A Smart Approach for GPT Cryptosystem Based on Rank Codes

Haitham Rashwan Department of Communications Department of Radio Engineering Department of Communications InfoLab21, South Drive Lancaster University Lancaster UK LA1 4WA Email: h.rashwan@lancaster.ac.uk

Ernst M. Gabidulin Moscow Institute of Physics and Technology (State University) 141700 Dolgoprudny, Russia Email: gab@mail.mipt.ru

Bahram Honary InfoLab21, South Drive Lancaster University Lancaster UK LA1 4WA Email: b.honary@lancaster.ac.uk

Abstract—The concept of Public- key cryptosystem was innovated by McEliece's cryptosystem. The public key cryptosystem based on rank codes was presented in 1991 by Gabidulin –Paramonov–Trejtakov (GPT). The use of rank codes in cryptographic applications is advantageous since it is practically impossible to utilize combinatoric decoding. This has enabled using public keys of a smaller size. Respective structural attacks against this system were proposed by Gibson and recently by Overbeck. Overbeck's attacks break many versions of the GPT cryptosystem and are turned out to be either polynomial or exponential depending on parameters of the cryptosystem. In this paper, we introduce a new approach, called the Smart approach, which is based on a proper choice of the distortion matrix X. The Smart approach allows for withstanding all known attacks even if the column scrambler matrix **P** over the base field \mathbb{F}_{q} .

I. INTRODUCTION

McEliece [1] has introduced the first code-based public-key cryptosystem (PKC). The system is based on Goppa codes in the Hamming metric, which is connected to the hardness of the general decoding problem. It is a strong cryptosystem but the size of a public key is too large (500 000 bits) for practical implementations to be efficient.

Neiderreiter [2] has introduced a new PKC based on a family of Generalized Reed-Solomon codes; its public key size is less than the McEliece cryptosystem, but still large for practical application.

Also, Gabidulin Paramonov and Trietakov have proposed a new public key cryptosystem, which is now called the GPT cryptosystem, based on rank error correcting codes in [3], [4]. The GPT cryptosystem has two advantages over McEliece's Cryptosystem. Firstly, it is more robust against decoding attacks than McEliece's Cryptosystem; secondly, the key size of the GPT is much smaller and more useful in terms of practical applications than McEliece's cryptosystem.

Rank codes are well structured. Subsequently in a series of works, Gibson [5], [6] developed attacks that break the GPT system for public keys of about 5 Kbits. The Gibson's attacks are efficient for practical values of parameters $n \leq 30$, where n is the length of rank code with the field \mathbb{F}_{2^N} as an alphabet.

Several proposals of the GPT PKC were introduced to withstand Gibson's attacks [7], [8]. One proposal is to use a rectangular row scramble matrix instead of a square matrix. The proposal allows working with subcodes of the rank codes which have much more complicated structure. Another proposal exploits a modification of Maximum Rank Distance (MRD) codes where the concept of a column scramble matrix was also introduced. A new variant, called reducible rank codes, is also implemented to modify the GPT cryptosystem [9], [10]. All these variants withstand Gibson's attack.

Recently, R. Overbeck [11], [12], and [13] has proposed new attacks, which are more effective than any of Gibson's attacks. His method is based on two factors : a) a column scrambler P that is defined over the base field, and b) the unsuitable choice of a distortion matrix X. However, Overbeck managed to break many instances of the GPT cryptosystem based on the general and developed ideas of Gibson.

Kshevetskiy in [19] suggested a secure approach towards the choice of parameters for avoiding Overbeck's attacks based on suitable choice of the distortion matrix X. Independently, Loidreau in [20] proposed similar method. Gabidulin [14] has offered a new approach called the Advanced approach, which makes the cryptographer define a proper column scrambler matrix over the extension field without violating the standard mode of the PKC. The Advanced approach allows the decryption of the authorised party, and prevents an unauthorized party from breaking the system by means of any known attacks. The two approaches withstand Overbeck and Gibson's attacks.

Recently, we have presented another variant of the GPT public key cryptosystem [21], based on a proper choice of column scrambler matrix over the extension field. This variant, which we call the Instrumental approach, is secure against all known attacks.

In this paper, we introduce a new approach called the Smart approach, which is based on a proper choice of the distortion matrix X. The Smart approach allows for withstanding all known attacks even if the column scrambler matrix **P** over the base field \mathbb{F}_q .

The rest of this paper is structured as follows. Section 2 gives a short introduction to rank codes. Section 3 describes the GPT cryptosystems. Section 4 discusses the Overbeck's attacks. Section 5 presents the Smart approach of GPT PKC cryptosystem with two examples. Finally, section 6 concludes the paper with some remarks.

II. RANK CODES

Let us introduce the basic notion of rank codes [3], [15]. Let \mathbb{F}_q be a finite field of q elements and let \mathbb{F}_{q^N} be an extension field of degree N. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be a vector with coordinates in \mathbb{F}_{q^N} .

The Rank norm of **x** is defined as the maximal number of x_i , which are linearly independent over the base field \mathbb{F}_q and is denoted $\operatorname{Rk}(\mathbf{x} \mid \mathbb{F}_q)$.

Similarly, for a matrix M with entries in \mathbb{F}_{q^N} , the columns rank is defined as the maximal number of columns, which are linearly independent over the base field \mathbb{F}_q , and is denoted $\operatorname{Rk}_{\operatorname{col}}(M|\mathbb{F}_q)$. We distinguish two ranks of the matrix:

We distinguish two ranks of the matrix:

- 1) The usual rank of matrix M over \mathbb{F}_{q^N} $\operatorname{Rk}(M \mid \mathbb{F}_{q^N})$.
- The column rank of a matrix M over the base field F_q − Rk_{col}(M | F_q).

The column rank of the matrix M depends on the field. In particular, $\operatorname{Rk}_{\operatorname{col}}(M \mid \mathbb{F}_q) \geq \operatorname{Rk}_{\operatorname{col}}(M \mid \mathbb{F}_{q^N})$

The Rank distance between \mathbf{x} and \mathbf{y} is defined as the rank norm of the difference $\mathbf{x} - \mathbf{y}$: $d(\mathbf{x}, \mathbf{y}) = \operatorname{Rk}_{\operatorname{col}}(\mathbf{x} - \mathbf{y} \mid \mathbb{F}_q).$

Any linear (n, k, d) code $\mathcal{C} \subset \mathbb{F}_{q^N}^n$ fulfils the Singleton-style bound [15] for the rank distance:

$$Nk \le Nn - (d - 1) \max\{N, n\}.$$
 (1)

A code C reaching that bound is called a Maximal Rank Distance (MRD) code.

The theory of optimal MRD (Maximal Rank Distance) codes is given in [15]. The notation $g[i] := g^{q^i \mod n}$ means the *i*-th

The notation $g[i] := g^q$ means the *i*-th Frobenius power of g. It allows to consider both positive and negative Frobenius powers *i*.

For $n \leq N$, a generator matrix \mathbf{G}_k of a (n, k, d)MRD code is defined by a matrix of the following form:

$$\mathbf{G}_{k} = \begin{bmatrix} g_{1} & g_{2} & \dots & g_{n} \\ g_{1}^{[1]} & g_{2}^{[1]} & \dots & g_{n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1}^{[k-1]} & g_{2}^{[k-1]} & \dots & g_{n}^{[k-1]} \end{bmatrix}$$
(2)

where g_1, g_2, \ldots, g_n are any set of elements of the extension field \mathbb{F}_{q^N} which are linearly independent over the base field \mathbb{F}_q .

A code with the generator matrix (2) is referred to as (n, k, d) code, where n is code length, kis the number of information symbols, d is code distance. For MRD codes, d = n - k + 1. Let $\mathbf{m} = (m_1, m_2, \dots, m_k)$ be an information vector of dimension k. The corresponding code vector is the *n*-vector

$$\mathbf{g}(\mathbf{m}) = \mathbf{m}\mathbf{G}_k.$$

If $\mathbf{y} = \mathbf{g}(\mathbf{m}) + \mathbf{e}$ and $\operatorname{Rk}(\mathbf{e}) = s \leq t = \frac{d-1}{2}$, then the information vector \mathbf{m} can be recovered uniquely from \mathbf{y} by some decoding algorithm. There exist fast decoding algorithms for MRD codes [15], [16]. A decoding procedure requires elements of the $(n-k) \times n$ parity check matrix \mathbf{H} such that $\mathbf{G}_k \mathbf{H}^T = 0$. For decoding, the matrix \mathbf{H} should be of the form

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & \dots & h_n \\ h_1^{[1]} & h_2^{[1]} & \dots & h_n^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{[d-2]} & h_2^{[d-2]} & \dots & h_n^{[d-2]} \end{bmatrix}, \quad (3)$$

where elements h_1, h_2, \ldots, h_n are in the extension field \mathbb{F}_{q^N} and are linearly independent over the base field \mathbb{F}_q .

The optimal code has the following design parameters: code length $n \leq N$; dimension k = n - d + 1, rank code distance d = n - k + 1.

III. THE GPT CRYPTOSYSTEM

Description of the standard GPT cryptosystem. The GPT cryptosystem is described as follows: **Plaintext**: A Plaintext is any k-vector $\mathbf{m} = (m_1, m_2, \ldots, m_k), m_s \in \mathbb{F}_{q^N}, s = 1, 2, \ldots, k$. In previous works, different representations of the public key are given. All of them can be reduced to the following form.

The Public key is a $k \times (n + t_1)$ generator matrix

$$\mathbf{G}_{pub} = \mathbf{S} \begin{bmatrix} \mathbf{X} & \mathbf{G}_k \end{bmatrix} \mathbf{P}. \tag{4}$$

Let us explain roles of the factors.

- The main matrix G_k is given by 2. It is used to correct rank errors. Errors of rank not greater than ^{n-k}/₂ can be corrected.
- A matrix S is a row scrambler. This matrix is a non singular square matrix of order k over *F*_{q^N}.
- A matrix **X** is a distortion $(k \times t_1)$ matrix over \mathbb{F}_{q^N} with *full* column rank $\operatorname{Rk}_{\operatorname{col}}(X \mid \mathbb{F}_q) = t_1$ and rank $\operatorname{Rk}(\mathbf{X} \mid \mathbb{F}_{q^N}) = t_X, t_X \leq t_1$. The matrix $[\mathbf{X} \quad \mathbf{G}_k]$ has full column rank $\operatorname{Rk}_{\operatorname{col}}([\mathbf{X} \quad \mathbf{G}_k] \mid \mathbb{F}_q) = n + t_1$.
- A matrix **P** is a square column scramble matrix of order $(t_1 + n)$ over \mathbb{F}_q .
- $t_1 + n$ may be greater than N, but $n \leq N$.

The Private keys are matrices S, G_k , X, P separately and (explicitly) a fast decoding algorithm of an MRD code. Note also, that the matrix X is not used to decrypt a ciphertext and can be deleted after calculating the Public key.

Encryption: Let $\mathbf{m} = (m_1, m_2, \dots, m_k)$ be a plaintext. The corresponding ciphertext is given by

$$\mathbf{c} = \mathbf{m}\mathbf{G}_{\text{pub}} + \mathbf{e} = \mathbf{m}\mathbf{S}\begin{bmatrix}\mathbf{X} & \mathbf{G}_k\end{bmatrix}\mathbf{P} + \mathbf{e},$$
 (5)

where e is an artificial vector of errors of rank t_2 or less. It is assumed that $t_1 + t_2 \le t = \lfloor \frac{n-k}{2} \rfloor$

Decryption: The legitimate receiver upon receiving c calculates

$$\mathbf{c}' = (c_1', c_2', \dots, c_{t_1+n}') =$$

 $\mathbf{c}\mathbf{P}^{-1} = \mathbf{m}\mathbf{S} \begin{bmatrix} \mathbf{X} & \mathbf{G}_k \end{bmatrix} + \mathbf{e}\mathbf{P}^{-1}$

Then from \mathbf{c}' he extracts the subvector

$$\mathbf{c}'' = (c'_{t_1+1}, c'_{t_1+2}, \dots, c'_{t_1+n}) = \mathbf{mSG}_k + \mathbf{e}'',$$
(6)

where $e^{''}$ is the subvector of eP^{-1} . Then the legitimate receiver applies the fast decoding algorithm to correct the error $e^{''}$, extracts **mS** and recovers m as $\mathbf{m} = (\mathbf{mS})\mathbf{S}^{-1}$.

In this system, the size of the public key is $V = k(t_1 + n)N$ bits, and the information rate is $R = \frac{k}{t_1+n}$.

IV. OVERBECK'S ATTACK

In [11], [12], and [13], new attacks are proposed on the GPT PKC described in the form of 4. It is claimed, that similar attacks can be proposed on all the variants of GPT PKC.

We recall briefly this attack.

We need some notations.

For $x \in \mathbb{F}_{q^N}$ let $\sigma(x) = x^q$ be the Frobenius automorphism.

For the matrix $\mathbf{T} = (t_{ij})$ over \mathbb{F}_{q^N} , let $\sigma(\mathbf{T}) = (\sigma(t_{ij})) = (t_{ij}^q)$.

For any integer s, let $\sigma^{s}(\mathbf{T}) = \sigma(\sigma^{s-1}(\mathbf{T}))$.

It is clear that $\sigma^N = \sigma$. Thus the inverse exists $\sigma^{-1} = \sigma^{N-1}$.

The following simple properties if σ are useful:

- $\sigma(a+b) = \sigma(a) + \sigma(b).$
- $\sigma(ab) = \sigma(a)\sigma(b)$.
- In general, for matrices $\sigma(\mathbf{T}) \neq \mathbf{T}$.
- If **P** is a matrix over the *base* field \mathbb{F}_q , then $\sigma(\mathbf{P}) = \mathbf{P}$.

Description of Overbeck's attack: To break a system, a cryptanalyst constructs from the public key $\mathbf{G}_{\text{pub}} = \mathbf{S} \begin{bmatrix} \mathbf{X} & \mathbf{G}_k \end{bmatrix} \mathbf{P}$ the *extended* public key $\mathbf{G}_{\text{ext,pub}}$ as follows:

$$\mathbf{G}_{\text{ext,pub}} = \begin{vmatrix} \mathbf{G}_{\text{pub}} \\ \sigma(\mathbf{G}_{\text{pub}}) \\ \cdots \\ \sigma^{u}(\mathbf{G}_{\text{pub}}) \end{vmatrix} = \\ \mathbf{S}_{\mathbf{G}} \begin{bmatrix} \mathbf{X} & \mathbf{G}_{k} \\ \cdots \\ \mathbf{G}_{k} \end{bmatrix} = \mathbf{P} \begin{vmatrix} \mathbf{F}_{k} \end{vmatrix}$$

$$\begin{vmatrix} \sigma(\mathbf{S}) & [\sigma(\mathbf{X}) & \sigma(\mathbf{G}_k) \end{bmatrix} & \mathbf{P} \\ \dots & \dots & \dots \\ \sigma^u(\mathbf{S}) & [\sigma^u(\mathbf{X}) & \sigma^u(\mathbf{G}_k) \end{bmatrix} & \mathbf{P} \end{vmatrix} .$$
(7)

The property that $\sigma(\mathbf{P}) = \mathbf{P}$, if \mathbf{P} is a matrix over the *base* field \mathbb{F}_q , is used in (7).

Rewrite this matrix as

$$\mathbf{G}_{\mathrm{ext,pub}} = \mathbf{S}_{\mathrm{ext}} \begin{bmatrix} \mathbf{X}_{\mathrm{ext}} & \mathbf{G}_{\mathrm{ext}} \end{bmatrix} \mathbf{P},$$
 (8)

where

$$\mathbf{S}_{\text{ext}} = \text{Diag} \begin{bmatrix} \mathbf{S} & \sigma(\mathbf{S}) & \dots & \sigma^{u}(\mathbf{S}) \end{bmatrix}$$
$$\mathbf{X}_{\text{ext}} = \begin{bmatrix} \mathbf{X} \\ \sigma(\mathbf{X}) \\ \vdots \\ \sigma^{u}(\mathbf{X}) \end{bmatrix}, \quad \mathbf{G}_{\text{ext}} = \begin{bmatrix} \mathbf{G}_{k} \\ \sigma(\mathbf{G}_{k}) \\ \vdots \\ \sigma^{u}(\mathbf{G}_{k}) \end{bmatrix}. \tag{9}$$

Choose

$$u = n - k - 1. \tag{10}$$

For a $k \times t_1$ matrix **X**, let **X**₁ be the $(k-1) \times t_1$ matrix, obtained from **X** by deleting the *last* row. Similarly, let **X**₂ be the $(k-1) \times t_1$ matrix, obtained from **X** by deleting the *first* row.

Define a linear mapping $T : \mathbb{F}_{q^N}^{k \times t_1} \to \mathbb{F}_{q^N}^{(k-1) \times t_1}$ by the rule: if $\mathbf{X} \in \mathbb{F}_{q^N}^{k \times t_1}$, then $T(\mathbf{X}) = \mathbf{Y} = \sigma(\mathbf{X}_1) - \mathbf{X}_2$. Let

$$\mathbf{Y}_{\text{ext}} = \begin{bmatrix} \mathbf{Y} & \sigma(\mathbf{Y}) & \sigma^2(\mathbf{Y}) & \dots & \sigma^{u-1}(\mathbf{Y}) \end{bmatrix}^\top$$
 (11)

Using this and other suitable transformations of rows, one can rewrite for analysis (8) and (9) in the form

$$\tilde{\mathbf{G}}_{\text{pub,ext}} = \tilde{\mathbf{S}}_{\text{ext}} \begin{bmatrix} \mathbf{Z} & | & \mathbf{G}_{n-1} \\ \mathbf{Y}_{\text{ext}} & | & 0 \end{bmatrix} \mathbf{P} \qquad (12)$$

where G_{n-1} is the generator matrix of the (n, n-1, 2) MRD code.

Let us try to find a solution **u** of the system

$$\tilde{\mathbf{S}}_{ext} \begin{bmatrix} \mathbf{Z} & | & \mathbf{G}_{n-1} \\ \mathbf{Y}_{ext} & | & 0 \end{bmatrix} \mathbf{P} \mathbf{u}^T = \mathbf{0}, \qquad (13)$$

where **u** is a vector-row over the extension field \mathbb{F}_{q^N} of length $t_1 + n$. Represent the vector $\mathbf{P}\mathbf{u}^T$ as

$$\mathbf{P}\mathbf{u}^T = \begin{bmatrix} \mathbf{y} & \mathbf{h} \end{bmatrix}^T$$

where the subvector y has length t_1 and h has length n. Then the system (13) is equivalent to the following system:

$$\mathbf{Z}\mathbf{y}^T + \mathbf{G}_{n-1}\mathbf{h}^T = \mathbf{0},\tag{14}$$

$$\mathbf{Y}_{ext}\mathbf{y}^T = \mathbf{0}.$$
 (15)

Assume that the next condition is valid:

$$\operatorname{Rk}(\mathbf{Y}_{ext}|\mathbb{F}_{q^N}) = t_1.$$
(16)

Then the equation (15) has only the trivial solution $\mathbf{y}^T = \mathbf{0}$. The equation (14) becomes

$$\mathbf{G}_{n-1}\mathbf{h}^T = \mathbf{0}.$$
 (17)

It allows to find the first row of the parity check matrix for the code with the generator matrix (12) (see,[11], [12], and [13], for details). Hence this solution breaks a GPT cryptosystem in polynomial time. The Overbeck's attack requires $O((n + t_1)^3)$ operation over \mathbb{F}_{q^N} since all the steps of the attack have at most cubic complexity on $n + t_1$.

V. SMART APPROACH

To withstand Overbeck's attack, the cryptographer should choose the matrix \mathbf{X} in such a manner that

$$\operatorname{Rk}(\mathbf{Y}_{ext} \mid \mathbb{F}_{q^N}) = t_1 - a, \tag{18}$$

where $a \ge 2$. In this case, the system (15) has q^{aN} solutions \mathbf{y}^T . Hence the exhaustive search over \mathbf{y}^T is needed. The work function has order $O(q^{aN}(n+t_1)^3)$ and Overback's attack fails.

 $t_1)^3$) and Overback's attack fails. One method to provide the condition (18) is proposed in [19], [20]. Choose the matrix **X** over the extension field \mathbb{F}_{q^N} in such a manner that the following conditions are satisfied:

$$\begin{aligned} t_1 &= \operatorname{Rk}_{\operatorname{col}}(\mathbf{X} \mid \mathbb{F}_q) > n-k. \\ r_X &= \operatorname{Rk}(\mathbf{X} \mid \mathbb{F}_{q^N}) = \left\lfloor \frac{t_1-a}{n-k} \right\rfloor \leq k. \end{aligned}$$
(19)

Overbeck's attack is exponential on a and has the minimum complexity at least $O(q^{aN}(n+t_1)^3)$.

We propose an alternative Smart approach. The point is to choose the matrix X in such a manner that the corresponding matrix $\mathbf{Y} = T(\mathbf{X})$ has column rank $\operatorname{Rk}(\mathbf{Y} \mid \mathbb{F}_q)$ not greater than $t_1 - a, a \geq 2$.

The following result is evident.

Lemma 1: If $\operatorname{Rk}(\mathbf{Y} \mid \mathbb{F}_q) = s$, then $\operatorname{Rk}(\mathbf{Y}_{ext} \mid \mathbb{F}_q) = s$.

Corollary 1: $\operatorname{Rk}(\mathbf{Y}_{\operatorname{ext}} | \mathbb{F}_{q^N}) \leq \operatorname{Rk}(\mathbf{Y}_{\operatorname{ext}} | \mathbb{F}_q) = s = \operatorname{Rk}(\mathbf{Y} | \mathbb{F}_q).$

a) **The simple case**: Let a matrix **X** be of the following form:

$$\mathbf{X} = \begin{bmatrix} \mathbf{m} \\ \mathbf{m}^{[1]} \\ \vdots \\ \mathbf{m}^{[k-1]} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{k-1} \end{bmatrix}.$$
(20)

Here **m** is a random vector over the extension field \mathbb{F}_{q^N} with full column rank t_1 and vectors \mathbf{s}_i , $i = 1, \ldots, k-1$, are random vectors over the *base* field \mathbb{F}_q such that the matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{s}_1 & \dots & \mathbf{s}_{k-1} \end{bmatrix}^\top$$

has rank $t_1 - a$. Then the matrix $\mathbf{Y} = T(\mathbf{X})$ has the form

$$\mathbf{Y} = \begin{bmatrix} -\mathbf{s}_1 & \mathbf{s}_1 - \mathbf{s}_2 & \dots & \mathbf{s}_{k-1} - \mathbf{s}_k \end{bmatrix}^\top.$$
(21)

This matrix is a matrix over the *base* field \mathbb{F}_q and has rank $t_1 - a$ too. It follows that

$$\sigma(\mathbf{Y}) = \begin{bmatrix} \sigma(-\mathbf{s}_1) \\ \sigma(\mathbf{s}_1 - \mathbf{s}_2) \\ \vdots \\ \sigma(\mathbf{s}_{k-1} - \mathbf{s}_k) \end{bmatrix} = \begin{bmatrix} -\mathbf{s}_1 \\ \mathbf{s}_1 - \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_{k-1} - \mathbf{s}_k \end{bmatrix} = \mathbf{Y}.$$
 (22)

Hence

$$\mathbf{Y}_{ext} = \begin{bmatrix} \mathbf{Y} \\ \sigma(\mathbf{Y}) \\ \dots \\ \sigma^{u-1}(\mathbf{Y}) \end{bmatrix} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{Y} \\ \dots \\ \mathbf{Y} \end{bmatrix}.$$
 (23)

Therefore $\operatorname{Rk}(\mathbf{Y}_{ext} | \mathbb{F}_{q^N}) = \operatorname{Rk}(\mathbf{Y} | \mathbb{F}_{q^N}) = t_1 - a$, and the condition (18) is satisfied.

As in the previous case, the proposed Smart approach shows that Overbeck's attack is exponential on a and has the bit complexity at least $O(q^{aN}(n+t_1)^3)$.

It has been shown that the Smart approach presented above is secure against all known attacks including the recent attack presented by Overbeck in [13].

Example 1: Let n = 8, k = 4, N = 8, t = 5, $t_1 = 4$, q = 2, a = 2Let the extension field \mathbb{F}_{2^8} be defined by the primitive polynomial $r(x) = 1 + x^2 + x^3 + x^4 + x^8$, and let α be a primitive element of the field. Choose the matrix **X** as in (20). A vector **m** of full column rank $t_1 = 4$ is defined as $\mathbf{m} = \begin{bmatrix} \alpha^3 & \alpha^5 & \alpha^6 & \alpha^2 \end{bmatrix}$. Choose vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ as $\mathbf{s}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$. Then we obtain

$$\mathbf{X} = \begin{bmatrix} \alpha^{3} & \alpha^{5} & \alpha^{6} & \alpha^{2} \\ \alpha^{6} & \alpha^{10} & \alpha^{12} & \alpha^{4} \\ \alpha^{12} & \alpha^{20} & \alpha^{24} & \alpha^{8} \\ \alpha^{24} & \alpha^{40} & \alpha^{48} & \alpha^{16} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \\\begin{bmatrix} \alpha^{3} & \alpha^{5} & \alpha^{6} & \alpha^{2} \\ \alpha^{6} + 1 & \alpha^{10} + 1 & \alpha^{12} & \alpha^{4} \\ \alpha^{12} + 1 & \alpha^{20} + 1 & \alpha^{24} + 1 & \alpha^{8} + 1 \\ \alpha^{24} & \alpha^{40} & \alpha^{48} + 1 & \alpha^{16} + 1 \end{bmatrix}$$
(24)

The corresponding matrix **Y** is as follows:

$$\mathbf{Y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$
 (25)

It has rank $t_1 - a = 2$. The attack is exponential on a and has the bit complexity at least $O(q^{aN}(n+t_1)^3) = O(2^{37})$ bite operations.

b) The general case: Let X be a matrix consisting of a Frobenius-type columns and $t_1 - a$ non-Frobenius columns. A column w is called Frobenius-type if it has the form $\mathbf{w} = (w \ w^{[1]} \ \dots \ w^{[k-1]})^{\top}$. It is clear that $T(\mathbf{w}) = 0$. Hence the matrix $\mathbf{Y} = T(\mathbf{X})$ will have a all zero columns and column rank $t_1 - a$ and by Corollary 1 the matrix \mathbf{Y}_{ext} has rank not greater than $t_1 - a$. The result is valid also if suitable linear combinations of non-Frobenius columns are added to Frobenius-type columns.

Example 2: In conditions of the previous example, let matrix \mathbf{X} be as follows:

$$\mathbf{X} = \begin{bmatrix} \alpha^3 + \alpha^6 & \alpha^5 + \alpha^2 & \alpha^6 & \alpha^2 \\ \alpha^6 + \alpha^{12} & \alpha^{10} + \alpha^5 & \alpha^{12} & \alpha^5 \\ \alpha^{12} + \alpha^{12} & \alpha^{20} + \alpha^5 & \alpha^{12} & \alpha^5 \\ \alpha^{24} + \alpha^{12} & \alpha^{40} + \alpha^2 & \alpha^{12} & \alpha^2 \end{bmatrix}.$$

The third column is added to the first Frobenius-type, and the fourth is added to the second Frobenius-type, so a = 2. Column rank of **X** is $t_1 = 4$. The corresponding matrix $\mathbf{Y} = T(\mathbf{X})$ is of the form:

$$\mathbf{Y} = \begin{bmatrix} 0 & \alpha^4 + \alpha^5 & 0 & \alpha^4 + \alpha^5 \\ \alpha^{24} + \alpha^{12} & \alpha^4 + \alpha^5 & \alpha^{24} + \alpha^{12} & \alpha^4 + \alpha^5 \\ \alpha^{24} + \alpha^{12} & \alpha^{10} + \alpha^5 & \alpha^{24} + \alpha^{12} & \alpha^{10} + \alpha^5 \end{bmatrix}$$

It has rank $t_1 - a = 2$.

In general, Overbeck's attack fails when $aN \ge 60$.

VI. CONCLUSION

We have introduced the Smart approach as a technique of withstanding Overbeck's attack on the GPT Public key cryptosystem, which is based on rank codes.

It is shown that proper choice of the distortion matrix **X** over the extension field \mathbb{F}_{q^N} allows the decryption by the authorized party and prevents the unauthorized party from breaking the system by means of any known attacks.

References

- R.J. McEliece, "A Public Key Cryptosystem Based on Algebraic Coding Theory," JPL DSN Progress Report 42– 44, Pasadena, CA, pp. 114–116, 1978.
- [2] H. Niederreiter, (1986), Knapsack-Type Cryptosystem and Algebraic Coding Theory, Probl. Control and Inform. Theory, vol. 15, pp. 19-34,1986.

- [3] E.M. Gabidulin, A.V. Paramonov, O.V. Tretjakov, "Ideals over a Non-commutative Ring and Their Application in Cryptology", in: Advances in Cryptology — Eurocrypt '91, LNCS 547, 1991, pp. 482–489.
- [4] E.M. Gabidulin, "Public-Key Cryptosystems Based on Linear Codes over Large Alphabets: Efficiency and Weakness," in: *Codes and Ciphers*, Editor: P.G. Farrell, pp. 17– 32, Essex: Formara Limited, 1995.
- [5] J. K. Gibson, "Severely denting the Gabidulin version of the McEliece public key cryptosystem," // Designs, Codes and Cryptography, 6(1), 1995, pp. 37–45.
- [6] J. K. Gibson, "The security of the Gabidulin publickey cryptosystem," in: U. M. Maurer, ed. // Advances in Cryptology – EUROCRYPT'96, LNCS 1070, 1996, pp. 212–223.
- [7] E.M. Gabidulin, A.V. Ourivski, "Improved GPT Public Key Cryptosystems." // In: P. Farrell, M. Darnell, B. Honary (Ed's), "Coding, Communications, and Broadcasting", Research Studies Press, 2000, pp. 73-102.
- [8] A. V. Ourivski, E. M. Gabidulin, "Column Scrambler for the GPT Cryptosystem." // Discrete Applied Mathematics. 128(1): 207-221 (2003).
- [9] E. M. Gabidulin, A. V. Ourivski, B. Honary, B. Ammar, "Reducible Rank Codes and Their Applications to Cryptography." // IEEE Transactions on Information Theory. 49(12): 3289-3293 (2003).
- [10] A. S. Kshevetskiy, E. M. Gabidulin, "High-weight errors in public-key cryptosystems based on reducible rank codes." // In: Proc. of ISCTA, 2005.
- [11] Overbeck, R.: A new structural attack for GPT and variants. In: Proc. of Mycrypt2005, vol. 3517 of LNCS, pp. 563. Springer-Verlag (2005).
- [12] Overbeck R.: Extending Gibson's attacks on the GPT cryptosystem. In Proc. of WCC 2005, volume 3969 of LNCS, pp. 178-188, Springer Verlag,2006.
- [13] Overbeck R : Structural Attacks for Public Key Cryptosystems based on Gabidulin Codes, Journal of Cryptology, volume 21, number 2, April 2008
- [14] E. M. Gabidulin, "Attacks and counter-attacks on the GPT public key cryptosystem," *Designs, Codes and Cryptography.* V. 48, No. 2/ August 2008. Pp. 171-177, Springer Netherlands, DOI 10.1007/s10623-007-9160-8.
- [15] E.M. Gabidulin, "Theory of Codes with Maximum Rank Distance," *Probl. Inform. Transm.*, vol. 21, No. 1, pp. 1– 12, July, 1985.
- [16] E. M. Gabidulin, "A Fast Matrix Decoding Algorithm For Rank-Error-Correcting Codes." In: (Eds G. Cohen, S. Litsyn, A. Lobstein, G. Zemor), *Algebraic coding*, pp. 126-132, Lecture Notes in Computer Science No. 573, Springer-Verlag, Berlin, 1992.
- [17] T. Johansson, A.V. Ourivski, "New technique for decoding codes in the rank metric and its cryptography applications," *Problems Inform. Transm.* 38(3), 237246 (2002).
- [18] F. Levy-dit-Vehell, J.-Ch. Jean-Charles Faug'ere, and L. Perret, "Cryptanalysis of MinRank." Advances in Cryptology - CRYPTO 2008, 28th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2008, Proceedings. Series: Lecture Notes in Computer Science. Subseries: Security and Cryptology, Vol. 5157. Wagner, David (Ed.). 2008. Pp. 280-296.
- [19] Kshevetskiy A.S.: Security of GPT-like cryptosystems based on linear rank codes. Signal Design and Its Applications in Communications, 2007. IWSDA 2007. On page(s): 143-147.
- [20] P. Loidreau, "Designing a rank metric based McEliece cryptosystem." PQCrypto 2010. The Third International Workshop on Post-Quantum Cryptography. Darmstadt, Germany, May 25-28, 2010.
- [21] E. M. Gabidulin, H.Rashwan and B. Honary., "On improving security of GPT cryptosystems." IEEE International Symposium on Information Theory, June 2009.