

# Entropy Measures vs. Algorithmic Information

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**Abstract**— We further study the connection between Algorithmic Entropy and Shannon and Rényi Entropies. It is given an example for which the difference between the expected value of algorithmic entropy and Shannon Entropy meets the known upper-bound and, for Rényi Entropy, proving that all other values of the parameter ( $\alpha$ ), the same difference can be big. We also prove that for a particular type of distributions Shannon Entropy is able to capture the notion of computationally accessible information by relating it to time-bounded algorithmic entropy. In order to better study this unexpected relation it is investigated the behavior of the different entropies (Shannon, Rényi and Tsallis) under the distribution based on the time-bounded algorithmic entropy.

## I. INTRODUCTION

Algorithmic Entropy, the size of the smallest program that generates a string, denoted by  $K(x)$ , is a rigorous measure of the amount of information, or randomness, in an individual object  $x$ . Algorithmic entropy and Shannon entropy are conceptually very different, as the former is based on the size of programs and the later in probability distributions. Surprisingly, they are, however, closely related. The expectation of the algorithmic entropy equals (up to a constant depending on the distribution) the Shannon entropy.

Shannon entropy measures the amount of information in situations where unlimited computational power is available. However this measure does not provide a satisfactory framework for the analysis of public key cipher systems which are based on the limited computational power of the adversary. The public key and the cipher text together contain all the Shannon information concerning the plaintext, but the information is computationally inaccessible. So, we face this intriguing question: what is accessible information?

By considering the time-bounded algorithmic entropy (length of the program limited to run in time  $t(|x|)$ ) we can take into account the computational difficulty (time) of extracting information. Under some computational restrictions on the distributions we show (Theorem 15) that Shannon entropy equals (up to a constant that depends only on the distribution) the time-bound algorithmic information. This result partially solves, for this type of distributions, the problem of finding a measure that captures the notion of computationally accessible information. This result is unexpected since it states that for the class of probability distribution such that its cumulative probability distribution is computable in time  $t(n)$ , the Shannon entropy captures the notion of computational difficulty of extracting information within this time bound.

With this result in mind we further study the relation of the probability distribution based on time-bounded algorithmic

entropy with several entropy measures (Shannon, Rényi and Tsallis).

## II. PRELIMINARIES

All strings used are elements of  $\Sigma^* = \{0, 1\}^*$ .  $\Sigma^n$  denotes the set of strings of length  $n$  and  $| \cdot |$  denotes the length of a string. It is assumed that all strings are ordered by lexicographic ordering. When  $x - 1$  is written, where  $x$  is a string, it means the predecessor of  $x$  in the lexicographic order. The function  $\log$  is the function  $\log_2$ . The real interval between  $a$  and  $b$ , including  $a$  and excluding  $b$  is represented by  $[a, b)$ .

### A. Algorithmic Information Theory

We give essential definitions and basic results which will be need in the rest of the paper. A more detailed reference is [LV97]. The model of computation used is the prefix free Turing machine. A set of strings  $A$  is prefix-free if no string in  $A$  is prefix of another string of  $A$ . Notice that Kraft inequality guarantees that for any prefix-free set  $A$ ,  $\sum_{x \in A} 2^{-|x|} \leq 1$ .

**Definition 1.** Let  $U$  be a fixed prefix free universal Turing machine. For any string  $x \in \Sigma^*$ , the Kolmogorov complexity or algorithmic entropy of  $x$  is  $K(x) = \min_p \{ |p| : U(p) = x \}$ . For any time constructible  $t$ , the  $t$ -time-bounded algorithmic entropy (or  $t$ -time-bounded Kolmogorov complexity) of  $x \in \Sigma^*$  is,  $K^t(x) = \min_p \{ |p| : U(p) = x \text{ in at most } t(|x|) \text{ steps} \}$ .

The choice of the universal Turing machine affects the running time of a program at most by a logarithmic factor and the program length at most a constant number of extra bits.

**Proposition 2.** For all  $x$  and  $y$  we have:

- 1)  $K(x) \leq K^t(x) \leq |x| + O(1)$ ;
- 2)  $K(x|y) \leq K(x) + O(1)$  and  $K^t(x|y) \leq K^t(x) + O(1)$ ;

**Definition 3.** A string  $x$  is said algorithmic-random or Kolmogorov-random if  $K(x) \geq |x|$ .

A simple counting argument shows the existence of algorithmic-random strings of any length.

**Definition 4.** A semi-measure over a space  $X$  is a function  $f : X \rightarrow [0, 1]$  such that  $\sum_{x \in X} f(x) \leq 1$ . We say that a semi-measure is a measure if the equality holds. A semi-measure is called constructive if it is semi-computable from below.

The function  $\mathbf{m}(x) = 2^{-K(x)}$  is a semi-measure which is constructible and dominates any other constructive semi-measure  $\mu$  ([Lev74] and [Gac74]), in the sense that there is

a constant  $c_\mu = 2^{K(\mu)}$  such that for all  $x$ ,  $\mathbf{m}(x) \geq c_\mu \mu(x)$ . For this reason, this semi-measure is called universal. Since it is natural to consider time bounds on the Kolmogorov complexity we can define a time bounded version of  $\mathbf{m}(x)$ .

**Definition 5.** The  $t$ -time bounded universal distribution, denoted by  $\mathbf{m}^t$  is  $\mathbf{m}^t(x) = c2^{-K^t(x)}$ , where  $c$  is a fixed constant such that  $\sum_{x \in \Sigma^*} \mathbf{m}^t(x) = 1$ .

In [LV97], Claim 7.6.1, the authors prove that  $\mathbf{m}^{t(\cdot)}$  dominates every distribution  $\mu$  such that  $\mu^*$ , the cumulative probability distribution of  $\mu$ , is computable in time  $t(\cdot)$ .

**Theorem 6.** If  $\mu^*$  is computable in time  $t(n)$  then there exists a constant  $c$  such that, for all  $x \in \Sigma^*$ ,  $\mathbf{m}^{nt(n)}(x) \geq 2^{-K^{nt(n)}(\mu)} \mu(x)$ .

## B. Entropies

We consider several types of entropies. Shannon information theory was introduced in 1948 by C.E. Shannon [Sha48]. Information theory quantifies the uncertainty about the results of an experiment. It is based on the concept of entropy which measures the number of bits necessary to describe an outcome from an ensemble.

**Definition 7** (Shannon Entropy [Sha48]). Let  $\mathcal{X}$  be a finite or infinitely countable set and let  $X$  be a random variable taking values in  $\mathcal{X}$  with distribution  $P$ . The Shannon Entropy of random variable  $X$  is given by

$$H(X) = - \sum_{x \in \mathcal{X}} P(x) \log P(x).$$

The Rényi entropy is a generalization of Shannon entropy. Formally the Rényi entropy is defined as follows:

**Definition 8** (Rényi Entropy [Ren61]). Let  $\mathcal{X}$  be a finite or infinitely countable set and let  $X$  be a random variable taking values in  $\mathcal{X}$  with distribution  $P$  and let  $\alpha \neq 1$  be a positive real number. The Rényi Entropy of order  $\alpha$  of the random variable  $X$  is defined as:

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \left( \sum_{x \in \mathcal{X}} P(x)^\alpha \right).$$

It can be shown that  $\lim_{\alpha \rightarrow 1} H_\alpha(X) = H(X)$ .

**Definition 9** (Min-Entropy). Let  $\mathcal{X}$  be a finite or infinitely countable set and let  $X$  be a random variable taking values in  $\mathcal{X}$  with distribution  $P$ . We define the Min-Entropy of  $P$  by:

$$H_\infty(P) = - \log \max_{x \in \mathcal{X}} P(x).$$

It is easy to see that  $H_\infty(P) = \lim_{\alpha \rightarrow \infty} H_\alpha(P)$ .

**Definition 10** (Tsallis Entropy [Ts88]). Let  $\mathcal{X}$  be a finite or infinitely countable set and let  $X$  be a random variable taking values in  $\mathcal{X}$  with distribution  $P$  and let  $\alpha \neq 1$  be a positive

real number. The Tsallis Entropy of order  $\alpha$  of the random variable  $X$  is defined as:

$$T_\alpha(P) = \frac{1 - \sum_{x \in \Sigma^*} P(x)^\alpha}{\alpha - 1}.$$

## C. Algorithmic Information vs. Entropy Information

Given the conceptual differences in the definition of Algorithmic Information Theory and Information Theory, it is surprising that under some weak restrictions on the distribution of the strings, they are closely related, in the sense that the expectation of the algorithmic entropy equals the entropy of the distribution up to a constant that depends only on that distribution.

**Theorem 11.** Let  $P(x)$  be a recursive probability distribution. Then:

$$0 \leq \sum_x P(x)K(x) - H(P) \leq K(P)$$

*Proof.* (Sketch, see [LV97] for details) The first inequality follows directly from the well known Noiseless Coding Theorem, that, for this distributions, states

$$H(P) \leq \sum_x P(x)K(x)$$

Since  $\mathbf{m}$  is universal,  $P(x) \leq 2^{K(P)} \mathbf{m}(x)$ , for all  $x$ , which is equivalent to  $\log P(x) \leq K(P) - K(x)$ . Thus, we have:

$$\begin{aligned} \sum_x P(x)K(x) - H(P) &= \sum_x (P(x)(K(x) + \log P(x))) \\ &\leq \sum_x (P(x)(K(x) + K(P) - K(x))) = K(P) \quad \square \end{aligned}$$

## III. ALGORITHMIC ENTROPY VS. ENTROPY: HOW CLOSE?

Given the surprising relationship between algorithmic entropy and entropy, in this section we investigate how close they are. We study also the relation between algorithmic entropy and Rényi entropy. In particular, we will find the values of  $\alpha$  for which the same relation as in Theorem 11 holds for the Rényi entropy. We also prove that for a particular type of distributions, entropy is able to capture the notion of computationally accessible information.

First we show that the interval  $[0, K(P)]$  of the inequalities of Theorem 11 is tight:

**Proposition 12.** There exist distributions  $P$ , with  $K(P)$  large such that:

- 1)  $\sum_x P(x)K(x) - H(P) = K(P) - O(1)$ .
- 2)  $\sum_x P(x)K(x) - H(P) = O(1)$ .

*Proof.* 1) Fix  $x_0 \in \Sigma^n$ . Consider the following probability distribution:

$$P_n(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

Notice that describing the distribution is equivalent to describe  $x_0$ . So,  $K(P_n) = K(x_0) + O(1)$ . On the other

hand,  $\sum_x P_n(x)K(x) - H(P_n) = K(x_0)$ . So, if  $x_0$  is Kolmogorov-random then  $K(P_n) \approx n$ .

- 2) Let  $y$  be a string of length  $n$  such that  $K(y) = n - O(1)$  and consider the following probability distribution over  $\Sigma^*$ :

$$P_n(x) = \begin{cases} 0.y & \text{if } x = x_0 \\ 1 - 0.y & \text{if } x = x_1 \\ 0 & \text{otherwise} \end{cases}$$

where  $0.y$  represent the real number between 0 and 1 which binary representation is  $y$ . Notice that we can choose  $x_0$  and  $x_1$  such that  $K(x_0) = K(x_1) \leq c$  where  $c$  is a constant that does not depend on  $n$ .

Thus we have:

- $K(P_n) \approx n$ , since describing  $P_n$  is equivalent to describe  $x_0, x_1$  and  $y$ ;
- $\sum_x P_n(x)K(x) = (0.y)K(x_0) + (1 - 0.y)K(x_1) \leq 0.y \times c + (1 - 0.y) \times c = c$ ;
- $H(P_n) = -0.y \log 0.y - (1 - 0.y) \log(1 - 0.y) \leq 1$

Thus  $\sum P_n(x)K(x) - H(P_n) \leq c \ll K(P_n) \approx n$ .  $\square$

Now we address the question if the same relations as in Theorem 11 holds for the Rényi entropy. We show, in fact, the Shannon entropy is the “smallest” entropy that verify these properties.

Since, for every  $0 < \varepsilon < 1$ ,

$$H_\infty \leq H_{1-\varepsilon}(X) \leq H(X) \leq H_{1+\varepsilon}(X) \leq H_0(X)$$

it follows that

$$0 \leq \underbrace{\sum_x P(x)K(x) - H_\alpha(P)}_{\alpha \geq 1} \leq \underbrace{K(P)}_{\alpha \leq 1}$$

In the next result we show that the inequalities above are, in general, false for different values of  $\alpha$ .

**Theorem 13.** For every  $\Delta > 0$  and  $\alpha > 1$  there exists a recursive distribution  $P$  such,

- $\sum_x P(x)K(x) - H_\alpha(P) \geq (K(P))^\alpha$
- $\sum_x P(x)K(x) - H_\alpha(P) \geq K(P) + \Delta$

The proof of this Theorem is similar to the proof of the following Corollary:

**Corollary 14.** There exists a recursive probability distribution  $P$  such that:

- $\sum_x P(x)K(x) - H_\alpha(P) > K(P)$ , where  $\alpha > 1$ ;
- $\sum_x P(x)K(x) - H_\alpha(P) < 0$ , where  $\alpha < 1$ .

*Proof.* For  $x \in \{0, 1\}^n$ , consider the following probability distribution:

$$P_n(x) = \begin{cases} 1/2 & \text{if } x = 0^n \\ 2^{-n} & \text{if } x = 1x', x' \in \{0, 1\}^{n-1} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that this distribution is recursive.

- 1) First observe that

$$\begin{aligned} H(P_n) &= -\sum_x P_n(x) \log P_n(x) \\ &= -\left(\frac{1}{2} \log \frac{1}{2} + \frac{1}{2^n} \log \frac{1}{2^n} 2^{n-1}\right) \\ &= -\left(-\frac{1}{2} - n \frac{1}{2^n} 2^{n-1}\right) \\ &= \frac{n+1}{2} \end{aligned}$$

Notice also that  $K(P_n) = O(\log n)$ .

We want to prove that, for every  $\alpha > 1$ ,

$$(\exists n_0)(\forall n \geq n_0) \sum_x P_n(x)K(x) - H_\alpha(P_n) > K(P_n)$$

Fix  $\alpha$  such that  $\alpha - 1 = \frac{1}{(n_0 - 1)^{1/\varepsilon}}$ .

$$\begin{aligned} H_\alpha(P_n) &= \frac{1}{1-\alpha} \log \sum_x P_n(x)^\alpha \\ &= \frac{1}{1-\alpha} \log \left( \frac{1}{2^\alpha} + 2^{n-1} \times \frac{1}{2^{n\alpha}} \right) \\ &= \frac{1}{1-\alpha} \left( \log(2^{(n-1)\alpha} + 2^{n-1}) - n\alpha \right) \end{aligned}$$

Now we calculate  $\log(2^{(n-1)\alpha} + 2^{n-1})$ . To simplify notation consider:

$$\begin{cases} x &= n-1 \\ \alpha &= 1+\varepsilon, \text{ with } \varepsilon > 0 \end{cases}$$

Thus,

$$\begin{aligned} \log(2^{(n-1)\alpha} + 2^{n-1}) &= \log(2^{x(1+\varepsilon)} + 2^x) = \\ &= \log(2^x (2^{x\varepsilon} + 1)) \\ &= x + \log(2^{x\varepsilon} + 1) \end{aligned}$$

Consider  $\delta = x\varepsilon$ . It is clear that

$$2^\delta = e^{\ln 2 \cdot \delta} = 1 + \ln 2 \cdot \delta + \frac{(\ln 2)^2 \cdot \delta^2}{2} + \dots$$

then,

$$2^\delta + 1 = 2 + \ln 2 \cdot \delta + \frac{(\ln 2)^2 \cdot \delta^2}{2} + \dots$$

and hence,

$$\log(2^\delta + 1) = \log \left( 2 + \underbrace{\ln 2 \cdot \delta + \frac{(\ln 2)^2 \cdot \delta^2}{2} + \dots}_\beta \right)$$

Notice that  $\lim_{\alpha \rightarrow 1} \beta = 0$ .

$$\begin{aligned} \log(2 + \beta) &= \frac{1}{\ln 2} \ln(2 + \beta) \\ &= \frac{\ln 2}{\ln 2} \ln\left(2\left(1 + \frac{\beta}{2}\right)\right) = \\ &= \frac{1}{\ln 2} \left( \ln 2 + \frac{\beta}{2} - \frac{\beta^2}{8} + \dots \right) \\ &= 1 + \frac{\beta}{2 \ln 2} - \frac{\beta^2}{8 \ln 2} + \dots \end{aligned}$$

Then,

$$\begin{aligned} & \log\left(2 + \ln 2 \cdot \delta + \frac{(\ln 2)^2 \cdot \delta^2}{2} + \dots\right) = \\ & = 1 + \frac{\delta}{2} + \frac{\ln 2}{8} \delta^2 + \dots - \frac{(\ln 2)^2}{8} \delta^3 - \frac{(\ln 2)^3}{32} \delta^4 - \dots \end{aligned}$$

So we have:

$$\log(2^{x\varepsilon} + 1) = 1 + \frac{x\varepsilon}{2} + \frac{\ln 2}{8} (x\varepsilon)^2 + \dots$$

which means,

$$x + \log(2^{x\varepsilon} + 1) = x + 1 + \frac{x\varepsilon}{2} + \frac{\ln 2}{8} (x\varepsilon)^2 + \dots$$

Thus

$$\begin{aligned} H_\alpha(P_n) &= \frac{-1}{\alpha - 1} (\log(2^{(n-1)\alpha} + 2^{n-1}) - n\alpha) \\ &= n - \frac{n-1}{2} - \frac{\ln 2}{8} (n-1)^2 (\alpha - 1) - \dots \end{aligned}$$

Notice that the rest of elements in the series expansion

$$c_1(n-1)^3(\alpha-1)^2 + c_2(n-1)^4(\alpha-1)^3 + \dots, c_1, c_2 \in \mathbb{R}$$

can be ignored in the limit since  $\alpha - 1 = \frac{1}{(n_0-1)^{1.8}}$ .

So, for all  $n \geq n_0$ :

$$H_\alpha(P_n) = \frac{n+1}{2} - \frac{\ln 2}{8} (n-1)^{0.2}$$

It is known that  $\lim_{\alpha \rightarrow 1} H_\alpha(P_n) = H(P_n)$ . In fact, we have

$$H_\alpha(P_n) = H(P_n) - \frac{\ln 2}{8} (n_0 - 1)^{0.2}.$$

Now, the first item of the Theorem is proved by contradiction. Assume by contradiction that

$$\sum_x P_n(x) K(x) - H_\alpha(P_n) \leq c \log n, \text{ with } c \in \mathbb{R}$$

i.e., for all  $n \geq n_0$

$$\sum_x P_n(x) K(x) - H(P_n) + \frac{\ln 2}{8} (n-1)^{0.2} \leq c \log n$$

Since,  $\sum_x P_n(x) K(x) - H(P_n) \geq 0$ , we would have  $\frac{\ln 2}{8} (n-1)^{0.2} \leq c \log n$ , which is impossible for all  $n \geq n_0$ . So, we conclude that

$$\sum_x P_n(x) K(x) - H_\alpha(P_n) > c \log n.$$

- 2) Analogous to the proof of the previous item, but now fixing  $\alpha - 1 = \frac{-1}{(n-1)^{1.8}}$ .  $\square$

If instead of considering  $K(P)$  and  $K(x)$  in the inequalities of Theorem 11 we use the time bounded version and imposing some computational restrictions on the distributions we obtain a similar result. Notice that for the class of distributions on the following Theorem the entropy equals (up to a constant) the time-bounded algorithmic entropy.

**Theorem 15.** Let  $P$  be a probability distribution such that  $P^*$ , the cumulative probability distribution of  $P$ , is computable in time  $t(n)$ . Then:

$$0 \leq \sum_x P(x) K^{nt(n)} - H(P) \leq K^{nt(n)}(P)$$

*Proof.* The first inequality follows directly from Theorem 11 and from the fact that  $K^t(x) \geq K(x)$ .

By Theorem 6, if  $P$  is a probability distribution such that  $P^*$  is computable in time  $t(n)$ , then for all  $x \in \Sigma^n$

$$K^{nt(n)}(x) + \log P(x) \leq K^{nt(n)}(P)$$

Then, summing over all  $x$  we get

$$\sum_x P(x) (K^{nt(n)}(x) + \log P(x)) \leq \sum_x P(x) K^{nt(n)}(P)$$

which is equivalent to

$$\sum_x P(x) K^{nt(n)}(x) - H(P) \leq K^{nt(n)}(P) \quad \square$$

This result partially solves, for this type of distributions, the problem of finding a measure that captures the notion of computationally accessible information. This is an important open problem with applications and consequences in cryptography.

#### IV. ON THE ENTROPY OF THE TIME-BOUNDED ALGORITHMIC UNIVERSAL DISTRIBUTION

We now focus our attention on the universal distribution. Its main drawback is the fact that it is not computable. In order to make it computable, one can impose restrictions on the time that a program can use to produce a string obtaining the time-bounded universal distribution ( $\mathbf{m}^t(x) = c2^{-K^t(x)}$ ). We investigate the behavior of the different entropies under this distribution. The proof of the following Theorem uses some ideas from [KT].

**Theorem 16.** The Shannon entropy of the distribution  $\mathbf{m}^t$  diverges.

*Proof.* If  $x \geq 2$  then  $f(x) = x2^{-x}$  is a decreasing function. Let  $A$  be the set of strings such that  $-\log \mathbf{m}^t(x) \geq 2$ . Since  $\mathbf{m}^t$  is computable,  $A$  is recursively enumerable. Notice also that  $A$  is infinite and contains arbitrarily large Kolmogorov-random strings.

$$\begin{aligned} \sum_{x \in \Sigma^*} -\mathbf{m}^t(x) \log \mathbf{m}^t(x) &\geq \sum_{x \in A} -\mathbf{m}^t(x) \log \mathbf{m}^t(x) \\ &= \sum_{x \in A} c2^{-K^t(x)} (K^t(x) - \log c) \\ &= -c \log c \sum_{x \in A} 2^{-K^t(x)} + c \sum_{x \in A} K^t(x) 2^{-K^t(x)} \end{aligned}$$

So if we prove that  $\sum_{x \in A} K^t(x) 2^{-K^t(x)}$  diverges the result follows.

Assume, by contradiction, that  $\sum_{x \in A} K^t(x) 2^{-K^t(x)} < d$  for some  $d \in \mathbb{R}$ . Then, considering  $r(x) = \frac{1}{d} K^t(x) 2^{-K^t(x)}$  if

$s \in A$  and  $r(x) = 0$  otherwise, we conclude that  $r$  is a semi-measure. Thus, there exists a constant  $c'$  such that, for all  $x$ ,  $r(x) \leq c' \mathbf{m}(x)$ . Hence, for  $x \in A$ , we have

$$\frac{1}{d} K^t(x) 2^{-K^t(x)} \leq c' 2^{-K(x)}$$

So,  $K^t(x) \leq c' d 2^{K^t(x) - K(x)}$ . This is a contradiction since  $A$  contains Kolmogorov - random strings of arbitrarily large size. The contradiction results from assuming that  $\sum_{x \in A} K^t(x) 2^{-K^t(x)}$  converges. So,  $H(\mathbf{m}^t)$  diverges.  $\square$

Now we show that, similarly to the behavior of entropy of universal distribution,  $T_\alpha(\mathbf{m}^t) < \infty$  iff  $\alpha > 1$  and  $H_\alpha(\mathbf{m}^t) < \infty$  iff  $\alpha < 1$ . First observe that we have the following ordering relationship between these two entropies for all probability distribution  $P$ :

- 1) If  $\alpha > 1$ ,  $T_\alpha(P) \leq \frac{1}{\alpha - 1} + H_\alpha(P)$ ;
- 2) If  $\alpha < 1$ ,  $T_\alpha(P) \geq \frac{1}{\alpha - 1} + H_\alpha(P)$ ;

**Theorem 17.** *Let  $\alpha \neq 1$  be a real computable number. Then we have,  $T_\alpha(\mathbf{m}^t) < \infty$  iff  $\alpha > 1$ .*

*Proof.* From Theorem 8 of [KT], it is known that  $\sum_{x \in \Sigma^*} (\mathbf{m}(x))^\alpha$  converges iff  $\alpha > 1$ . Since  $\mathbf{m}^t$  is a probability measure there exists a constant  $\lambda$  such that, for all  $x$ ,  $\mathbf{m}^t(x) \leq \lambda \mathbf{m}(x)$ . So,  $(\mathbf{m}^t(x))^\alpha \leq (\lambda \mathbf{m}(x))^\alpha$ , which implies that  $\sum_{x \in \Sigma^*} (\mathbf{m}^t(x))^\alpha \leq \lambda^\alpha \sum_{x \in \Sigma^*} (\mathbf{m}(x))^\alpha$ , from where we conclude that, for  $\alpha > 1$ ,  $T_\alpha(\mathbf{m}^t)$  converges.

For  $\alpha < 1$ , the proof is analogous to the proof of Theorem 16. Suppose that  $\sum_{x \in \Sigma^*} (\mathbf{m}^t(x))^\alpha < d$  for some  $d \in \mathbb{R}$ .

Hence,  $r(x) = \frac{1}{d} (\mathbf{m}^t(x))^\alpha$  is a computable semi-measure. Then, there exists a constant  $\tau$  such that for all  $x \in \Sigma^*$ ,  $r(x) = \frac{1}{d} (c 2^{-K^t(x)})^\alpha \leq \tau 2^{-K(x)}$  which is equivalent to  $\frac{c^\alpha}{d\tau} \leq 2^{\alpha K^t(x) - K(x)}$ . For example, if  $x$  is random it follows that  $\frac{c^\alpha}{d\tau} \leq 2^{(\alpha-1)|x|}$ , which is false.  $\square$

**Theorem 18.** *The Rényi entropy of order  $\alpha$  of time bounded universal distribution converges for  $\alpha < 1$  and diverges if  $\alpha > 1$ .*

*Proof.* Consider  $\alpha = 1 + \varepsilon$ , where  $\varepsilon > 0$ . Since for all  $x \in \Sigma^*$ ,  $K^t(x) \leq |x| + c'$  then  $2^{-|x|+c} \leq 2^{-K^t(x)}$ . Since  $f(y) = y^{1+\varepsilon}$  increases in  $[0, 1]$ , it is also true that for all  $x \in \Sigma^*$ ,  $(2^{-|x|+c})^{1+\varepsilon} \leq (2^{-K^t(x)})^{1+\varepsilon}$ . So, summing up over all  $x \in \Sigma^*$  and applying  $-\log$  we conclude that

$$-\log \sum_x (2^{-K^t(x)})^{1+\varepsilon} \leq -\log \sum_x (2^{-|x|+c})^{1+\varepsilon}$$

If we prove that the series  $\sum_{x \in \Sigma^*} (2^{-|x|+c})^{1+\varepsilon}$  converges, then the Rényi entropy of order  $1 + \varepsilon$  of  $\mathbf{m}^t$  also converges.

$$\begin{aligned} \sum_{x \in \Sigma^*} (2^{-|x|+c})^{1+\varepsilon} &= \sum_{n=1}^{\infty} \sum_{x \in \Sigma^n} (2^{-n+c})^{1+\varepsilon} \\ &= \sum_{n=1}^{\infty} \sum_{x \in \Sigma^n} 2^{-n-n\varepsilon+c+c\varepsilon} \\ &= \sum_{n=1}^{\infty} 2^n \times 2^{-n-n\varepsilon} \times 2^{c+c\varepsilon} \\ &= 2^{c+c\varepsilon} \sum_{n=1}^{\infty} 2^{-n\varepsilon} \\ &= 2^{c+c\varepsilon} \times \frac{2^\varepsilon}{2^\varepsilon - 1} < \infty \end{aligned}$$

Now, assume that  $\alpha < 1$ . Since the Rényi entropy is non increasing with  $\alpha$ , for any distribution  $P$  we have  $H(P) \leq H_\alpha(P)$ . So, in particular,  $H(\mathbf{m}^t) \leq H_\alpha(\mathbf{m}^t)$ . As  $H(\mathbf{m}^t)$  diverges we conclude that the Rényi entropy of order  $\alpha < 1$  for the time bounded universal distribution diverges.  $\square$

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