Collecting Coded Coupons over Generations

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Abstract—To reduce computational complexity and delay in randomized network coded content distribution (and for some other practical reasons), coding is not performed simultaneously over all content blocks but over much smaller subsets known as generations. A penalty is throughput reduction. We model coding over generations as the coupon collector's brotherhood problem. This model enables us to theoretically compute the expected number of coded packets needed for successful decoding of the entire content, as well as a bound on the probability of decoding failure, and further, to quantify the tradeoff between computational complexity and throughput. Interestingly, with a moderate increase in the generation size, throughput quickly approaches link capacity. As an additional contribution, we derive new results for the generalized collector's brotherhood problem which can also be used for further study of many other aspects of coding over generations.

I. INTRODUCTION

In P2P systems, such as BitTorrent, content distribution involves fragmenting the content at its source, and using swarming techniques to disseminate the fragments among peers. Acquiring a file by collecting its fragments can be to a certain extent modeled by the classic coupon collector problem, which indicates some problems such systems may have. For example, probability of acquiring a novel fragment drops rapidly as the number of those already collected increases. In addition, as the number of peers increases, it becomes harder to do optimal scheduling of distributing fragments to receivers. One possible solution is to use a heuristic that prioritizes exchanges of locally rarest fragments. But, when peers have only local information about the network, such fragments often fail to match those that are globally rarest. The consequences are, among others, slower downloads and stalled transfers. Consider file at a server that consists of N packets, and a client that chooses a packet from the server uniformly at random with replacement. Then the process of downloading the file is the classical coupon collection, and the expected number of samplings needed to acquire all N coupons is $\mathcal{O}(N \log N)$ (see for example [1]), which does not scale well for large N(large files).

Randomized coding systems, such as Microsoft's Avalanche [2], attempt to lessen such problems in the following way. Instead of distributing the original file fragments, peers produce linear combinations of the fragments they already hold. These combinations are distributed together with a tag that describes the coefficients used for the combination. When a peer has enough linearly independent combinations of the

original fragments, it can decode and build up the original file. The information packets can be decoded when N linearly independent equations have been collected. For a large field, this strategy reduces the expected number of required samplings from $\mathcal{O}(N \log N)$ to almost N (see for example [3]). However, the expense to pay is the computational complexity. If the information packets consist of d symbols in GF(q), it takes $\mathcal{O}(Nd)$ operations in GF(q) to form a linear combination per coded packet, and $\mathcal{O}(N^3 + N^2d)$ operations, or, equivalently, $\mathcal{O}(N^2 + Nd)$ operations per information packet, to decode the information packets by solving linear equations.

We refer to the number of information packets combined in a coded packet as the degree of the coded packet. The complexity of computations required for solving the equations of information packets depends on the average degree of coded packets. To reduce computational complexity while maintaining the throughput gain brought by coding, several approaches have been introduced seeking to decrease the average degree of coded packets [4], [5], [6]. Nevertheless, it is hard to design distributed coding schemes with good throughput/complexity tradeoff that further combine coded packets.

Chou et al. [7] proposed to partition information packets into disjoint *generations*, and combine only packets from the same generation. The performance of codes with random scheduling of disjoint generations was first theoretically analyzed by Maymounkov et al. in [8], who referred to them as *chunked codes*. Chunked codes allow convenient encoding at intermediate nodes, and are readily suitable for peer-to-peer file dissemination. In [8], the authors used an adversarial schedule as the network model and measured the code performance by estimating the loss of "code density" as packets are combined at intermediate nodes throughout the network.

In this work, we propose a way to analyze coding with generations from a coupon collection perspective. Here, we view the generations as coupons, and model the receiver who needs to acquire multiple linear equations for each generation as a collector seeking to collect multiple copies of the same coupon. This collecting model is sometimes refereed to as the *collector's brotherhood* problem, as in [9]. As a classical probability model which studies random sampling of a population of distinct elements (coupons) with replacement [1], the *coupon collector's* problem finds application in a wide range of fields [10], from testing biological cultures for contamination [11] to probabilistic-packet-marking (PPM) schemes for IP traceback problem in the Internet [12]. We here use the ideas from the collector's brotherhood problem, to derive refined

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results for expected throughput for finite information length in unicast scenarios. We describe the tradeoff in throughput versus computational complexity of coding over generations. Our results include the asymptotic code performance, by proportionally increasing either the generation size or the number of generations. The code performance mean concentration leads us to a lower bound for the probability of decoding error. Our paper is organized as follows: Sec. II describes the unicast file distribution with random scheduling and introduces pertaining results under the coupon collector's model. Sec. III studies the throughput performance of coding with disjoint generations, including the expected performance and the probability of decoding failure. Sec. IV concludes.

II. CODING OVER GENERATIONS

A. Coding over Generations in Unicast

We consider the transmission of a file from a source to a receiver over a unicast link using network coding over generations.

The file is divided into N information packets, p_1 , p_2 , ..., p_N . Each packet $p_i(i = 1, 2, ..., N)$ is represented as a column vector of d information symbols in Galois field GF(q). Packets are sequentially partitioned into $n = \frac{N}{h}$ generations of size h, denoted as $G_1, G_2, ..., G_n$. $G_j = [p_{(i-1)h+1}, p_{(i-1)h+2}, ..., p_{ih}]$ for j = 1, 2, ..., n. The receiver collects coded packets from the n generations. The coding scheme is described as follows:

a) Encoding: In each transmission, the source first selects one of the n generations with equal probability. Assume G_j is chosen. Then the source chooses a coding vector e of length h, with each entry chosen independently and equally probably from GF(q). A new packet \bar{p} is then formed by linearly combining packets from G_j by e: $\bar{p} = G_j e$. The coded packet \bar{p} is then sent over the communication link to the receiver along with the coding vector e and the generation index j.

b) Decoding: The receiver gathers coded packets and their coding vectors. We say that a receiver collects one more degree of freedom for a generation if it receives a coded packet from that generation with a coding vector linearly independent of previously received coding vectors from that generation. Once the receiver has collected h degrees of freedom for a certain generation, it can decode that generation by solving a system of h linear equations in h unknowns over GF(q).

Note that the two extremes, generation sizes h = N and h = 1, correspond respectively to full random linear network coding(i.e. without generations) and not using coding at all(but with random piece selection).

Since in the coding scheme described above, both the coding vector and the generation whose packets are combined are chosen uniformly at random, the code is inherently rateless. We measure the throughput by the number of coded packets necessary for decoding all the information packets.

c) Computational Complexity: It takes $\mathcal{O}(hd)$ operations to form each linear combination of h length-d vectors(packets) in GF(q). The computational cost for encoding is then $\mathcal{O}(hd)$ per coded packet. Meanwhile, it takes $\mathcal{O}(h^3 + h^2 d)$ operations in GF(q) to solve h linearly independent equations for the h packets of one generation. Thus, the cost for decoding is $\mathcal{O}(h^2 + hd)$ per information packet.

B. Extension to a General Network

In a general network, intermediate nodes follow a similar coding scheme except that previously received coded packets are combined instead of the original information packets. We leave out the details since the analysis of code performance in general network topologies is beyond the scope of this paper.

C. Collector's Problem and Coding Over Generations

In a *coupon collector's problem*, a set of distinct coupons are sampled with replacement. Consider the n generations as n distinct coupons, and collecting degrees of freedom for generation G_i is analogous to collecting copies of the *i*th element in the coupon set. In the next section, we will characterize the throughput performance of the code under the coupon collector's probability model.

III. THROUGHPUT OF CODING OVER GENERATIONS

We next study the throughput performance of coding over generations in the simplest single-server-single-user scenario by using the *collector's brotherhood* model [9]. Recall that, to successfully recover the file, the receiver has to collect hdegrees of freedom for each generation. For i = 1, ..., n, let N_i be the number of coded packets sampled from G_i until hdegrees of freedom are collected. Then, N_i s are i.i.d. random variables with the expected value [3]

$$E[N_i] = \sum_{j=0}^{h-1} \frac{1}{1 - q^{j-h}}.$$
(1)

Approximating summation by integration, from (1) we get

$$\begin{split} E[N_i] &\lesssim \int_0^{h-1} \frac{1}{1-q^{x-h}} dx + \frac{1}{1-q^{-1}} \\ &= h + \frac{q^{-1}}{1-q^{-1}} + \log_q \frac{1-q^{-h}}{1-q^{-1}}. \end{split}$$

Note that $N_i \leq s$ means the $h \times s$ matrix formed by s coding vectors of length h as columns is of full row rank. For s < h, $\Pr[N_i \leq s] = 0$. For $s \geq h$, $\Pr[N_i \leq s]$ equals the probability that for each k = 1, 2, ..., h, the kth row in the matrix is linearly independent of rows 1 through (k - 1). Hence,

$$\Pr[N_i \le s] = \prod_{k=0}^{h-1} \left(\left(q^s - q^k \right) / q^s \right) = \prod_{k=0}^{h-1} (1 - q^{k-s}).$$
(2)

We have the following Lemma 1 upper bounding the complementary cumulative distribution function (CCDF) of N_i .

Lemma 1: There exist positive constants $\alpha_{q,h}$ and $\alpha_{2,\infty}$ such that, for $s \ge h$,

$$\Pr[N_i > s] = 1 - \prod_{k=0}^{h-1} (1 - q^{k-s})$$

< $1 - \exp(-\alpha_{q,h}q^{-(s-h)}) < 1 - \exp(-\alpha_{2,\infty}q^{-(s-h)}).$

Proof: Please refer to Appendix. Let T(n,m) be the number of coded packets collected when for the first time there are at least $m(\geq 1)$ coded packets from every generation in the collection at the receiver. The total number of coded packets needed for accumulating $N_i(\geq h)$ coded packets of each generation G_i is then greater or equal to T(n,h).

The collector's brotherhood problem[9], also referred to as the double dixie cup problem[13], investigates the stochastic quantities associated to acquiring $m(\geq 1)$ complete sets of n distinct elements by random sampling.

A. Results From The Collector's Brotherhood Problem

For any $m \in \mathbb{N}$, we define $S_m(x)$ as follows:

$$S_m(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{m-1}}{(m-1)!} \quad (m \ge 1)$$
 (3)

$$S_{\infty}(x) = \exp(x) \text{ and } S_0(x) = 0.$$
 (4)

Theorem 2: Consider uniformly random sampling of n distinct coupons with replacement. Suppose for some $A \in \mathbb{N}$, integers k_1, \ldots, k_A and m_1, \ldots, m_A satisfy $1 \leq k_1 < \cdots < k_A \leq n$ and $m_1 > \cdots > m_A \geq 1$. For convenience of notation, let $m_0 = \infty$ and $m_{A+1} = 0$. Then, the expected number of samplings needed to acquire at least m_1 copies of at least k_1 coupons, at least m_2 copies of at least k_2 coupons, and so on, at least m_A copies of at least k_A coupons in the collection is

$$n \int_{0}^{\infty} \left\{ e^{nx} - \left(5 \right) \right\}_{\substack{(i_0,i_1,\dots,i_{A+1}):\\i_0=0,i_{A+1}=n\\k_j \le i_j \le i_{j+1}\\j=1,2,\dots,A}} \prod_{j=0}^{A} \binom{i_{j+1}}{i_j} \left[S_{m_j}(x) - S_{m_{j+1}}(x) \right]^{i_{j+1}-i_j} \right\} e^{-nx} dx$$

Proof: Our proof generalizes the symbolic method of [13]. Please refer to Appendix.

Setting A = 1, $k_1 = k$ and $m_1 = m$ in Theorem 2 gives the following corollary:

Corollary 3: The expected number of samplings needed to collect at least k of the n distinct coupons for at least m times is $n \int_0^\infty \left\{ \sum_{i=0}^{k-1} {n \choose i} S_m^{n-i}(x) \left[e^x - S_m(x) \right]^i \right\} e^{-nx} dx$. In the context of coding over generations, this corollary

In the context of coding over generations, this corollary gives us an estimation of the growth of the size of decodable information. It is also helpful to the study of a coding scheme with a "precode" as discussed in [8].

Furthermore, by setting k = n in Corollary 3, we recover the following result of [13]:

Corollary 4: The expected sampling size to acquire m complete sets of n coupons is

$$E[T(n,m)] = n \int_0^\infty \left[1 - (1 - S_m(x)e^{-x})^n \right] dx \quad (6)$$

(6) can be numerically evaluated for finite m and n.

The asymptotic of E[T(n,m)] for large n has been discussed in literature such as [13], [14] and [15].

Theorem 5: ([14]) When $n \to \infty$,

$$E[T(n,m)] = n \log n + (m-1)n \log \log n + C_m n + o(n),$$

where $C_m = \gamma - \log(m-1)!$, γ is Euler's constant and $m \in \mathbb{N}$. For $m \gg 1$, on the other hand, we have [13]

$$E[T(n,m)] \to nm.$$
 (7)

What is worth mentioning is that, as the number of coupons $n \to \infty$, for the first complete set of coupons, the number of samplings needed is $\mathcal{O}(n \log n)$, while the additional number of samplings needed for each additional set is only $\mathcal{O}(n \log \log n)$.

In addition to the expected value of T(n, m), the concentration of T(n, m) around its mean is also of great interest to us. We can derive from it an estimate of the probability of successful decoding after gathering a certain number of coded packets. The generating function of T(n, m) and its probability distribution are given in [9], but it is quite difficult to evaluate them numerically. We will instead look at the asymptotic case where the number of coupons $n \to \infty$. Erdös and Rényi have proven in [16] the limit law of T(n, m) as $n \to \infty$. Here we restate Lemma B from [14] by Flatto, which in addition expresses the rate of convergence to the limit law. Lemma 6: [14] Let

$$Y(n,m) = \frac{1}{n} (T(n,m) - n \log n - (m-1)n \log \log n).$$

Then,

$$\Pr[Y(n,m) \le y] = \exp\left(-\frac{e^{-y}}{(m-1)!}\right) + \mathcal{O}\left(\frac{\log\log n}{\log n}\right).$$

Remark 1: (Remark 2, [14]) The estimation in Lemma 6 is understood to hold uniformly on any finite interval $-a \le y \le a$. i.e., for any a > 0,

$$\Pr[Y(n,m) \le y] - \exp\left(-\frac{\exp(-y)}{(m-1)!}\right) \le C(m,a) \frac{\log\log n}{\log n},$$

 $n \ge 2$ and $-a \le y \le a$. C(m, a) is a positive constant depending on m and a, but independent of n.

B. Throughput

The number of coded packets T the receiver needs to collect for successful decoding is lower bounded by T(n, h), and so E[T] is lower bounded by E[T(n, h)]. Also, by Lemma 1, N_i is well concentrated near h for large q, and so T(n, h) could be a good estimate for T for finite n. In addition, from Thm. 5 and (7) we observe that, when $n \gg 1$ or $m \gg 1$, E[T(n, m)]is linear in m. Thus, we could substitute $E[N_i]$ for m in these expressions to roughly estimate the asymptotic expected number of coded packets needed for successful decoding.

From Lemma 6, we obtain the following lower bound to the probability of decoding failure as $n \to \infty$:

Theorem 7: When $n \to \infty$, the probability of decoding failure when t coded symbols have been collected is greater than $1 - \exp\left[-\frac{1}{(h-1)!}n(\log n)^{h-1}\exp\left(-\frac{t}{n}\right)\right] + \mathcal{O}\left(\frac{\log \log n}{\log n}\right)$.

Proof: The probability of decoding failure after acquiring t coded packets equals Pr[T > t]. Since $T \ge T(n, h)$,

$$\begin{aligned} \Pr[T > t] \geq \Pr[T(n, h) > t] \\ = 1 - \Pr\left[Y(n, h) \leq \frac{t}{n} - \log n - (m - 1) \log \log n\right] \end{aligned}$$

The result in Theorem 7 follows directly from Lemma 6.



Fig. 1. (a)Estimates of T, the number of coded packets needed for successful decoding when the total number of information packets N = 1000: E[T(n,h)]; average of T in simulation (q = 256); $n \to \infty$ asymptotic Theorem 5; $m \gg 1$ asymptotics (7). (b)Estimates of probability of decoding failure versus the number of coded packets collected: Theorem 7 along with simulation results (q = 256).

Figure 1(a) shows several estimates of T for fixed N = nh = 1000 versus generation size h. It is worth noting that, for fixed N, E[T(n, h)] drops significantly as h is increased to a relatively small value. For the example N = 1000, when each generation includes 1/10 of all the information packets, the expected overhead required for successful decoding is below 16%. In such scenarios as assumed in our model, where feedback is minimal, the benefit of coding on throughput is pronounced even when it is done in relatively small information block length. In effect, communication overhead due to the exchange of control messages makes way for moderate computational complexity at individual nodes.

This is particularly meaningful to communication networks in shared media, in which contention occurs frequently, and also to networks of nodes with medium computing power.

Figure 1(b) shows the estimate of the probability of decoding failure versus T. As pointed out in Remark 1, the deviation of the CDF of T(n, m) from the limit law for $n \to \infty$ depends on m and is on the order of $\mathcal{O}(\frac{\log \log n}{\log n})$, which is quite slow. This is also implied in our observation from Figure 1(a) that Thm. 5 gives a good estimate of E[T(n, m)] only for very small values of m(compared to n).

IV. CONCLUSION AND FUTURE WORK

We investigated the throughput performance of coding over generations in the unicast scenario under the classical yet everuseful coupon collector's model. We derived a general formula (Theorem 2) to compute the expected number of samplings necessary for collecting multiple copies of n distinct coupons. The formula can be applied in various ways to analyze a number of different aspects of coding over generations. Here in particular, we used a special case of this result, namely, the expected number of samplings E[T(n, h)] needed to collect h complete sets of n distinct coupons to estimate the expected number of coded packets necessary for successful decoding when N information packets are encoded over n generations of size h each. Apart from results for finite information length N, the asymptotics of E[T(n, h)] can also be used to estimate the performance of coding when either the number of generations n or the generation size h go to inifinity. We also gave a lower bound for the probability of decoding failure by using the limit law for T(n,h) as $n \to \infty$.

The general result expressed in Theorem 2 has proved its usefulness to the analysis of coding over overlapping generations in our recent work [17]. Further, we have been able to derive in [18] the exact expression, as well as an upper bound, for the expected number of coded packets necessary for successful decoding. We expect to extend our analysis under the coupon collection framework to many other aspects of coding over generations.

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APPENDIX

Proof of Lemma 1

For $i = 1, 2, \ldots, n$ and any $s \ge h$, we have

$$\ln \Pr\{N_i \le s\} = \sum_{k=0}^{h-1} \ln(1-q^{k-s}) = -\sum_{k=0}^{h-1} \sum_{j=1}^{\infty} \frac{1}{j} q^{(k-s)j}$$
$$= -\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=0}^{h-1} q^{j(k-s)} = -\sum_{j=1}^{\infty} \frac{1}{j} q^{-js} \frac{q^{jh}-1}{q^j-1}$$
$$= -q^{-(s-h)} \sum_{j=1}^{\infty} \frac{1}{j} q^{-(j-1)(s-h)} \frac{1-q^{-jh}}{q^j-1}$$
$$> q^{-(s-h)} \sum_{j=1}^{\infty} \frac{1}{j} \frac{1-q^{-jh}}{1-q^j} = q^{-(s-h)} \ln \Pr\{N_i \le h\}$$
$$> q^{-(s-h)} \lim_{h \to \infty, q=2} \ln \Pr\{N_i \le h\}$$

The claim is obtained by setting

$$\alpha_{q,h} = -\ln \Pr\{N_i \le h\}, \ \alpha_{2,\infty} = -\lim_{h \to \infty, q=2} \ln \Pr\{N_i \le h\}.$$

Proof of Theorem 2

Our proof generalizes the symbolic method of [13]. Let E be the event that, in the acquired collection, there are at least m_1 copies of at least k_1 coupons, at least m_2 copies of at least k_2 coupons, and so on, at least m_A copies of at least k_A coupons. For $t \ge 0$, let E(t) be the event that E has occurred after tsamplings, and let $\mathbb{1}_{\bar{E}(t)}$ be the indicator that takes value 1 if E(t) does not occur and 0 otherwise. Then random variable $W = \mathbb{1}_{\bar{E}(0)} + \mathbb{1}_{\bar{E}(1)} + \ldots$ equals the waiting time for E to occur. For $t \ge 0$, let $\pi_t = \operatorname{Prob}[\bar{E}(t)]$; we then have

$$E[W] = \sum_{t \ge 0} \pi_t.$$
(8)

To derive π_t , we introduce an operator f acting on an n-variable polynomial g. For a monomial $x_1^{w_1} \dots x_n^{w_n}$, let i_j be the number of exponents w_u among w_1, \dots, w_n satisfying $w_u \ge m_j$, for $j = 1, \dots, A$. f removes all monomials $x_1^{w_1} \dots x_n^{w_n}$ in g satisfying $i_1 \ge k_1, \dots, i_A \ge k_A$ and $i_1 \le \dots \le i_A$. Note that f is a linear operator, i.e., if g_1 and g_2 are two polynomials in the same n-variables, and a and b two scalars, we have $af(g_1) + bf(g_2) = f(ag_1 + bg_2)$.

Each monomial in $(x_1 + \cdots + x_n)^t$ corresponds to one of the n^t possible outcomes after t samplings, with the exponent of x_i being the number of collected copies of the *i*th coupon. Hence, the number of outcomes counted in $\overline{E}(t)$ equals $f((x_1 + \cdots + x_n)^t)$ evaluated at $x_1 = \cdots = x_n = 1$.

$$\pi_t = \frac{f((x_1 + \dots + x_n)^t)}{n^t}|_{x_1 = \dots = x_n = 1}$$

and thus, by (8), $E[W] = \sum_{t\geq 0} \frac{f((x_1+\dots+x_n)^t)}{n^t}|_{x_1=\dots=x_n=1}$. Making use of the identity $\frac{1}{n^t} = n \int_0^\infty \frac{1}{t!} y^t e^{-ny} dy$, and because of the linearity of operator f, we further have

$$E[W] = n \int_{0}^{\infty} \sum_{t \ge 0} \frac{f\left((x_{1} + \dots + x_{n})^{t}\right)}{t!} y^{t} e^{-ny} dy$$
$$= n \int_{0}^{\infty} f\left(\sum_{t \ge 0} \frac{(x_{1}y + \dots + x_{n}y)^{t}}{t!}\right) e^{-ny} dy$$
$$= n \int_{0}^{\infty} f\left(\exp(x_{1}y + \dots + x_{n}y)\right) e^{-ny} dy \qquad (9)$$

evaluated at $x_1 = \cdots = x_n = 1$.

We next need to find the sum of the monomials in the polynomial expansion of $\exp(x_1 + \cdots + x_n)$ that should be removed under f. If we choose integers $0 = i_0 \le i_1 \le \cdots \le i_A \le i_{A+1} = n$, such that $i_j \ge k_j$ for $j = 1, \ldots, A$, and then partition indices $\{1, \ldots, n\}$ into (A+1) subsets $\mathcal{I}_1, \ldots, \mathcal{I}_{A+1}$, where $\mathcal{I}_j(j = 1, \ldots, A+1)$ has $i_j - i_{j-1}$ elements. Then

$$\prod_{j=1}^{A+1} \prod_{i \in \mathcal{I}_j} (S_{m_{j-1}}(x_i) - S_{m_j}(x_i))$$
(10)

equals the sum of all monomials in $\exp(x_1 + \dots + x_n)$ with $(i_j - i_{j-1})$ of the *n* exponents smaller than m_{j-1} but greater than or equal to m_j , for $j = 1, \dots, A+1$. (Here *S* is as defined by (3-4).) The number of such partitions of $\{1, \dots, n\}$ is equal to $\binom{n}{n-i_A, \dots, i_2-i_1, i_1} = \prod_{j=0}^{A} \binom{i_{j+1}}{i_j}$. Finally, we need to sum the terms of the form (10) over all partitions of all choices of i_1, \dots, i_A satisfying $k_j \leq i_j \leq i_{j+1}$ for $j = 1, \dots, A$:

$$f\left(\exp(x_{1}y + \dots + x_{n}y)\right)|_{x_{1}=\dots=x_{n}=1} = \exp(ny) - \sum_{\substack{(i_{0},i_{1},\dots,i_{A+1}=n\\i_{0}=0,i_{A+1}=n\\k_{j}\leq i_{j}\leq i_{j+1}\\j=1,2,\dots,A}} \prod_{j=0}^{A} \binom{i_{j+1}}{i_{j}} \left[S_{m_{j}}(y) - S_{m_{j+1}}(y)\right]^{i_{j+1}-i_{j}}.$$
(11)

Bringing (11) into (9) gives our result in Theorem 2.