling policies are throughput-optimal [8]. However, these conditions are quite restricted and do not include all throughput-optimal scheduling policies (e.g. exponential rule, QPS, ISPS).

In this paper, we consider scheduling policies in a broadcast system and give general conditions for their throughputoptimality. The scheduling policy is formulated as the solution of a weighted sum rate maximization problem differing only in the choice of the weight factors. Then we show that throughput-optimality can be verified solely by checking characteristics of the weight factors. It is shown that the weight factors of a throughput-optimal scheduling policy only need to satisfy two conditions:

- 1) If the total queue length in the system becomes large, the change of the weight factors between two time slots tends to zero.
- If the total queue length in the system becomes large, the weight factors of "nonurgent" users, whose queues are bounded while the other queues expand, tend to zero.

The proofs are derived by using some special Lyapunov functions in connection with theorems in differential geometry. It is proven that the presented conditions are also necessary and indeed cover all queue-length based throughput-optimal scheduling policies. We apply these results to well-known scheduling policies and show their throughput-optimality.

The remainder of this paper is organized as follows: Section II describes the system model including some assumptions on the rate and arrival processes. In this section we also introduce the definition of stability and throughput-optimality. General sufficient conditions for throughput-optimality are presented in Section III. The necessity of these conditions is proven in Section IV. Some applications of our results are shown in Section V. Finally we conclude in Section VI.

**Notations:** We use boldface letters to denote vectors and common letters with subscript are the elements.  $\|\mathbf{x}\|_i$  denotes the  $l_i$ -norm of the vector  $\mathbf{x}$  and  $\|\mathbf{x}\|$  is a arbitrary norm of  $\mathbf{x}$ .  $\mathbb{E}\{x\}$  denotes the expected value of random variable x. Furthermore we use  $\mathcal{A}^c$  to denote the complement of a set  $\mathcal{A}$ .

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The probability function is denoted as  $Pr\{\cdot\}$ . The indicator  $\mathbb{I}\{\cdot\}$  equals 1 if the argument is true and equals 0 otherwise.

## II. SYSTEM MODEL

## A. Physical layer

We consider a single cell downlink system in which a base station simultaneously supplies M mobile users. The channel between the base station and each user is assumed to be constant within a time slot and varies from one time slot to another in a stationary i.i.d. manner. The channel state of user i in the n-th time slot is denoted as  $h_i(n) \in S$ , where S is an arbitrary countable or uncountable set, and all channel states of the user set  $\mathcal{M} := \{1, ..., M\}$  are collected in the vector  $\mathbf{h}(n) \in \mathcal{S}^M$ . Here the set  $\mathcal{S}$  is used to indicate that the general approach is not restricted to a specific transmission scheme. For example in a MIMO system the channel state can be described as a matrix of complex channel gains such that  $h_i(n) \in \mathbb{C}^{n_r n_t}$  where  $n_r$ ,  $n_t$  are the number of transmit and receive antennas at the base station and mobiles, respectively. Likewise for an OFDM system the channel state can be defined as a vector of complex channel gains on each subcarrier  $h_i(n) \in \mathbb{C}^K$  where K is the number of subcarriers. Furthermore we assume that the channel state information is, either perfectly or partially, accessible at the receiver and the transmitter. In the n-th time slot the data is transmitted through the channel at the rate  $\mathbf{r}(n) \in \mathbb{R}^M_+$  with the transmission power  $\mathbf{p}(n) \in \mathbb{R}^M_+$ .

We define a resource allocation policy  $\mathcal{P}$  with rate allocation  $\mathbf{r}^{\mathcal{P}}(n)$  and power allocation  $\mathbf{p}^{\mathcal{P}}(n)$ . The allocated rate satisfies  $\mathbf{r}^{\mathcal{P}}(n) \in \mathcal{C}(\mathbf{h}(n), \mathbf{p}^{\mathcal{P}}(n))$ , where  $\mathcal{C}(\mathbf{h}(n), \mathbf{p}(n))$  is the achievable rate region of the broadcast system with the channel state  $\mathbf{h}(n)$  and the transmission power  $\mathbf{p}(n)$ . We don't specify the achievable rate region  $\mathcal{C}(\mathbf{h}(n), \mathbf{p}(n))$  so that the results in the paper can be applied to different transmission schemes.

The entire achievable rate region of a system with maximum power constraint  $\hat{P}$  in time slot n is given by

$$\mathcal{C}\left(\mathbf{h}(n),\hat{P}\right) := \bigcup_{\mathcal{P}\in\Omega} \mathcal{C}\left(\mathbf{h}(n),\mathbf{p}^{\mathcal{P}}(n)\right),\tag{1}$$

where  $\Omega$  is the set of feasible policies

$$\Omega := \left\{ \tilde{\mathcal{P}} : \left\| \mathbf{p}^{\tilde{\mathcal{P}}} \left( n \right) \right\|_{1} \leq \hat{P}, \forall n \right\}.$$

In general, the rate region  $C(\mathbf{h}(n), \hat{P})$  might be non-convex; then we consider the convex hull of the achievable rate region. Any point on the convex hull of the region  $\underline{\mathsf{Shag}}_{nephacisments}$  solution of the maximization problem

$$\mathbf{r}(\boldsymbol{\mu}, \mathbf{h}(n)) = \underset{\tilde{\mathbf{r}} \in \mathcal{C}(\mathbf{h}(n), \hat{P})}{\operatorname{arg\,max}} \boldsymbol{\mu}^{T} \tilde{\mathbf{r}},$$
(2)

where  $\boldsymbol{\mu} \in \mathbb{R}^M_+$  is the set of weight factors. It is important to stress that even though the weight factors are fixed and independent of the channel state  $\mathbf{h}(n)$ , the rate allocation  $\mathbf{r}(\boldsymbol{\mu}, \mathbf{h}(n))$  depends on the channel state due to the maximization problem over  $\mathcal{C}(\mathbf{h}(n), \hat{P})$  in (2). Observe that weight factors  $\mu$  also represent a normal vector of a supporting hyperplane which is tangential to the convex hull at the point  $\mathbf{r}(\mu, \mathbf{h}(n))$ .

Then, the ergodic achievable rate region is defined as

$$\mathcal{C}(\hat{P}) := \bigcap_{\|\boldsymbol{\mu}\|=1} \left\{ r_1, ..., r_M : \boldsymbol{\mu}^T \mathbf{r} \le \mathbb{E} \left\{ \max_{\tilde{\mathbf{r}} \in \mathcal{C}(\mathbf{h}(n), \hat{P})} \boldsymbol{\mu}^T \tilde{\mathbf{r}} \right\} \right\}.$$
(3)

Note that the definition (3) coincides with the capacity region given in [9], [10], when  $\mathcal{C}(\mathbf{h}(n), \mathbf{p}(n))$  is the capacity region of parallel degraded channels. Furthermore, it is shown in [3], [4], [11] that the region is also the maximal achievable stability region. In general the characterization of the achievable rate region  $\mathcal{C}(\mathbf{h}(n), \hat{P})$  and the solution of the weighted sum rate maximization problem is complicated. Considering practical constraints such as finite code and modulation scheme and imperfect channel state information, the ergodic achievable rate region  $\mathcal{C}(\hat{P})$  can be much smaller than the informationtheoretical capacity region. Thereby the instantaneous achievable rate region  $\mathcal{C}(\mathbf{h}(n), \hat{P})$  might be non-convex or even be restricted to a set of some discrete rate points. The results in this paper can also be applied to these practical systems as long as a solution of the problem in (2) exists (i.e. for an discrete OFDMA system described in [12]).

It is easy to show that the ergodic achievable region given in (3) is convex. According to the convexity, any boundary point of the region  $C(\hat{P})$  is a solution of the problem

$$\mathbf{r}_E(\boldsymbol{\mu}) = \underset{\tilde{\mathbf{r}} \in \mathcal{C}(\hat{P})}{\arg \max \boldsymbol{\mu}^T \tilde{\mathbf{r}}}.$$
(4)

Likewise here the weight factors  $\mu$  also represents the normal vector of the boundary at the point  $\mathbf{r}_{E}(\mu)$  (see Fig.1).



Fig. 1. The solution of the weight maximization problem (4) is a point on the boundary of the ergodic achievable rate region  $C(\hat{P})$ . The vector of weight factors  $\mu$  can be interpreted as the normal vector of the boundary at the obtained point

#### B. Medium Access Control (MAC) layer

Assuming that the transmission is time-slotted, data packets arrive randomly at the MAC and queue up in a buffer reserved for each user  $i \in \mathcal{M}$ . Simultaneously the data is read out from the buffers according to the system state, i.e., the random channel state and the current queue lengths. Thus, the system can be modeled as a queueing system with random processes reflecting the arrival and the departure of data packets.

Denoting the queue state of the *i*-th buffer in time slot  $n \in \mathbb{N}$ by  $q_i(n)$  and arranging all queue states in the vector  $\mathbf{q}(n) \in \mathbb{R}^M_+$ , the evolution of the queueing system can be written as

$$\mathbf{q}(n+1) = \left[\mathbf{q}(n) - \mathbf{r}(n) + \mathbf{a}(n)\right]^{+}, \quad (5)$$

where  $[x]_i^+ = \max\{0, x_i\}, \forall i \in \mathcal{M}$ . Vector  $\mathbf{a}(n) \in \mathbb{R}^M_+$  is a random vector denoting the amount of arrival packets during the *n*-th time slot and vector  $\mathbf{r}(n) \in \mathbb{R}^M_+$  is the amount of transmitted data.

Without loss of generality we set the length of a time slot T = 1 so that  $\mathbf{a}(n)$  and  $\mathbf{r}(n)$  are equal to the arrival and transmission rate during the time slot n. We assume that the size of a data packet is constant. To simplify the notation we set the packet size to 1 unit without loss of generality.

Eqn.(5) can also be formulated as

$$\mathbf{q}(n+1) = \mathbf{q}(n) + \mathbf{a}(n) - \mathbf{r}(n) + \mathbf{z}(n), \quad (6)$$

with

$$z_i(n) = \begin{cases} 0 & q_i(n) + a_i(n) - r_i(n) \ge 0\\ r_i(n) - q_i(n) - a_i(n) & \text{otherwise} \end{cases}$$

We assume that the sequence of arrival bits forms an i.i.d. sequence of variables over time. For technical reasons we assume that the arrival bits  $a_i(n)$  are uniformly bounded by some real constant  $C_a > 0$ .

The transmission rate  $\mathbf{r}(n)$  is determined by the applied scheduling policy. We consider scheduling policies which are independent of the time index and define the policies as the mapping from the cartesian product of the set of channel gains  $\mathbf{h}(n)$  and queue lengths  $\mathbf{q}(n)$  to the set of transmission rates. The rate allocated by policy  $\mathcal{P}$  is denoted as  $\mathbf{r}^{\mathcal{P}}(\mathbf{h}(n), \mathbf{q}(n))$ . Further we make the technical assumption that the maximum transmission rate  $\mathbf{r}^{\mathcal{P}}(\mathbf{h}(n), \mathbf{q}(n))$  are uniformly bounded by some real constant  $C_r > 0$ . Under these assumptions, the considered queueing system can be modeled as a  $\psi$ -irreducible Markov chain with  $\psi$ -irreducible measure  $\delta_0$  where  $\delta_0$  denotes a Dirac measure at zero [13].

## C. Stability

The stability of an  $\psi$ -irreducible Markov chain can be defined in different manners. We first introduce the definitions of *recurrent* and *transient* Markov chain as given in [13]. These definitions are based on the measure of the occupation time

$$\eta_{\mathcal{A}} := \sum_{n=1}^{\infty} \mathbb{I}\left(\mathbf{q}\left(n\right) \in \mathcal{A}\right)$$

which gives the number of visits in a set  $\mathcal{A} \in \mathbb{R}^M_+$  by a Markov chain after time zero.

Definition 1: A Markov chain is recurrent, if it holds  $\mathbb{E} \{\eta_A\} = +\infty$ ,  $\forall x \in A$  for any set  $A \in \mathbb{R}^M_+$ . Additionally, if the Markov chain admits an invariant probability measure  $\pi$ , then it is *positive recurrent*.

If the Markov chain is positive recurrent, it is also *weakly stable* [14] so that it holds

$$\lim_{n \to +\infty} \Pr(\|\mathbf{q}(n)\| > B) < \epsilon$$

for any  $\epsilon > 0$  and some constant B > 0.

Definition 2: A Markov chain is *transient*, if there is a countable cover of  $\mathbb{R}^M_+$  with uniformly transient sets, i.e. there is a constant C with  $\mathbb{E} \{\eta_A\} \leq C, \forall x \in A$ .

In this paper we also apply another stability definition as it is used in [8]:

Definition 3: A Markov chain is called *f-stable*, if there is an unbounded function  $f : \mathbb{R}^M_+ \to \mathbb{R}_+$  so that for any  $0 < B < +\infty$  the set  $\mathcal{B} := \{\mathbf{x} : f(\mathbf{x}) \leq B\}$  is compact, and furthermore it holds

$$\limsup \mathbb{E}\left\{f\left(\mathbf{q}\left(n\right)\right)\right\} < +\infty.$$
(7)

The function f is unbounded in all positive directions so that  $f(\mathbf{q}(n))$  goes to infinity when  $\|\mathbf{q}\|$  goes to infinity. Choosing directly  $f(\mathbf{q}) = \|\mathbf{q}\|$ , Definition 3 is equivalent to the definition of *strongly stable* [14]. Moreover, it is easy to show that for any  $f(\mathbf{q})$  which grows faster than  $\|\mathbf{q}\|$ , inequality (7) implies that the Markov chain is strongly stable.

Denoting the mean of the arrival bits  $a_i(n)$  per time slot as  $\rho_i$  collected in the vector  $\rho \in \mathbb{R}^M_+$ , we call a vector of arrival rates  $\rho$  stabilizable under  $\mathcal{P}$  when the corresponding queueing system driven by some specific scheduling policy  $\mathcal{P}$ is positive recurrent.

It is well-known that any vector of arrival rates inside the ergodic achievable rate region  $C(\hat{P})$  is stabilizable (e.g. under MWM policy) and any vector of arrival rate outside  $C(\hat{P})$  is not stabilizable [8], [11]. Thus a scheduling policy is now called *throughput-optimal* if it keeps the Markov chain positive recurrent for any vector of arrival rates  $\rho \in int(C(\hat{P}))$ , where  $int(C(\hat{P}))$  denotes the interior of the ergodic achievable rate region  $C(\hat{P})$ .

#### **III. STABILITY CONDITIONS**

We consider scheduling policies which solve the weighted sum rate maximization problem

$$\mathbf{r}^{\mathcal{P}}(\mathbf{h}(n), \mathbf{q}(n)) = \underset{\tilde{\mathbf{r}} \in \mathcal{C}(\mathbf{h}(n), \hat{P})}{\arg \max} \boldsymbol{\mu}^{\mathcal{P}}(\mathbf{q}(n)) \, \tilde{\mathbf{r}}, \qquad (8)$$

where  $\mu^{\mathcal{P}}(\mathbf{q})$  denotes the weight vector for some queue state  $\mathbf{q}$  determined by a scheduling policy  $\mathcal{P}$ . Note that the weight factors  $\mu^{\mathcal{P}}(\mathbf{q})$  depends solely on the queue state. In Section IV we argue for the necessity of this assumption. It is worth noting that the solution of the optimization problem in (8) is a boundary point of the convex hull of  $\mathcal{C}(\mathbf{h}(n), \hat{P})$ . However, such a rate allocation is possibly not uniquely defined by the weight vector  $\mu^{\mathcal{P}}(\mathbf{q})$  (which is often the case). Nevertheless

we can enforce uniqueness by invoking e.g. additional constraints on the allocated rate vector which, by the way, do not affect the line of proof in Theorem 1.

One of the well-known throughput-optimal scheduling policies is the MWM policy, which uses the weight vector

$$\boldsymbol{\mu}^{MWM}(\mathbf{q}) = \mathbf{q}.$$
(9)

The throughput-optimality of the MWM policy in general multiple-access and broadcast channels is proven in [3], [4]. Despite its simple form the MWM policy has satisfactory delay and fairness properties and is applied in several systems including MIMO [15] and OFDM.

A further class of throughput-optimal scheduling policies use the exponential rule [5]. Here, the weight vector is given by

$$\mu_i^{EXP}(\mathbf{q}) = \gamma_i exp\left(\frac{\alpha_i q_i}{\beta + \left(\frac{1}{M} \sum_{j \in \mathcal{M}} \alpha_j q_j\right)^{\eta}}\right), \quad (10)$$

where  $\gamma_1, ..., \gamma_M, \alpha_1, ..., \alpha_M$  are arbitrary sets of positive constants and the positive constants  $\beta$  and  $\eta \in (0, 1)$  are fixed.

A more generalized class of throughput-optimal scheduling policies is presented by Eryilmaz et al. in [8]. The weight factor  $\mu_i(q_i)$  is given by a function of  $q_i$  satisfying the following conditions:

- 1)  $\mu_i(q_i)$  is a nondecreasing, continuous function with  $\lim_{q_i \to +\infty} \mu_i(q_i) = +\infty.$
- 2) Given any  $C_1 > 0$ ,  $C_2 > 0$  and  $0 < \epsilon < 1$  there exists a  $B < +\infty$ , such that for all  $q_i > B$  and  $\forall i \in \mathcal{M}$ ,

$$(1-\epsilon)\mu_i(q_i) \le \mu_i(q_i - C_1)$$
$$\le \mu_i(q_i + C_2) \le (1+\epsilon)\mu_i(q_i).$$
(11)

Condition (11) implies that the relative difference  $\frac{|\mu_i(q_i \pm C) - \mu_i(q_i)|}{|\mu_i(q_i)|}$  tends toward zero for constant C if  $q_i$  is large. Hence the scheduling policies using weight functions such as  $\mu_i^{\mathcal{P}} = e^{q_i}$  do not belong to this class. Actually, it can be proven that these scheduling policies are not throughputoptimal.

The conditions given in [8] cover quite a large class of throughput-optimal scheduling policies. However, the weight factor  $\mu_i(q_i)$  is exclusively calculated by  $q_i$  and independent of the queue length of other users, which is a rather strict constraint. In general the weight factor is determined by the queue state of all users, i.e.

$$\mu_i : \mathbb{R}^M_+ \to \mathbb{R}_+, \mathbf{q} \mapsto \mu_i(\mathbf{q}).$$

Some examples are the aforementioned policies using exponential rule, the QPS and the ISPS. Unfortunately the results in [8] can not be applied in these cases.

Here, we give generalized sufficient conditions for throughput-optimality. The conditions are presented by characterizing the corresponding weight vector  $\mu^{\mathcal{P}}$  of the scheduling policies. In the following we consider the normalized weight vector

$$\bar{\boldsymbol{\mu}}^{\mathcal{P}}(\mathbf{q}) := \frac{\boldsymbol{\mu}^{\mathcal{P}}(\mathbf{q})}{\|\boldsymbol{\mu}^{\mathcal{P}}(\mathbf{q})\|_{1}}$$
(12)

and hence  $\left\| \bar{\mu}^{\mathcal{P}}(\mathbf{q}) \right\|_{1} = 1$ . Since the magnitude of the weight vector does not affect the solution of the maximization problem (2), namely the scheduling decision, we only need to consider the direction of the vector. Thus the normalization of weight factors  $\mu^{\mathcal{P}}(\mathbf{q})$  can be done without lose of generality.

Theorem 1: Any vector of arrival rates  $\rho \in int(\mathcal{C}(\hat{P}))$  is stabilizable under the scheduling policy  $\mathcal{P}$ , if its corresponding normalized weight vector  $\bar{\mu}(\mathbf{q})$  given in Eqn.(12) fulfills the following conditions:

- 1) Given any  $0 < \epsilon_1 < 1$  and  $C_1 > 0$ , there is some  $B_1 > 0$  so that for any  $\Delta \mathbf{q} \in \mathbb{R}^M$  with  $\|\Delta \mathbf{q}\| < C_1$ , we have  $|\bar{\mu}_i(\mathbf{q} + \Delta \mathbf{q}) - \bar{\mu}_i(\mathbf{q})| \leq \epsilon_1$  for any  $\mathbf{q} \in \mathbb{R}^M_+$  with  $\|\mathbf{q}\| > B_1, \forall i \in \mathcal{M}.$
- 2) Given any  $0 < \epsilon_2 < 1$  and  $C_2 > 0$ , there is some  $B_2 > 0$  so that for any  $\mathbf{q} \in \mathbb{R}^M_+$  with  $\|\mathbf{q}\| > B_2$  and  $q_i < C_2$ , we have  $\bar{\mu}_i(\mathbf{q}) \leq \epsilon_2$ , for any  $i \in \mathcal{M}$ .

Moreover, for any arrival process with  $\rho \in int(\mathcal{C}(P))$ , the queueing system is f-stable under the given policy  $\mathcal{P}$ , where f is an unbounded function as defined in Definition 3. The exact formulation of f depends on the weight function  $\bar{\mu}(q)$ .

In order to simplify the notation, provided the limit exists, we can also write the two conditions as

$$\lim_{\|\mathbf{q}\| \to +\infty} \left| \bar{\mu}_i \left( \mathbf{q} + \Delta \mathbf{q} \right) - \bar{\mu}_i \left( \mathbf{q} \right) \right| = 0, \qquad \|\Delta \mathbf{q}\| < C_1$$
$$\lim_{\|\mathbf{q}\| \to +\infty} \left| \bar{\mu}_i \left( \mathbf{q} \right) \right| = 0, \qquad q_i < C_2,$$

where  $\|\mathbf{q}\| \to +\infty$  is any path in  $\mathbb{R}^M_+$  with unbounded norm. The conditions are interpreted in the next section and their necessity is proven in Theorem 3. Before we give a proof of Theorem 1, we compare the conditions in Theorem 1 and the conditions given by Eryilmaz et al. To this end, let  $\mu_i(q_i)$ be the function dependent only on  $q_i$  and satisfies Eryilmaz's conditions. We consider two different cases:  $q_i < +\infty$  and  $q_i \to +\infty$ . If  $q_i$  is bounded and  $\|\mathbf{q}\|$  goes to infinity, then we have some  $j \neq i$  with  $q_j \rightarrow +\infty$ . According to Eryilmaz's conditions, it follows that  $\mu_i(q_i) < +\infty$  and  $\mu_j(q_j) \to +\infty$ . We normalize the weight vector so that we have

$$\lim_{\|\mathbf{q}\| \to +\infty} \bar{\mu}_i(\mathbf{q})$$
$$= \lim_{q_j \to +\infty} \frac{\mu_i(q_i)}{\mu_j(q_j) + \sum_{k \neq j} \mu_k(q_k)} = 0$$

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and Condition 2) in Theorem 1 is fulfilled. It holds also that  $\lim_{\|\mathbf{q}\|\to+\infty} \bar{\mu}_i(\mathbf{q}+\Delta \mathbf{q}) = 0$  as long as  $\|\Delta \mathbf{q}\|$  is bounded. Thus  $\lim_{\|\mathbf{q}\|\to+\infty} |\bar{\mu}_i(\mathbf{q}+\Delta \mathbf{q})-\bar{\mu}_i(\mathbf{q})|=0$  and Condition 1) is satisfied.

If  $q_i \to +\infty$  we only need to check the first condition in Theorem 1. Suppose  $\|\Delta \mathbf{q}\| < C$  for some constant C > 0, we have  $\Delta q_i < C, \forall i \in \mathcal{M}$ . After the normalization we have

$$\bar{\mu}_i(\mathbf{q} + \Delta \mathbf{q}) = \frac{1}{1 + \sum_{j \neq i} \frac{\mu_j(q_j + \Delta q_j)}{\mu_i(q_i + \Delta q_i)}}.$$
(13)

As  $\|\mathbf{q}\| \to +\infty$ , if there are other users  $j \neq i$  with  $q_j \to +\infty$ , then according to (11) we have

$$\lim_{|\mathbf{q}|| \to +\infty} \frac{\mu_j(q_j + \Delta q_j)}{\mu_i(q_i + \Delta q_i)} = \lim_{\|\mathbf{q}\| \to +\infty} \frac{\mu_j(q_j)}{\mu_i(q_i)}.$$

Otherwise if  $q_j$  is bounded, it holds

$$\lim_{\|\mathbf{q}\|\to+\infty}\frac{\mu_j(q_j+\Delta q_j)}{\mu_i(q_i+\Delta q_i)} = 0$$

Considering the both situations and substitute them in (13), we have  $\lim_{\|\mathbf{q}\|\to+\infty} |\bar{\mu}_i(\mathbf{q}+\Delta \mathbf{q})-\bar{\mu}_i(\mathbf{q})| = 0$  and the condition is satisfied. Hence we conclude that the class of policies in [8] is indeed included in the theorem.

*Proof:* The proof is given in Appendix A.

So far we considered scheduling policies based on the current queue state. In some situations the queue state information might be imprecise or delayed. These cases occur even more frequently in an uplink system, where the queue state information has to be quantized and transmitted from the mobile terminal to the base station through a signaling channel. In this paper we mainly consider the downlink system, however we emphasize that the physical layer described in Section II can be generalized as the achievable rate region is independent of the transmission schemes. Thus our results can also be applied to multiple-access channels in uplink systems if we replace the term downlink achievable rate region with uplink achievable rate region in our system model.

Theorem 1 has an interesting interpretation: suppose the scheduling policy determines the rate allocation based on the quantized queue state information  $\bar{\mathbf{q}}(n)$  with some quantization error  $\varepsilon(\mathbf{q}(n))$ , then we have  $\bar{\mathbf{q}}(n) = \mathbf{q}(n) + \varepsilon(\mathbf{q}(n))$ . If the weight vector  $\bar{\boldsymbol{\mu}}(\bar{\mathbf{q}}(n))$  determined by the scheduling policy satisfies the conditions given in Theorem 1 when  $\|\bar{\mathbf{q}}(n)\|$  is sufficiently large, it is easy to show that

$$\left|\bar{\mu}_{i}\left(\mathbf{q}+\Delta\mathbf{q}+\boldsymbol{\varepsilon}\left(\mathbf{q}+\Delta\mathbf{q}\right)\right)-\bar{\mu}_{i}\left(\mathbf{q}+\boldsymbol{\varepsilon}\left(\mathbf{q}\right)\right)\right|\leq\epsilon_{1}$$

for any  $0 < \epsilon_1 < 1$  and bounded  $\Delta \mathbf{q}$ , and if  $q_i$  is bounded, it holds

$$\bar{\mu}_{i}(\mathbf{q}+\boldsymbol{\varepsilon}(\mathbf{q})) \leq \epsilon_{2}$$

for any  $0 < \epsilon_2 < 1$  as  $\|\mathbf{q}(n)\|$  is sufficiently large. Thus the scheduling policy is also throughput-optimal.

If the obtained queue state information has  $\Delta n$  time slots delay, we have  $\bar{\mathbf{q}}(n) = \mathbf{q}(n - \Delta n)$ . Since the transmission and arrival rate are bounded, we have  $\bar{\mathbf{q}}(n) = \mathbf{q}(n) + \varepsilon_d$  where the error  $\varepsilon_d$  caused by delay is also bounded. Similarly it can be shown that the stability conditions can also be applied in this case.

## IV. NECESSITY OF STABILITY CONDITIONS

In the previous section we presented sufficient conditions for throughput-optimal scheduling policies. To do so, the scheduling problem is formulated as a weighted sum rate maximization problem, and, hence, the rate allocation of these scheduling policies always lies on the convex hull of the instantaneous achievable rate region  $C(\mathbf{h}(n), \hat{P})$ . Additionally, the weight factors  $\mu^{\mathcal{P}}$  are independent of the instantaneous channel state. Actually, we can observe that all existing throughput-optimal policies have these general characters (although it is not explicitly noticed in previous works). In fact it is an inherent necessity of any throughput-optimal scheduling policy which is expressed as the following theorem. Theorem 2: If the queue lengths are sufficiently large,

- a throughput-optimal policy always allocates the rate vector on the convex hull of the instantaneous rate region thus the rate allocation can be formulated as a weighted sum rate maximization problem.
- 2) Furthermore, the weight vector  $\mu^{\mathcal{P}}$  in the maximization problem is independent of the current fading state  $\mathbf{h}(n)$ .

The theorem is proven in [7]. Next we consider the necessity of the stability conditions given in Theorem 1. It is immediately clear that the conditions are not universally necessary. In some specific scenario the achievable rate region has no unique supporting hyperplane for some point on the boundary. Hence, the weight vector on these points is not unique. A typical example is a rate region with only two available rate points on the boundary. In this case a throughput-optimal scheduling policy can be characterized by a weight function where the image consists of two points in  $\mathbb{R}^M_+$  only. Obviously this weight function does not satisfy Condition 1) in Theorem 1. To fix this problem, we argue that in the wireless case the achievable rate regions are varying over time and further the policies must be defined for any possible configuration of  $\mathcal{C}(\mathbf{h}(n), P)$ . Hence, to prove the necessity we will only considered those achievable rate regions where the weight vector is unique for every boundary point.

Before we prove the necessity under above assumptions, we want to give some intuition about why these conditions must hold in general. Recall that a throughput-optimal policy should keep the queues stable for any mean arrival rate  $\rho$  inside the ergodic rate region  $\mathcal{C}(P)$ ; so if the arrival rate vector  $\rho$  lies close to some boundary point  $\mathbf{r}^*$  of  $\mathcal{C}(\hat{P})$  corresponding to a weight vector  $\bar{\mu}^*$ , heuristically, the weight vector determined by a throughput-optimal scheduling  $\bar{\mu}^{\mathcal{P}}$  should be in the close neighborhood of  $\bar{\mu}^*$  for almost all time slots. Condition 1) in Theorem 1 ensures now that the weight vector varies smoothly between two time slots if the queue lengths become large. Thus for the above situation, it guarantees that the weight vector  $\bar{\mu}^{\mathcal{P}}$ does not leave the neighborhood of  $\bar{\mu}^*$  in almost all time slots. Condition 2) in Theorem 1 guarantees that no rate is wasted on "nonurgent" users. If the queues of some users are bounded while the other queues expand, the scheduler should reduce the weights on these users and save these rate resources for other users.

Theorem 3: A scheduling policy  $\mathcal{P}$  is not throughputoptimal, namely there exists some arrival process with  $\rho \in$  $\operatorname{int}(\mathcal{C}(\hat{P}))$  which is not stabilizable under the policy  $\mathcal{P}$ , if the policy has one of the following characteristics:

 The change of the weight vector between two time slots is not negligible, i.e., there is some constant 0 < γ ≤ 1 and ε > 0 so that it holds

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\left\{ \| \bar{\boldsymbol{\mu}} \left( \mathbf{q}(n+1) \right) - \bar{\boldsymbol{\mu}} \left( \mathbf{q}(n) \right) \| \ge \epsilon \right\} \ge \gamma$$
(14)

for any  $\mathbf{q}(n)$ ,  $n \in \mathbb{N}$ , with probability 1.

There is some user i ∈ M, whose weight factor is not negligible, i.e., there is some constant 0 < γ ≤ 1 and ε > 0 so that it holds

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\left\{\bar{\mu}_{i}\left(\mathbf{q}(n)\right) \geq \epsilon\right\} \geq \gamma \qquad (15)$$

for any  $\mathbf{q}(n)$ ,  $n \in \mathbb{N}$ , with probability 1.

*Proof:* The proof is given in Appendix B. Applying the results in Theorem 3, it can be proven that the policies using certain exponential functions as weight factors are not throughput-optimal. The details are given in Section V.

## V. APPLICATIONS

In this section we prove the throughput-optimality of some well-known scheduling policies. Note that the throughputoptimality of these policies are already proven in the previous works, here we use our results to perform the proof in a different way and show the applicability of our results. In this section we also use the necessary conditions in Theorem 3 and verify some policies which are not throughput-optimal.

Normalizing the weight factors given in (9), it is easy to show that the MWM policy satisfies the conditions in Section III, and, hence, it is throughput-optimal. In the following we use the results to check the throughput-optimality of several other scheduling policies.

## A. Exponential Rule

Normalizing the weight factors given in (10), we have

$$\bar{\mu}_{i}(\mathbf{q}) = \frac{\gamma_{i}}{\sum_{j \in \mathcal{M}} \gamma_{j} exp\left(\frac{\alpha_{j}q_{j} - \alpha_{i}q_{i}}{\beta + \left(\frac{1}{M}\sum_{k \in \mathcal{M}} \alpha_{k}q_{k}\right)^{\eta}}\right)}$$
$$= \frac{\gamma_{i}}{\sum_{j \in \mathcal{M}} \gamma_{j} exp\left(\frac{\alpha_{j}q_{j} - \alpha_{i}q_{i}}{\beta + \tilde{\alpha}(\mathbf{q}) \|\mathbf{q}\|_{2}^{\eta}}\right)}.$$
(16)

Recall that  $\gamma_1, ..., \gamma_M$ ,  $\alpha_1, ..., \alpha_M$ ,  $\beta$ ,  $\eta$  are predefined constants in the scheduler and we define  $\widetilde{\alpha}(\mathbf{q}) := \left(\frac{1}{M} \cdot \frac{\boldsymbol{\alpha}^T \mathbf{q}}{\|\boldsymbol{\alpha}\|_2 \|\mathbf{q}\|_2}\right)^{\eta}$ . In order to check the first condition in Theorem 1, we need

In order to check the first condition in Theorem 1, we need to show that PSfrag replacements

$$\lim_{\|\mathbf{q}\|\to+\infty} \left|\bar{\mu}_i\left(\mathbf{q}+\Delta\mathbf{q}\right)-\bar{\mu}_i\left(\mathbf{q}\right)\right|=0.$$
 (17)

If  $\|\Delta \mathbf{q}\|$  is bounded, it holds  $\lim_{\|\mathbf{q}\|\to+\infty} \widetilde{\alpha} (\mathbf{q} + \Delta \mathbf{q}) = \lim_{\|\mathbf{q}\|\to+\infty} \widetilde{\alpha} (\mathbf{q})$  and

$$\lim_{\|\mathbf{q}\|\to+\infty} \frac{\alpha_j \left(q_j - \Delta q_j\right) - \alpha_i \left(q_i - \Delta q_i\right)}{\beta + \widetilde{\alpha} \left(\mathbf{q} + \Delta \mathbf{q}\right) \|\mathbf{q} + \Delta \mathbf{q}\|_2^{\eta}}$$
$$= \lim_{\|\mathbf{q}\|\to+\infty} \frac{\alpha_j q_j - \alpha_i q_i}{\beta + \widetilde{\alpha} \left(\mathbf{q}\right) \|\mathbf{q}\|_2^{\eta}}$$

for all  $i, j \in \mathcal{M}$  so that the Eqn. (17) follows.

Considering the second condition in Theorem 1, if  $q_i$  is bounded as  $\|\mathbf{q}\|$  increases, there is another user j who has the longest queue so that  $q_j \geq \frac{\|\mathbf{q}\|}{M}$ , then we have

$$\lim_{\|\mathbf{q}\|\to+\infty} \frac{\alpha_j q_j - \alpha_i q_i}{\beta + \widetilde{\alpha} \left(\mathbf{q}\right) \|\mathbf{q}\|_2^{\eta}} = \lim_{\|\mathbf{q}\|\to+\infty} \frac{\alpha_j q_j}{\widetilde{\alpha} \left(\mathbf{q}\right) \|\mathbf{q}\|_2^{\eta}} = +\infty$$
(18)

since  $\eta < 1$ .

Substituting (18) in Eqn.(16), it follows

$$\lim_{\|\mathbf{q}\| \to +\infty} \bar{\mu}_i \left( \mathbf{q} \right) = 0$$

which fulfills the Condition 2) in Theorem 1 and the throughput-optimality is proven.

## B. QPS

QPS from [6] is a scheduling policy which has good delay and fairness performance in the downlink. Applying QPS in a broadcast system with random arrivals, each user's queueing delay becomes equal as  $n \to +\infty$ . Additionally, if the queue state is initialized by  $\mathbf{q}(0) > 0$  and there is no new packet arrivals after n = 0, which can be considered as a draining problem, the QPS minimizes the expected draining time until all the buffers are cleared.

The rate vector is allocated so that

$$\mathbb{E}\left\{\mathbf{r}^{\mathcal{P}}\left(\mathbf{h}\left(n\right),\mathbf{q}\left(n\right)\right)\right\}=\mathbf{q}\left(n\right)\max_{x\mathbf{q}^{n}\in\mathcal{C}\left(\hat{P}\right)}x,$$

where x is a scalar. According to the policy, the weight vector is chosen as the norm at the boundary point of  $C(\hat{P})$  where  $\mathbb{E} \{ \mathbf{r}^{\mathcal{P}}(\mathbf{h}, \mathbf{q}) | \mathbf{q} \}$  is proportional to  $\mathbf{q}$ . Fig.2 shows the expected rate vector allocated by QPS compared to MWM policy and the ergodic achievable rate region in a 2-user scenario.



Fig. 2. The weight vector of QPS and MWM policy. For MWM  $\mu^{MWM} = \mathbf{q}$  and for QPS the weight vector  $\mu^{QPS}$  is the norm at the boundary point of  $\mathcal{C}(\hat{P})$  where  $\mathbb{E} \{ \mathbf{r}^{\mathcal{P}}(\mathbf{h}, \mathbf{q}) | \mathbf{q} \}$  is proportional to  $\mathbf{q}$ 

The weight vector in QPS is found by *Geometric Programming*. Unfortunately, there exists no explicit function of the weight factor  $\mu$  and the rate allocation  $\mathbf{r}^{\mathcal{P}}$  in QPS. Therefore the verification of its throughput-optimality is not easy using standard methods. The presented general approach in this paper is particularly suitable in this case.

Since the weight vector is determined by the normalized and queue state  $\frac{\mathbf{q}}{\|\mathbf{q}\|_2}$ , for bounded  $\|\Delta \mathbf{q}\|$  we have

$$\lim_{\|\mathbf{q}\| \to +\infty} \bar{\mu}_i(\mathbf{q} + \Delta \mathbf{q})$$
$$= \lim_{\|\mathbf{q}\| \to +\infty} \bar{\mu}_i(\frac{\mathbf{q} + \Delta \mathbf{q}}{\|\mathbf{q}\|_2})$$
$$= \lim_{\|\mathbf{q}\| \to +\infty} \bar{\mu}_i(\mathbf{q}).$$

If  $q_i < +\infty$  we have

$$\lim_{\|\mathbf{q}\| \to +\infty} \frac{q_i}{\|\mathbf{q}\|_2} = 0$$

and hence

$$\lim_{\|\mathbf{q}\|\to+\infty}\bar{\mu}_i(\mathbf{q})=0$$

Both conditions in Theorem 1 are satisfied and the QPS policy is throughput-optimal.

## C. ISPS

Another scheduling policy which also provides superior delay performance is the ISPS in [7]. If there is no new packet arrivals in the system and the queue state is initialized by  $\mathbf{q}^0 > 0$ , the policy minimizes the average queueing delay

$$\bar{D}(N) := \frac{1}{M} \sum_{i=1}^{M} D_i(N) = \frac{1}{MN} \sum_{i=1}^{M} \sum_{n=1}^{N} \frac{q_i(n)}{\mathbb{E}\{a_i(n)\}}, \quad (19)$$

where N is the length of the observed time slots.

The weight factor  $\bar{\mu}_i(\mathbf{q})$  is determined by the estimated service time  $\eta_i$  such that

$$\bar{\mu}_i(\mathbf{q}) = \frac{\frac{\eta_i(\mathbf{q})}{\overline{a}_i}}{\sum_{j \in \mathcal{M}} \frac{\eta_j(\mathbf{q})}{\overline{a}_j}},\tag{20}$$

where  $\overline{a}_i, \overline{a}_j \in \mathbb{R}_+$  are some predefined constants. Parameter  $\eta_i(\mathbf{q})$  is the expected service time of user *i* if the ISPS is applied in the system and no new packet arrival occurs. Then in the time slot  $n > \eta_i(\mathbf{q})$  the buffer of user *i* is completely emptied and the corresponding transmitter is in idle state. The estimated service time  $\eta$  is obtained with an iterative algorithm. Hence there is also no explicit function of  $\mu_i(\mathbf{q})$  in ISPS.

However, we can see that if  $||\Delta \mathbf{q}|| < +\infty$ , it follows  $\Delta \eta_i := |\eta_i(\mathbf{q} + \Delta \mathbf{q}) - \eta_i(\mathbf{q})| < +\infty$ . If  $q_i \to +\infty$ , then  $\eta_i(\mathbf{q}) \to +\infty$ . It holds

$$\lim_{\|\mathbf{q}\| \to +\infty} \left| \bar{\mu}_i \left( \mathbf{q} + \Delta \mathbf{q} \right) - \bar{\mu}_i \left( \mathbf{q} \right) \right|$$
$$= \lim_{\|\mathbf{q}\| \to +\infty} \left| \frac{\frac{\Delta \eta_i}{\overline{a}_i}}{\frac{\eta_i(\mathbf{q})}{\overline{a}_i} + \sum_{j \neq i} \frac{\eta_j(\mathbf{q})}{\overline{a}_j}} \right| = 0$$

If  $q_i$  is bounded we have  $\eta_i(\mathbf{q}) < +\infty$  and

$$\lim_{\|\mathbf{q}\|\to+\infty}\sum_{j\neq i}\eta_j(\mathbf{q})=+\infty$$

$$\lim_{\|\mathbf{q}\|\to+\infty} \bar{\mu}_i(\mathbf{q}) = \lim_{\|\mathbf{q}\|\to+\infty} \frac{\frac{\eta_i(\mathbf{q})}{\bar{a}_i}}{\frac{\eta_i(\mathbf{q})}{\bar{a}_i} + \sum_{j\neq i} \frac{\eta_j(\mathbf{q})}{\bar{a}_i}} = 0.$$

Thus the throughput-optimality is proven.

## D. Exponential Functions as Weight factors

Applying Theorem 3, it can easily be shown that the policy using exponential weight function such as  $\mu_i^{\mathcal{P}}(\mathbf{q}) = e^{q_i}$  is not throughput-optimal. We consider a point  $\mathbf{r}^*$  on the boundary of ergodic achievable rate region  $\mathcal{C}(\hat{P})$  and its corresponding normal vector  $\boldsymbol{\mu}^*$  with  $\mu_i^* > 0$ ,  $\forall i \in \mathcal{M}$ . Suppose the expected arrival rate vector  $\boldsymbol{\rho}$  lies close to the boundary point  $\mathbf{r}^*$ , in order to keep the system stable, the weight factors  $\boldsymbol{\mu}$  should be frequently chosen close to  $\boldsymbol{\mu}^*$  so that there is some constant  $\gamma_1 > 0$  with

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\left\{ \left| \bar{\mu}_{i} \left( \mathbf{q}(n) \right) - \mu_{i}^{*} \right| < \theta, \forall i \in \mathcal{M} \right\} \geq \gamma_{1}.$$

for arbitrary  $\theta > 0$  and any  $\mathbf{q}(n)$ ,  $n \in \mathbb{N}$ , with probability 1.

If  $|\bar{\mu}_i(\mathbf{q}(n)) - \mu_i^*| < \theta$ , according to the definition of  $\bar{\mu}(\mathbf{q})$  we have

$$C_1 \le \frac{\mu_i^* - \theta}{\mu_j^* + \theta} \le \frac{\bar{\mu}_i\left(\mathbf{q}(n)\right)}{\bar{\mu}_j\left(\mathbf{q}(n)\right)} = \frac{e^{q_i}}{e^{q_j}} \le \frac{\mu_i^* + \theta}{\mu_j^* - \theta} \le C_2, \quad (21)$$

for some constant  $C_1, C_2 > 0$ . We denote by

$$\Delta \mathbf{q} := \mathbf{q}(n+1) - \mathbf{q}(n) = \mathbf{a}(n) - \mathbf{r}^{\mathcal{P}}(\mathbf{h}(n), \mathbf{q}(n)).$$

Due to the randomness of  $\mathbf{a}(n)$  and  $\mathbf{h}(n)$ , for a particular user  $i \in \mathcal{M}$ , there is some probability  $\gamma_2 > 0$  that

$$\Pr\left\{C_3 < \Delta q_i < C_4, \Delta q_j \le 0, \forall j \ne i\right\} \ge \gamma_2 \qquad (22)$$

for some constants  $C_3, C_4 > 0$ . Then it holds

$$\bar{\mu}_{i}\left(\mathbf{q}(n+1)\right) - \bar{\mu}_{i}\left(\mathbf{q}(n)\right) \\
= \frac{e^{q_{i}+\Delta q_{i}}}{e^{q_{i}+\Delta q_{i}} + \sum_{j\neq i} e^{q_{j}+\Delta q_{j}}} - \frac{e^{q_{i}}}{e^{q_{i}} + \sum_{j\neq i} e^{q_{j}}} \\
\geq \frac{\left(e^{q_{i}+\Delta q_{i}} - e^{q_{i}}\right) \sum_{j\neq i} e^{q_{j}}}{\left(e^{q_{i}+\Delta q_{i}} + \sum_{j\neq i} e^{q_{j}}\right) \left(e^{q_{i}} + \sum_{j\neq i} e^{q_{j}}\right)} \\
\geq \frac{\left(e^{\Delta q_{i}} + \sum_{j\neq i} \frac{e^{q_{j}}}{e^{q_{i}}}\right) \left(\frac{e^{q_{i}}}{\sum_{j\neq i} e^{q_{j}}} + 1\right)}.$$

According to (21) we have

$$\bar{\mu}_{i} \left( \mathbf{q}(n+1) \right) - \bar{\mu}_{i} \left( \mathbf{q}(n) \right) \\
\geq \frac{e^{C_{3}} - 1}{\left( e^{C_{4}} + \frac{M-1}{C_{1}} \right) \left( \frac{C_{2}}{M-1} + 1 \right)} = \epsilon$$

for some constant  $\epsilon > 0$ . Then, combining with (22) and our i.i.d. assumption the inequality (14) in Theorem 3 holds with

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\left\{ \bar{\mu}_{i} \left( \mathbf{q}(n+1) \right) - \bar{\mu}_{i} \left( \mathbf{q}(n) \right) > \epsilon, \exists i \in \mathcal{M} \right\} \\ \geq \gamma_{1} \gamma_{2}$$

with probability 1 and the queueing system is not stable.

#### VI. CONCLUSIONS

We have presented sufficient and necessary conditions for throughput-optimality of queue-length based scheduling policies. For a wide class of arrival and channel models these conditions guarantee that if the mean arrival rate lies inside the ergodic achievable rate region, the system is stable. With application examples such as Queue Proportional Scheduling and Idle State Prediction Scheduling we have shown that the stability can even been proven in cases where conventional proof techniques fail.

## VII. APPENDIX

## A. Proof of Theorem 1

Stability can be proven by checking the so-called *Lyapunov* drift criteria as given in [1], [3], [14]. That is to say if we can find some non-negative  $V(\mathbf{q}) : \mathbb{R}^M_+ \to \mathbb{R}_+$ , some  $\theta > 0$  and a compact region  $\overline{\mathcal{B}} := {\mathbf{q} : ||\mathbf{q}|| \le B}$  such that

$$\mathbb{E}\left\{V\left(\mathbf{q}\left(n+1\right)\right)|\mathbf{q}\left(n\right)\right\} < +\infty \qquad \forall \mathbf{q}\left(n\right) \in \overline{\mathcal{B}}$$
(23)

$$\Delta V\left(\mathbf{q}\left(n\right)\right) < -\theta \qquad \qquad \forall \mathbf{q}\left(n\right) \notin \bar{\mathcal{B}},$$

the queueing system is positive recurrent. Here,  $\Delta V(\mathbf{q}(n))$  is the one-step drift defined as

$$\Delta V(\mathbf{q}(n)) := \mathbb{E}\left\{ V(\mathbf{q}(n+1)) - V(\mathbf{q}(n)) | \mathbf{q}(n) \right\}.$$

Furthermore, if for some  $\theta > 0$ , it satisfies

$$\Delta V\left(\mathbf{q}\right) \le -\theta f\left(\mathbf{q}\right), \qquad \forall \|\mathbf{q}\| > B \tag{25}$$

for some B > 0 and unbounded positive function  $f(\mathbf{q})$ , it can be shown that the queueing system is f-stable.

We carry out the proof in two steps. First, we prove the throughput-optimality for those policies, whose weight factors  $\bar{\mu}(\mathbf{q})$  fulfill the integrability condition in Eqn.(26). The weight factors in those policies can be regarded as the normalized gradient of a certain potential field  $V(\mathbf{q})$ . We show that the expected drift  $\Delta V(\mathbf{q})$  satisfies the inequality (25) and hence the system driven by those policies is stable. In the second step, we extend the results to all other policies whose weight factors are not integrable. It is shown that if the policies fulfill the condition given in the theorem, their weight factors  $\bar{\mu}(\mathbf{q})$  can be approximated by some functions  $\tilde{\mu}(\mathbf{q})$  which are integrable. Then we prove the drift condition  $\Delta V(\mathbf{q})$  for those policies and establish the stability.

Firstly, we analyze the subclass of weight functions whose  $\bar{\mu}_i(\mathbf{q})$  are continuously differentiable. Furthermore, we assume that the weight functions satisfy the integrability condition, i.e.,

$$\frac{\partial\left(\bar{\mu}_{i}\left(\mathbf{q}\right)\right)}{\partial q_{i}} = \frac{\partial\left(\bar{\mu}_{j}\left(\mathbf{q}\right)\right)}{\partial q_{i}}, \qquad \forall i, j \in \mathcal{M}.$$
 (26)

For scheduling policies with this kind of weight functions, we have the following lemma.

*Lemma 1:* If Eqn. (26) holds for all  $\mathbf{q} \in \mathbb{R}^M_+$ , then any vector of arrival rates  $\boldsymbol{\rho} \in \operatorname{int}(\mathcal{C}(\hat{P}))$  is stabilizable under the

corresponding scheduling policy as long as  $\bar{\mu}(\mathbf{q})$  fulfills the conditions given in Theorem 1.

*Proof:* Condition (26) implies that the vector field defined by  $\bar{\mu}(\mathbf{q})$  has the path independence property, namely the integral of  $\bar{\mu}(\mathbf{q})$  along a path depends only on the start and end points of that path, not the particular route taken. According to *Poincaré lemma*, the vector field  $\bar{\mu}(\mathbf{q})$  is completely integrable and it is the gradient of a scalar field, that is to say, there exist some function  $f(\mathbf{q}) : \mathbb{R}^M_+ \to \mathbb{R}_+$  with

$$\frac{\partial f(\mathbf{q})}{\partial q_i} = \bar{\mu}_i(\mathbf{q}). \tag{27}$$

Setting the value of f(q) at the origin equal zero, f(q) at the point q can be calculated by

$$f(\mathbf{q}) = \int_0^{\|\mathbf{q}\|_2} \bar{\boldsymbol{\mu}} \left(t\bar{\mathbf{q}}\right)^T \bar{\mathbf{q}} dt, \qquad (28)$$

where  $\bar{\mathbf{q}} := \frac{\mathbf{q}}{\|\mathbf{q}\|_2}$  is the normalized vector of  $\mathbf{q}$ . Since each element of  $\bar{\boldsymbol{\mu}}(\mathbf{q})$  is larger than or equal to zero,  $f(\mathbf{q})$  is a positive function. Moreover, if  $\|\mathbf{q}\|$  becomes large, according to Condition 2) in the Theorem 1, for *i*-th queue with bounded  $q_i, \bar{q}_i \to 0$  results in  $\bar{\mu}_i(\mathbf{q}) \to 0$ . Then for other queues with  $\bar{\mu}_j(\mathbf{q}) > C_{\mu}, q_j$  grows with increasing  $\|\mathbf{q}\|$  and we have  $\bar{q}_j > C_q$  for some  $C_q$  and  $C_{\mu} > 0$ . Thus it holds

$$\bar{\mu}(\mathbf{q})^T \bar{\mathbf{q}} > C$$

for some C > 0 if  $\|\mathbf{q}\|$  is sufficiently large. Considering Eqn.(28), it follows that  $f(\mathbf{q}) \to +\infty$  as  $\|\mathbf{q}\| \to +\infty$ . Therefore,  $f(\mathbf{q})$  is a positive, unbounded function as we used in Definition 3.

Observing a new vector field defined by  $\boldsymbol{\nu}(\mathbf{q}) = f(\mathbf{q}) \bar{\boldsymbol{\mu}}(\mathbf{q}),$  we have

$$\frac{\nu_{i}(\mathbf{q})}{\partial q_{j}} = \frac{\partial \left(f\left(\mathbf{q}\right)\bar{\mu}_{i}\left(\mathbf{q}\right)\right)}{\partial q_{j}} \\
= \bar{\mu}_{j}\left(\mathbf{q}\right)\bar{\mu}_{i}\left(\mathbf{q}\right) + \frac{\partial\bar{\mu}_{j}\left(\mathbf{q}\right)}{\partial q_{i}} \\
= \bar{\mu}_{j}\left(\mathbf{q}\right)\bar{\mu}_{i}\left(\mathbf{q}\right) + \frac{\partial\bar{\mu}_{i}\left(\mathbf{q}\right)}{\partial q_{j}} \\
= \frac{\partial \left(f\left(\mathbf{q}\right)\bar{\mu}_{j}\left(\mathbf{q}\right)\right)}{\partial q_{i}} \\
= \frac{\partial \left(\nu_{j}\left(\mathbf{q}\right)\right)}{\partial q_{i}}, \quad \forall i, j \in \mathcal{M}.$$
(29)

Condition (29) ensures that  $\nu(\mathbf{q})$  is also the gradient of a scalar field and there is a function  $V(\mathbf{q}) : \mathbb{R}^M_+ \to \mathbb{R}_+$  with

$$\frac{\partial V(\mathbf{q})}{\partial q_i} = f(\mathbf{q})\bar{\mu}_i(\mathbf{q}),$$

where  $f(\mathbf{q})$  is the magnitude of the gradient and  $\bar{\boldsymbol{\mu}}(\mathbf{q})$  is the direction of the gradient. Set  $V(\mathbf{0}) = 0$  and  $V(\mathbf{q})$  at the point  $\mathbf{q}$  is

$$V(\mathbf{q}) = \int_0^{\|\mathbf{q}\|_2} f(t\bar{\mathbf{q}}) \,\bar{\boldsymbol{\mu}} \left(t\bar{\mathbf{q}}\right)^T \bar{\mathbf{q}} dt$$

It is easy to shown that the function  $V(\mathbf{q})$  is also a positive, unbounded function. We use the function  $V(\mathbf{q})$  as our Lyapunov function in the proof. The first condition of the Lyapunov function given in (23) is satisfied as long as the arrival rates  $a_i(n)$  and transmission rates  $r_i(n)$  are bounded. Next we analyze the second condition, namely the drift of  $V(\mathbf{q})$  of the queueing system. For convenience we use the superscript to denote the time slot in the following.

Using the *mean value theorem* of differential calculus we have for some  $\tilde{\mathbf{q}}^n$  between  $\mathbf{q}^n$  and  $\mathbf{q}^{n+1}$  i.e.  $\tilde{q}^n_i = \alpha_i q^n_i + (1 - \alpha_i) q^{n+1}_i$ ,  $\forall i \in \mathcal{M}$ , for some  $\alpha_i \in [0, 1]$ 

$$\Delta V(\mathbf{q}^{n})$$
(30)  
= $\mathbb{E}\left\{ \sum_{i=1}^{M} f(\tilde{\mathbf{q}}^{n}) \bar{\mu}_{i}(\tilde{\mathbf{q}}^{n}) \left(a_{i}^{n} - r_{i}^{n}\right) \middle| \mathbf{q}^{n} \right\}$ + $\mathbb{E}\left\{ \sum_{i=1}^{M} f(\tilde{\mathbf{q}}^{n}) \bar{\mu}_{i}(\tilde{\mathbf{q}}^{n}) z_{i}^{n} \middle| \mathbf{q}^{n} \right\}$ (31)

Considering the first part in (31), we have

$$\mathbb{E}\left\{\sum_{i=1}^{M} f(\tilde{\mathbf{q}}^{n})\bar{\mu}_{i}(\tilde{\mathbf{q}}^{n})\left(a_{i}^{n}-r_{i}^{n}\right)\middle|\mathbf{q}^{n}\right\}$$
(32)  

$$\leq f(\mathbf{q}^{n})\left(\sum_{i=1}^{M}\bar{\mu}_{i}(\mathbf{q}^{n})\rho_{i}-\sum_{i=1}^{M}\bar{\mu}_{i}(\mathbf{q}^{n})\mathbb{E}\left\{r_{i}^{n}\middle|\mathbf{q}^{n}\right\}\right)$$
$$+\mathbb{E}\left\{\sum_{i=1}^{M}\left|f(\tilde{\mathbf{q}}^{n})\bar{\mu}_{i}(\tilde{\mathbf{q}}^{n})-f(\mathbf{q}^{n})\bar{\mu}_{i}(\mathbf{q}^{n})\right|\left|a_{i}^{n}-r_{i}^{n}\right|\middle|\mathbf{q}^{n}\right\}.$$
(32)

Since

$$\mathbb{E}\left\{\left.\mathbf{r}^{n}\right|\mathbf{q}^{n}
ight\}=rg\max_{\widetilde{\mathbf{r}}\in\mathcal{C}\left(\hat{P}
ight)}ar{oldsymbol{\mu}}(\mathbf{q}^{n})^{T}\widetilde{\mathbf{r}}.$$

for any  $\rho \in int(\mathcal{C}(\hat{P}))$ , we can always find some  $\Gamma > 0$ , so that

$$\mathbb{E}\left\{\left.\sum_{i=1}^{M} \bar{\mu}_{i}(\mathbf{q})(\rho_{i}-r_{i}^{n})\right|\mathbf{q}^{n}\right\} \leq -\Gamma.$$

Hence the first part in (33)

$$f(\mathbf{q}^n)\left(\sum_{i=1}^M \bar{\mu}_i(\mathbf{q}^n)\rho_i - \sum_{i=1}^M \bar{\mu}_i(\mathbf{q}^n)\mathbb{E}\left\{r_i^n | \mathbf{q}^n\right\}\right)$$
  
$$\leq -\Gamma f(\mathbf{q}^n).$$

For the second part in (33), we define  $\Delta \mathbf{q} = \tilde{\mathbf{q}}^n - \mathbf{q}^n$ . Then

$$f(\mathbf{q}^{n} + \Delta \mathbf{q}) - f(\mathbf{q}^{n})$$
  
= 
$$\int_{0}^{1} \bar{\boldsymbol{\mu}}(\mathbf{q}^{n} + t\Delta \mathbf{q})\Delta \mathbf{q} dt$$
$$\leq \int_{0}^{1} \|\Delta \mathbf{q}\|_{1} dt = \|\Delta \mathbf{q}\|_{1}$$

Since  $a_i^n$  and  $r_i^n$  are bounded, we choose some  $C_3 > 1$  so that  $a_i^n < C_3$  and  $r_i^n < C_3$  for all *i*. Then  $\|\Delta \mathbf{q}\|_1$  is bounded by  $2MC_3$  and we have

$$\left|f\left(\tilde{\mathbf{q}}^{n}\right) - f\left(\mathbf{q}^{n}\right)\right| < \epsilon_{3}f\left(\mathbf{q}\right)$$

for any given  $\epsilon_3 > 0$  and sufficiently large  $||\mathbf{q}||$ . According to Condition 1) in Theorem 1, we also have

$$\left|\bar{\mu}_i(\tilde{\mathbf{q}}^n) - \bar{\mu}_i(\mathbf{q}^n)\right| < \epsilon_1.$$

Then if  $\|\mathbf{q}^n\|$  is sufficiently large,

$$\mathbb{E}\left\{\sum_{i=1}^{M} |f(\tilde{\mathbf{q}}^{n})\bar{\mu}_{i}(\tilde{\mathbf{q}}^{n}) - f(\mathbf{q}^{n})\bar{\mu}_{i}(\mathbf{q}^{n})||a_{i}^{n} - r_{i}^{n}| \left| \mathbf{q}^{n} \right.\right\} \\
\leq 2C_{3}\mathbb{E}\left\{\sum_{i=1}^{M} (f(\mathbf{q}^{n}) + \epsilon_{3}f(\mathbf{q}^{n}))\left(\bar{\mu}_{i}(\mathbf{q}^{n}) + \epsilon_{1}\right) \left| \mathbf{q}^{n} \right.\right\} \\
- 2C_{3}\mathbb{E}\left\{\sum_{i=1}^{M} f(\mathbf{q}^{n})\bar{\mu}_{i}(\mathbf{q}^{n}) \left| \mathbf{q}^{n} \right.\right\} \\
= \underbrace{\left(2MC_{3}\epsilon_{1} + 2C_{3}\epsilon_{3} + 2MC_{3}\epsilon_{1}\epsilon_{3}\right)}_{\sigma_{1}}f(\mathbf{q}^{n}) \qquad (34)$$

holds for any  $\epsilon_1, \epsilon_3 > 0$ . Hence we have  $\sigma_1 \to 0$  when  $\|\mathbf{q}\| \to +\infty$ .

Now we consider the second part in (31).

$$\mathbb{E}\left\{\sum_{i=1}^{M} f(\tilde{\mathbf{q}}^{n}) \bar{\mu}_{i}(\tilde{\mathbf{q}}^{n}) z_{i}^{n} \middle| \mathbf{q}^{n}\right\}$$

$$\leq \mathbb{E}\left\{\sum_{i=1}^{M} f(\mathbf{q}^{n}) \bar{\mu}_{i}(\mathbf{q}^{n}) z_{i}^{n} \middle| \mathbf{q}^{n}\right\}$$

$$+ \mathbb{E}\left\{\sum_{i=1}^{M} \left|f(\tilde{\mathbf{q}}^{n}) \bar{\mu}_{i}(\tilde{\mathbf{q}}^{n}) - f(\mathbf{q}^{n}) \bar{\mu}_{i}(\mathbf{q}^{n})\right| z_{i}^{n} \middle| \mathbf{q}^{n}\right\}$$

$$(35)$$

For the first part in (36), since  $z_i^n \leq r_i^n$  is bounded by the current rate region,  $\mathbb{E} \{ z_i^n | \mathbf{q}^n \}$  is bounded by the ergodic achievable rate region so that for some  $C_4 > 0$  we have

$$\mathbb{E}\left\{z_i(t)\right\} \le C_4. \tag{37}$$

We define the set  $\mathcal{G} := \{i : z_i > 0, i \in \mathcal{M}\}$ . Since  $r_i^n < C_3$  is bounded by  $C_3$ , then  $q_i^n < C_3$ ,  $\forall i \in \mathcal{G}$ . If  $\|\mathbf{q}^n\|$  is sufficiently large so that  $\|\mathbf{q}^n\| > MC_3$ , we can exclude the case  $\mathcal{G} = \mathcal{M}$ . According to Condition 2) we have  $\bar{\mu}_i(\mathbf{q}^n) \leq \epsilon_2$ ,  $\forall i \in \mathcal{G}$  for arbitrarily small  $\epsilon_2$ . Then

$$\mathbb{E}\left\{\left.\sum_{i\in\mathcal{G}}f(\mathbf{q}^n)\bar{\mu}_i(\mathbf{q}^n)z_i^n\right|\mathbf{q}^n\right\} < MC_4\epsilon_2 f(\mathbf{q}^n)$$
(38)

holds.

Using the same proof method as for (34) it can be shown that the second part in (36) can be bounded by  $\sigma_2 f(\mathbf{q}^n)$  for any  $\sigma_2 > 0$ .

Define  $\theta = \Gamma - \sigma_1 - MC_4\epsilon_2 - \sigma_2$  and choose  $\sigma_1, \sigma_2, \epsilon_2$ so that  $\theta > 0$  we have the drift

$$\Delta V(\mathbf{q}^n) \le -\theta f(\mathbf{q}^n) \tag{39}$$

and which is negative and the Markov chain is positive recurrent.

Lemma 1 is applied to weight functions which are completely integrable. In general the weight functions don't have to meet the integrability condition (26) or can be even not continuously differentiable. However, it can be shown that if the weight function  $\bar{\mu}(\mathbf{q})$  has the properties described in Theorem 1, it can be approximated by some (at least piecewise integrable) function  $\tilde{\mu}(\mathbf{q})$ . The following lemma help us to achieve our main result.

Lemma 2: If the function  $\bar{\mu}(\mathbf{q})$  fulfills the Condition 1), 2) in Theorem 1, then there exists a positive, unbounded function  $f : \mathbb{R}^M_+ \to \mathbb{R}_+$  as given in Definition 3, and a positive, continuous, piecewise differentiable function  $V : \mathbb{R}^M_+ \to \mathbb{R}_+$ , such that it holds

$$\frac{\partial V(\mathbf{q})}{\partial q_i} = f(\mathbf{q})\widetilde{\mu}_i(\mathbf{q}), \forall i \in \mathcal{M}$$
(40)

on each differentiable subdomain of V, and

$$|\tilde{\mu}_i(\mathbf{q}) - \bar{\mu}_i(\mathbf{q})| < \epsilon_4, \forall i \in \mathcal{M}, \tag{41}$$

for any  $\epsilon_4 > 0$  if  $\|\mathbf{q}\|$  is sufficiently large.

*Proof:* In the following we show how to construct the function  $V(\mathbf{q})$ ,  $f(\mathbf{q})$  and  $\tilde{\mu}(\mathbf{q})$  based on  $\bar{\mu}(\mathbf{q})$ . Since we only need to ensure that  $|\tilde{\mu}_i(\mathbf{q}) - \bar{\mu}_i(\mathbf{q})| < \epsilon_4$  for large  $||\mathbf{q}||$ , it is sufficient to construct the functions on the domain where  $||\mathbf{q}|| \ge B$  for sufficiently large B. The function V and f on the domain  $||\mathbf{q}|| \le B$  can be defined as any positive, bounded, continuously differentiable function, which is continuous on the boundary  $||\mathbf{q}|| = B$ .

## replacements



Fig. 3. Orthogonal grid (irregular) in M = 3-dimension. The line integral between two points on the grid line (e.g. along the two paths marked by dashed line) depends only on the start point  $q^0$  and end point  $q^*$ . It is independent of the chosen pathes

In the domain of  $||\mathbf{q}|| \ge B$ , we at first construct a orthogonal grid such that each cell in the grid is a rectangle (see Fig.3 for an example in M = 3-dimension). Start by a point  $\mathbf{q}^a = \mathbf{Q} \in \mathbb{R}^M_+$ , the next cell in the dimensions i, j (see Fig.4) has

the grid points

$$\begin{aligned} \mathbf{q}^{a} &= [Q_{1}, ..., Q_{i}, ..., Q_{j}, ..., Q_{M}]^{T}, \\ \mathbf{q}^{b} &= [Q_{1}, ..., Q_{i} + \Delta Q_{i}, ..., Q_{j}, ..., Q_{M}]^{T}, \\ \mathbf{q}^{c} &= [Q_{1}, ..., Q_{i}, ..., Q_{j} + \Delta Q_{j}, ..., Q_{M}]^{T}, \\ \mathbf{q}^{d} &= [Q_{1}, ..., Q_{i} + \Delta Q_{i}, ..., Q_{j} + \Delta Q_{j}, ..., Q_{M}]^{T} \end{aligned}$$

The length of the cell  $\Delta Q_i, \Delta Q_j$  is determined by the equation

$$\int_{0}^{\Delta Q_{i}} \bar{\mu}_{i}(...,q_{i},Q_{j},...) - \bar{\mu}_{i}(...,q_{i},Q_{j} + \Delta Q_{j},...)dq_{i}$$
  
= 
$$\int_{0}^{\Delta Q_{j}} \bar{\mu}_{j}(...,Q_{i},q_{j},...) - \bar{\mu}_{j}(...,Q_{i} + \Delta Q_{i},q_{j},...)dq_{j}.$$
(42)

Condition 2) in Theorem 1 implies that in the region  $\|\mathbf{q}\| \ge B$  for some large constant B, the function  $\bar{\mu}_i(\mathbf{q})$  decreases with increasing  $q_j$  and  $\bar{\mu}_j(\mathbf{q})$  decreases with increasing  $q_i$  as well. Hence

$$\bar{\mu}_i(...,q_i,Q_j,...) - \bar{\mu}_i(...,q_i,Q_j + \Delta Q_j,...) > 0$$
  
$$\bar{\mu}_j(...,Q_i,q_j,...) - \bar{\mu}_j(...,Q_i + \Delta Q_i,q_j,...) > 0$$

and Eqn.(42) has positive general solutions with  $\Delta Q_i$ ,  $\Delta Q_j > 0$ . Iteratively take  $q^b$ ,  $q^c$  and  $q^d$  as start point, we can extend the grid until it covers the subdomain in the dimensions *i*, *j*. Based on the existing grid lines in the dimensions *i*, *j* (e.g. the line  $\mathbf{q}^{\bar{a}}\mathbf{q}^{\bar{b}}$  in Fig. 3), we can repeat the process in a further dimension *k* and construct the grid in this dimension (the grid  $\mathbf{q}^a \cdot \mathbf{q}^b \cdot \mathbf{q}^e \cdot \mathbf{q}^f$ ). Since relationship of  $\Delta Q_i$  and  $\Delta Q_j$  is determined by the definition of  $\bar{\mu}_i(\mathbf{q})$  on the particular points, each rectangle in the grid has different height and width so that the constructed grid has a irregular pattern.

Denote the path starts at  $q^a$  via  $q^b$  to  $q^d$  as  $S_{abd}$  and the path starts at  $q^a$  via  $q^c$  to  $q^d$  as  $S_{abd}$ , Eqn.(42) ensures that the integral of the function  $\bar{\mu}(\mathbf{q})$  along the path  $S_{abd}$  equals the integral along the path  $S_{acd}$ , which is

$$\int_{S_{abd}} \bar{\mu}(\mathbf{q}) \cdot d\mathbf{s} 
= \int_{0}^{\Delta Q_{i}} \bar{\mu}_{i}(...,q_{i},Q_{j},...)dq_{i} 
+ \int_{0}^{\Delta Q_{j}} \bar{\mu}_{j}(...,Q_{i}+\Delta Q_{i},q_{j},...)dq_{j} 
= \int_{0}^{\Delta Q_{j}} \bar{\mu}_{j}(...,Q_{i},q_{j},...)dq_{j} 
+ \int_{0}^{\Delta Q_{i}} \bar{\mu}_{i}(...,q_{i},Q_{j}+\Delta Q_{j},...)dq_{i} 
= \int_{S_{acd}} \bar{\mu}(\mathbf{q}) \cdot d\mathbf{s}.$$
(43)

Since Eqn.(43) holds for all cells of the grid, the integral between arbitrary two grid points along any grid line has the same value. Hence the vector field  $\bar{\mu}(\mathbf{q})$  can be considered as "path-independent" along the grid lines. Then we define a

function  $f(\mathbf{q})$  whose value on the grid line as the integral of where  $\bar{\mu}(\mathbf{q})$  along the grid lines, i.e.

$$f(\mathbf{q}^*) := f(\mathbf{Q}^0) + \int_S \bar{\boldsymbol{\mu}}(\mathbf{q}) \cdot d\mathbf{s},$$

where  $q^*$  is a point on the grid line and S is an arbitrary path between  $q^*$  and the initial point  $Q^0$  along the grid lines.

Define a new vector field by  $\nu(\mathbf{q}) := f(\mathbf{q})\bar{\mu}(\mathbf{q})$ , the line integral of  $\nu(\mathbf{q})$  along the path  $S_{abc}$  is

$$\begin{split} \int_{S_{abd}} \boldsymbol{\nu}(\mathbf{q}) \cdot d\mathbf{s} &= \int_{S_{abd}} f(\mathbf{q}) \bar{\boldsymbol{\mu}}(\mathbf{q}) \cdot d\mathbf{s} \\ &= \int_{S_{abd}} f(\mathbf{q}) df(\mathbf{q}) \\ &= \frac{1}{2} \left( f^2 \left( \mathbf{q}^d \right) - f^2 \left( \mathbf{q}^a \right) \right) \\ &= \int_{S_{acd}} \boldsymbol{\nu}(\mathbf{q}) \cdot d\mathbf{s}. \end{split}$$

Thus the integral of the vector field  $\nu(\mathbf{q})$  between two grid points along the grid lines is also independent of the chosen paths. Then we define a scalar field  $V(\mathbf{q})$  whose value on the grid line is given by

$$V(\mathbf{q}^*) := V(\mathbf{Q}^0) + \int_S f(\mathbf{q})\bar{\boldsymbol{\mu}}(\mathbf{q}) \cdot d\mathbf{s}.$$

The value of  $f(\mathbf{Q}^0)$  and  $V(\mathbf{Q}^0)$  at the initial point  $\mathbf{Q}^0$  can be choose as an arbitrary positive constant. Since  $\bar{\boldsymbol{\mu}}_i(\mathbf{q}) \geq 0$ ,  $\forall i \in \mathcal{M}$ , we have  $f(\mathbf{q}^*) \to +\infty$  and  $V(\mathbf{q}^*) \to +\infty$  as  $\|\mathbf{q}^*\| \to +\infty$ .

Once the value of  $V(\mathbf{q}^*)$  is fixed on the grid lines, we obtain the value of V inside a grid cell by the linear interpolation of  $V(\mathbf{q}^*)$  along the lines parallel to the diagonal line (see Fig.4), i.e. in the lower triangle with  $\frac{\Delta q_i}{\Delta Q_i} + \frac{\Delta q_j}{\Delta Q_j} < 1$ , V is defined as

$$V(\dots,Q_i + \Delta q_i, Q_j + \Delta q_j, \dots) = K_i V(\mathbf{q}^I) + K_j V(\mathbf{q}^J),$$
(44)

where

$$\begin{split} K_i &= \frac{\Delta Q_j \Delta q_i}{\Delta Q_j \Delta q_i + \Delta Q_i \Delta q_j}, \\ K_j &= \frac{\Delta Q_i \Delta q_j}{\Delta Q_j \Delta q_i + \Delta Q_i \Delta q_j}, \\ \mathbf{q}^I &= [Q_1, ..., Q_i + \Delta q_i + \frac{\Delta Q_i}{\Delta Q_j} \Delta q_j, Q_j, ..., Q_M]^T, \\ \mathbf{q}^J &= [Q_1, ..., Q_i, Q_j + \Delta q_j + \frac{\Delta Q_j}{\Delta Q_i} \Delta q_i, ..., Q_M]^T \end{split}$$

and in the higher triangle with  $\frac{\Delta q_i}{\Delta Q_i} + \frac{\Delta q_j}{\Delta Q_j} \ge 1$ , V is defined as

$$V(\dots,Q_i + \Delta q_i,Q_j + \Delta q_j,\dots) = K_i V(\mathbf{q}^I) + K_j V(\mathbf{q}^J),$$
(45)

$$K_{i} = \frac{\Delta Q_{j} \Delta Q_{i} - \Delta Q_{j} \Delta q_{i}}{2\Delta Q_{i} \Delta Q_{j} - \Delta Q_{j} \Delta q_{i} - \Delta Q_{i} \Delta q_{j}},$$
  

$$K_{j} = \frac{\Delta Q_{j} \Delta Q_{i} - \Delta Q_{i} \Delta q_{j}}{2\Delta Q_{i} \Delta Q_{j} - \Delta Q_{j} \Delta q_{i} - \Delta Q_{i} \Delta q_{j}},$$
  

$$\mathbf{q}^{I} = [..., Q_{i} + \Delta q_{i} + \frac{\Delta Q_{i}}{\Delta Q_{j}} \Delta q_{j} - \Delta Q_{i}, Q_{j} + \Delta Q_{j}, ...]^{T},$$
  

$$\mathbf{q}^{J} = [..., Q_{i} + \Delta Q_{i}, Q_{j} + \Delta q_{j} + \frac{\Delta Q_{j}}{\Delta Q_{i}} \Delta q_{i} - \Delta Q_{j}, ...]^{T}.$$

Eqn.(44) and (45) determine the value of  $V(\mathbf{q})$  on the orthogonal planes stretched by the grid, then the value of  $V(\mathbf{q})$  in the space between these planes is calculated by the linear interpolation of the existing value in further dimensions. Similarly, we can also define the value of  $f(\mathbf{q})$  in the entire domain.



Fig. 4. The value of  $V(\mathbf{q}) := V(..., Q_i + \Delta q_i, Q_j + \Delta q_j, ...)$  is calculated by the linear interpolation between the value  $V(\mathbf{q}^I)$  and  $V(\mathbf{q}^I)$  defined on the grid lines. The line  $\overline{\mathbf{q}^I \mathbf{q}^J}$  is parallel to the diagonal  $\overline{\mathbf{q}^b \mathbf{q}^c}$ 

Observing the function  $V(\mathbf{q})$ , we can see that it is continuous in  $\mathbb{R}^{M}_{+}$  and differentiable in each subspace bounded by the grid lines and diagonal lines. For two points  $\mathbf{q}$  and  $\mathbf{q}'$  which lie in the same cell, under Condition 1) in Theorem 1 we have  $|\bar{\mu}_{i}(\mathbf{q}) - \bar{\mu}_{i}(\mathbf{q}')| \leq \epsilon_{1}$  and hence  $|f(\mathbf{q}) - f(\mathbf{q}')| \leq \epsilon_{1} f(\mathbf{q})$ for arbitrarily small  $\epsilon_{1} > 0$ . Then for Eqn.(44) it holds

$$V(\mathbf{q}^{I}) = V(\mathbf{q}^{a}) + f(\mathbf{q}) \left( \bar{\mu}_{i}(\mathbf{q}) + \varepsilon_{i}(\mathbf{q}) \right) \left( \Delta q_{i} + \frac{\Delta Q_{i}}{\Delta Q_{j}} \Delta q_{j} \right),$$
  
$$V(\mathbf{q}^{J}) = V(\mathbf{q}^{a}) + f(\mathbf{q}) \left( \bar{\mu}_{j}(\mathbf{q}) + \varepsilon_{j}(\mathbf{q}) \right) \left( \Delta q_{j} + \frac{\Delta Q_{j}}{\Delta Q_{i}} \Delta q_{i} \right).$$

and further

$$V(\mathbf{q}) = V(\mathbf{q}^{a}) + f(\mathbf{q}) \left(\bar{\mu}_{i}(\mathbf{q}) + \varepsilon_{i}(\mathbf{q})\right) \Delta q_{i}$$
$$+ f(\mathbf{q}) \left(\bar{\mu}_{j}(\mathbf{q}) + \varepsilon_{j}(\mathbf{q})\right) \Delta q_{j},$$

where the deviation  $\varepsilon_i(\mathbf{q}), \varepsilon_j(\mathbf{q}) \to 0$  as  $\|\mathbf{q}\| \to +\infty$ . Similarly we can also obtain the same result for Eqn.(45).



Fig. 5. The Lyapunov function  $V(\mathbf{q})$  is differentiable inside the subdomain between  $\mathbf{q}^a, \mathbf{q}^b, \mathbf{q}^c$  and the subdomain between  $\mathbf{q}^b, \mathbf{q}^c, \mathbf{q}^d$ 

Then the partial derivative of V is

$$\frac{\partial V(\mathbf{q})}{\partial q_{i}} = f(\mathbf{q}) \left( \bar{\mu}_{i}(\mathbf{q}) + \epsilon_{4} \right)$$

for arbitrarily small  $\epsilon_4 > 0$  and we obtain the Lemma 2.

It can be shown that  $f(\mathbf{q})$  and  $V(\mathbf{q})$  constructed in Lemma 2 are positive and grow to infinity as  $\|\mathbf{q}\| \to +\infty$ . Now we use the function  $V(\mathbf{q})$  and  $f(\mathbf{q})$  in Lemma 2 as the Lyapunov function and the stability measure function respectively. It can also be shown that  $\Delta V(\mathbf{q^n})$  is bounded if  $\mathbf{q^n}$  lies in some compacted region  $\overline{\mathcal{B}}$  and the arrival rates  $a_i^n$  and transmission rates  $r_i^n$  are bounded. Hence the Lyapunov condition (23) is satisfied.



Fig. 6. The drift  $\Delta V$  crosses 5 subdomains, which can be written as the sum of the difference between  $V(\mathbf{q}^n)$ ,  $V(\mathbf{q}^{(1)})$ , ..., and  $V(\mathbf{q}^{n+1})$ 

Next we consider the drift  $\Delta V(\mathbf{q}^n)$  in Lyapunov condition (24) where  $\mathbf{q}^n \notin \overline{\mathcal{B}}$ . The connection between  $\mathbf{q}^n$  and  $\mathbf{q}^{n+1}$  probably pass through multiple differentiable subspaces of  $V(\mathbf{q})$  (see Fig.6), so we denote the intersection of the connecting line and the boundary of the subspaces as  $\mathbf{q}^{(1)}, ..., \mathbf{q}^{(L)}$  and the difference as  $\Delta \mathbf{q}^{(1)} = \mathbf{q}^{(1)} - \mathbf{q}^n, ..., \Delta \mathbf{q}^{(l)} = \mathbf{q}^{(1+1)} - \mathbf{q}^{(l)}$ . The drift is written as:

$$\begin{aligned} \Delta V(\mathbf{q}^{n}) \\ = & \mathbb{E} \left\{ V(\mathbf{q}^{n+1}) - V(\mathbf{q}^{(L)}) + \sum_{l=2}^{L} V(\mathbf{q}^{l+1}) - V(\mathbf{q}^{(l)}) \\ &+ V(\mathbf{q}^{(1)}) - V(\mathbf{q}^{n}) \middle| \mathbf{q}^{n} \right\} \\ = & \mathbb{E} \left\{ \sum_{l=1}^{L+1} f(\tilde{\mathbf{q}}^{(l)}) \tilde{\boldsymbol{\mu}}(\tilde{\mathbf{q}}^{(l)}) \cdot \Delta \mathbf{q}^{(l)} \middle| \mathbf{q}^{n} \right\} \\ \leq & \mathbb{E} \left\{ \sum_{l=1}^{L+1} f(\tilde{\mathbf{q}}^{(l)}) \bar{\boldsymbol{\mu}}(\tilde{\mathbf{q}}^{(l)}) \cdot \Delta \mathbf{q}^{(l)} + \epsilon_{4} \left\| \Delta \mathbf{q}^{(l)} \right\| f(\tilde{\mathbf{q}}^{(l)}) \middle| \mathbf{q}^{n} \right\}, \end{aligned}$$

Where  $\tilde{\mathbf{q}}^{(l)}$  is some point in the *l*-th subspace. Since the arrival rates  $a_i^n$  and the transmission rates  $r_i^n$  are bounded for all  $i \in \mathcal{M}$ , the difference  $\|\Delta \mathbf{q}\| = \|\mathbf{q}^{n+1} - \mathbf{q}^n\|$  is bounded. Thus according to Condition 1) in Theorem 1 we have  $|\bar{\mu}_i(\tilde{\mathbf{q}}^{(l)}) - \bar{\mu}_i(\tilde{\mathbf{q}}^{(1)})| < \epsilon_1$  and  $|f(\tilde{\mathbf{q}}^{(l)}) - f(\tilde{\mathbf{q}}^{(1)})| < \epsilon_1 f(\tilde{\mathbf{q}}^{(1)})$  for arbitrary  $\epsilon_1 > 0$  if  $\|\tilde{\mathbf{q}}^{(1)}\|$  is large. The drift

$$\Delta V(\mathbf{q}^{n})$$

$$\leq \mathbb{E} \left\{ \left. f(\tilde{\mathbf{q}}^{(1)}) \bar{\boldsymbol{\mu}}(\tilde{\mathbf{q}}^{(1)}) \cdot \sum_{l=1}^{L+1} \Delta \mathbf{q}^{(l)} + \sigma_{3} f(\mathbf{q}^{n}) \right| \mathbf{q}^{n} \right\}$$

$$\leq \mathbb{E} \left\{ \left. f(\tilde{\mathbf{q}}^{(1)}) \bar{\boldsymbol{\mu}}(\tilde{\mathbf{q}}^{(1)}) \cdot \left( \mathbf{q}^{n+1} - \mathbf{q}^{n} \right) \right| \mathbf{q}^{n} \right\} + \sigma_{3} f(\mathbf{q}^{n}),$$

where  $\sigma_3$  is some small constant.

Using the previous result in (39), it holds

$$\begin{aligned} &\Delta V(\mathbf{q}^n) \\ \leq & \mathbb{E}\left\{ \left. \sum_{i=1}^M f(\tilde{\mathbf{q}}^n) \bar{\mu}_i(\tilde{\mathbf{q}}^n) \left( a_i^n - r_i^n + z_i^n \right) \right| \mathbf{q}^n \right\} + \sigma_3 f(\mathbf{q}^n) \\ \leq & -\theta f(\mathbf{q}^n) + \sigma_3 f(\mathbf{q}^n) \\ \leq & -\theta' f(\mathbf{q}^n) \end{aligned}$$

for some  $\theta' > 0$  if  $\|\mathbf{q}^n\| > B$ , for some B > 0. The drift is negative thus the Markov chain is positive recurrent.

At last, we prove that the chain is also f-stable for the magnitude function  $f(\mathbf{q})$ . We can write

$$\mathbb{E}\left\{V(\mathbf{q}^{n+1}) \middle| \mathbf{q}^{n}\right\}$$
  

$$\leq \mathbb{E}\left\{V(\mathbf{q}^{n+1}) \middle| \mathbf{q}^{n} > B\right\} \Pr\left(\mathbf{q}^{n} > B\right)$$
  

$$+ \mathbb{E}\left\{V(\mathbf{q}^{n+1}) \middle| \mathbf{q}^{n} \le B\right\} \Pr\left(\mathbf{q}^{n} \le B\right)$$
  

$$\leq \mathbb{E}\left\{V(\mathbf{q}^{n}) - \theta' f(\mathbf{q}^{n}) \middle| \mathbf{q}^{n} > B\right\} \Pr\left(\mathbf{q}^{n} > B\right)$$
  

$$+ \mathbb{E}\left\{V(\mathbf{q}^{n+1}) \middle| \mathbf{q}^{n} \le B\right\} \Pr\left(\mathbf{q}^{n} \le B\right)$$
  

$$\leq \mathbb{E}\left\{V(\mathbf{q}^{n})\right\} - \theta' f(\mathbf{q}^{n}) + C_{5},$$

where  $C_5$  is some constant satisfying

$$C_5 \ge \mathbb{E}\left\{ V(\mathbf{q}^{n+1}) \,\middle|\, \mathbf{q}^n \le B \right\} \Pr\left(\mathbf{q}^n \le B\right) \\ + \mathbb{E}\left\{ \theta' f(\mathbf{q}^n) \,\middle|\, \mathbf{q}^n \le B \right\} \Pr\left(\mathbf{q}^n \le B\right).$$

Using the telescoping machinery, the summation of the drift states on some even time slots N = 2, 4, ..., it holds over N time slots yields T = (N)

$$\mathbb{E}\left\{V(\mathbf{q}^{N})\right\} \leq \mathbb{E}\left\{V(\mathbf{q}^{1})\right\} - \theta' \sum_{n=1}^{N} \mathbb{E}\left\{f(\mathbf{q}^{n})\right\} + N \cdot C_{5}.$$

since  $V(\mathbf{q})$  is non-negative function, it holds

$$\sum_{n=1}^{N} \mathbb{E}\left\{f(\mathbf{q}^{n})\right\} \leq \frac{\mathbb{E}\left\{V(\mathbf{q}^{1})\right\}}{\theta'} + \frac{N \cdot C_{5}}{\theta'}.$$

Hence we have

$$\limsup_{n \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\{ f(\mathbf{q}^n) \right\} \le \frac{\mathbb{E} \left\{ V(\mathbf{q}^1) \right\}}{N\theta'} + \frac{C_5}{\theta'} < +\infty$$

which completes the proof.

#### B. Proof of Theorem 3

Considering the first case in Theorem 3, we define the set of time slots in which the change of  $\bar{\mu}_i(\mathbf{q}(n))$  is not negligible as

$$\mathcal{N}_{V} := \left\{ n : \left\| \bar{\boldsymbol{\mu}} \left( \mathbf{q}^{n+1} \right) - \bar{\boldsymbol{\mu}} \left( \mathbf{q}^{n} \right) \right\| \ge \epsilon \right\}$$

for some constant  $\epsilon > 0$ , where the superscript is again used to denote the index of the time slot. Suppose there is some constant  $\gamma$  with  $0 < \gamma \leq 1$  and

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{I}\left\{n\in\mathcal{N}_{V}\right\}\geq\gamma+\varepsilon(N),$$

where  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow +\infty$ . If the difference between  $\bar{\mu}(\mathbf{q}^{n+1})$  and  $\bar{\mu}(\mathbf{q}^n)$  is larger than  $\epsilon$ , then the expected rate allocation  $\tilde{\mathbf{r}}_E^n := \mathbb{E}\{\mathbf{r}^{\mathcal{P}}(\mathbf{h}^n, \mathbf{q}^n)\}$  and  $\tilde{\mathbf{r}}_E^{n+1} := \mathbb{E}\{\mathbf{r}^{\mathcal{P}}(\mathbf{h}^{n+1}, \mathbf{q}^{n+1})\}$ , which are determined by  $\bar{\boldsymbol{\mu}}(\mathbf{q}^{n+1})$  and  $\bar{\mu}(\mathbf{q}^n)$ , also have non-negligible difference. Note that this assumption is valid only if the normal vector  $\bar{\mu}$  is unique on every boundary point of  $\mathcal{C}(\hat{P})$ . Generally two different normal vector  $\bar{\mu}$  and  $\bar{\mu}'$  might lead to the same boundary point of  $\mathcal{C}(P)$ , where the boundary has no unique supporting hyperplane. Fortunately, to disprove the throughput-optimality we only need to consider certain rate region  $\mathcal{C}(\hat{P})$  whose boundary is differentiable everywhere and the normal vector is always unique. Since both of  $\tilde{\mathbf{r}}_{E}^{n}$  and  $\tilde{\mathbf{r}}_{E}^{n+1}$  lie inside  $\mathcal{C}(\hat{P})$ , for a boundary point  $\mathbf{r}^*$  of  $\mathcal{C}(\hat{P})$  and the corresponding normal vector  $\mu^*$  we have

$$\boldsymbol{\mu}^{*T} \tilde{\mathbf{r}}_{E}^{n} + \boldsymbol{\mu}^{*T} \tilde{\mathbf{r}}_{E}^{n+1}$$

$$= \boldsymbol{\mu}^{*T} \cdot \arg \max_{\mathbf{r}' \in \mathcal{C}(\hat{P})} \bar{\boldsymbol{\mu}} (\mathbf{q}^{n})^{T} \cdot \mathbf{r}'$$

$$+ \boldsymbol{\mu}^{*T} \cdot \arg \max_{\mathbf{r}' \in \mathcal{C}(\hat{P})} \bar{\boldsymbol{\mu}} (\mathbf{q}^{n+1})^{T} \cdot \mathbf{r}'$$

$$\leq 2\boldsymbol{\mu}^{*T} \cdot \mathbf{r}^{*} - \theta(\epsilon),$$

where  $\theta(\epsilon)$  is determined by the difference  $\epsilon$  between  $\bar{\mu}(\mathbf{q}^{n+1})$  and  $\bar{\mu}(\mathbf{q}^n)$  with  $\theta(\epsilon) > 0$ . Considering the queue

$$\boldsymbol{\mu}^{*T} \mathbb{E} \left\{ \mathbf{q}^{N} \right\}$$

$$= \boldsymbol{\mu}^{*T} \sum_{n=1}^{N/2} \mathbb{E} \left\{ \mathbf{a}^{n} - \mathbf{r}^{\mathcal{P}}(\mathbf{h}^{2n}, \mathbf{q}^{2n}) + \mathbf{a}^{2n+1} - \mathbf{r}^{\mathcal{P}}(\mathbf{h}^{2n+1}, \mathbf{q}^{2n+1}) \right\}$$

$$= \sum_{n=1}^{N/2} \left( 2\boldsymbol{\mu}^{*T} \boldsymbol{\rho}^{*} - \boldsymbol{\mu}^{*T} \left( \tilde{\mathbf{r}}_{E}^{2n} + \tilde{\mathbf{r}}_{E}^{2n+1} \right) \mathbb{I} \left\{ 2n \notin \mathcal{N}_{V} \right\} - \boldsymbol{\mu}^{*T} \left( \tilde{\mathbf{r}}_{E}^{2n} + \tilde{\mathbf{r}}_{E}^{2n+1} \right) \mathbb{I} \left\{ 2n \in \mathcal{N}_{V} \right\} \right)$$

$$\geq \frac{N}{2} \left( 2\boldsymbol{\mu}^{*T} \boldsymbol{\rho}^{*} - 2\boldsymbol{\mu}^{*T} \mathbf{r}^{*} + (\gamma + \varepsilon(N)) \theta(\varepsilon) \right) \qquad (46)$$

Suppose the expected arrival rate  $\rho^*$  is close to the boundary point  $\mathbf{r}^*$  so that

$$\boldsymbol{\mu}^{*T}\mathbf{r}^* - \boldsymbol{\mu}^{*T}\boldsymbol{\rho}^* < \theta'$$

for some  $\theta' > 0$ . Combining with (46), it holds

$$\boldsymbol{\mu}^{*T} \mathbb{E} \left\{ \mathbf{q}(N) \right\} > \frac{N}{2} \left( \gamma \boldsymbol{\theta}(\epsilon) - 2\boldsymbol{\theta}' + \boldsymbol{\varepsilon}(N) \boldsymbol{\theta}(\epsilon) \right)$$

Since  $\varepsilon(N) \to 0$  as  $N \to +\infty$ , if  $\theta' < \frac{\gamma}{2}\theta(\epsilon)$ , we have

$$\lim_{N \to +\infty} \boldsymbol{\mu}^{*T} \mathbb{E} \left\{ \mathbf{q}(N) \right\} = +\infty$$

and the Markov chain is not strongly stable. Suppose the variance  $\mathbf{a}(n)$  and  $\mathbf{h}(n)$  is sufficiently small, so that for some constant  $C_A > 0$  the probability

$$Pr\left\{\boldsymbol{\mu}^{*T}\mathbf{q}(N) < C_A\right\}$$

decreases sufficiently fast when  $\Psi(N) := \mu^{*T} \mathbb{E} \{ \mathbf{q}(N) \}$ increases, i.e.  $\exists C_B, K > 0$  with

$$K\frac{1}{\Psi(N)^{1+\Gamma}} \ge Pr\left\{\boldsymbol{\mu}^{*T}\mathbf{q}(N) < C_A\right\}, \qquad \forall \Psi(N) > C_B,$$

for some constant  $\Gamma > 0$ . Define  $N_c := \min\{N \in \mathbb{N} :$  $\Psi(N) > C_B$ , the expected occupation time of the set  $\mathcal{A} := \{\mathbf{q} : \boldsymbol{\mu}^{*T} \mathbf{q} < C_A\}$  is given by

$$\mathbb{E} \{\eta_{\mathcal{A}}\} = \sum_{N=1}^{\infty} Pr \{ \boldsymbol{\mu}^{*T} \mathbf{q}(N) < C_A \}$$
  
$$\leq N_c + \sum_{N=N_c}^{\infty} K \left( \frac{1}{\Psi(N)^{1+\Gamma}} \right)$$
  
$$= N_c + \sum_{N=N_c}^{\infty} K \left( \frac{1}{\left( \left( \gamma \theta - \theta' \right) N \right)^{1+\Gamma}} \right)$$
  
$$< +\infty$$

and the Markov chain is transient.

For the second case in Theorem 3, we choose the expected arrival rate vector  $\rho^*$  close to the *j*-th corner of the ergodic achievable rate region with

$$r_j^* - \rho_j^* < \theta'$$

for the user  $j \neq i$  and some constant  $\theta' > 0$ , where  $r_j^* := \max_{\tilde{\mathbf{r}} \in \mathcal{C}(\hat{P})} \tilde{r}_j$ . According to (15) we have for some constant  $\epsilon > 0$  and  $\gamma$  with  $0 < \gamma \leq 1$ 

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left\{\bar{\mu}_{j}\left(\mathbf{q}(n)\right) \leq 1-\epsilon\right\} \geq \gamma + \varepsilon(N),$$

where  $\varepsilon(N) \to 0$  as  $N \to +\infty$ . It implies that for some  $\theta(\epsilon) > 0$ , it holds

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{I}\left\{r_{j}^{*}-r_{j}^{\prime}\left(n\right)\geq\theta(\epsilon)\right\}\geq\gamma+\varepsilon(N),$$

where  $r'_j(n) := \mathbb{E}\{\mathbf{r}^{\mathcal{P}}(\mathbf{h}^n, \mathbf{q}^n)\}$  and the corner point  $[0, ..., r^*_i, ...]$  can not be achieved. Then we have

$$\mathbb{E}\left\{q_{j}^{N}\right\} = \sum_{n=1}^{N} \mathbb{E}\left\{a_{j}^{n} - r_{j}^{\mathcal{P}}(\mathbf{h}^{n}, \mathbf{q}^{n})\right\} \\
= \sum_{n=1}^{N} \left\{\rho_{j}^{*} - r_{j}^{\prime}\left(n\right) \mathbb{I}\left\{r_{j}^{*} - r_{j}^{\prime}\left(n\right) \ge \theta(\epsilon)\right\} \\
- r_{j}^{\prime}\left(n\right) \mathbb{I}\left\{r_{j}^{*} - r_{j}^{\prime}\left(n\right) < \theta(\epsilon)\right\}\right) \\
> N\left(\gamma\theta(\epsilon) - \theta^{\prime} + \varepsilon(N)\theta(\epsilon)\right).$$
(47)

Choose  $\theta' < \gamma \theta(\epsilon)$ , we have

$$\lim_{N \to +\infty} \mathbb{E}\left\{q_j^N\right\} = +\infty$$

and the Markov chain is not strongly stable. Similarly we can also show that the Markov chain is transient if the variances of a(n) and h(n) are sufficiently small.

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