# EXIT Chart Approximations using the Role Model Approach

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Abstract-Extrinsic Information Transfer (EXIT) functions can be measured by statistical methods if the message alphabet size is moderate or if messages are true a-posteriori distributions. We propose an approximation we call mixed information that constitutes a lower bound for the true EXIT function and can be estimated by statistical methods even when the message alphabet is large and histogram-based approaches are impractical, or when messages are not true probability distributions and timeaveraging approaches are not applicable. We illustrate this with the hypothetical example of a rank-only message passing decoder for which it is difficult to compute or measure EXIT functions in the conventional way. We show that the role model approach [9] can be used to optimize post-processing for the decoder and that it coincides with Monte Carlo integration in the non-parametric case. It is guaranteed to tend towards the optimal Bayesian postprocessing estimator and can be applied in a blind setup with unknown code-symbols to optimize the check-node operation for non-binary Low-Density Parity-Check (LDPC) decoders.

#### I. INTRODUCTION

Extrinsic Information Transfer (EXIT) charts [5] are a well known tool to analyze the convergence of iterative decoders and receivers. While in some cases they can be computed analytically, one often has to resort to statistical estimation to measure the mutual informations plotted in these charts. There are two statistical approaches for measuring mutual information: the histogram-based approach [4] and the timeaveraging approach [7], [8]. While the former approach works in all cases but is highly impractical when the message space is large, the latter approach can only be applied when the messages correspond to true extrinsic probability distributions over the code alphabet.

Consider the following example. The sum-product algorithm for non-binary Low-Density Parity-Check (LDPC) codes over GF(q) works by passing probability distributions over GF(q)along the edges of a factor graph. We would like to design a simplified decoder where messages are ranked lists of symbols from GF(q) rather than probability distributions, a sort of "Gallager A" [3] for non-binary codes. Ideally, we would design check and variable node operations in such a way that the order of decreasing probabilities in the sumproduct messages is retained in the ranked list messages of our simplified algorithm. We are not actually able to design such an algorithm, but we can analyze the performance achievable by such a hypothetical algorithm using EXIT charts. In other words, we are trying to figure out how much information in a ranked list of probabilities is contained in the rank, or how much information we stand to loose if we throw away the probability values and retain only the ranks. In principle, given a measured mutual information, there exists an algorithm that can exploit it and achieve the performance we predict in theory, but we concede that it may be impractical to implement such an algorithm and we currently have no systematic way of designing it. Besides, the problem of knowing how much information is contained in the rank of an a-posteriori probability distribution is interesting in its own right.

Let X be a transmitted code symbol from GF(q), Y be the corresponding message in the sum-product algorithm, and Zbe the corresponding message in our hypothetical simplified algorithm, retaining only the orders in the ranked values of Y. For the EXIT chart, we need to determine the mutual information I(X; Z). The value of this mutual information will of course depend on the exact conditional density  $p_{Y|X}$ and it is not realistic to compute it analytically. We can however design a simple experimental setup to measure the mutual informations required for the EXIT charts via statistical methods. However, for q = 16 there are 16! possible orderings of the code alphabet. Using the histogram method would require us to measure  $16 \times 16!$  frequencies in order to estimate the corresponding conditional probabilities  $P_{Z|X}$ , which is clearly not practical. The time-averaging method, on the other hand, works by computing the expectation of the entropy  $H(P_{X|Z=z})$  over all realizations z of the message Z. By repeating this experiment independently in time, ergodicity allows us to replace the expectation by an average over time. This supposes however that we know how to compute the aposteriori distribution  $P_{X|Z=z}$  for every observed message z. In our example, computing the a-posteriori probability of Xgiven a specific probability ranking z is a difficult problem for which we cannot think of a computable closed-form solution.

In this paper, we will propose an approximation of the EXIT function that provides a lower bound for the true mutual information I(X; Z). This function is computed using a parametric model a-posteriori distribution of X given Z and the true distribution of X given Y. It can be optimized in an adaptive and blind manner, i.e., without requiring known code symbols X, in such a manner as to refine the model distribution so that it will approach the true unknown a-posteriori distribution of X given Z. Our approach uses the role model framework introduced in [9]. Its application to

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EXIT chart approximation was first described in [10]. This paper adds a fresh perspective to this approach by treating the example of the hypothetical rank-decoder, and draws parallels between the role model approach, the EM algorithm, and Monte Carlo integration.

### **II. MEASUREMENT SETUP**

Figure I represents our measurement setup for mutual information. Symbols X are emitted by a discrete memoryless source over the code alphabet GF(q) and transmitted over a "super-channel" consisting of a communications channel, processing performed at previous iterations, and the check or variable node operation for which we wish to draw the EXIT curve. The sum-product algorithm computes the optimal Bayesian message  $P_{X|Y}$  at the output of this channel. It may seem confusing to see this a-posteriori distribution among the signals in the block diagram of our measurement setup. In our context, we consider this probability-valued message to be itself a random variable. If the probability distribution computed by the node is indeed an a-posteriori probability vector, it is a sufficient statistic for Y.

Following this, we have the rank retainer which discards the probability values from the sum-product message and retains only the ranked list Z of symbols in GF(q), as explained in the introduction. The rightmost box in the figure is an additional operator that we will use in the mutual information measurement, labeled "post-processing". Its role is to convert the message Z into the space of probability distributions over X, but not necessarily into the true a-posteriori distribution  $P_{X|Z}$ . This is why we use the letter Q instead of P to denote this message.

If our aim was to compute I(X; Y), then we could use the following approach

$$I(X;Y) = H(X) - H(X|Y)$$
  
=  $H(X) - \sum_{y} P_Y(y)H(X|Y=y).$ 

H(X) is known and the expectation in the rightmost term can be computed through time-averaging due to the ergodicity of i.i.d. random processes. This allows us to compute the entropy of each message  $H(P_{X|Y_i=y_i})$ , then averaging it over time, which is essentially the approach described in [7], [8].

In our scenario, our aim is to compute I(X; Z). Since we do not know how to compute  $P_{X|Z}$  easily, we cannot apply the same trick. We can attack the problem the other way around by writing out the mutual information as

$$I(X;Z) = H(Z) - H(Z|X)$$

but this involves measuring histograms for every possible value of Z, whose alphabet is factorial in q as laid out in the introduction.

Note that if our post-processing element computes fake aposteriori probability distributions  $Q_{X|Z} \neq P_{X|Z}$ , we may be tempted to use the same trick as above with these fake messages. This however gives totally random results as shown in [10]. In particular, we could choose our post-processor to always return a probability of 1 of getting the symbol 0, irrespective of its input message, yielding a maximum mutual information even though the resulting message is independent of X and therefore in reality  $I(X; Q_{X|Z}) = 0$ .

# III. MIXED INFORMATION

#### We define the following quantity

Definition 1: The mixed information<sup>1</sup> is defined as

$$I'(X;Z) \stackrel{\text{def}}{=} \sum_{x} \sum_{z} P(x,z) \log_2 \left( Q(x|z)/P(x) \right). \quad (1)$$

Mixed information can be measured via statistical measurement using an approach parallel to the time averaging outlined in the previous section:

$$I'(X;Z) = \sum_{x,y,z} P(x,y,z) \log_2 \frac{Q(x|z)}{P(x)} = H(X) + \sum_{z,y} P(yz) \sum_x P(x|y) \log_2 Q(x|z)$$
(2)

where, again, the expected value in the rightmost term can be computed via time averaging due to the ergodicity of the i.i.d. processes involved. Unlike the measurement of the true mutual information, this statistical measurement can be implemented accurately even when  $Q_{X|Z} \neq P_{X|Z}$ . It is practical even when Z is defined over a large alphabet since it does not require the computation of histograms for Z.

While it is good to know that we can measure mixed information via time averaging, a crucial question is how it relates to the true mutual information. Building on (2), we can write

$$I'(X;Z) = H(X) + \sum_{y,z} P(yz) \sum_{x} P(x|y) \left[ \log_2 \frac{Q(x|z)}{P(x|y)} + \log_2 P(x|y) \right]$$
  
=  $H(X) - H(X|Y) -$ (3)  
 $\sum_y \sum_z P(yz) \sum_x P(x|y) \log_2 \frac{P(x|y)}{Q(x|z)}$   
=  $I(X;Y) - E_{P_{YZ}} \left[ D(P_{X|Y} ||Q_{X|Z}) \right]$ 

<sup>1</sup> Note that this quantity was called "mismatched information" in [9]. Meanwhile, we have become aware of the large body of literature on mismatched decoding, where mismatched information is a well-known term and is defined in a different manner, e.g., [11], so we refrain from using that term

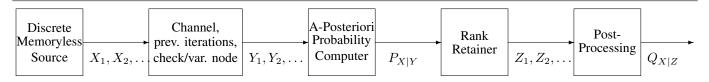


Fig. 1. Mutual Information Measurement Setup

where we have used a notation from [2] and expanded in [9] for the expected divergence. This allows us to state the following theorem:

Theorem 1:

$$I'(X;Z) \le I(X;Y) \tag{4}$$

with equality if and only if  $Q_{X|Z} = P_{X|Y}$  for all observations Y, i.e., if the decoder under scrutiny is an optimal Bayesian decoder.

*Proof:* the proof follows directly from the non-negativity of information divergence.  $\Box$ 

In other words, mixed information is always smaller or equal than the channel mutual information. This is unlike the "fake" mutual information that we mentioned at the end of the previous section, which could be higher than the channel mutual information, in violation of the data processing theorem.

Theorem 1 relates mixed information to the mutual information over the channel, but it does not tell us how mixed information relates to the mutual information I(X; Z), which is the one we are after. For this purpose, we use the following theorem, introduced in [9]:

Theorem 2 (The "role model" theorem): If X, Y and Z form a Markov chain X - Y - Z, then

$$ED(P_{X|Y}||Q_{X|Z}) = H(X|Z) - H(X|Y) + ED(P_{X|Z}||Q_{X|Z}).$$
(5)

In particular,

$$ED(P_{X|Y}||Q_{X|Z}) \ge H(X|Z) - H(X|Y)$$
(6)

with equality if and only if  $Q_{X|Z=z} = P_{X|Z=z}$  for all z for which P(z) > 0.

A direct consequence of this theorem is that mixed information is maximized for  $Q_{X|Z} = P_{X|Z}$ , giving the following result: *Theorem 3:* 

$$I'(X;Z) \le I(X;Z) \tag{7}$$

with equality if and only if the post-processing is optimal, i.e.,  $Q_{X|Z} = P_{X|Z}$ .

The theorem shows that mixed information is a tight lower bound for the mutual information I(X; Z) and that mixed information is maximized by the optimal post-processing function.

#### IV. ROLE MODEL ESTIMATION

The combination of theorems 2 and 3 give us a recipe for computing a lower bound approximation to the EXIT curve in cases where histogram-based approaches are impractical. In our simplified ranked-list decoder example, all we need to do is to devise a heuristic probabilistic model  $Q_{X|Z}$  of X given the ranked lists Z. Once we establish this model and compute its corresponding mixed information, Theorem 2 gives us a method to refine the model and increase the mixed information, whereby the mixed information tends towards the mutual information I(X;Z) and the model tends towards the true a-posteriori probability distribution. The optimization problem to be solved is a simple divergence minimization, which is convex in the full set of parameters  $Q_{X|Z}$ .

There are some parallels and differences to be drawn between this approach and Expectation-Maximization (EM) algorithms [12], [13]. Both have in common that they design an estimator based on incomplete observations, in our case Z, by bringing the problem back to an estimation problem based on superior or complete observations, in our case Y, where a Markov condition X - Y - Z ensures the superiority of Y over Z with respect to estimating X. The difference however is that the EM algorithm "fabricates" the complete data and uses an iterative process to produce an estimator based on the incomplete data where the complete data has been factored out. In contrast, we assume in our setting that the superior data is available for training purposes in order to train and refine our estimator based on the inferior data. By doing this, we automatically inherit the statistical model for X available for observations Y and any prior on X that results from this model. Consequently, the role model approach results in a simple divergence minimization, mirroring the E step in the EM algorithm rather than the M step where a divergence is maximized. Note also that the divergence maximized in the M step of the EM algorithm is from the model to the true probability rather than vice versa as is our case.

Let us now assume that we can adapt the full set of parameters of the model family of probability distributions  $Q_{X|Z}$  and see what the role model approach gives. If we set up the Karush-Kuhn-Tucker (KKT) conditions for the divergence minimization problem, we obtain that the role model estimator boils down to an evaluation by Monte Carlo integration of the sum

$$P(x|z) = \sum_{y} P(x|y)P(y|z).$$
(8)

The unknown in this equation is P(y|z), which is why Monte Carlo integration is necessary and constitutes the best we can hope to achieve in terms of estimating P(x|z) by observing realizations of Y and Z.

#### V. EXIT CHARTS

As a first step, we were interested in the problem of finding out how much is lost by retaining the rank of the a-posteriori probability message in a simple channel. For this, we are using exactly the setup described in Figure I with a simple communication channel used as the "super-channel". We are generated symbols from GF(64) uniformly at random and transmitted them over an Addititive White Gaussian Noise (AWGN) channel, once modulated as six BPSK symbols per source symbol using natural mapping, and once modulated using 64-QAM. The loss of mutual information measured is plotted in Figure 2. The measurement shows that surprisingly little information is contained in the actual values of the probabilities in the a-posteriori distributions computed, whereas most of the information is in the ranked list of symbols. The loss of mutual information when retaining only the ranked list

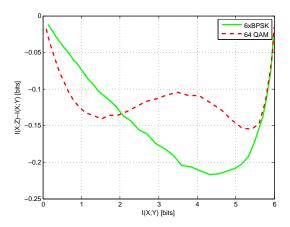


Fig. 2. Mutual information loss after rank retainer for bitwise BPSK and for 64-QAM for symbols X over GF(64)

is less than 1/4 bit in all cases. It is generally smaller for 64-QAM than for bitwise BPSK, except at very low SNR where the loss is higher for 64-QAM.

For the EXIT chart measurement for variable nodes, we used the 64-QAM modulated AWGN channel with a signal to noise ratio of  $E_s/N0 = 16.3 dB \ (E_b/N0 = 8.5 dB)$ . All our EXIT curves are for check node degree  $d_c = 4$ and variable node degree  $d_v = 2$  regular LDPC codes over GF(64). The quality of the EXIT analysis depends on the availability of a realistic parametric model for the distributions of messages coming from previous iterations and there is much literature covering this issue, e.g., [14], [15]. Since we are measuring a hypothetical algorithm, we opted at this stage to use simple 64-QAM modulated symbols through an AWGN channel as our incoming messages. We are aware that this casts some doubt over the validity of any performance predictions, but any approximation related to our message distribution is overshadowed by the far greater approximation related to the fact that we are measuring a hypothetical algorithm that we cannot actually implement and are currently investigating a concept rather than aiming to predict exact performance.

For the EXIT chart of the sum-product algorithm, we measure the mutual information of our generated incoming messages using time averaging, then apply the variable node rule of the sum-product algorithm and measure the mutual information of our extrinsic messages again using time averaging. For the EXIT chart of our hypothetical algorithm, we apply the rank retainer to the a-posteriori messages computed based on our generated graph messages. We then use our role model framework to optimize a post-processing distribution  $Q_{X|Z}$  of the symbols given the ranked lists. This distribution is used both to compute a lower bound on the EXIT curve and in the variable node operation, since we do not know any other way of implementing a variable node operation at this stage that processes incoming ranked lists and computes a ranked list. Instead, we implement the usual sum-product variable node rule but use the optimized  $Q_{X|Z}$  instead of

the original  $P_{X|Y}$  as the incoming messages. The resulting message computed by the variable node is again passed through a rank retainer so as to get a mapping from lists to lists as intended, and the mutual information of the final extrinsic list-valued message is measured again using time-averaging and the role model framework.

The EXIT curves measured for variable nodes are plotted in Figure 3. Surprisingly, the curve for the rank-based variable

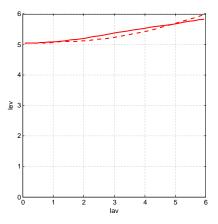


Fig. 3. EXIT chart for variable nodes  $d_v = 2$  over GF(64),  $E_b/N_0 = 8.5dB$ , 64-QAM over AWGN, for the sum-product algorithm (dashed curve) and for the hypothetical rank-based algorithm (continuous curve)

node is in part above the sum-product curve, in apparent breach of the data processing theorem. This is merely an illusion because the mutual informations on the x-axis do not correspond to the same incoming messages. The x-axis for the rank-based algorithm is in effect "warped" to its advantage, i.e., you need to start off with more informative messages if you want the resulting ranks to contain as much information as the equivalent full messages used by the sumproduct algorithm. Figure 4 shows an equivalent EXIT curve

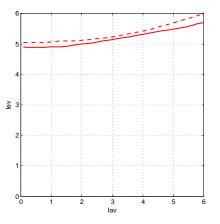


Fig. 4. EXIT curve as in Figure 3 but where the extrinsic information for the rank-based decoder is plotted in function of the same mutual information as for the sum-product decoder

where the extrinsic information for the rank-based decoder is

plotted in function of the mutual information of the original incoming messages before the rank retainer and shows that there is no breach of the data processing theorem as we feared. More interestingly, both figures show that the extrinsic mutual information for the rank-based decoder does not converge to 6 bits as the a-priori information goes to 6. This implies that any decoder that uses rank-based variable nodes is bound to have a decoder-induced error floor.

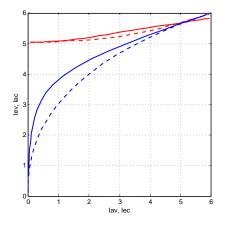


Fig. 5. Full EXIT chart for regular (4.2) LDPC over GF(64),  $E_b/N_0 = 8.5dB$ , 64-QAM over AWGN, for the sum-product algorithm (dashed curve) and for the hypothetical rank-based algorithm (continuous curve)

The resulting full EXIT chart with all measured EXIT functions including the check node curves for  $d_c = 4$  is in Figure 5. The curve of the rank-based check node shows a significant loss with respect to the sum-product check node. However, in the important region where the decoder converges to the error-free case, the curves have the same slope and become barely distinguishable. Our results are an indication that perhaps a hybrid decoder that somehow implements a rank-based operation in the check nodes while remaining in the probability domain in the variable nodes may be able to attain good performance. Since at this point we have neither a method for designing such a check node nor a precise setup for predicting performance, this indication is to be taken as an encouragement for futher research rather than an accurate prediction.

## VI. CONCLUSION

We have introduced *mixed information* and shown that it is a tight lower bound both for the mutual information over the channel and for the mutual information between the transmitted symbols and the output of a sub-optimal decoder. Mixed information can be measured effectively via time averaging. This allows us to draw a lower bound for the EXIT function, enabling the design of codes that are guaranteed to converge and matched to a practical simplified decoder.

Furthermore, since mixed information is maximized by an optimal post-processing function, i.e., when the output of the post-processor is the true a-posteriori probability distribution given the sub-optimal decoder output, it can serve as a design tool for the post-processing stage. By choosing the parameters of the post-processing function so as to maximize mixed information, we are guaranteed to approach the optimal postprocessor that gives the best possible performance for a given choice of the simplified decoder component.

We have illustrated the application of mixed information and the role model approach by using it to estimate the EXIT function of a hypothetical check node that retains only the symbol ranks in an ordered list of probablities in the messages of the sum-product algorithm. We have shown how for nonparametric estimation the role-model framework is equivalent to Monte Carlo integration, but whereas the latter cannot be applied to parametric models the former can.

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