

# Simultaneous Code/Error-Trellis Reduction for Convolutional Codes Using Shifted Code/Error-Subsequences

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**Abstract**—In this paper, we show that the code-trellis and the error-trellis for a convolutional code can be reduced simultaneously, if reduction is possible. Assume that the error-trellis can be reduced using shifted error-subsequences. In this case, if the identical shifts occur in the subsequences of each code path, then the code-trellis can also be reduced. First, we obtain pairs of transformations which generate the identical shifts both in the subsequences of the code-path and in those of the error-path. Next, by applying these transformations to the generator matrix and the parity-check matrix, we show that reduction of these matrices is accomplished simultaneously, if it is possible. Moreover, it is shown that the two associated trellises are also reduced simultaneously.

## I. INTRODUCTION

In this paper, we always assume that the underlying field is  $F = \text{GF}(2)$ . Let  $G(D)$  and  $H(D)$  be the generator matrix and the parity-check matrix of an  $(n, n - m)$  convolutional code  $C$ , respectively. Ariel and Snyders [1] presented a construction of error-trellises based on the scalar check matrix derived from  $H(D)$ . They showed that when some ( $j$ th) “column” of  $H(D)$  has a factor  $D^l$ , there is a possibility that state-space reduction can be realized. Being motivated by their work, we also examined the same case. The time- $k$  error  $e_k = (e_k^{(1)}, \dots, e_k^{(n)})$  and syndrome  $\zeta_k = (\zeta_k^{(1)}, \dots, \zeta_k^{(m)})$  are connected with the relation  $\zeta_k = e_k H^T(D)$  ( $T$  means transpose). From this relation, we noticed [9] that the transformation  $e_k^{(j)} \rightarrow D^l e_k^{(j)} = e_{k-l}^{(j)}$  is equivalent to dividing the  $j$ th column of  $H(D)$  by  $D^l$ . That is, reduction can be realized by shifting the “subsequence”  $\{e_k^{(j)}\}$  of the original error-path  $e$ . It is stated [1] that their construction can be used also to obtain code-trellises. However, it is not described in the paper. On the other hand, our construction is based on an equivalent modification of the relation  $\zeta_k = e_k H^T(D)$ . Hence, our method can be directly extended to code-trellises. That is, in the case of code-trellises, the construction is based on the relation  $\mathbf{y}_k = \mathbf{u}_k G(D)$  and its equivalent modifications, where  $\mathbf{u}_k$  and  $\mathbf{y}_k$  are the time- $k$  information and code symbols, respectively. Note that there exists a one-to-one correspondence between the code-paths in a code-trellis and the error-paths in the corresponding error-trellis. Accordingly, it is reasonable

to think that the two trellises can be reduced simultaneously, if reduction is possible. Here, consider the situation that the identical shifts occur both in the components of  $\mathbf{y}_k$  and in those of  $e_k$ . In this case, if one trellis is reduced, then the other trellis should be equally reduced. In this paper, based on this idea, we discuss the simultaneous reduction of a code-trellis and the corresponding error-trellis. First, we obtain the general transformations which generate the identical shifts both in the subsequences of  $\mathbf{y}$  and in those of  $e$ . Next, we show that these transformations preserve the relation that *one is a generator matrix and the other is the corresponding parity-check matrix*. (In this paper, we call this relation the “*GH Relation*” and if  $G(D)$  and  $H(D)$  have this relation, then it is denoted as  $G(D) \Leftrightarrow H(D)$ ). Using this property, it is shown that  $G(D)$  and  $H(D)$  are reduced simultaneously, if reduction is possible. Moreover, it is shown that the corresponding two trellises are also reduced simultaneously. These results again imply that a code/error-trellis construction using shifted code/error-subsequences is very effective.

## II. TRELLIS CONSTRUCTION USING SHIFTED PATH-SUBSEQUENCES

### A. Error-trellis construction using shifted error-subsequences

Let  $H(D)$  be the parity-check matrix for an  $(n, n - m)$  convolutional code  $C$ . Consider the error-trellis based on the syndrome former  $H^T(D)$ . In this case, the adjoint-obvious realization of  $H^T(D)$  is assumed unless otherwise specified. Assume that the  $j$ th column of  $H(D)$  has the form

$$\left( D^{l_j} h'_{1j}(D) \quad D^{l_j} h'_{2j}(D) \quad \dots \quad D^{l_j} h'_{mj}(D) \right)^T, \quad (1)$$

where  $l_j \geq 1$ . Let  $H'(D)$  be the modified version of  $H(D)$  with the  $j$ th column being replaced by

$$\left( h'_{1j}(D) \quad h'_{2j}(D) \quad \dots \quad h'_{mj}(D) \right)^T. \quad (2)$$

Also, let  $e'_k \triangleq (e_k^{(1)}, \dots, e_k^{(j)}, \dots, e_k^{(n)})$ , where  $e_k^{(j)} \triangleq D^{l_j} e_k^{(j)} = e_{k-l_j}^{(j)}$ . Then we have

$$\zeta_k = e'_k H'^T(D). \quad (3)$$

Hence, in the case where the  $j$ th column of  $H(D)$  has a factor  $D^{l_j}$ , there is a possibility that an error-trellis with reduced number of states can be constructed by shifting the  $j$ th error-subsequence by  $l_j$  time units [9]. Assume that the corresponding code-trellis is terminated in the all-zero state at  $t = N$ . Then  $e_k^{(j)} = e_{k-l_j}^{(j)}$  is modified as  $e_k^{(j)} = e_{\langle k-l_j \rangle}^{(j)}$ , where  $\langle t \rangle$  denotes  $t \bmod (N + l_j)$  (i.e., “cyclic shift”).

### B. Error-trellis construction using backward-shifted error-subsequences

The construction using shifted error-subsequences is further extended [9], [10]. That is, a reduced error-trellis can be equally constructed using “backward-shifted” error-subsequences. Consider the transformation  $e_k^{(j)} \rightarrow D^{-l_j} e_k^{(j)} = e_{k+l_j}^{(j)}$ . We see that this is equivalent to “multiplying” the  $j$ th column of  $H(D)$  by  $D^{l_j}$ . Let  $H'(D)$  be the parity-check matrix after modification. If  $H'(D)$  is reduced to an equivalent  $H''(D)$  with overall constraint length less than that of  $H(D)$ , then reduction can be realized. We remark that the power  $l_j$  of  $D$  has to be determined properly for each  $j$ . For the purpose, we can use the *reciprocal dual encoder* [6]  $\tilde{H}(D)$  associated with  $H(D)$ .

*Example 1 ([9]):* Consider the canonical parity-check matrix

$$H_1(D) = \begin{pmatrix} D^2 & D^2 & 1 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}. \quad (4)$$

Since all the columns of  $H_1(D)$  are delay free, any further reduction seems to be impossible. In fact, it follows from Theorem 1 of [1] that the dimension  $d_1$  of the state space of the error-trellis based on  $H_1^T(D)$  is 4. However, a corresponding generator matrix is given by  $G_1(D) = (1 + D + D^2, 1, D^3 + D^4)$ . Observe that the third “column” of  $G_1(D)$  has a factor  $D^2$ . (*Remark:* It suffices to divide the third column by  $D^2$  in order to obtain a reduced code-trellis.) This fact implies that a reduced error-trellis can be constructed [1], [9]. Then consider the reciprocal dual encoder

$$\tilde{H}_1(D) = \begin{pmatrix} 1 & 1 & D^2 \\ D^2 & 1 + D + D^2 & 0 \end{pmatrix}. \quad (5)$$

Note that the third column of  $\tilde{H}_1(D)$  has a factor  $D^2$ . Accordingly, dividing the third column of  $\tilde{H}_1(D)$  by  $D^2$ , we can construct an error-trellis with 4 states (i.e.,  $d_1 = 2$ ) [1], [9]. Here, notice that each error-path in the error-trellis based on  $H_1^T(D)$  can be represented in time-reversed order using the error-trellis based on  $\tilde{H}_1^T(D)$ . Hence, a factor  $D^2$  in the column of  $\tilde{H}_1(D)$  corresponds to backward-shifting by two time units (i.e.,  $D^{-2}$ ) in terms of the original  $H_1(D)$ . Hence, multiply the third column  $H_1(D)$  by  $D^2$ . Then we have

$$H'_1(D) = \begin{pmatrix} D^2 & D^2 & D^2 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}. \quad (6)$$

We see that this matrix can be reduced to an equivalent canonical parity-check matrix

$$H''_1(D) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix} \quad (7)$$

by dividing the first “row” by  $D^2$ . Hence, the dimension  $d_1$  can be reduced to 2.

### C. Code-trellis construction using shifted code-subsequences

Note that the relation  $\mathbf{y}_k = \mathbf{u}_k G(D)$  holds with respect to a generator matrix  $G(D)$ , where  $\mathbf{u}_k = (u_k^{(1)}, \dots, u_k^{(n-m)})$  and  $\mathbf{y}_k = (y_k^{(1)}, \dots, y_k^{(n)})$  are the time- $k$  information and code symbols, respectively. In the same way as for  $H(D)$ , by dividing the  $j$ th column of  $G(D)$  by  $D^{l_j}$  or by multiplying the  $j$ th column of  $G(D)$  by  $D^{l_j}$ , reduction of  $G(D)$  can be realized. We see that the former corresponds to the backward-shift  $y_k^{(j)} \rightarrow y_{k+l_j}^{(j)}$ , whereas the latter corresponds to the forward-shift  $y_k^{(j)} \rightarrow y_{k-l_j}^{(j)}$ . Note that the shift directions are reversed compared to  $H(D)$ .

## III. TRANSFORMATIONS GENERATING THE IDENTICAL SHIFTS BOTH IN $\mathbf{y}$ AND IN $\mathbf{e}$

### A. General case

Consider the transformations which generate the identical shifts both in the components of  $\mathbf{y}_k$  and in those of  $\mathbf{e}_k$ . Now, assume that the relation  $G(D) \Leftrightarrow H(D)$  holds. Consider a pair of transformations:

- 1) divide the  $j$ th column of  $G(D)$  by  $D^{l_j^{(d)}}$  and multiply the same column by  $D^{l_j^{(m)}}$ ,
- 2) divide the  $j$ th column of  $H(D)$  by  $D^{l_j^{(d)}}$  and multiply the same column by  $D^{l_j^{(m)}}$ .

Then

- 1) the  $j$ th component of  $\mathbf{y}_k$  becomes

$$y_k^{(j)} \rightarrow y_{k+l_j^{(d)}-l_j^{(m)}}^{(j)}, \quad (8)$$

- 2) the  $j$ th component of  $\mathbf{e}_k$  becomes

$$e_k^{(j)} \rightarrow e_{k-l_j^{(d)}+l_j^{(m)}}^{(j)}. \quad (9)$$

After shifting  $e_{k-l_j^{(d)}+l_j^{(m)}}^{(j)}$  by  $l$  time units ( $l$  is independent of  $j$ ), compare the time-index of  $e_{k+l-l_j^{(d)}+l_j^{(m)}}^{(j)}$  and that of  $y_{k+l_j^{(d)}-l_j^{(m)}}^{(j)}$ . If the two time-indices coincide, then  $y_k^{(j)}$  and  $e_k^{(j)}$  have “relatively” the identical shift. This condition is written as

$$l = (l_j^{(d)} + l_j^{(d)}) - (l_j^{(m)} + l_j^{(m)}) \quad (1 \leq j \leq n), \quad (10)$$

where  $l$  is a constant independent of  $j$  ( $1 \leq j \leq n$ ). (In the following, this condition is denoted as “ $C_{SR}$ ”.)

### B. Special cases

*Case 1:* Only division is applied both to the columns of  $G(D)$  and to those of  $H(D)$ .

From the assumption,  $l_j^{(m)} = \tilde{l}_j^{(m)} = 0$ . Hence, we have

$$l = l_j^{(d)} + \tilde{l}_j^{(d)}. \quad (11)$$

Here, assume that either  $l_j^{(d)}$  or  $\tilde{l}_j^{(d)}$  is 0. Define the sets  $L_G$  and  $L_H$  as

$$L_G \triangleq \{j : l_j^{(d)} = l\} = \{j : \tilde{l}_j^{(d)} = 0\} \quad (12)$$

$$L_H \triangleq \{j : \tilde{l}_j^{(d)} = l\} = \{j : l_j^{(d)} = 0\}. \quad (13)$$

In words,  $L_G$  is the set of columns of  $G(D)$  from which  $D^l$  is factoring out, whereas  $L_H$  is the set of columns of  $H(D)$  from which  $D^l$  is factoring out. Note that  $L_G$  and  $L_H$  are disjoint and the relation

$$L_G \cup L_H = \{1, 2, \dots, n\} \quad (14)$$

holds. In the following, we call this kind of transformations “type-1”.

*Example 2:* Consider the relation

$$\begin{aligned} G_2(D) &= (D + D^2, D^2, 1 + D) \\ \Leftrightarrow H_2(D) &= \begin{pmatrix} 1 & 0 & D \\ D & 1 + D & 0 \end{pmatrix}. \end{aligned} \quad (15)$$

Choosing  $l = 1$ ,  $L_G = \{1, 2\}$ , and  $L_H = \{3\}$ , we have

$$\begin{aligned} G'_2(D) &= (1 + D, D, 1 + D) \\ \Leftrightarrow H'_2(D) &= \begin{pmatrix} 1 & 0 & 1 \\ D & 1 + D & 0 \end{pmatrix}. \end{aligned} \quad (16)$$

*Case 2:* Division and multiplication are separately applied either to the columns of  $G(D)$  or to the columns of  $H(D)$ .

Without loss of generality, assume that division is applied to the columns of  $G(D)$ , whereas multiplication is applied to the columns of  $H(D)$ . From the assumption,  $l_j^{(m)} = \tilde{l}_j^{(d)} = 0$ . Hence, we have

$$l = l_j^{(d)} - \tilde{l}_j^{(m)}. \quad (17)$$

In particular, set  $l = 0$ . Then we have

$$l_j^{(d)} = \tilde{l}_j^{(m)} \triangleq l_j. \quad (18)$$

This is equivalent to dividing the  $j$ th column of  $G(D)$  by  $D^{l_j}$  and multiplying the  $j$ th column of  $H(D)$  by  $D^{l_j}$ . In the following, we call this kind of transformations “type-2”.

*Example 3:* Consider the relation

$$\begin{aligned} G_3(D) &= (1 + D, 1, D + D^2) \\ \Leftrightarrow H_3(D) &= \begin{pmatrix} D & 0 & 1 \\ 1 & 1 + D & 0 \end{pmatrix}. \end{aligned} \quad (19)$$

Choosing  $l_3^{(d)} = \tilde{l}_3^{(m)} = 1$ , we have

$$\begin{aligned} G'_3(D) &= (1 + D, 1, 1 + D) \\ \Leftrightarrow H'_3(D) &= \begin{pmatrix} D & 0 & D \\ 1 & 1 + D & 0 \end{pmatrix}. \end{aligned} \quad (20)$$

Note that  $H'_3(D)$  can be reduced to

$$H''_3(D) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 + D & 0 \end{pmatrix}. \quad (21)$$

Type-1 and type-2 transformations form a subclass of general transformations defined in Section III-A. However, these transformations are quite effective.

### C. Property of transformations

Observe that in Example 2 and Example 3, the GH Relation is preserved after type-1 and type-2 transformations. It is shown that this property holds in general. Assume that the relation  $G(D) \Leftrightarrow H(D)$  holds. Also, assume that a pair of transformations which satisfies the condition  $C_{SR}$  is applied to  $G(D)$  and  $H(D)$ . Let  $G'(D)$  and  $H'(D)$  be the resulting matrices, respectively. Then we have the following.

*Proposition 1:* The relation  $G'(D) \Leftrightarrow H'(D)$  holds.

*Proof:* Fix  $p, q$  ( $1 \leq p \leq n - m, 1 \leq q \leq m$ ) arbitrarily.

Let

$$(g_{p1}(D), \dots, g_{pj}(D), \dots, g_{pn}(D)) \quad (22)$$

be the  $p$ th row of  $G(D)$ . Then the  $(p, j)$  element of  $G'(D)$  is given by

$$g_{pj}(D) \frac{D^{l_j^{(m)}}}{D^{l_j^{(d)}}}. \quad (23)$$

Similarly, defining the  $q$ th row of  $H(D)$  as

$$(h_{q1}(D), \dots, h_{qj}(D), \dots, h_{qn}(D)), \quad (24)$$

the  $(q, j)$  element of  $H'(D)$  is given by

$$h_{qj}(D) \frac{D^{\tilde{l}_j^{(m)}}}{D^{\tilde{l}_j^{(d)}}}. \quad (25)$$

Then the  $(p, q)$  element  $h'_{pq}$  of  $G'(D)H'^T(D)$  is given by

$$\begin{aligned} h'_{pq} &= \sum_{j=1}^n g_{pj}(D) \frac{D^{l_j^{(m)}}}{D^{l_j^{(d)}}} h_{qj}(D) \frac{D^{\tilde{l}_j^{(m)}}}{D^{\tilde{l}_j^{(d)}}} \\ &= \sum_{j=1}^n g_{pj}(D) h_{qj}(D) D^{(l_j^{(m)} + \tilde{l}_j^{(m)}) - (l_j^{(d)} + \tilde{l}_j^{(d)})} \\ &= \frac{1}{D^l} \sum_{j=1}^n g_{pj}(D) h_{qj}(D). \end{aligned} \quad (26)$$

Since  $G(D) \Leftrightarrow H(D)$ ,  $\sum_{j=1}^n g_{pj}(D) h_{qj}(D) = 0$ . Hence, we have  $h'_{pq} = 0$ . ■

### IV. SIMULTANEOUS REDUCTION OF $G(D)$ AND $H(D)$

The discussion in the previous section implies that  $G(D)$  and  $H(D)$  can be reduced simultaneously, if reduction is possible. Assume that the relation  $G(D) \Leftrightarrow H(D)$  holds. Let  $\nu$  and  $\nu^\perp$  be the overall constraint lengths of  $G(D)$  and  $H(D)$ , respectively. If both  $G(D)$  and  $H(D)$  are *canonical* [4], [5], then we have  $\nu = \nu^\perp$ . Here, apply a pair of transformations which satisfies the condition  $C_{SR}$  to  $G(D)$  and  $H(D)$ . Denote by  $\nu'$  and  $\nu'^\perp$  the overall constraint lengths of the modified matrices  $G'(D)$  and  $H'(D)$ , respectively. Note that the relation  $G'(D) \Leftrightarrow H'(D)$  still holds from Proposition 1. Hence, if necessary, by modifying equivalently, we have  $\nu' = \nu'^\perp$ . Therefore, if the strict inequality  $\nu' < \nu$  ( $\nu'^\perp < \nu^\perp$ ) holds, then  $G(D)$  and  $H(D)$  are reduced simultaneously. That is, we have the following.

*Proposition 2:* Assume that the relation  $G(D) \Leftrightarrow H(D)$  holds. Also, assume that a pair of transformations which

satisfies the condition  $C_{SR}$  is applied to  $G(D)$  and  $H(D)$ . In this case, if  $G(D)$  is reduced, then  $H(D)$  is equally reduced, and vice versa.

*Example 4:* Assume that

$$\begin{aligned} G_4(D) &= (1 + D + D^2, D, D^4 + D^5) \\ \Leftrightarrow H_4(D) &= \begin{pmatrix} D^3 & D^2 & 1 \\ D & 1 + D + D^2 & 0 \end{pmatrix}. \end{aligned} \quad (27)$$

Note that both  $G_4(D)$  and  $H_4(D)$  are canonical and the equality  $\nu = \nu^\perp = 5$  holds. Choosing  $l = 1$ ,  $L_G = \{2, 3\}$ , and  $L_H = \{1\}$ , let us apply a type-1 transformation. Then we have

$$\begin{aligned} G'_4(D) &= (1 + D + D^2, 1, D^3 + D^4) \\ \Leftrightarrow H'_4(D) &= \begin{pmatrix} D^2 & D^2 & 1 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}. \end{aligned} \quad (28)$$

Also, let us apply a type-2 transformation with  $l_3^{(d)} = \tilde{l}_3^{(m)} = 2$ . Then we have

$$\begin{aligned} G''_4(D) &= (1 + D + D^2, 1, D + D^2) \\ \Leftrightarrow H''_4(D) &= \begin{pmatrix} D^2 & D^2 & D^2 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}. \end{aligned} \quad (29)$$

Since  $H''_4(D)$  is reduced to

$$H'''_4(D) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}, \quad (30)$$

we finally have

$$\begin{aligned} G''_4(D) &= (1 + D + D^2, 1, D + D^2) \\ \Leftrightarrow H'''_4(D) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + D + D^2 & 0 \end{pmatrix}. \end{aligned} \quad (31)$$

In this example, the overall constraint lengths are reduced from  $\nu = \nu^\perp = 5$  to  $\nu' = \nu'^\perp = 2$ .

*Remark:* The reduction process is not unique. In the above example, if a type-2 transformation is applied to  $G_4(D)$  and  $H_4(D)$  with  $l_3^{(d)} = \tilde{l}_3^{(m)} = 3$ , then we have

$$\begin{aligned} G_4^*(D) &= (1 + D + D^2, D, D + D^2) \\ \Leftrightarrow H_4^*(D) &= \begin{pmatrix} D^3 & D^2 & D^3 \\ D & 1 + D + D^2 & 0 \end{pmatrix} \\ \simeq H_4^{**}(D) &= \begin{pmatrix} D & 1 & D \\ D & 1 + D + D^2 & 0 \end{pmatrix}, \end{aligned} \quad (32)$$

where “ $\simeq$ ” means equivalent. Here, choosing  $l = 1$ ,  $L_G = \{2\}$ , and  $L_H = \{1, 3\}$ , let us apply a type-1 transformation. Then we have  $G_4^*(D) \Leftrightarrow H_4^{**}(D)$ .

## V. SIMULTANEOUS CODE/ERROR-TRELLIS REDUCTION

Assume that the relation  $G(D) \Leftrightarrow H(D)$  holds. Let  $T_c$  be the code-trellis associated with  $G(D)$ . It is assumed that  $T_c$  is terminated in the all-zero state at  $t = N$ . Denote by  $T_e$  the corresponding error-trellis. Note that each code-path  $\mathbf{y}$  in  $T_c$  corresponds to the unique error-path  $\mathbf{e}$  in  $T_e$  by way of the received data  $\mathbf{z}$ . Here, apply a pair of transformations which satisfies the condition  $C_{SR}$  to  $G(D)$  and  $H(D)$ . (Let  $G'(D)$  and  $H'(D)$  be the resulting matrices.) Then from

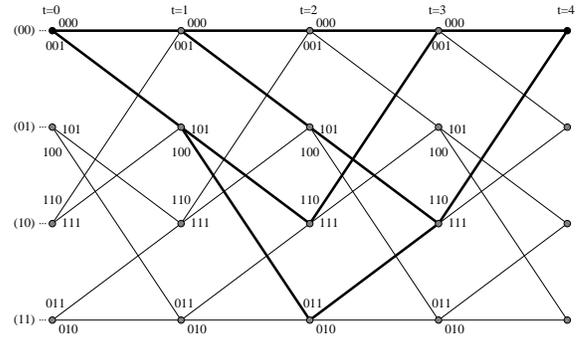


Fig. 1. Example code-trellis associated with  $G_2(D)$ .

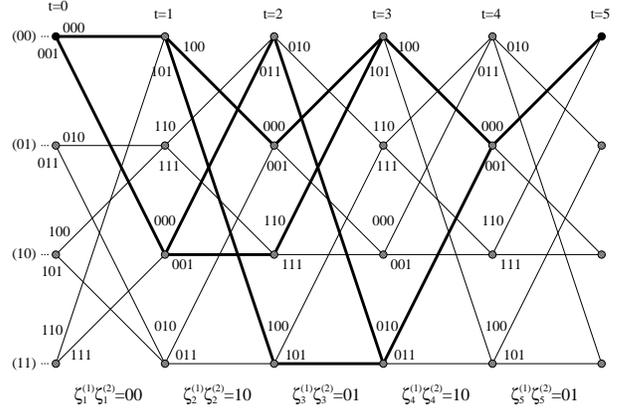


Fig. 2. Example error-trellis based on  $H_2^T(D)$ .

Proposition 2, it is reasonable to think that  $T_c$  and  $T_e$  are reduced simultaneously. In fact, we have the following.

*Proposition 3:* Assume that a pair of transformations which satisfies the condition  $C_{SR}$  is applied to  $G(D)$  and  $H(D)$ . In this case, if the code-trellis associated with  $G(D)$  is reduced, then the error-trellis based on  $H^T(D)$  is equally reduced, and vice versa.

*Proof:* Denote by  $\mathbf{e}'$  the shifted version of  $\mathbf{e}$ . Assume that the set of shifted error-paths  $\{\mathbf{e}'\}$  is represented using the reduced error-trellis  $T'_e$  based on  $H'^T(D)$ . Note that there exists a one-to-one correspondence between the code-paths  $\{\mathbf{y}\}$  and the error-paths  $\{\mathbf{e}\}$ . Also, from the assumption of the transformations, the identical shifts are generated both in the subsequences of a code-path  $\mathbf{y}$  and in those of the corresponding error-path  $\mathbf{e}$ . Hence, the set of shifted code-paths  $\{\mathbf{y}'\}$  is also represented using the reduced code-trellis  $T'_c$  associated with  $G'(D)$ . That is, if one trellis is reduced, then the other trellis is equally reduced. ■

*Example 5:* Consider the relation  $G_2(D) \Leftrightarrow H_2(D)$ . Fig.1 shows the code-trellis associated with  $G_2(D)$ . Note that the trellis is terminated in the all-zero state (00) at  $t = 4$ . The corresponding error-trellis based on  $H_2^T(D)$  is shown in Fig.2. A received data  $\mathbf{z}$  is assumed to be

$$\mathbf{z} = \mathbf{z}_1 \mathbf{z}_2 \mathbf{z}_3 \mathbf{z}_4 \mathbf{z}_5 = 001 \ 000 \ 011 \ 010 \ 000, \quad (33)$$

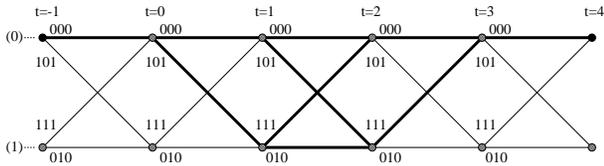


Fig. 3. Reduced code-trellis associated with  $G'_2(D)$ .

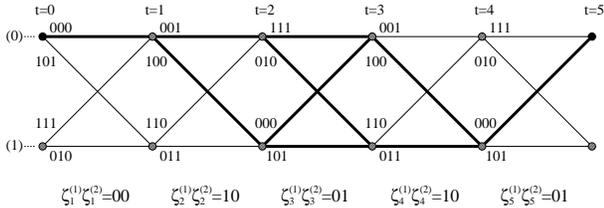


Fig. 4. Reduced error-trellis based on  $H_2^T(D)$ .

where  $z_5 = 000$  is the “imaginary” received data at  $t = 5$ . The syndrome sequence is given as

$$\zeta = \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = 00 \ 10 \ 01 \ 10 \ 01. \quad (34)$$

As we have already seen in Example 2, if the first and second components of  $\mathbf{y}_k$  are shifted left by the unit time and if the third component of  $\mathbf{e}_k$  is shifted right by the unit time, then  $G_2(D)$  and  $H_2(D)$  are reduced simultaneously. Denote by  $G'_2(D)$  and  $H'_2(D)$  the modified generator and parity-check matrices after transformation, respectively. The corresponding code and error-trellises are shown in Fig.3 and Fig.4, respectively.

First, consider the reduced error-trellis in Fig.4. In this example, it is defined as  $e'_k \triangleq e_{\langle k-1 \rangle}^{(3)}$ , where  $\langle t \rangle$  denotes  $t \bmod 5$ . Since  $e_5 = 000$ , we have  $e'_1 = e_{\langle 0 \rangle}^{(3)} = e_5^{(3)} = 0$  using the relation  $e'_k = e_{\langle k-1 \rangle}^{(3)}$ . That is, the third error-bit of the branch from  $t = 0$  to  $t = 1$  must be 0. Similarly, the first two error-bits of the branch from  $t = 4$  to  $t = 5$  must be 00. Then we have four admissible error-paths:

$$\begin{aligned} e'_{p_1} &= 000 \ 001 \ 010 \ 011 \ 000 \\ e'_{p_2} &= 000 \ 001 \ 111 \ 100 \ 000 \\ e'_{p_3} &= 000 \ 100 \ 101 \ 011 \ 000 \\ e'_{p_4} &= 000 \ 100 \ 000 \ 100 \ 000. \end{aligned}$$

Here, noting the relation  $e'_k = e_{\langle k-1 \rangle}^{(3)}$ , we cyclically shift the third bit of each  $z_k$  to the right by the unit time and make the modified received data  $\mathbf{z}'$  for  $H_2^T(D)$ .  $\mathbf{z}'$  is given by

$$\begin{aligned} \mathbf{z}' &= z'_1 \ z'_2 \ z'_3 \ z'_4 \ z'_5 \\ &= 000 \ 001 \ 010 \ 011 \ 000. \end{aligned} \quad (35)$$

Note that if  $\mathbf{z}'$  is inputted to  $H_2^T(D)$ , then the same syndrome sequence  $\zeta = 00 \ 10 \ 01 \ 10 \ 01$  as for  $H_2^T(D)$  is obtained.

Next, consider the reduced code-trellis in Fig.3. Since  $\mathbf{y}_0 = 000$ , we have  $y_4^{(i)} = y_{\langle 5 \rangle}^{(i)} = y_0^{(i)} = 0$  ( $i = 1, 2$ ). That is, the first two code-bits of the branch from  $t = 3$  to  $t = 4$  must be

00. Similarly, the third code-bit of the branch from  $t = -1$  to  $t = 0$  must be 0. Here, to each of admissible error-paths in Fig.4, we add the modified received data  $\mathbf{z}'$ . Then we have

$$\begin{aligned} \mathbf{y}'_{p_1} &= 000 \ 000 \ 000 \ 000 \ 000 \\ \mathbf{y}'_{p_2} &= 000 \ 000 \ 101 \ 111 \ 000 \\ \mathbf{y}'_{p_3} &= 000 \ 101 \ 111 \ 000 \ 000 \\ \mathbf{y}'_{p_4} &= 000 \ 101 \ 010 \ 111 \ 000. \end{aligned}$$

We observe that the obtained paths completely coincide with those in Fig.3. That is, the two trellises associated with  $G_2(D)$  and  $H_2^T(D)$  have been reduced simultaneously.

## VI. CONCLUSION

We have shown that the code-trellis and the error-trellis for a convolutional code can be reduced simultaneously. The proposed method is based on the fact that if the identical shifts occur both in the components of  $\mathbf{y}_k$  and in the components of  $\mathbf{e}_k$ , then the two trellises are reduced simultaneously, if reduction is possible. We have obtained the general transformations which generate the identical shifts both in the subsequences of  $\mathbf{y}$  and in those of  $\mathbf{e}$ . We have shown that these transformations preserve the GH Relation. Using this property, we have shown that reduction of  $G(D)$  and  $H(D)$  is accomplished simultaneously, if it is possible. Moreover, we have shown that the corresponding two trellises are also reduced simultaneously. These results again imply that a code/error-trellis construction using shifted code/error-subsequences is very effective. We remark that a parity-check matrix with the form described in the paper appears in [11] in connection with a class of LDPC convolutional codes. We think [10] that the proposed method is useful for reducing the state complexity of the code/error-trellis for such an LDPC convolutional code.

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