Superposition Noisy Network Coding

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***Abstract*—We present a superposition coding scheme for communication over a network, which combines partial decode and forward and noisy network coding. This hybrid scheme is termed as superposition noisy network coding. The scheme is designed and analyzed for single relay channel, single source multicast network and multiple source multicast network. The achievable rate region is determined for each case. The special cases of Gaussian single relay channel and two way relay channel are analyzed for superposition noisy network coding. The achievable rate of the proposed scheme is higher than the existing schemes of noisy network coding and compress-forward.**

I. Introduction

IN an N-node Discrete Memoryless Network (DMN), each node transmits its message to a set of destination nodes and acts as a relay to help transmit messages from other nodes. It is an important network model in multi-user information theory. This general network model includes many important class of channels as special cases. Noiseless, erasure and deterministic networks are few examples [1], [2], [3]. The DMN also includes the relay, broadcast, interference and multiple access channels which are the fundamental building blocks for any network. This DMN model can also be modified to include Gaussian networks and networks with state.

The capacity of the discrete memoryless network is not known in general. The best known upper bound is the cut-set bound [4]. Cover and El Gamal [5] introduced coding schemes for the general discrete memoryless single relay channel. The schemes introduced in brief are decode-forward, compress-forward and superposition-forward. Superposition-forward is the combination of decode-forward and compress-forward. Decode-forward and compress-forward are special cases of superposition-forward. The superposition-forward scheme achieves the optimal rate for all the special cases where capacity is known.

Lim et al. [6] introduced a general lower bound for the discrete memoryless network using the equivalent characterization of compress-forward. This new scheme is termed "noisy network coding". Noisy network coding combines network coding [7] with the compress-forward scheme. The key ideas used are message repetition encoding, no Wyner-Ziv [8] binning at the relay and joint decoding at the destination. This scheme achieves a higher rate than the better known compress-forward scheme for networks with multiple relays [9]. The noisy network coding scheme naturally extends to single and multiple source multicast networks.

In this paper, we improve the achievable rates of the noisy network coding scheme by allowing the nodes to decode a part of message and use it to make a better compressed signal to be relayed. The superposition noisy network coding scheme combines superposition-forward with network coding. Modifications are made to the superposition-forward scheme to make it applicable to the network coding scenario. Specifically, the input distributions at each node are chosen to be independent.

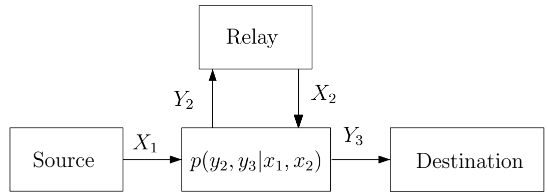

Fig. 1. The three node relay channel

Superposition noisy network coding splits the message at each node into two parts. A part of the message is required to be decoded at each relay after every block. The other part of the message is transmitted over b blocks using repetition coding. The relay nodes use compress-forward to transmit this message. The destination nodes decode the messages after b blocks of transmission using joint decoding. Similar to noisy network coding, our scheme does not use Wyner-Ziv encoding at the relay, employs repetition encoding and joint decoding. These techniques have been shown to improve the achievable rates as in the case of network coding.

For simplicity and ease of understanding, the superposition noisy network coding scheme is explained for a simple 3 node relay channel first. In section II, the scheme is designed and achievable rates derived for a single relay channel. In Section III, the scheme is further extended to single source multicast network where there is only a single source node transmitting information to a set of destination nodes. All other nodes can act as relays. In Section IV, the superposition noisy network coding scheme is designed for multiple source multicast networks. This scheme is then applied to AWGN single and two-way relay channel to quantify performance.

II. Superposition noisy network coding for single relay channel

Consider the discrete memoryless relay channel $p(y_2, y_3|x_1, x_2)$ shown in Fig. 1. The source node is terminal 1, relay node is terminal 2, and terminal 3 is the destination node. x_k denotes the transmitted symbol at terminal k. y_k denotes the received symbol at terminal k.

The rate achieved by superposition-forward scheme [5, Theorem 7] for discrete memoryless relay channel is

$$R_{SF} = \sup(\min\{I(X_1; Y_3, \hat{Y}_2|X_2, U) + I(U; Y_2|X_2, V),$$
$$I(X_1, X_2; Y_3) - I(\hat{Y}_2; Y_2|U, X_1, X_2, Y_3)\}) \quad (1)$$

where the supremum is over all joint probability distributions of the form

$$p(u, v, x_1, x_2, \hat{y}_2, y_3, y_2) = p(v)p(u|v)p(x_1|u)p(x_2|v)p(y_2, y_3|x_1, x_2)p(\hat{y}_2|x_2, y_2, u) \quad (2)$$

subject to the constraint

$$I(X_2;Y_3|V) \geq I(\hat{Y}_2;Y_2|X_2,Y_3,U). \tag{3}$$

To facilitate network coding we will restrict the superposition-forward strategy such that the auxiliary random variables U and V are generated independent of each other. This will lead to a rate loss compared to original scheme. Nevertheless, the rates achieved would be higher than the compress-forward or noisy network coding scheme.

The rate achieved by superposition noisy network coding scheme for a single relay channel is stated in Theorem 1.

Theorem 1: For any discrete memoryless relay channel, the rate $\sup_P R' + R''$ is achievable, where

$$R' < \min\{I(U_1;Y_2|X_2),\ I(U_1,V_2;Y_3)\}, \tag{4}$$

$$R'' < \min\{I(X_1;\hat{Y}_2,Y_3|X_2,U_1),\ I(X_1,X_2;Y_3|U_1,V_2) - I(\hat{Y}_2;Y_2|U_1,X_1,X_2,Y_3)\}, \tag{5}$$

and the supremum is taken over all joint probability distributions of the form

$$p(u_1,v_2,x_1,x_2,y_2,y_3,\hat{y}_2) = p(u_1)p(v_2)p(x_1|u_1)\cdot p(x_2|v_2)p(y_2,y_3|x_1,x_2)p(\hat{y}_2|x_2,y_2,u_1) \tag{6}$$

Proof: The message $m' \in [1:2^{nR'}]$ is transmitted over every block j and the message $m'' \in [1:2^{nbR''}]$ is transmitted over b blocks of transmission. The source node transmits $\mathbf{x}_{1j}(m''|m'_j)$ for each block $j \in [1:b]$. After block j, the relay decodes the message m'_j and maps it to a $\mathbf{v}_{2,j+1j}(m'_j)$ codeword. It also finds a "compressed" version $\hat{\mathbf{y}}_{2j}(l_j|l_{j-1},m'_j,m'_{j-1})$ of the relay output $\mathbf{y}_{2j}$ conditioned on $\mathbf{x}_{2j}$ and $\mathbf{u}_{1j}$. The relay transmits a codeword $\mathbf{x}_{2,j+1}(l_j|m'_j)$ in the next block. After b blocks of transmission, the decoder finds the correct message $m'' \in [1:2^{nbR}]$ using $(\mathbf{y}_{31},\ldots,\mathbf{y}_{3b})$ and joint decoding for each of the b blocks simultaneously. The decoder has decoded all the messages m'_j by sliding window decoding for each block $j \in [1:b]$. The details are as follows.

Codebook generation: Fix $p(u_1)p(x_1|u_1)p(v_2)p(x_2|v_2)\cdot p(\hat{y}_2|y_2,x_2,u_1)$.

1) For each $j \in [1:b]$, randomly and independently generate $2^{nR'}$ sequences $\mathbf{u}_{1j}(m'_j)$, $m' \in [1:2^{nR'}]$, each according to $\prod_{i=1}^{n} p_{U_1}(u_{1,(j-1)n+i})$.
2) For each $\mathbf{u}_{1j}(m'_j)$, randomly and independently generate $2^{nbR''}$ sequences $\mathbf{x}_{1j}(m''|m'_j)$, $m'' \in [1:2^{nbR''}]$, each according to $\prod_{i=1}^{n} p_{X_1|U_1}(x_{1,(j-1)n+i}|u_{1,(j-1)n+i}(m'_j))$.
3) Similarly, randomly and independently generate $2^{nR'}$ sequences $\mathbf{v}_{2j}(m'_{j-1})$, $m'_{j-1} \in [1:2^{nR'}]$, each according to $\prod_{i=1}^{n} p_{V_2}(v_{2,(j-1)n+i})$.
4) For each $\mathbf{v}_{2j}(m'_{j-1})$, randomly and independently generate $2^{n\hat{R}_2}$ sequences $\mathbf{x}_{2j}(l_{j-1}|m'_{j-1})$, $l_{j-1} \in [1:2^{n\hat{R}_2}]$, $m'_{j-1} \in [1:2^{nR'}]$, each according to $\prod_{i=1}^{n} p_{X_2|V_2}(x_{2,(j-1)n+i}|v_{2,(j-1)n+i}(m'_{j-1}))$.
5) For each $\mathbf{x}_{2j}(l_{j-1}|m'_{j-1})$, $l_{j-1} \in [1:2^{n\hat{R}_2}]$ and $\mathbf{u}_{1j}(m'_j)$, $m'_j, m'_{j-1} \in [1:2^{nR'}]$, randomly and independently generate $2^{n\hat{R}_2}$ sequences $\hat{\mathbf{y}}_{2j}(l_j|l_{j-1},m'_{j-1},m'_j)$, $l_j \in [1:2^{n\hat{R}_2}]$, each according to $\prod_{i=1}^{n} p_{\hat{Y}_2|X_2,U_1}(\hat{y}_{2,(j-1)n+i}|x_{2,(j-1)n+i}(l_{j-1},m'_{j-1}), u_{1,(j-1)n+i}(m'_j))$.

This defines the codebook

$$\mathcal{C}_j = \big\{\mathbf{u}_{1j}(m'_j), \mathbf{v}_{1j}(m'_{j-1}), \mathbf{x}_{1j}(m''|m'_j), \mathbf{x}_{2j}(l_{j-1}|m'_{j-1}), \hat{\mathbf{y}}_{2j}(l_j|l_{j-1},m'_{j-1},m'_j) : m'_j, m'_{j-1} \in [1:2^{nR'}], m'' \in [1:2^{nbR''}], l_j, l_{j-1} \in [1:2^{n\hat{R}_2}]\big\} \tag{7}$$

for $j \in [1:b]$.

Encoding and decoding are explained with the help of Table I.

Encoding: Let m'_j be the message to be sent in block j and m'' be the message to be sent over b blocks. The relay, upon receiving $\mathbf{y}_{2j}$ at the end of block $j \in [1:b]$, finds an index m'_j such that

$$(\mathbf{u}_{1j}(m'_j), \mathbf{y}_{2j}, \mathbf{x}_{2j}(l_{j-1}|\hat{m}'_{j-1})) \in \mathcal{T}_{\epsilon'}^{(n)},$$

and then finds an index l_j such that

$$(\mathbf{u}_{1j}(\hat{m}'_{j-1}), \hat{\mathbf{y}}_{2j}(l_j|l_{j-1},\hat{m}'_j,\hat{m}'_{j-1}), \mathbf{y}_{2j}, \mathbf{x}_{2j}(l_{j-1}|\hat{m}'_{j-1})) \in \mathcal{T}_{\epsilon'}^{(n)},$$

where $l_0 = 1$ by convention. If there is more than one such index, choose one of them at random. If there is no such index, choose an arbitrary index at random from $[1:2^{n\hat{R}_2}]$. The codeword pair $(\mathbf{x}_{1j}(m''|m'_j), \mathbf{x}_{2j}(l_{j-1}|m'_{j-1}))$ is transmitted in block $j \in [1:b]$.

Decoding: Let $\epsilon > \epsilon'$. After block j, the decoder uses $\mathbf{y}_{3(j-1)}$ and $\mathbf{y}_{3j}$ to find a unique message $\hat{m}'_{j-1} \in [1:2^{nR'}]$. The unique message satisfies the following two conditions simultaneously

$$\begin{aligned}(\mathbf{u}_{1j}(m'_{j-1}), \mathbf{v}_{2j}(\hat{m}'_{j-2}), \mathbf{y}_{3j-1}) &\in \mathcal{T}_{\epsilon}^{(n)}\\ (\mathbf{v}_{2j}(m'_{j-1}), \mathbf{y}_{3j}) &\in \mathcal{T}_{\epsilon}^{(n)}\end{aligned}$$

At the end of block b, after decoding all the messages m'_j, $j \in [1:(b-1)]$ the decoder finds a unique message $\hat{m}'' \in [1:2^{nbR''}]$ such that

$$(\mathbf{u}_{1j}(\hat{m}'_j), \mathbf{v}_{2j}(\hat{m}'_{j-1}), \hat{\mathbf{y}}_{2j}(l_j|l_{j-1},\hat{m}'_{j-1},\hat{m}'_j), \mathbf{x}_{1j}(m''_j|\hat{m}'_{j-1}), \mathbf{x}_{2j}(l_{j-1}|\hat{m}'_{j-1}), \mathbf{y}_{3j}) \in \mathcal{T}_{\epsilon}^{(n)}, \text{ for all } j \in [1:b]$$

for some $l_1, l_2, \ldots, l_b$. If there is none or more than one such message, it declares an error.

Analysis of the probability of error: Let M'_j, denote the messages sent at the source node for $j \in [1:(b-1)]$, M'' be the message sent at the source node over b blocks and L_j denote the indices chosen by the relay at block $j \in [1:b]$. Define

$$\mathcal{E}_{m'(0)} := \bigcup_{j=1}^{b} \big\{(\mathbf{U}_{1j}(m'_j), \mathbf{Y}_{2j}, \mathbf{X}_{2j}(l_{j-1}|m'_{j-1})) \notin \mathcal{T}_{\epsilon'}^{(n)}\big\},$$

$$\mathcal{E}_{m'(1)} := \big\{(\mathbf{u}_{1,j-1}(m'_{j-1}), \mathbf{v}_{2,j-1}(\hat{m}'_{j-2}), \mathbf{y}_{3j-1}) \in \mathcal{T}_{\epsilon}^{(n)}, j \in [1:b]\big\} \bigcup \big\{(\mathbf{v}_{2j}(m'_{j-1}), \mathbf{y}_{3j}) \in \mathcal{T}_{\epsilon}^{(n)}, j \in [1:b]\big\}.$$

To bound the probability of error in decoding message m'_{j-1}, assume without loss of generality that $M'_{j-2} = M'_{j-1} = 1$. The probability of error is upper bounded by

$$\mathsf{P}(\mathcal{E}) \leq \mathsf{P}(\mathcal{E}_{m'(0)}) + \mathsf{P}(\mathcal{E}^c_{m'(0)} \cap \mathcal{E}^c_{m'(1)} \cap M'_{j-1} = 1) + \mathsf{P}(\mathcal{E}^c_{m'(0)} \cap \mathcal{E}_{m'(1)} \cap M'_{j-1} \neq 1).$$

By the conditional typicality lemma [10], $\mathsf{P}(\mathcal{E}_{m'(0)}) \to 0$ as $n \to \infty$ if $R' < I(U_1;Y_2|X_2)$ for sufficiently large n,

Block	1	2	3	$\cdots$	$b-1$	b
U_1	$\mathbf{u}_{11}(m'_1)$	$\mathbf{u}_{12}(m'_2)$	$\mathbf{u}_{13}(m'_3)$	$\cdots$	$\mathbf{u}_{1,b-1}(m'_{b-1})$	$\mathbf{u}_{1b}(m'_b)$
V_2	$\mathbf{v}_{21}(1)$	$\mathbf{v}_{22}(m'_1)$	$\mathbf{v}_{23}(m'_2)$	$\cdots$	$\mathbf{v}_{2,b-1}(m'_{b-2})$	$\mathbf{v}_{2b}(m'_{b-1})$
X_1	$\mathbf{x}_{11}(m''\|m'_1)$	$\mathbf{x}_{12}(m''\|m'_2)$	$\mathbf{x}_{13}(m''\|m'_3)$	$\cdots$	$\mathbf{x}_{1,b-1}(m''\|m'_{b-1})$	$\mathbf{x}_{1b}(m''\|m'_b)$
$\hat{Y}_2$	$\hat{\mathbf{y}}_{21}(l_1\|1,1,m'_1)$	$\hat{\mathbf{y}}_{22}(l_2\|l_1,m'_1,m'_2)$	$\hat{\mathbf{y}}_{23}(l_3\|l_2,m'_2,m'_3)$	$\cdots$	$\hat{\mathbf{y}}_{2,b-1}(l_{b-1}\|l_{b-2},m'_{b-2},m'_{b-1})$	$\hat{\mathbf{y}}_{2b}(l_b\|l_{b-1},m'_{b-1},m'_b)$
X_2	$\mathbf{x}_{21}(1\|1)$	$\mathbf{x}_{22}(l_1\|m'_1)$	$\mathbf{x}_{23}(l_2\|m'_2)$	$\cdots$	$\mathbf{x}_{2,b-1}(l_{b-2}\|m'_{b-2})$	$\mathbf{x}_{2b}(l_{b-1}\|m'_{b-1})$
Y_3	$\emptyset$	$\hat{m}'_1$	$\hat{m}'_2$	$\cdots$	$\hat{m}'_{b-2}$	$\hat{m}'_{b-1},\hat{m}''$

TABLE I
SUPERPOSITION NOISY NETWORK CODING FOR THE RELAY CHANNEL.

and $\mathsf{P}(\mathcal{E}^c_{m'(0)} \cap \mathcal{E}^c_{m'(1)} \cap M'_{j-1} = 1) \to 0$ as $n \to \infty$. Since the codebooks are generated independently for each block, the two events of $\mathcal{E}_{m'(1)}$ are independent. Thus by the law of large numbers and joint typicality lemma [10] $\mathsf{P}(\mathcal{E}^c_{m'(0)} \cap \mathcal{E}_{m'(1)} \cap M'_{j-1} \neq 1) \to 0$ as $n \to \infty$ if

$$R' < I(U_1; Y_3|V_2) + I(V_2; Y_3) = I(U_1, V_2; Y_3)$$

and n is sufficiently large. So the message m'_j can be decoded correctly at the destination provided

$$R' < \min\{I(U_1; Y_2|X_2), I(U_1, V_2; Y_3)\}$$

After decoding the messages m'_j for $j \in [1 : (b-1)]$, the destination decodes the message M'' after b blocks. The probability of error analysis for message M'' is similar to the noisy network coding scheme [6], given the partial information of the messages m'_j. It can be shown that when

$$R'' < \min\{I(X_1; \hat{Y}_2, Y_3|X_2, U_1),\ I(X_1, X_2; Y_3|U_1, V_2) - I(\hat{Y}_2; Y_2|U_1, X_1, X_2, Y_3)\} - \delta(\epsilon) - \delta(\epsilon'),$$

the probability of error of detecting M'' can be made arbitrarily small. The probability of error analysis is omitted here due to limited space. ■

III. SUPERPOSITION NOISY NETWORK CODING FOR MULTICAST NETWORKS

We now describe the superposition noisy network coding scheme for single-source discrete memoryless networks with multicast (DMN-MC) $p(y_2, \ldots, y_N|x^N)$, where terminal 1 is the source node. We assume that there is no feedback to terminal 1. Source terminal 1 splits the message in two parts m' and m'' and transmits using superposition forwarding. The message m' is transmitted in the same fashion decode-forward is extended to multicast relay networks [11], [12]. The scheme is modified to make the input distributions at each node independent of each other. After decoding the partial information m', the message m'' is decoded using noisy network coding [6] given the partial information.

Theorem 2: For a discrete memoryless multicast network $p(y_2, \ldots, y_N|x^N)$, the rate $R' + R''$ is achievable, where

$$R' < \min_k I(V^{k-1}; Y_k|X_k, V_k^N)$$

$$R'' < \min_{\mathcal{S}} \Big(I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_k|X(\mathcal{S}^c), V) - I(\hat{Y}(\mathcal{S}); Y(\mathcal{S})|X^N, \hat{Y}(\mathcal{S}^c), Y_k, V_{k-1}^N) \Big)$$

and $k \in \mathcal{D}$ the set of destination nodes. The minimum is over all possible cut-sets for node k. The random variables satisfy a joint pmf of the form $p(v_1)p(x_1|v_1)\prod_{k=2}^N p(v_k)p(x_k|v_k)p(\hat{y}_k|y_k, x_k, v_k^N)$.

Sketch of Proof: The encoding and decoding process is similar to superposition noisy network coding for single relay channel. The relay nodes use an extension of decode-forward to multicast networks. The partial message is decoded at each of the nodes $\{1 : (k-1)\}$, and coherently transmitted to node k. The node k waits for $k-1$ transmissions to decode the partial information. After decoding the partial message, the remaining message is decoded using noisy network coding. Due to space limit, we omit details of the encoding and decoding processes, and the error probability analysis. ■

IV. SUPERPOSITION NOISY NETWORK CODING FOR MULTIPLE SOURCE MULTICAST NETWORKS

The superposition noisy network coding scheme can also be generalized to an N node discrete memoryless multiple source multicast network $p(y^N|x^N)$, [6]. In the general setup each node sends its independent message to a set of destination nodes while acting as relays for messages from other sources.

We make a general assumption to make the application of superposition noisy network coding easier. The source nodes are restricted not to act as relays. Two-way relay channel and interference relay channel are two examples where such an assumption holds. With this assumption, the channel model is now similar to single source multicast network with a replacement of the source node with many independent nodes. The partial information is transmitted the same way decode-forward is extended for the single source multicast network in Section III. The relay decodes the message from all the sources using an m-user multiple access channel [4]. After decoding the partial information from the source nodes, the relay uses binning to transmit the decoded information [13]. Further relays and destination nodes decode the message in the same multiple access fashion. The relays that have decoded the messages act as source nodes and coherently transmit the partial information. The remaining message is superimposed and decoded using noisy network coding. The following theorem provides an achievable rate for this network, using superposition noisy network coding.

Theorem 3: For an N node discrete memoryless multiple source multicast network with k_0 source nodes, the following rate is achievable using superposition noisy network coding

$$R'(\mathcal{S}) < \min_{\mathcal{S}} I(V(\mathcal{S}); Y_k|X_k, V(\mathcal{S}^c)) \tag{8}$$

$$R''(\mathcal{S}) < \min_{\mathcal{S}} \big(I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_N|X(\mathcal{S}^c), V_1^N) - I(\hat{Y}(\mathcal{S}); Y(\mathcal{S})|X^N, \hat{Y}(\mathcal{S}^c), Y_N, V_{k-1}^N) \big) \tag{9}$$

where the random variables are jointly distributed according to $\prod_{k=1}^{k_0} p(v_k)p(x_k|v_k) \prod_{k=k_0+1}^N p(v_k)p(x_k|v_k)p(\hat{y}_k|y_k, x_k, u_1)$, and the maximum is over all possible cut-sets $\mathcal{S}$.

Sketch of Proof:

Codebook generation: Fix $\prod_{k=1}^{k_0} p(v_k)p(x_k|v_k) \prod_{k=k_0+1}^N p(v_k)p(x_k|v_k)p(\hat{y}_k|y_k, x_k, u_k^N)$.

1) For each $j \in [1:b]$ and $k \in [1:k_0]$, randomly and independently generate $2^{nR'_k}$ sequences $\mathbf{v}_{k,j}(m'_k)$, $m'_k \in [1:2^{nR'_k}]$, each according to $\prod_{i=1}^n p_{V_k}(v_{k,(j-1)n+i})$.
2) For each $\mathbf{v}_{k,j}(m'_k)$, $j \in [1:b]$ and $k \in [1:k_0]$, randomly and conditionally independently generate $2^{nbR''_k}$ sequences $\mathbf{x}_{k,j}(m''_k|m'_k)$, such that $m''_k \in [1:2^{nbR''_k}]$, $m'_k \in [1:2^{nbR'_k}]$. The sequences are generated independently according to the distribution $\prod_{i=1}^n p_{X_k|V_k}(x_{k,(j-1)n+i}|v_{k,(j-1)n+i}(m'_k))$
3) For all nodes $k \in [k_0+1:N]$ randomly and independently generate $2^{n\tilde{R}'_k}$ codewords $\mathbf{v}_{k,j}(\kappa(m_1^{'k_0}))$. The rate $\tilde{R}'_k$ is chosen such that
$$\tilde{R}'_k \geq \max_{d\in\mathcal{D}} I(V_k; Y_d|V_1^{k_0})$$
The maximum is over $\mathcal{D}$ the set of all destination nodes. $\kappa(m_1^{'k_0})$ is the bin index of the messages $m_1^{'k_0}$.
4) For each $\mathbf{v}_{k,j}(\kappa(m_1^{'k_0}))$ and $k \in [k_0+1:N]$, randomly and independently generate $2^{n\hat{R}_k}$ sequences $\mathbf{x}_{k,j}(l_{k,j-1}|\kappa(m_1^{'k_0}))$, such that $m'_k \in [1:2^{nR'_k}]$, $l_{k,j-1} \in [1:2^{n\hat{R}_k}]$, each according to the probability distribution $\prod_{i=1}^n p_{X_k|V_k}(x_{k,(j-1)n+i}|v_{k,(j-1)n+i}(\kappa(m_1^{'k_0})))$.
5) For each node $k \in [k_0+1:N]$ and each $\mathbf{x}_{k,j}(l_{k,j-1}|\kappa(m_1^{'k_0}))$ $\mathbf{v}_{kj}(\kappa(m_1^{'k_0})), \ldots, \mathbf{v}_{Nj}(\kappa(m_1^{'k_0}))$, such that $m''_k \in [1:2^{nbR''_k}]$, $m'_k \in [1:2^{nR'_k}]$, $l_{k,j-1} \in [1:2^{n\hat{R}_k}]$, randomly and conditionally independently generate $2^{n\hat{R}_k}$ sequences $\hat{\mathbf{y}}_{kj}(l_{kj}|m''_k, l_{k,j-1}, \kappa(m_1^{'k_0}))$, $l_{kj} \in [1:2^{n\hat{R}_k}]$, each according to $\prod_{i=1}^n p_{\hat{Y}_k|X_k,V_k}(\hat{y}_{k,(j-1)n+i}|x_{k,(j-1)n+i}(m''_k|m'_k), v_{k,(j-1)n+i}(\kappa(m_1^{'k_0})))$.

This defines the codebook
$$\begin{aligned}\mathcal{C}_j \;=\; & \big\{\mathbf{v}_{k,j}(m'_k), \mathbf{x}_{k,j}(m''_k|m'_k), k\in[1:k_0],\\ & \mathbf{v}_{k,j}(\kappa(m_1^{'k_0})), \mathbf{x}_{k,j}(l_{k,j-1}|\kappa(m_1^{'k_0})),\\ & \quad \hat{\mathbf{y}}_{kj}(l_{kj}|m''_k, l_{k,j-1}, \kappa(m_1^{'k_0})), k\in[k_0+1:N]\\ & : m'_k \in [1:2^{nR'_k}], m''_k \in [1:2^{nbR''_k}],\\ & l_{kj}, l_{k,j-1} \in [1:2^{n\hat{R}_k}]\big\}\end{aligned}$$
for $j \in [1:b]$.

Encoding: Let $(m'_1, \ldots, m'_{k_0}, m''_1, \ldots, m''_{k_0})$ be the messages to be sent. Each relay node $k \in [k_0+1:N]$, upon receiving $\mathbf{y}_{kj}$ at the end of block $j \in [1:b]$, decode the messages $m_1^{'k_0}$ as shown in the decoding step. After finding $m_1^{'k_0}$, the node finds an index l_{kj} such that
$$\begin{aligned}&(\hat{\mathbf{y}}_{kj}(l_{kj}|m''_k, l_{k,j-1}, \kappa(m_1^{'k_0})), \mathbf{y}_{kj}, \mathbf{x}_{k,j}(l_{k,j-1}|\kappa(m_1^{'k_0})),\\ &\qquad \mathbf{v}_{k,j}(\kappa(m_1^{'k_0}))) \in \mathcal{T}_{\epsilon'}^{(n)},\end{aligned}$$
where $l_{k0} = 1$, $k \in [k_0+1:N]$, by convention. If there is more than one such index, choose one of them at random. If there is no such index, choose an arbitrary index at random from $[1:2^{n\hat{R}_k}]$. Then each node $k \in [k_0+1:N]$ transmits the codeword $\mathbf{x}_{k,j}(l_{k,j-1}|\kappa(m_1^{'k_0}))$ in block $j \in [1:b]$.

Decoding: Let $\epsilon > \epsilon'$. After each block, the decoder $d \in \mathcal{D}$ decodes the messages $m_1^{'k_0}$. The messages are decoded as a k_0 user multiple access channel. The probability of error of decoding the messages $m_1^{'k_0}$ can be arbitrarily small if the (8) is satisfied [4]. After b blocks, the decoder $d \in \mathcal{D}$ finds a unique index tuple $(\hat{m}''_{1d}, \ldots, \hat{m}''_{k_0d})$, where $\hat{m}''_{kd} \in [1:2^{nbR''_k}]$, such that there exist some $(\hat{l}_{1j}, \ldots, \hat{l}_{Nj})$, $\hat{l}_{kj} \in [1:2^{n\hat{R}_k}]$, and $j \in [1:b]$, satisfying
$$\begin{aligned}&(\mathbf{v}_{1,j}(m'_1), \ldots, \mathbf{v}_{k_0,j}(m'_{k_0}), \mathbf{v}_{k_0+1,j}(\kappa(m_1^{'k_0})), \ldots,\\ &\quad \mathbf{v}_{Nj}(\kappa(m_1^{'k_0})), \mathbf{x}_{1,j}(m''_1|m'_1), \ldots, \mathbf{x}_{k_0,j}(m''_{k_0}|m'_{k_0}),\\ &\quad \mathbf{x}_{(k_0+1),j}(l_{(k_0+1),j-1}|\kappa(m_1^{'k_0})), \ldots, \mathbf{x}_{N,j}(l_{N,j-1}|\kappa(m_1^{'k_0})),\\ &\quad \hat{\mathbf{y}}_{(k_0+1),j}(l_{(k_0+1),j}|l_{k_0+1,j-1}, \kappa(m_1^{'k_0})), \ldots\\ &\qquad \hat{\mathbf{y}}_{Nj}(l_{Nj}|l_{k,j-1}, \kappa(m_1^{'k_0})), \mathbf{y}_{Nj}) \in \mathcal{T}_\epsilon^{(n)}\end{aligned}$$
for all $j \in [1:b]$, given that the messages $m_1^{'k_0}$ have been decoded correctly. The probability of error goes to 0 as $n \to \infty$ if (9) is satisfied. The detailed analysis is similar to that in [6] and is omitted here. ■

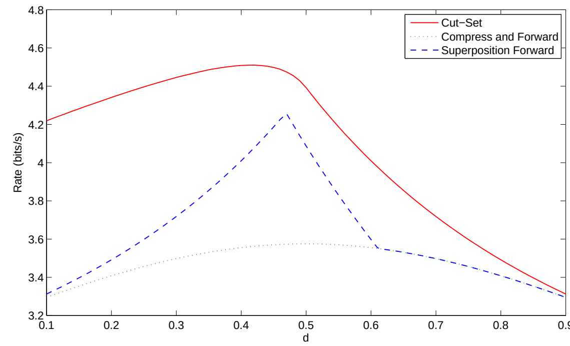

Fig. 2. Achievable rates for an AWGN single relay channel

V. Numerical Results

In this section, we apply the superposition noisy network coding scheme to additive white Gaussian noise (AWGN) three-node relay channel and the two-way relay channel. We compare the achievable rates to the existing schemes of noisy network coding, compress-forward and the cut-set upper bound.

Consider a Gaussian relay channel model [11]
$$Y_2 = aX_1 + Z_1 \tag{10}$$
$$Y_3 = X_1 + bX_2 + Z_2 \tag{11}$$
where the noise terms Z_1 and Z_2 are uncorrelated zero mean Gaussian random variables with variances N_1 and N_2 respectively, and a and b are the channel gain constants. The power constraints at the transmitters are $\frac{1}{n}\sum_{i=1}^n x_{1i}^2(k) \leq P_1, \quad \forall k \in \mathcal{M}$, and $\frac{1}{n}\sum_{i=1}^n x_{2i}^2 \leq P_2, \quad \forall y_2^n \in \mathcal{R}^n$.

All the terminals are aligned in a line. The source and destination are at unit distance. The relay is at distance d from the source and distance $1-d$ from the destination. We assume $a = 1/d$ and $b = 1/(1-d)$. Fig. 2 plots the rates achieved by superposition noisy network coding for $P1 = P2 = 5$. They are compared to those achieved by noisy network coding, compress-forward and the cut-set bound. Noisy network coding achieves the same rate as compress-forward scheme for a single relay channel.

It is observed that the superposition noisy network coding scheme has an advantage over the noisy network coding

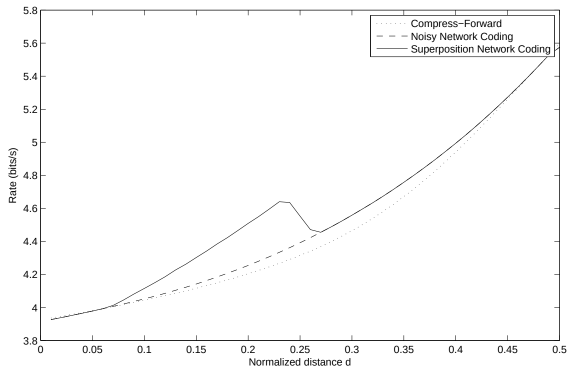

Fig. 3. Achievable rates for an AWGN two-way relay channel

scheme when the relay is close to the source. This advantage arises due to a strong source-relay link.

A. Two-Way Relay Channel

The two-way relay channel was first introduced by Shannon [14]. The two-way relay channel is a fundamental building block for multi-user information theory. Rankov et al. [15] derived the achievable rates for the two-way relay channel using the schemes decode-forward and compress-forward. The rates achieved by superposition noisy network coding is derived for the two way relay channel and compared to the existing rates.

Consider the AWGN two-way relay channel [15]

$$\begin{aligned} Y_1 &= g_{21}X_2 + g_{31}X_3 + Z_1, \\ Y_2 &= g_{12}X_1 + g_{32}X_3 + Z_2, \\ Y_3 &= g_{13}X_1 + g_{23}X_2 + Z_3, \end{aligned} \tag{12}$$

where the channel gains are $g_{12} = g_{21} = 1$, $g_{13} = g_{31} = d^{-\gamma/2}$ and $g_{23} = g_{32} = (1-d)^{-\gamma/2}$, and $d \in [0,1]$ is the location of the relay node between nodes 1 and 2 (which are unit distance apart). Source nodes 1 and 2 wish to exchange messages reliably with the help of relay node 3. Specializing the Theorem 3 to the two-way relay channel gives the inner bound that consists of all rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1' &\leq \min\{I(U_1;Y_2|U_2,V_3,X_3), \\ &\qquad I(U_1,V_3;Y_2|U_2,X_2)\} \\ R_2' &\leq \min\{I(U_2;Y_1|U_1,V_3,X_3), \\ &\qquad I(U_2,V_3;Y_1|U_1,X_1)\} \\ R_1'+R_2' &\leq I(U_1,U_2;Y_3|V_3,X_3) \\ R_1'' &\leq \min\{I(X_1;Y_2,\hat{Y}_3|X_2,X_3,U_1,U_2), \\ &\qquad I(X_1,X_3;Y_2|X_2,U_1,V_3) - \\ &\qquad I(Y_3;\hat{Y}_3|X_1,X_2,X_3,Y_2,U_1,U_2)\} \\ R_2'' &\leq \min\{I(X_2;Y_1,\hat{Y}_3|X_1,X_3,U_1,U_2), \\ &\qquad I(X_2,X_3;Y_1|X_1,U_2,V_3) - \\ &\qquad I(Y_3;\hat{Y}_3|X_1,X_2,X_3,Y_1,U_1,U_2)\} \end{aligned}$$

for some $p(q)p(u_1)p(u_2)p(v_3)p(x_1|u_1,q)p(x_2|u_2,q)p(x_3|v_3,q)$ $p(\hat{y}_3|y_3,x_3,q)$.

Fig. 3 compares the achievable rates of the schemes derived as a function of relay distance. The power constraints at the nodes are $P_1 = P_2 = P_3 = 10$. It is observed that superposition noisy network coding provides higher rates than both compress-forward and noisy network coding. Noisy network coding is a special case of superposition noisy network coding scheme. The superposition scheme performs better when the relay is close to either of the sources and decoding partial information is advantageous to the sum rate.

VI. Conclusions

The noisy network coding for discrete memoryless channel is improved by superimposing partial decode and forward of the messages. The encoding and decoding strategies are first derived for the superposition noisy network coding in a three-node relay channel. The rates achieved by superposition noisy network coding is higher than the rates achieved by noisy network coding, when the channel from the source to the relay nodes are strong. We then derive the superposition noisy network coding scheme for both single-source and multiple-source multicast networks. We specialized the result to a two-way relay channel. For Gaussian three-node and two-way relay channels, it is numerically observed that the superposition noisy network coding scheme provides higher rates than noisy network coding or compress-forward.

References

[1] A. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *Information Theory, IEEE Transactions on*, vol. 52, no. 3, pp. 789 –804, Mar. 2006.

[2] N. Ratnakar and G. Kramer, "The multicast capacity of deterministic relay networks with no interference," *Information Theory, IEEE Transactions on*, vol. 52, no. 6, pp. 2425 – 2432, Jun. 2006.

[3] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "Wireless network information flow: A deterministic approach," *CoRR*, vol. abs/0906.5394, 2009.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. John Wiley & Sons, Inc., 2006.

[5] T. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 5, pp. 572–584, May 1979.

[6] S. H. Lim, Y.-H. Kim, A. El Gamal, and S.-Y. Chung, "Noisy network coding," in *Information Theory Workshop (ITW), 2010 IEEE*, Jan. 2010, pp. 1 –5.

[7] R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, "Network information flow," *Information Theory, IEEE Transactions on*, vol. 46, no. 4, pp. 1204 –1216, Jul. 2000.

[8] A. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," *Information Theory, IEEE Transactions on*, vol. 22, no. 1, pp. 1 – 10, Jan. 1976.

[9] X. Wu and L.-L. Xie, "On the optimality of successive decoding in compress-and-forward relay schemes," in *Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on*, Oct. 2010, pp. 534 –541.

[10] A. E. Gamal and Y.-H. Kim, "Lecture notes on network information theory," *CoRR*, vol. abs/1001.3404, 2010.

[11] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *Information Theory, IEEE Transactions on*, vol. 51, no. 9, pp. 3037 – 3063, Sep. 2005.

[12] L.-L. Xie and P. R. Kumar, "A network information theory for wireless communication: scaling laws and optimal operation," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 748–767, May 2004.

[13] L.-L. Xie, "Network coding and random binning for multi-user channels," in *Information Theory, 2007. CWIT '07. 10th Canadian Workshop on*, Jun. 2007, pp. 85 –88.

[14] C. E. Shannon, "Two-way Communication Channels," in *Proc. 4th Berkeley Symp. Math. Stat. Prob.*, vol. 1. Univ. California Press, 1961.

[15] B. Rankov and A. Wittneben, "Achievable rate regions for the two-way relay channel," in *Proc. IEEE Int. Symposium on Information Theory (ISIT)*, Jul. 2006. [Online]. Available: http://www.nari.ee.ethz.ch/wireless/pubs/p/isit2006