

Maximum Rate of 3- and 4-Real-Symbol ML Decodable Unitary Weight STBCs

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Abstract—It has been shown recently that the maximum rate of a 2-real-symbol (single-complex-symbol) maximum likelihood (ML) decodable, square space-time block codes (STBCs) with unitary weight matrices is $\frac{2a}{2^a}$ complex symbols per channel use (cspcu) for 2^a number of transmit antennas [1]. These STBCs are obtained from Unitary Weight Designs (UWDs). In this paper, we show that the maximum rates for 3- and 4-real-symbol (2-complex-symbol) ML decodable square STBCs from UWDs, for 2^a transmit antennas, are $\frac{3(a-1)}{2^a}$ and $\frac{4(a-1)}{2^a}$ cspcu, respectively. STBCs achieving this maximum rate are constructed. A set of sufficient conditions on the signal set, required for these codes to achieve full-diversity are derived along with expressions for their coding gain.

I. INTRODUCTION

Consider an N transmit antenna, N_r receive antenna quasi-static Rayleigh flat fading MIMO channel given by

$$Y = XH + W \quad (1)$$

where $H \in \mathbb{C}^{N \times N_r}$ is the channel matrix with the entries assumed to be i.i.d., circularly symmetric Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$, $X \in \mathbb{C}^{T \times N}$ is the matrix of transmitted signal, $W \in \mathbb{C}^{T \times N_r}$ is a complex white Gaussian noise matrix with i.i.d., entries $\sim \mathcal{N}_{\mathbb{C}}(0, N_0)$ and $Y \in \mathbb{C}^{T \times N_r}$ is the matrix of received signal ($\mathbb{C}$ is the field of complex numbers). Throughout this paper, we assume $T = N$.

Definition 1 (LSTD [2]): An $N \times N$ Linear Space-Time Design (LSTD) or simply a design X in K real variables $x_1, \dots, x_K$ is a matrix $\sum_{i=1}^K x_i A_i$, where $A_i \in \mathbb{C}^{N \times N}$, $i = 1, \dots, K$ and $A_1, \dots, A_K$ are linearly independent over the field of real numbers $\mathbb{R}$. The matrices A_i are known as the weight matrices.

Definition 2 (Rate): The rate of an $N \times N$ design X in K real variables is $R = \frac{K}{2N}$ complex symbols per channel use (cspcu).

Definition 3 (STBC): An $N \times N$ Space-Time Block Code (STBC) $\mathcal{C}$ is a finite subset of $\mathbb{C}^{N \times N}$.

An STBC can be obtained from a design X by letting the vector $(x_1, \dots, x_K)$ take values from a finite set $\mathcal{A} \subset \mathbb{R}^K$. The set $\mathcal{A}$ is called the signal set. Denote the STBC obtained this way by $\mathcal{C}(X, \mathcal{A})$. Let $\mathbf{s} = [x_1 \ x_2 \ \dots \ x_K]^T$ and $S(\mathbf{s}) = \sum_{i=1}^K x_i A_i$. Then, we have

$$\mathcal{C}(X, \mathcal{A}) = \{S(\mathbf{s}) | \mathbf{s} \in \mathcal{A}\}. \quad (2)$$

An STBC $\mathcal{C}$, whose encoding symbols $(x_1, \dots, x_K)$ are chosen from a set $\mathcal{A}$ is said to offer *full-diversity* iff for every possible codeword pair $(S, \hat{S})$ ($S, \hat{S} \in \mathcal{C}$), with $S \neq \hat{S}$, the codeword difference matrix $S - \hat{S}$ is full-ranked [3]. In general,

the diversity offered by a code, depends on the constellation it employs. A code can offer full-diversity for certain signal set $\mathcal{A}$ but not for another signal set. The CODs are special in this aspect since they offer full-diversity for any arbitrary signal set. The coding gain δ of an STBC $\mathcal{C}$ is defined as

$$\delta = \min_{S \neq \hat{S}} \left(\prod_{i=1}^r \lambda_i \right)^{\frac{1}{r}},$$

where λ_i , $i = 1, 2, \dots, r$ are the non-zero eigenvalues of the matrix $(S - \hat{S})^H (S - \hat{S})$ and r is the minimum of the rank of $(S - \hat{S})^H (S - \hat{S})$ for all possible codeword pairs $(S, \hat{S})$ ($S, \hat{S} \in \mathcal{C}$), with $S \neq \hat{S}$ [3].

A. Encoding complexity and group ML Decoding

One of the important aspects in the design of STBCs is their ML decoding complexity. This depends on their encoding complexity [2]. If we use (2) for encoding an STBC from a LSTD, we see that, in general, one needs to choose an element from $\mathcal{A}$ and then substitute for the real variables $x_1, \dots, x_K$ in the LSTD. This method of encoding clearly requires a look-up table (memory) with $|\mathcal{A}|$ entries. However, if the signal set $\mathcal{A}$ is a Cartesian product of g smaller signal sets in $\frac{K}{g}$ real variables, then the encoding complexity can be reduced (to memory with $g|\mathcal{A}|^{\frac{1}{g}}$ entries). Moreover, if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_g$ where each $\mathcal{A}_i \subset \mathbb{R}^{\frac{K}{g}}$ with cardinality $|\mathcal{A}_i|^{\frac{1}{g}}$, then the STBC $\mathcal{C}$ itself decomposes into a sum of g different STBCs as follows.

Let $K = g\lambda$. Then, by appropriately reordering/relabeling the real variables, we can assume without loss of generality¹ that $S(\mathbf{s}) = \sum_{i=1}^K x_i A_i = S_1(\mathbf{s}_1) + S_2(\mathbf{s}_2) + \dots + S_g(\mathbf{s}_g)$, where $S_i(\mathbf{s}_i) = \sum_{j=(i-1)\lambda+1}^{i\lambda} x_j A_j$ and $\mathbf{s}_i = [x_{(i-1)\lambda+1} \ x_{(i-1)\lambda+2} \ \dots \ x_{i\lambda}]^T$, for $i = 1, 2, \dots, g$. Hence, the STBC decomposes as $\mathcal{C} = \sum_{i=1}^g \mathcal{C}_i$, where

$$\begin{aligned} \mathcal{C}_1 &= \{S_1(\mathbf{s}_1) | \mathbf{s}_1 \in \mathcal{A}_1\} \\ \mathcal{C}_2 &= \{S_2(\mathbf{s}_2) | \mathbf{s}_2 \in \mathcal{A}_2\} \\ &\vdots \\ \mathcal{C}_g &= \{S_g(\mathbf{s}_g) | \mathbf{s}_g \in \mathcal{A}_g\}. \end{aligned}$$

For the given channel (1) the ML decoder is given by

$$\hat{X} = \arg \min_{X \in \mathcal{C}} \|Y - XH\|_F^2.$$

¹Here we have assumed that the first λ variables belong to first group and second λ variables belong to second group and last λ variables belong to the g -th group. In general, the partitioning of real variables into g -groups can be arbitrary.

For a g -group encodable STBC $\mathcal{C}$, $\mathbf{X} = \sum_{i=1}^g \mathbf{X}_i$ for some $\mathbf{X}_i \in \mathcal{C}_i$. Let, $\mathcal{C}_i = \{A_{(i-1)\lambda+1}, A_{(i-1)\lambda+2}, \dots, A_{i\lambda}\}$, where, $\mathcal{C}_i$ is the set of weight matrices corresponding to STBC $\mathcal{C}_i$. It is shown in [4]- [6] that, the ML decoder decomposes as

$$\hat{\mathbf{X}} = \sum_{i=1}^g \arg \min_{\mathbf{X}_i \in \mathcal{C}_i} \|\mathbf{Y} - \mathbf{X}_i \mathbf{H}\|_F^2,$$

if the weight matrices $A_i, i = 1, \dots, K$ satisfy the conditions

$$A_k^H A_l + A_l^H A_k = 0 \quad \forall A_k \in \mathcal{C}_k, A_l \in \mathcal{C}_l, \quad k \neq l. \quad (3)$$

In other words, the component STBCs $\mathcal{C}_i$'s can then be decoded independently.

Definition 4 ([7]): A STBC $\mathcal{C} = \{S(s) | s \in \mathcal{A} \subset \mathbb{R}^K\}$ is said to be g -group decodable or $\frac{K}{g}$ real symbol decodable (or $\frac{K}{2g}$ complex symbol ML decodable) if $\mathcal{C}$ is g -group encodable and if the associated weight matrices satisfy (3).

B. Contributions

In [1], an achievable upper bound on the rate of unitary-weight single-complex-symbol-decodable (SSD) code is derived to be $\frac{2a}{2^a}$ cspcu for 2^a antennas. The maximum rate of 3- and 4-real symbol ML decodable $2^a \times 2^a$ ($a \geq 2$) Unitary Weight Designs (UWDs) (LSTDs with unitary weight matrices) has not been reported so far in the literature, to the best of our knowledge.

The contributions of this paper are as follows:

- We show that the maximum rate of 3- and 4-real symbol ML decodable $2^a \times 2^a$ ($a \geq 2$) UWDs are $\frac{3(a-1)}{2^a}$ cspcu and $\frac{4(a-1)}{2^a}$ cspcu, respectively. (Section III)
- Codes which achieve this maximum rate are presented (Explicit construction in the proof of Theorem 2).
- For our explicitly constructed codes, signal sets achieving full-diversity have been identified along with expressions for their coding gain (Section IV).

Organization: In Section II, we define 3- and 4-real symbol decodable unitary weight STBCs and explain the notion of normalization and its use in our analysis. In Section III, we present the main result of this paper, a tight upper bound on the rates of 3- and 4-real symbol decodable $2^a \times 2^a$ UWDs. In Section IV, signal sets achieving full-diversity have been identified for the STBCs given in Section III. Concluding remarks and scope for further work constitute Section V.

Notations: $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively. The set of purely imaginary numbers is represented by $\text{img}(\mathbb{C})$. $GL(n, \mathbb{C})$ denotes the group of invertible matrices of size $n \times n$ with complex entries. For any complex matrix A , A^T and A^H represent the Transpose and Hermitian of A respectively. I_n and 0_n represent the $n \times n$ identity matrix and the zero matrix, respectively. For a set S , $|S|$ denotes the cardinality of S . The Frobenius norm is denoted by $\|\cdot\|_F$. For sets A_1 and A_2 , the Cartesian product of A_1 and A_2 is denoted by $A_1 \times A_2$. For a complex number Z , complex conjugate is Z^* . Also, j represents $\sqrt{-1}$ unless it is used as a subscript or index of some quantity or as a running variable. Bold face small letters (ex: $\mathbf{a}$) represent vectors.

II. REPRESENTATION OF λ -REAL SYMBOL DECODABLE UNITARY WEIGHT STBCS

In this section, we give a representation of λ -real symbol or g -group decodable STBCs. Any $n \times n$ codeword matrix S of a linear STBC $\mathcal{S}$ with g groups is represented as

$$S = \sum_{i=0}^{g-1} \sum_{j=1}^{\lambda} x_{ij} A_{ij}$$

for λ -real symbol decodable STBCs, where $\lambda = \frac{K}{g}$. We consider $\lambda = 3$ and 4. All the K matrices ($A_{ij}, 0 \leq i \leq g-1, 1 \leq j \leq \lambda$) have to be linearly independent over $\mathbb{R}$.

For a g -group decodable STBC $\mathcal{S}$, a set of necessary and sufficient conditions on the weight matrices are (from (3)),

$$A_{ij_1}^H A_{kj_2} + A_{kj_2}^H A_{ij_1} = 0, \quad (4)$$

for $0 \leq i \neq k \leq g-1$ and $1 \leq j_1, j_2 \leq \lambda$. UWDs also satisfy the following criteria

$$A_{ij}^H A_{ij} = I_n \text{ for } 0 \leq i \leq g-1 \text{ and } 1 \leq j \leq \lambda. \quad (5)$$

Lemma 1 ([1]): Let $\mathcal{S}$ be a unitary-weight STBC (i.e. obtained from a UWD) and consider the STBC $\mathcal{S}_U \triangleq \{US | S \in \mathcal{S}\}$, where U is any unitary matrix. Then if $\mathcal{S}$ satisfies conditions (4) and (5), then so does $\mathcal{S}_U$. Further, both the codes have the same coding gain for any signal set $\mathcal{A}$.

The STBCs $\mathcal{S}$ and $\mathcal{S}_U$ are said to be *equivalent*. To simplify our analysis of unitary weight STBCs, we make use of normalization as described below. Let $\mathcal{S}$ be a unitary weight STBC and let its codeword matrix S be expressed as

$$S = \sum_{i=0}^{g-1} \sum_{j=1}^{\lambda} x_{ij} \hat{A}_{ij}.$$

Consider the code $\mathcal{S}_N \triangleq \{\hat{A}_{01}^H S | S \in \mathcal{S}\}$. Clearly, from Lemma 1, $\mathcal{S}_N$ is equivalent to $\mathcal{S}$. The weight matrices of $\mathcal{S}_N$ are

$$A_{ij} = \hat{A}_{01}^H A'_{ij} \text{ for } 0 \leq i \leq g-1 \text{ and } 1 \leq j \leq \lambda.$$

We call the code $\mathcal{S}_N$ to be the normalized code of $\mathcal{S}$. In general, any unitary-weight STBC with one of its weight matrices being the identity matrix is called *normalized* unitary-weight STBC. Studying unitary-weight STBCs becomes simpler by studying the normalized unitary-weight STBCs. Now, the conditions presented in (4) and (5) can be written as

$$A_{ij}^H = -A_{ij} \text{ (equivalently } A_{ij}^2 = -I_n) \quad (6)$$

$$A_{0j_1}^H A_{ij_2} = A_{ij_2} A_{0j_1}, \text{ for } i \neq 0 \text{ and } 1 \leq j, j_1, j_2 \leq \lambda \quad (7)$$

and

$$A_{ij_1} A_{kj_2} = -A_{kj_2} A_{ij_1}, \quad (8)$$

for $1 \leq i \neq k \leq g-1$ and $1 \leq j_1, j_2 \leq \lambda$.

The grouping of weight matrices is shown below.

$A_{01} = I_n$	A_{11}	A_{21}	$\dots$	$A_{(g-1)1}$
A_{02}	A_{12}	A_{22}	$\dots$	$A_{(g-1)2}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$A_{0\lambda}$	$A_{1\lambda}$	$A_{2\lambda}$	$\dots$	$A_{(g-1)\lambda}$

III. AN UPPER BOUND ON THE RATE OF 3-AND 4-REAL SYMBOL DECODABLE UNITARY WEIGHT STBCS

In this section, we determine the upper bound on the rate of 3-and 4-real symbol decodable $2^a \times 2^a$ UWDs and also give a construction scheme to obtain designs meeting this upper bound. To do so, we make use of the following lemmas regarding matrices of size $n \times n$.

Lemma 2 ([8]): Consider $n \times n$ matrices with complex entries.

- 1) If $n = 2^a n_0$, with n_0 odd, then there are l elements of $GL(n, \mathbb{C})$ that anti-commute pairwise if and only if $l \leq 2a + 1$.
- 2) If $n = 2^a$ and matrices $F_1, \dots, F_{2a}$ anti-commute pairwise, then the set of products $F_{i_1} F_{i_2} \dots F_{i_s}$ with $1 \leq i_1 < \dots < i_s \leq 2a$ along with I_n forms a basis for the 2^{2a} dimensional space of all $n \times n$ matrices over $\mathbb{C}$. In each case F_i^2 is a scalar matrix (i.e. $F_i^2 = cI_n$, where $c \in \mathbb{C}$).

Let $F_1, \dots, F_{2a}$ be anti-commuting, anti-Hermitian, unitary matrices (so that $F_i^2 = -I_n$, $i = 1, 2, \dots, 2a$). We can get these matrices from matrix realizations of Clifford algebras and is given in [9].

Lemma 3 ([1]): The product $F_{i_1} F_{i_2} \dots F_{i_s}$ with $1 \leq i_1 < \dots < i_s \leq 2a$ squares to $(-1)^{\frac{s(s+1)}{2}} I_n$.

Lemma 4 ([1]): Let $\Omega_1 = \{F_{i_1}, F_{i_2}, \dots, F_{i_s}\}$ and $\Omega_2 = \{F_{j_1}, F_{j_2}, \dots, F_{j_r}\}$ with $1 \leq i_1 < \dots < i_s \leq 2a$ and $1 \leq j_1 < \dots < j_r \leq 2a$. Let $|\Omega_1 \cap \Omega_2| = p$. Then, the product matrix $F_{i_1} F_{i_2} \dots F_{i_s}$ commutes with $F_{j_1} F_{j_2} \dots F_{j_r}$, if exactly one of the following is satisfied, and anti-commutes otherwise.

- 1) r, s and p are all odd.
- 2) The product rs is even and p is even (including 0).

Lemma 5 ([1]): The maximum rate in cspcu of a $2^a \times 2^a$ unitary-weight SSD code is $\frac{a}{2^{a-1}}$.

The above Lemma is equivalent to showing that $2a$ is the maximum number of groups possible for 2-real symbol (1-complex symbol) decodable $2^a \times 2^a$ UWD.

Though the following theorem (proof given in Appendix A) does not give a tight bound, Theorem 1 and Theorem 2 are used in Theorem 3, to get a tight bound.

Theorem 1: For a 3-real symbol decodable $2^a \times 2^a$ UWD, the rate in cspcu is upper bounded by $\frac{3(2a-1)}{2^{a+1}}$, which is not tight.

Theorem 2: There exists a rate $\frac{a-1}{2^{a-2}}$ cspcu, 4-real symbol decodable $2^a \times 2^a$ UWD for $a \geq 2$.

Proof: Proof is by explicit construction. This construction is based on the proof of Theorem 6 in [2].

For $a \geq 2$ let $m = 2^{a-2}$ and $n = 2^a$. Then from Lemma 2, for $m \times m$ matrices, we can have $2(a-2) + 1$ anti-Hermitian and anti-commuting unitary matrices. Let them be

$E_1, E_2, \dots, E_{2a-3}$. Let $A_{01} = I_n$ and for $1 \leq i \leq 2a-3$,

$$A_{i1} = \begin{bmatrix} E_i & 0_m & 0_m & 0_m \\ 0_m & E_i & 0_m & 0_m \\ 0_m & 0_m & E_i & 0_m \\ 0_m & 0_m & 0_m & E_i \end{bmatrix}; A_{02} = \begin{bmatrix} I_m & 0_m & 0_m & 0_m \\ 0_m & -I_m & 0_m & 0_m \\ 0_m & 0_m & I_m & 0_m \\ 0_m & 0_m & 0_m & -I_m \end{bmatrix};$$

$$A_{03} = \begin{bmatrix} I_m & 0_m & 0_m & 0_m \\ 0_m & I_m & 0_m & 0_m \\ 0_m & 0_m & -I_m & 0_m \\ 0_m & 0_m & 0_m & -I_m \end{bmatrix}; A_{04} = \begin{bmatrix} I_m & 0_m & 0_m & 0_m \\ 0_m & -I_m & 0_m & 0_m \\ 0_m & 0_m & -I_m & 0_m \\ 0_m & 0_m & 0_m & I_m \end{bmatrix};$$

$A_{i2} = A_{02} A_{i1}$; $A_{i3} = A_{03} A_{i1}$ and $A_{i4} = A_{04} A_{i1}$.

It can be easily seen that these matrices satisfy the conditions (4), (6) and (8). So, we have constructed a 4-real symbol $2a-2$ group decodable UWD. The rate of this design is $\frac{4(2a-2)}{2^{a+1}} = \frac{a-1}{2^{a-2}}$ cspcu. ■

Corollary 1: There exists a rate $\frac{3(a-1)}{2^a}$ cspcu, 3-real symbol decodable $2^a \times 2^a$ UWD for $a \geq 2$.

Proof: Straightforward from Theorem 2, by removing A_{i4} matrices. ■

Theorem 3: For a 3-real symbol decodable $2^a \times 2^a$ UWD, the rate in cspcu is tightly upper bounded by $\frac{3(a-1)}{2^a}$.

Proof: Let g be the number of groups. From Theorem 1 and Corollary 1, it is enough to show that $g = 2a-1$ is not possible. To prove this, consider the following grouping of weight matrices:

I_n	F_1	F_2	$\dots$	F_{2a-2}
A_{02}	A_{12}	A_{22}	$\dots$	$A_{(2a-2)2}$
A_{03}	A_{13}	A_{23}	$\dots$	$A_{(2a-2)3}$

The theorem is proved in the following 5 steps, the proof for all of which is given in Appendix B:

- Step 1: Finding a relation between coefficients of A_{1i} and A_{2j} ($i, j \in \{2, 3\}$) when expanded in terms of the basis of Lemma 2.
- Step 2: Finding a relation between coefficients of A_{0i} , A_{1j} and A_{2k} ($i, j, k \in \{2, 3\}$) when expanded in terms of the basis of Lemma 2.
- Step 3: Showing that there is a possibility of 7 types of solutions that takes into account the relations in Step 1.
- Step 4: Showing that after including the relations from Step 2 also, there is a possibility of 7 types of solutions.
- Step 5: None of the 7 solutions in Step 4 is possible. ■

Theorem 4: For a 4-real symbol decodable $2^a \times 2^a$ UWD, the rate in cspcu is tightly upper bounded by $\frac{(a-1)}{2^{a-2}}$.

The proof is given in Appendix C.

Example 1: Consider a 4×4 UWD with weight matrices $A_{01} = I_4$, $A_{11} = jI_4$ and

$$A_{02} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} A_{03} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A_{04} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with $A_{12} = jA_{02}$, $A_{13} = jA_{03}$ and $A_{14} = jA_{04}$. The codeword matrix $S(\mathbf{x}_0, \mathbf{x}_1)$ is given by $(\mathbf{x}_i = [x_0 \ x_1 \ \dots \ x_\lambda])$

$$S(\mathbf{x}_0, \mathbf{x}_1) = \sum_{i=0}^1 \sum_{j=1}^4 x_{ij} A_{ij} = \begin{bmatrix} Z_1 & 0 & 0 & 0 \\ 0 & Z_2 & 0 & 0 \\ 0 & 0 & Z_3 & 0 \\ 0 & 0 & 0 & Z_4 \end{bmatrix},$$

for 4-real symbol decodable UWD and $S(\mathbf{x}_0, \mathbf{x}_1)$ is given by

$$S(\mathbf{x}_0, \mathbf{x}_1) = \sum_{i=0}^1 \sum_{j=1}^3 x_{ij} A_{ij} = \begin{bmatrix} Z_5 & 0 & 0 & 0 \\ 0 & Z_6 & 0 & 0 \\ 0 & 0 & Z_7 & 0 \\ 0 & 0 & 0 & Z_8 \end{bmatrix},$$

for 3-real symbol decodable UWD, where,

$$\begin{aligned} Z_1 &= x_{01} + x_{02} + x_{03} + x_{04} + jx_{11} + jx_{12} + jx_{13} + jx_{14} \\ Z_2 &= x_{01} - x_{02} + x_{03} - x_{04} + jx_{11} - jx_{12} + jx_{13} - jx_{14} \\ Z_3 &= x_{01} + x_{02} - x_{03} - x_{04} + jx_{11} + jx_{12} - jx_{13} - jx_{14} \\ Z_4 &= x_{01} - x_{02} - x_{03} + x_{04} + jx_{11} - jx_{12} - jx_{13} + jx_{14} \\ Z_5 &= x_{01} + x_{02} + x_{03} + jx_{11} + jx_{12} + jx_{13} \\ Z_6 &= x_{01} - x_{02} + x_{03} + jx_{11} - jx_{12} + jx_{13} \\ Z_7 &= x_{01} + x_{02} - x_{03} + jx_{11} + jx_{12} - jx_{13} \\ Z_8 &= x_{01} - x_{02} - x_{03} + jx_{11} - jx_{12} - jx_{13}. \end{aligned}$$

It is easily checked that the weight matrices above satisfy (6) to (8). So, for this 4-real symbol decodable UWD, rate is 1 cspcu and for 3-real symbol decodable UWD, rate is $\frac{3}{4}$ cspcu.

Example 2: Consider a 8×8 UWD with codeword matrix $S(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sum_{i=0}^3 \sum_{j=1}^4 x_{ij} A_{ij}$ given by

$$\begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 \\ Z_4 & -Z_3 & -Z_2 & Z_1 & Z_8 & -Z_7 & -Z_6 & Z_5 \\ Z_5 & Z_6 & Z_7 & Z_8 & Z_1 & Z_2 & Z_3 & Z_4 \\ Z_8 & -Z_7 & -Z_6 & Z_5 & Z_4 & -Z_3 & -Z_2 & Z_1 \\ Z_2^* & -Z_1^* & Z_4^* & -Z_3^* & Z_6^* & -Z_5^* & Z_8^* & -Z_7^* \\ Z_3^* & Z_4^* & -Z_1^* & -Z_2^* & Z_7^* & Z_8^* & -Z_5^* & -Z_6^* \\ Z_6^* & -Z_5^* & Z_8^* & -Z_7^* & Z_2^* & -Z_1^* & Z_4^* & -Z_3^* \\ Z_7^* & Z_8^* & -Z_5^* & -Z_6^* & Z_3^* & Z_4^* & -Z_1^* & -Z_2^* \end{bmatrix}.$$

where, $Z_1 = x_{01} + jx_{11}$, $Z_2 = x_{21} + jx_{31}$, $Z_3 = x_{22} + jx_{32}$, $Z_4 = x_{02} + jx_{12}$, $Z_5 = x_{03} + jx_{13}$, $Z_6 = x_{23} + jx_{33}$, $Z_7 = x_{24} + jx_{34}$ and $Z_8 = x_{04} + jx_{14}$. By calculating $S^H S$, we can easily say that, this design is 4-real symbol decodable UWD. Rate of this design is 1 cspcu. By assigning $x_{i4} = 0$ ($i \in \{0, 1, 2, 3\}$), we get 3-real symbol decodable UWD with rate $\frac{3}{4}$ cspcu.

IV. DIVERSITY AND CODING GAIN

In this Section, we show that for the code shown in Theorem 2, full diversity is achievable for 4-real symbol decodable STBCs with $n = 2^a$ antennas. Also, expressions for coding gain are presented.

Let, $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$ take values from a finite signal set $\mathcal{B}$. The differential signal set $\Delta\mathcal{B}$ of signal set $\mathcal{B}$ is defined as

$$\Delta\mathcal{B} = \{\Delta\mathbf{xx}' = \mathbf{x} - \mathbf{x}' | \mathbf{x}, \mathbf{x}' \in \mathcal{B}\}.$$

For a differential signal set $\Delta\mathcal{B}$, $\forall \Delta\mathbf{xx}' \in \Delta\mathcal{B}$ Let,

$$\begin{aligned} \Psi(\Delta\mathbf{xx}') &= [((\Delta\mathbf{xx}')_1 + (\Delta\mathbf{xx}')_2 + (\Delta\mathbf{xx}')_3 + (\Delta\mathbf{xx}')_4)^2]^{\frac{1}{4}} \\ &\quad [((\Delta\mathbf{xx}')_1 - (\Delta\mathbf{xx}')_2 + (\Delta\mathbf{xx}')_3 - (\Delta\mathbf{xx}')_4)^2]^{\frac{1}{4}} \\ &\quad [((\Delta\mathbf{xx}')_1 + (\Delta\mathbf{xx}')_2 - (\Delta\mathbf{xx}')_3 - (\Delta\mathbf{xx}')_4)^2]^{\frac{1}{4}} \\ &\quad [((\Delta\mathbf{xx}')_1 - (\Delta\mathbf{xx}')_2 - (\Delta\mathbf{xx}')_3 + (\Delta\mathbf{xx}')_4)^2]^{\frac{1}{4}}. \end{aligned}$$

Theorem 5: If $\mathcal{B}$ is the signal set, from which the variables of a group take values from, for the STBC of Theorem 2, then, the code achieves full-diversity if the differential signal set $\Delta\mathcal{B}$ satisfies

$$\Psi(\Delta\mathbf{xx}') > 0, \quad \forall (\Delta\mathbf{xx}' \neq 0) \in \Delta\mathcal{B}. \quad (9)$$

Proof: Proof available in Appendix D. ■

Now, we will find the signal set $\mathcal{B}$, which gives full-diversity for the STBC of Theorem 2.

Let us define a new vector variable $\Delta\mathbf{q} \triangleq [\Delta q_1 \ \Delta q_2 \ \Delta q_3 \ \Delta q_4]^T$ as $\Delta\mathbf{q} = P \Delta\mathbf{xx}'$, where

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Now, $\Delta_{\min}$ (from Appendix D) can be re-written as (since P is invertible $\Delta\mathbf{q} = 0$ iff $\Delta\mathbf{xx}' = 0$)

$$\Delta_{\min} = \min_{\Delta\mathbf{q} \neq 0} [(\Delta q_1)^2 (\Delta q_2)^2 (\Delta q_3)^2 (\Delta q_4)^2]^{\frac{1}{4}}.$$

To achieve full diversity we need $\Delta_{\min} > 0$. This can be achieved if $\Delta q_i \neq 0$, $\forall 1 \leq i \leq 4$. And, this can be guaranteed by letting $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$ take values from $P^{-1}\mathcal{G}_4\mathbb{Z}^4$, where, $\mathcal{G}_4$ is the generator matrix of a 4-dimensional lattice designed to maximize the product distance [10], [11], and the coding gain $\delta_{\min}$ is given by

$$\delta_{\min} = (\Delta_{\min})^{\frac{1}{n}} = \min_{\Delta\mathbf{q} \neq 0} [(\Delta q_1)^2 (\Delta q_2)^2 (\Delta q_3)^2 (\Delta q_4)^2]^{\frac{1}{4}}. \quad (10)$$

The right hand side of (10) can be obtained from [10], [11].

A. Calculation of Diversity and Coding gain with examples

Let, the signal set $\mathcal{B}$ be obtained for $[\pm 1 \ \pm 1 \ \pm 1 \ \pm 1]^T \in \mathbb{Z}^4$ from $P^{-1}\mathcal{G}_4\mathbb{Z}^4$. Here, $P^{-1} = \frac{1}{4}P$ and from [11]

$$\mathcal{G}_4 = \begin{bmatrix} -0.3664 & -0.7677 & 0.4231 & 0.3121 \\ -0.2264 & -0.4745 & -0.6846 & -0.5050 \\ -0.4745 & 0.2264 & -0.5050 & 0.6846 \\ -0.7677 & 0.3664 & 0.3121 & -0.4231 \end{bmatrix}.$$

Let $\mathbf{x} \in \mathcal{B}$, be written as $\mathbf{x} = eP^{-1}\mathcal{G}_4\mathbf{z}$, for some $\mathbf{z} \in [\pm 1 \ \pm 1 \ \pm 1 \ \pm 1]^T$, where, e is used for normalizing the average energy. Define $E(\mathbf{x}) = \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then

$$E(\mathbf{x}) = \frac{1}{4}e^2\mathbf{z}^T\mathcal{G}_4^T P P^{-1}\mathcal{G}_4\mathbf{z} = \frac{1}{4}e^2\mathbf{z}^T\mathbf{z} = e^2.$$

Here, we used the fact that $\mathcal{G}_4^T\mathcal{G}_4 = I_4$ from [11] and $\mathbf{z}^T\mathbf{z} = 4$. So, $E(\mathbf{x}) = e^2 = E$ for all $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{B}$.

Let, $\Delta u = \mathcal{G}_4 \Delta v$, where $\Delta u = [\Delta u_1 \ \Delta u_2 \ \Delta u_3 \ \Delta u_4]$ and $\Delta v = [\Delta v_1 \ \Delta v_2 \ \Delta v_3 \ \Delta v_4]$. Let $\Delta v_i \in \{-2, 0, 2\}$. Define,

$$\delta_u \triangleq \min_{\Delta u \neq 0} [(\Delta u_1)^2 (\Delta u_2)^2 (\Delta u_3)^2 (\Delta u_4)^2]^{\frac{1}{4}} = 0.6503.$$

Now, the following example calculates the coding gains for a 4×4 and 8×8 , 4-real symbol decodable Unitary-weight STBCs of Example 1 and Example 2.

Example 3: For the codeword S of Example 1 (for 4-real symbol decodable UWD), average energy E_{avg} is given by

$$\begin{aligned} E_{avg} &= \frac{1}{16} \mathbb{E}(\|S\|_F^2) = \frac{1}{16} \mathbb{E}(|Z_1|^2 + |Z_2|^2 + |Z_3|^2 + |Z_4|^2) \\ &= \frac{1}{4} \mathbb{E}(\|\mathbf{x}_0\|^2 + \|\mathbf{x}_1\|^2) = \frac{E}{2}, \end{aligned}$$

where $\mathbb{E}$ is over all possible information symbols. For $E_{avg} = 1$ to be satisfied, $E = 2$ and $e = \sqrt{2}$. From (10), coding gain is given by $e^2 \delta_u = 1.3006 = 1.1414 \text{ dB}$. Since $\delta_{min} > 0$, this STBC has full diversity.

For the codeword S of Example 2, the average energy E_{avg} is given by

$$E_{avg} = \frac{1}{64} \mathbb{E}(\|S\|_F^2) = \frac{1}{8} \mathbb{E}(\sum_{i=1}^8 |Z_i|^2) = \frac{1}{8} \mathbb{E}(\sum_{j=0}^3 \|\mathbf{x}_j\|^2) = \frac{1}{2} E,$$

where $\mathbb{E}$ is over all possible information symbols. For $E_{avg} = 1$ to be satisfied, $E = 2$ and $e = \sqrt{2}$.

Let, for a complex number $Z_i = \Re\{Z_i\} + j\Im\{Z_i\}$, $\Delta Z_i = \Re\{\Delta Z_i\} + j\Im\{\Delta Z_i\}$, where $\Re\{Z_i\}$, $\Im\{Z_i\}$ are the real and imaginary parts of Z_i respectively. To find coding gain we need to find $\det[(\Delta S)^H (\Delta S)] = \det[(\Delta S)(\Delta S)^H]$.

$$(\Delta S)(\Delta S)^H = \begin{bmatrix} \Phi(t, \alpha, \beta, \gamma) & 0_4 \\ 0_4 & \Phi(t, -\alpha, \beta, -\gamma) \end{bmatrix}.$$

Here,

$$\Phi(t, \alpha, \beta, \gamma) = \begin{bmatrix} t & \alpha & \beta & \gamma \\ \alpha & t & \gamma & \beta \\ \beta & \gamma & t & \alpha \\ \gamma & \beta & \alpha & t \end{bmatrix},$$

where,

$$\begin{aligned} t &= \sum_{i=1}^8 |\Delta Z_i|^2, \\ \alpha &= 2\Re\{\Delta Z_1 \Delta Z_4^* - \Delta Z_2 \Delta Z_3^* + \Delta Z_5 \Delta Z_8^* - \Delta Z_6 \Delta Z_7^*\}, \\ \beta &= 2\Re\{\Delta Z_1 \Delta Z_5^* + \Delta Z_2 \Delta Z_6^* + \Delta Z_3 \Delta Z_7^* + \Delta Z_4 \Delta Z_8^*\}, \\ \gamma &= 2\Re\{\Delta Z_1 \Delta Z_8^* - \Delta Z_2 \Delta Z_7^* - \Delta Z_3 \Delta Z_6^* + \Delta Z_4 \Delta Z_5^*\}. \end{aligned}$$

$$\begin{aligned} \det[\Phi(t, \alpha, \beta, \gamma)] &= (t + \gamma + \alpha + \beta)(t + \gamma - \alpha - \beta) \\ &\quad (t - \gamma + \alpha - \beta)(t - \gamma - \alpha + \beta) \\ &= \det[\Phi(t, -\alpha, \beta, -\gamma)] \end{aligned}$$

Now,

$$\begin{aligned} \det[(\Delta S)^H (\Delta S)] &= \det[\Phi(t, \alpha, \beta, \gamma)] \times \det[\Phi(t, -\alpha, \beta, -\gamma)] \\ &= (\det[\Phi(t, \alpha, \beta, \gamma)])^2. \end{aligned}$$

As in (30), here too $\det[(\Delta S)^H (\Delta S)]$ is a product of sum of squares of real numbers, so,

$$\begin{aligned} \det[(\Delta S)^H (\Delta S)] &\geq \\ &[(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4]^4 \\ &[(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4]^4 \\ &[(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4]^4 \\ &[(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4]^4, \end{aligned}$$

for some $0 \leq i \leq 3$. And, Δ_{min} occurs when all but one among $\Delta \mathbf{x}_i \mathbf{x}'_i$, $0 \leq i \leq 3$ are zeros. So, from (10), coding gain is given by $e^2 \delta_u = 1.3006 = 1.1414 \text{ dB}$. Since $\delta_{min} > 0$, this STBC has full diversity.

V. DISCUSSION AND CONCLUDING REMARKS

In this paper, we have shown that, the maximum rate achieved by 3- and 4-real symbol ML decodable $2^a \times 2^a$ UWDs is $\frac{3(a-1)}{2^a}$ and $\frac{4(a-1)}{2^a}$ cspcu respectively. We have also given a STBC which achieves the maximum rate. And, also shown that this STBC can achieve full-diversity for rotated lattice constellations. Possible directions for further research are:

- 1) A general upper bound on the rate of the λ real symbol decodable UWDs is yet to be found.
- 2) Even though maximum rate possible for g -group decodable CUWDs is found, it is not found for general UWDs.

These could possibly be the future direction of research.

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APPENDIX A

Proof of Theorem 1

Let g be the number of groups. The rate is given by $\frac{3g}{2a+1}$ cspcu. It is enough to show that g is upper bounded by $2a-1$. If g is more than $2a$ we can remove one weight matrix from each group and get 2-real symbol decodable UWD, which contradicts Lemma 5. So, $g \leq 2a$. To prove that g cannot be $2a$ let us consider the following weight matrices,

I_n	F_1	F_2	$\dots$	F_{2a-1}
A_{02}	A_{12}	A_{22}	$\dots$	$A_{(2a-1)2}$
A_{03}	A_{13}	A_{23}	$\dots$	$A_{(2a-1)3}$

where $A_{01} = I_n$ and $A_{j1} = F_j$ ($j = 1, 2, \dots, 2a-1$). From Case-3 in the proof of Theorem 1 of [1], A_{j2} can be written as $A_{j2} = \pm m \prod_{i=1, i \neq j}^{2a-1} F_i$, for $j = 1, 2, \dots, 2a-1$. Hence, A_{j3} must be equal to $a_{j,1}F_j + a_{j,2}F_{2a} + a_{j,4}F_1F_2 \dots F_{2a}$. But these A_{j3} s violate linear independence of weight matrices, so we cannot have A_{j3} s for $g = 2a$. This completes the proof.

APPENDIX B

Proof of Theorem 3

Step 1:

Let $A_{ik} = \sum_{j=1}^{2^{2a}} a_{ikj} F_1^{\lambda_{1,j}} F_2^{\lambda_{2,j}} \dots F_{2a}^{\lambda_{2a,j}}$, $\lambda_{m,j} \in \{0, 1\}$, $m = 1, 2, \dots, 2a$, $i = 0, 1, 2, \dots, 2a-2$, $k = 2, 3$ and $a_{ikj} \in \mathbb{C}$. This is possible because of Lemma 2. Considering A_{1k} , since A_{1k} anti-commutes with $F_2, F_3, \dots, F_{2a-2}$, every individual term of A_{1k} must anti-commute with $F_2, F_3, \dots, F_{2a-2}$. The only matrices from the set $\{F_1^{\lambda_1} F_2^{\lambda_2} \dots F_{2a}^{\lambda_{2a}}, \lambda_i \in \{0, 1\}, i = 1, 2, \dots, 2a\}$ that anti-commute with $F_2, F_3, \dots, F_{2a-2}$ are $F_1, F_{2a-1}, F_{2a}, F_1 F_{2a-1} F_{2a}, F_1 F_2 \dots F_{2a-2}, F_2 F_3 \dots F_{2a-1}, F_2 F_3 \dots F_{2a-2} F_{2a}$, and $F_1 F_2 \dots F_{2a}$. Hence, let (for $k \in \{2, 3\}$)

$$\begin{aligned}
 A_{1k} = & a_{k1}F_1 + a_{k2}F_{2a-1} + a_{k3}F_{2a} + a_{k4}F_1F_{2a-1}F_{2a} \\
 & + a_{k5}F_1F_2 \dots F_{2a-2} + a_{k6}F_2F_3 \dots F_{2a-1} \\
 & + a_{k7}F_2F_3 \dots F_{2a-2}F_{2a} + a_{k8}F_1F_2 \dots F_{2a}.
 \end{aligned}$$

Similarly, let (for $k \in \{2, 3\}$)

$$\begin{aligned}
 A_{2k} = & c_{k1}F_2 + c_{k2}F_{2a-1} + c_{k3}F_{2a} + c_{k4}F_1F_{2a-1}F_{2a} \\
 & + c_{k5}F_1F_2 \dots F_{2a-2} + c_{k6}F_1F_3 \dots F_{2a-1} \\
 & + c_{k7}F_1F_3 \dots F_{2a-2}F_{2a} + c_{k8}F_1F_2 \dots F_{2a}. \quad (11)
 \end{aligned}$$

Let, $(F_1F_2 \dots F_{2a})^2 = pI$, where $p = -1$ when, a is odd and $p = 1$ when, a is even. Further, from (6), $A_{1k}^2 = A_{2k}^2 =$

$-I_n$ and from the above equations we have

$$\begin{aligned} A_{1k}^2 &= -(a_{k1}^2 + a_{k2}^2 + a_{k3}^2 - a_{k4}^2 + p(a_{k5}^2 + a_{k6}^2 + a_{k7}^2 - a_{k8}^2))I_n \\ &\quad - 2(a_{k1}a_{k4} + pa_{k5}a_{k8})F_{2a-1}F_{2a} \\ &\quad + 2(a_{k1}a_{k6} + a_{k2}a_{k5})F_1 \cdots F_{2a-1} \\ &\quad + 2(a_{k1}a_{k7} + a_{k3}a_{k5})F_1 \cdots F_{2a-2}F_{2a} \\ &\quad + 2(a_{k2}a_{k4} - pa_{k6}a_{k8})F_1F_{2a} \\ &\quad - 2(a_{k2}a_{k7} - a_{k3}a_{k6})F_2F_3 \cdots F_{2a} \\ &\quad - 2(a_{k3}a_{k4} - pa_{k7}a_{k8})F_1F_{2a-1}, \\ A_{2k}^2 &= -(c_{k1}^2 + c_{k2}^2 + c_{k3}^2 - c_{k4}^2 + p(c_{k5}^2 + c_{k6}^2 + c_{k7}^2 - c_{k8}^2))I_n \\ &\quad - 2(c_{k1}c_{k4} + pc_{k5}c_{k8})F_{2a-1}F_{2a} \\ &\quad - 2(c_{k1}c_{k6} - c_{k2}c_{k5})F_1 \cdots F_{2a-1} \\ &\quad - 2(c_{k1}c_{k7} - c_{k3}c_{k5})F_1 \cdots F_{2a-2}F_{2a} \\ &\quad + 2(c_{k2}c_{k4} + pc_{k6}c_{k8})F_2F_{2a} \\ &\quad - 2(c_{k2}c_{k7} - c_{k3}c_{k6})F_1F_3 \cdots F_{2a-1}F_{2a} \\ &\quad - 2(c_{k3}c_{k4} + pc_{k7}c_{k8})F_2F_{2a-1}. \end{aligned}$$

Since $I_n, F_{2a-1}F_{2a}, F_1F_2 \cdots F_{2a-1}, F_1F_2 \cdots F_{2a-2}F_{2a}, F_1F_{2a}, F_2F_3 \cdots F_{2a}, F_1F_{2a-1}, F_2F_{2a}, F_1F_3 \cdots F_{2a}$ and F_2F_{2a-1} are linearly independent over $\mathbb{C}$, the following equations have to be satisfied.

$$\left. \begin{aligned} a_{k1}^2 + a_{k2}^2 + a_{k3}^2 - a_{k4}^2 + p(a_{k5}^2 + a_{k6}^2 + a_{k7}^2 - a_{k8}^2) &= 1; \\ c_{k1}^2 + c_{k2}^2 + c_{k3}^2 - c_{k4}^2 + p(c_{k5}^2 + c_{k6}^2 + c_{k7}^2 - c_{k8}^2) &= 1; \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} a_{k1}a_{k4} + pa_{k5}a_{k8} &= 0; \quad a_{k1}a_{k6} + a_{k2}a_{k5} = 0; \\ a_{k1}a_{k7} + a_{k3}a_{k5} &= 0; \quad a_{k2}a_{k4} - pa_{k6}a_{k8} = 0; \\ a_{k2}a_{k7} - a_{k3}a_{k6} &= 0; \quad a_{k3}a_{k4} - pa_{k7}a_{k8} = 0; \\ c_{k1}c_{k4} + pc_{k5}c_{k8} &= 0; \quad c_{k1}c_{k6} - c_{k2}c_{k5} = 0; \\ c_{k1}c_{k7} - c_{k3}c_{k5} &= 0; \quad c_{k2}c_{k4} + pc_{k6}c_{k8} = 0; \\ c_{k2}c_{k7} - c_{k3}c_{k6} &= 0; \quad c_{k3}c_{k4} + pc_{k7}c_{k8} = 0. \end{aligned} \right\} \quad (13)$$

Since $A_{ik}^H = -A_{ik}$, $i \in \{1, 2\}$ $k \in \{2, 3\}$, we need, $a_{k1}, a_{k2}, a_{k3}, c_{k1}, c_{k2}, c_{k3} \in \mathbb{R}$, $a_{k4}, c_{k4} \in \text{img}(\mathbb{C})$. Also if $p = 1$, $a_{k5}, a_{k6}, a_{k7}, c_{k5}, c_{k6}, c_{k7} \in \mathbb{R}$, $a_{k8} \in \text{img}(\mathbb{C})$, $c_{k8} \in \text{img}(\mathbb{C})$ and if $p = -1$, $a_{k5}, a_{k6}, a_{k7}, c_{k5}, c_{k6}, c_{k7} \in \text{img}(\mathbb{C})$, $a_{k8} \in \mathbb{R}$, $c_{k8} \in \mathbb{R}$.

For A_{1i} and A_{2j} ($i, j \in \{2, 3\}$) to anti-commute the following conditions need to be satisfied (we get these conditions by equating $A_{1i}A_{2j} + A_{2j}A_{1i} = 0$ and using the linear independence condition over $\mathbb{C}$).

$$\left. \begin{aligned} a_{i2}c_{j4} + pa_{i8}c_{j6} &= 0; \quad a_{i2}c_{j5} + a_{i5}c_{j2} = 0; \quad a_{i2}c_{j7} - a_{i3}c_{j6} = 0; \\ a_{i3}c_{j4} + pa_{i8}c_{j7} &= 0; \quad a_{i3}c_{j5} + a_{i5}c_{j3} = 0; \quad a_{i4}c_{j2} - pa_{i6}c_{j8} = 0; \\ a_{i4}c_{j3} - pa_{i7}c_{j8} &= 0; \quad a_{i4}c_{j6} + a_{i6}c_{j4} = 0; \quad a_{i4}c_{j7} + a_{i7}c_{j4} = 0; \\ a_{i5}c_{j8} + a_{i8}c_{j5} &= 0; \quad a_{i6}c_{j3} - a_{i7}c_{j2} = 0; \quad a_{i6}c_{j7} - a_{i7}c_{j6} = 0; \\ a_{i2}c_{j2} + a_{i3}c_{j3} + pa_{i5}c_{j5} - pa_{i8}c_{j8} &= 0. \end{aligned} \right\} \quad (14)$$

Step 2:

From Lemma 1, by multiplying all the weight matrices by $-A_{11}$ (i.e., $-F_1$), we get another equivalent UWD with weight matrices grouped as shown at the top of the next page, after interchanging the first and the second columns.

It should be noted that in the first row, except I_n , all are mutually anti-commuting matrices and all of them also anti-commute with F_1F_{2a-1} and F_1F_{2a} . So, $-F_1, -F_1F_2, -F_1F_3, \dots, -F_1F_{2a-1}$ and $-F_1F_{2a}$ are $2a$ pairwise anti-commuting matrices.

From (8), A'_{1k} ($k \in \{2, 3\}$) has to be anti-commuting with $A'_{21}, A'_{31}, \dots, A'_{(2a-2)1}$. The only matrices from the set $\{F_1^{\lambda_1}F_2^{\lambda_2} \cdots F_{2a}^{\lambda_{2a}}, \lambda_i \in \{0, 1\}, i = 1, 2, \dots, 2a\}$ that anti-commute with $F_1F_2, F_1F_3, \dots$ and F_1F_{2a-2} are $F_1, F_1F_{2a-1}, F_1F_{2a}, F_1F_{2a-1}F_{2a}, F_2 \cdots F_{2a-2}, F_2F_3 \cdots F_{2a-1}, F_2F_3 \cdots F_{2a-2}F_{2a}, F_2F_3 \cdots F_{2a}$. Hence, let (for $k \in \{2, 3\}$)

$$\begin{aligned} A'_{1k} &= b_{k1}F_1 + b_{k2}F_1F_{2a-1} + b_{k3}F_1F_{2a} + b_{k4}F_1F_{2a-1}F_{2a} \\ &\quad + b_{k5}F_2 \cdots F_{2a-2} + b_{k6}F_2F_3 \cdots F_{2a-1} \\ &\quad + b_{k7}F_2F_3 \cdots F_{2a-2}F_{2a} + b_{k8}F_2F_3 \cdots F_{2a}. \end{aligned}$$

From (6), $(A'_{1k})^2 = -I_n$ and we have

$$\begin{aligned} (A'_{1k})^2 &= -(b_{k1}^2 + b_{k2}^2 + b_{k3}^2 - b_{k4}^2 + p(b_{k5}^2 + b_{k6}^2 + b_{k7}^2 - b_{k8}^2))I_n \\ &\quad - 2(b_{k1}b_{k4} + pb_{k5}b_{k8})F_{2a-1}F_{2a} \\ &\quad - 2(b_{k2}b_{k4} + pb_{k6}b_{k8})F_{2a} \\ &\quad + 2(b_{k1}b_{k6} - b_{k2}b_{k5})F_1F_2 \cdots F_{2a-1} \\ &\quad + 2(b_{k1}b_{k7} - b_{k3}b_{k5})F_1F_2 \cdots F_{2a-2}F_{2a} \\ &\quad - 2(b_{k2}b_{k7} - b_{k3}b_{k6})F_1F_2F_3 \cdots F_{2a} \\ &\quad + 2(b_{k3}b_{k4} + pb_{k7}b_{k8})F_{2a-1}. \end{aligned}$$

Since, $I_n, F_{2a-1}F_{2a}, F_1F_2 \cdots F_{2a-1}, F_1F_2 \cdots F_{2a-2}F_{2a}, F_{2a}, F_1F_2F_3 \cdots F_{2a}, F_{2a-1}$ are linearly independent over $\mathbb{C}$ the following equations have to be satisfied.

$$\begin{aligned} b_{k1}^2 + b_{k2}^2 + b_{k3}^2 - b_{k4}^2 + p(b_{k5}^2 + b_{k6}^2 + b_{k7}^2 - b_{k8}^2) &= 1; \\ b_{k1}b_{k4} + pb_{k5}b_{k8} &= 0; \quad b_{k1}b_{k6} - b_{k2}b_{k5} = 0; \quad b_{k1}b_{k7} - b_{k3}b_{k5} = 0; \\ b_{k2}b_{k4} + pb_{k6}b_{k8} &= 0; \quad b_{k2}b_{k7} - b_{k3}b_{k6} = 0; \quad b_{k3}b_{k4} + pb_{k7}b_{k8} = 0. \end{aligned} \quad (15)$$

Since $A_{1k}^H = -A'_{1k}$ ($k \in \{2, 3\}$), we need $b_{k1}, b_{k2}, b_{k3} \in \mathbb{R}$, $b_{k4} \in \text{img}(\mathbb{C})$. If $p = 1$, $b_{k5}, b_{k6}, b_{k7} \in \mathbb{R}$, $b_{k8} \in \text{img}(\mathbb{C})$ and if $p = -1$, $b_{k5}, b_{k6}, b_{k7} \in \text{img}(\mathbb{C})$, $b_{k8} \in \mathbb{R}$.

For A'_{1k} and A'_{2j} ($j, k \in \{2, 3\}$) to anti-commute the following conditions need to be satisfied (we get these conditions by equating $A'_{1k}A'_{2j} + A'_{2j}A'_{1k} = 0$ and using the linear independence condition over $\mathbb{C}$)

$$\left. \begin{aligned} b_{k2}c_{j4} - pb_{k8}c_{j6} &= 0; \quad b_{k2}c_{j5} - b_{k5}c_{j2} = 0; \quad b_{k2}c_{j7} - b_{k3}c_{j6} = 0; \\ b_{k3}c_{j4} - pb_{k8}c_{j7} &= 0; \quad b_{k3}c_{j5} - b_{k5}c_{j3} = 0; \quad b_{k4}c_{j2} - pb_{k6}c_{j8} = 0; \\ b_{k4}c_{j3} - pb_{k7}c_{j8} &= 0; \quad b_{k4}c_{j6} + b_{k6}c_{j4} = 0; \quad b_{k4}c_{j7} + b_{k7}c_{j4} = 0; \\ b_{k5}c_{j8} + b_{k8}c_{j5} &= 0; \quad b_{k6}c_{j3} - b_{k7}c_{j2} = 0; \quad b_{k6}c_{j7} - b_{k7}c_{j6} = 0; \\ b_{k2}c_{j2} + b_{k3}c_{j3} - pb_{k5}c_{j5} + pb_{k8}c_{j8} &= 0. \end{aligned} \right\} \quad (16)$$

Similarly, by equating $A_{1k}^H A'_{0i} + A'_{0i} A'_{1k} = 0$ ($i, k \in \{2, 3\}$) and using the linear independence condition over $\mathbb{C}$ we get

I_n	$A'_{11} = -F_1$	$A'_{21} = -F_1 F_2$	$\dots$	$A'_{(2a-2)1} = -F_1 F_{2a-2}$
$A'_{02} = -F_1 A_{12}$	$A'_{12} = -F_1 A_{02}$	$A'_{22} = -F_1 A_{22}$	$\dots$	$A'_{(2a-2)2} = -F_1 A_{(2a-2)2}$
$A'_{03} = -F_1 A_{13}$	$A'_{13} = -F_1 A_{03}$	$A'_{23} = -F_1 A_{23}$	$\dots$	$A'_{(2a-2)3} = -F_1 A_{(2a-2)3}$

$$\begin{aligned}
b_{k2}a_{i4} + pb_{k8}a_{i6} &= 0; & b_{k2}a_{i5} - b_{k5}a_{i2} &= 0; & b_{k2}a_{i7} - b_{k3}a_{i6} &= 0; \\
b_{k3}a_{i4} + pb_{k8}a_{i7} &= 0; & b_{k3}a_{i5} - b_{k5}a_{i3} &= 0; & b_{k4}a_{i2} - pb_{k6}a_{i8} &= 0; \\
b_{k4}a_{i3} - pb_{k7}a_{i8} &= 0; & b_{k4}a_{i6} - b_{k6}a_{i4} &= 0; & b_{k4}a_{i7} - b_{k7}a_{i4} &= 0; \\
b_{k5}a_{i8} + b_{k8}a_{i5} &= 0; & b_{k6}a_{i3} - b_{k7}a_{i2} &= 0; & b_{k6}a_{i7} - b_{k7}a_{i6} &= 0; \\
b_{k2}a_{i2} + b_{k3}a_{i3} - pb_{k5}a_{i5} + pb_{k8}a_{i8} &= 0.
\end{aligned} \quad (17)$$

Step 3:

The conditions (13) to (17) can also be re-written as

$$\begin{aligned}
\frac{a_{i1}}{a_{i5}} &= \frac{-pa_{i8}}{a_{i4}} = \frac{-a_{i2}}{a_{i6}} = \frac{-a_{i3}}{a_{i7}} = z_{i1} \\
\frac{c_{j1}}{c_{j5}} &= \frac{-pc_{j8}}{c_{j4}} = \frac{c_{j2}}{c_{j6}} = \frac{c_{j3}}{c_{j7}} = z_{j2} \\
\frac{b_{k1}}{b_{k5}} &= \frac{-pb_{k8}}{b_{k4}} = \frac{b_{k2}}{b_{k6}} = \frac{b_{k3}}{b_{k7}} = z_{k9}
\end{aligned} \quad (18)$$

The relations between the coefficients in the representation of A_{1i} and A_{2j} , $i, j \in \{2, 3\}$ are

$$\begin{aligned}
\frac{a_{i2}}{pa_{i8}} &= \frac{a_{i6}}{a_{i4}} = \frac{-c_{j6}}{c_{j4}} = \frac{c_{j2}}{pc_{j8}} = z_3; & \frac{a_{i2}}{a_{i5}} &= \frac{-c_{j2}}{c_{j5}} = z_4; \\
\frac{a_{i2}}{a_{i3}} &= \frac{a_{i6}}{a_{i7}} = \frac{c_{j6}}{c_{j7}} = \frac{c_{j2}}{c_{j3}} = z_5; & \frac{a_{i5}}{pa_{i8}} &= \frac{-c_{j5}}{pc_{j8}} = z_8; \\
\frac{a_{i7}}{a_{i4}} &= \frac{a_{i3}}{pa_{i8}} = \frac{-c_{j7}}{c_{j4}} = \frac{c_{j3}}{pc_{j8}} = z_6; & \frac{a_{i3}}{a_{i5}} &= \frac{-c_{j3}}{c_{j5}} = z_7; \\
a_{i2}c_{j2} + a_{i3}c_{j3} + pa_{i5}c_{j5} - pa_{i8}c_{j8} &= 0.
\end{aligned} \quad (19)$$

The relations between the coefficients in the representation of A_{1i} and A_{0k} , (i.e., A'_{0i} and A'_{1k}) $i, k \in \{2, 3\}$ are

$$\begin{aligned}
\frac{a_{i2}}{pa_{i8}} &= \frac{a_{i6}}{a_{i4}} = \frac{b_{k6}}{b_{k4}} = \frac{-b_{k2}}{pb_{k8}} = z_{10}; & \frac{a_{i2}}{a_{i5}} &= \frac{b_{k2}}{b_{k5}} = z_{11}; \\
\frac{a_{i2}}{a_{i3}} &= \frac{a_{i6}}{a_{i7}} = \frac{b_{k6}}{b_{k7}} = \frac{b_{k2}}{b_{k3}} = z_{12}; & \frac{a_{i5}}{pa_{i8}} &= \frac{-b_{k5}}{pb_{k8}} = z_{15}; \\
\frac{a_{i7}}{a_{i4}} &= \frac{a_{i3}}{pa_{i8}} = \frac{b_{k7}}{b_{k4}} = \frac{-b_{k3}}{pb_{k8}} = z_{13}; & \frac{a_{i3}}{a_{i5}} &= \frac{b_{k3}}{b_{k5}} = z_{14}; \\
a_{i2}b_{k2} + a_{i3}b_{k3} - pa_{i5}b_{k5} + pa_{i8}b_{k8} &= 0.
\end{aligned} \quad (20)$$

The relations between the coefficients in the representation of A_{0k} and A_{2j} , (i.e., A'_{1k} and A'_{2j}) $i, k \in \{2, 3\}$ are

$$\begin{aligned}
\frac{b_{k2}}{pb_{k8}} &= \frac{-b_{k6}}{b_{k4}} = \frac{c_{j6}}{c_{j4}} = \frac{-c_{j2}}{pc_{j8}} = z_{16}; & \frac{b_{k2}}{b_{k5}} &= \frac{c_{j2}}{c_{j5}} = z_{17}; \\
\frac{b_{k2}}{b_{k3}} &= \frac{b_{k6}}{b_{k7}} = \frac{c_{j6}}{c_{j7}} = \frac{c_{j2}}{c_{j3}} = z_{18}; & \frac{b_{k5}}{pb_{k8}} &= \frac{-c_{j5}}{pc_{j8}} = z_{21}; \\
\frac{-b_{k7}}{b_{k4}} &= \frac{b_{k3}}{pb_{k8}} = \frac{c_{j7}}{c_{j4}} = \frac{-c_{j3}}{pc_{j8}} = z_{19}; & \frac{b_{k3}}{b_{k5}} &= \frac{c_{j3}}{c_{j5}} = z_{20}; \\
b_{k2}c_{j2} + b_{k3}c_{j3} - pb_{k5}c_{j5} + pb_{k8}c_{j8} &= 0.
\end{aligned} \quad (21)$$

Let $\mathbf{a}_k = [a_{k1} \ a_{k2} \ a_{k3} \ a_{k4} \ a_{k5} \ a_{k6} \ a_{k7} \ a_{k8}]$, $\mathbf{b}_k = [b_{k1} \ b_{k2} \ b_{k3} \ b_{k4} \ b_{k5} \ b_{k6} \ b_{k7} \ b_{k8}]$ and $\mathbf{c}_k = [c_{k1} \ c_{k2} \ c_{k3} \ c_{k4} \ c_{k5} \ c_{k6} \ c_{k7} \ c_{k8}]$ for $k \in \{2, 3\}$. Let $\mathbf{a}_1 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$ (according to notation for F_1 in $\mathbf{a}_k$). Similarly, let $\mathbf{b}_1 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$ and $\mathbf{c}_1 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$. To construct the weight matrices, it is enough to find linearly independent $\mathbf{a}_k$ s, linearly independent $\mathbf{b}_k$ s and linearly independent $\mathbf{c}_k$ s ($k = 1, 2, 3$), which satisfy conditions (13) to (17). Since, any combination of $(\mathbf{a}^{(i)}, \mathbf{c}^{(j)}, \mathbf{b}^{(k)}, i, j, k \in \{1, 2\})$ has to satisfy above conditions, z_i ($3 \leq i \leq 20, i \neq 9$) are forced to be constants.

Now, we find the possibilities under which solution exists for $\mathbf{a}_i, \mathbf{c}_j, i, j \in \{2, 3\}$.

From the above conditions assuming all coefficients are non zero, we can write $\mathbf{a}_i$ and $\mathbf{c}_j$ using a_{i2} and c_{j2} as (for $i, j \in \{2, 3\}$)

$$\left. \begin{aligned}
a_{i1} &= \frac{a_{i2}z_{i1}}{z_4}; & a_{i2} &= a_{i2}; & a_{i3} &= \frac{a_{i2}}{z_5}; & a_{i4} &= \frac{-a_{i2}}{z_{i1}z_3}; \\
a_{i5} &= \frac{a_{i2}}{z_4}; & a_{i6} &= \frac{-a_{i2}}{z_{i1}}; & a_{i7} &= \frac{-a_{i2}}{z_{i1}z_5}; & pa_{i8} &= \frac{a_{i2}}{z_3}; \\
c_{j1} &= \frac{-c_{j2}z_{j2}}{z_4}; & c_{j2} &= c_{j2}; & c_{j3} &= \frac{c_{j2}}{z_5}; & c_{j4} &= \frac{-c_{j2}}{z_{j2}z_3}; \\
c_{j5} &= \frac{-c_{j2}}{z_4}; & c_{j6} &= \frac{c_{j2}}{z_{j2}}; & c_{j7} &= \frac{-c_{j2}}{z_3z_5}; & pc_{j8} &= \frac{c_{j2}}{z_3}
\end{aligned} \right\} \quad (22)$$

$$\text{and } z_4 = \frac{a_{i2}z_{i1}}{a_{i1}} = \frac{-c_{j2}z_{j2}}{c_{j1}}. \quad (23)$$

From (14), (22) and (23)

$$a_{i2}c_{j2} \left(1 + \frac{1}{z_5^2} - \frac{p}{z_3^2} \right) = \frac{-pa_{i1}c_{j1}}{z_{i1}z_{j2}}.$$

$$\text{So, } 1 + \frac{1}{z_5^2} - \frac{p}{z_3^2} = \frac{-pa_{i1}c_{j1}}{z_{i1}z_{j2}a_{i2}c_{j2}} = \frac{pc_{j1}^2}{c_{j2}^2z_{j2}^2} = \frac{pa_{i1}^2}{a_{i2}^2z_{i1}^2}. \quad (24)$$

From (12) and (22) to (24)

$$\begin{aligned}
1 &= \left(1 + \frac{p}{z_{i1}^2} \right) (a_{i1}^2 + a_{i2}^2 + a_{i3}^2 - pa_{i8}^2) \\
&= \left(1 + \frac{p}{z_{i1}^2} \right) \left(a_{i1}^2 + a_{i2}^2 \left(1 + \frac{1}{z_5^2} - \frac{p}{z_3^2} \right) \right) = \left(1 + \frac{p}{z_{i1}^2} \right)^2 a_{i1}^2 \\
&\text{and similarly } \left(1 + \frac{p}{z_{j2}^2} \right)^2 c_{j1}^2 = 1.
\end{aligned} \quad (25)$$

For a given p , by choosing the values of $z_{i1}, z_{j2}, z_3, z_5, a_{i1}, a_{i2}, c_{j1}, c_{j2}$ we can get $\mathbf{a}_i$ and $\mathbf{c}_j$. For example, for $p = 1$, let $z_3 = \frac{1}{2}, z_4 = \frac{-1}{3}, z_5 = \frac{1}{2}, c_2 = -a_2 = \frac{1}{6}$. Now, for $z_{21} = 1 = z_{22}$ and $a_{21} = c_{21} = \frac{1}{2}$

we get

$$\mathbf{a}_2 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{6} & \frac{-1}{3} & \frac{-i}{3} & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{i}{3} \end{bmatrix};$$

$$\mathbf{c}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{i}{3} & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{-i}{3} \end{bmatrix},$$

and, for $z_{31} = -1 = z_{32}$ and $a_{31} = c_{31} = \frac{-1}{2}$, we get

$$\mathbf{a}_3 = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{6} & \frac{-1}{3} & \frac{i}{3} & \frac{1}{2} & \frac{-1}{6} & \frac{-1}{3} & \frac{i}{3} \end{bmatrix};$$

$$\mathbf{c}_3 = \begin{bmatrix} \frac{-1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{-i}{3} & \frac{1}{2} & \frac{-1}{6} & \frac{-1}{3} & \frac{-i}{3} \end{bmatrix}.$$

Now we consider the cases where some of the coefficients a_{ik} s or c_{jk} s are zero. For these cases we use (12) to (25).

Case 1: Let $a_{21} = 0$. Then, $a_{25} = 0$ or $a_{22} = a_{23} = a_{28} = 0$.

Let $a_{22} = a_{23} = a_{28} = 0$ and $a_{25} \neq 0$. Then, we get $c_{j2} = c_{j3} = c_{j8} = c_{j5} = 0$, $j = 2, 3$. So, $c_{j1}c_{j4} = c_{j1}c_{j6} = c_{j1}c_{j7} = 0$. If $c_{j1} \neq 0$, $A_{2j} = \pm F_2$ is clearly not a solution. This implies $c_1 = 0$. From (14), we have (for $i, j \in \{2, 3\}$)

$$\frac{a_{i4}}{a_{i6}} = \frac{-c_{j4}}{c_{j6}} = y_1; \quad \frac{a_{i4}}{a_{i7}} = \frac{-c_{j4}}{c_{j7}} = y_2; \quad \frac{a_{i6}}{a_{i7}} = \frac{c_{j6}}{c_{j7}} = y_3,$$

for some constants y_1, y_2, y_3 . From these equations, $\mathbf{c}_2$ and $\mathbf{c}_3$ are linearly dependent. So, assuming $a_{22} = a_{23} = a_{28} = 0$ and $a_{25} \neq 0$ is not valid.

Now, let $a_{22} = a_{23} = a_{28} = 0$ and $a_{25} = 0$. Consider $z_4, z_7, z_8, z_{11}, z_{14}, z_{15}, z_{17}, z_{20}, z_{21}$. Let $a_{35} \neq 0$. Now, if $a_{32} \neq 0$, then $c_{j2} = c_{j5} = 0$ or $b_{j2} = b_{j5} = 0$, $j = 2, 3$. With out loss of generality assume $c_{j2} = c_{j5} = 0$, $j = 2, 3$. Then, $c_{j3} = c_{j8} = 0$, $j = 2, 3$. Then c_{js} , $j = 1, 2, 3$ cannot be linearly independent. This happens even if $a_{33} \neq 0$ or $a_{38} \neq 0$. So $a_{32} = a_{33} = a_{38} = 0$, when $a_{35} \neq 0$. Then, again $c_{j2} = c_{j3} = c_{j5} = c_{j8} = 0$, $j = 2, 3$, which is not valid. So $a_{35} = 0$. If $a_{32} = a_{33} = a_{38} = 0$, then for $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ to be linearly independent, we need to have $c_{j2} = c_{j3} = c_{j4} = c_{j6} = c_{j7} = c_{j8} = 0$, $j = 2, 3$. In that case, $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ cannot satisfy linear independence conditions (because only $c_{j1} \neq 0$ and $c_{j5} \neq 0$). So, with out loss of generality let $a_{32} \neq 0$. Then $c_{j5} = 0$, $j = 2, 3$. Since $a_{35} = 0$, $a_{31} = 0$ or $a_{34} = a_{36} = a_{37} = 0$. Let $a_{34} = a_{36} = a_{37} = 0$. Since $c_{j5} = 0$, $c_{j1} = 0$ or $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$. Let $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$. Then, for $\mathbf{c}_js$, $j = 1, 2, 3$ to be linearly independent $a_{24} = a_{26} = a_{27} = 0$, which is not valid. So, let $c_{24} = c_{26} = c_{27} = 0$ and $c_{31} = 0$. From 19, if $a_{32} \neq 0$, $a_{33} \neq 0$ and $a_{38} \neq 0$, then, $c_{22} = c_{23} = c_{28} = 0$ which is not possible. If $a_{38} = 0$, then $c_{34} = 0$, then $a_{24} = 0$, then $c_{28} = c_{38} = 0$, which gives $c_{22}^2 + c_{23}^2 = 0$. So $c_{22} = c_{23} = 0$, because $c_{22}, c_{23} \in \mathbb{R}$. So, let $c_{j1} = 0$, $j = 2, 3$. Now, as in previous assumption $c_{j2} = c_{j3} = c_{j8} = 0$, $j = 2, 3$. Now, only c_{j4}, c_{j6}, c_{j7} are possibly non-zero. So, we cannot get $\mathbf{a}_is$ and $\mathbf{c}_is$ satisfying linear independence conditions. So, consider $a_{31} = 0$. Again, since $c_{j5} = 0$, $c_{j1} = 0$ or $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$. If $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$, we cannot get $\mathbf{a}_is$ and $\mathbf{c}_is$ satisfying linear independence conditions. So, let $c_{24} = c_{26} = c_{27} = 0$ and $c_{31} = 0$. From 19, if $a_{32} \neq 0$,

$a_{33} \neq 0$ and $a_{38} \neq 0$, then, $c_{22} = c_{23} = c_{28} = 0$ which is not possible. If $a_{38} = 0$, since $a_{32} \neq 0$ then $c_{34} = 0$, then $a_{24} = a_{34} = 0$, then $c_{28} = c_{38} = 0$. If $a_{32} \neq 0$ and $a_{33} \neq 0$, then $c_{22} = c_{23} = 0$, so not a solution. So $a_{33} = 0$. Then, $c_{22} = c_{32} = 0$, $c_{37} = 0$, $a_{37} = 0$, $a_{27} = 0$ and $c_{23} = c_{33} = 0$, which is not a solution. So, let $c_{j1} = 0$, $j = 2, 3$. Now, as in the previous assumption $a_{32} \neq 0$, $a_{33} \neq 0$ and $a_{38} \neq 0$ is not possible and $a_{32} \neq 0$, $a_{33} \neq 0$ and $a_{38} = 0$ is also not possible. So, let $a_{32} \neq 0$, $a_{33} = 0$ and $a_{38} = 0$. Then $c_{j4} = 0$, $a_{j4} = 0$ and $c_{j8} = 0$, $j = 2, 3$. And, $c_{j2} = 0$, $c_{j7} = 0$, $a_{j7} = 0$ and $c_{j3}c_{j6} = 0$, $j = 2, 3$. So, this is not a valid option.

Now, let $a_{25} = 0$. Then, we get $a_{22}c_{j5} = a_{23}c_{j5} = a_{28}c_{j5} = 0$, $j = 2, 3$. Since, for $a_{22} = a_{23} = a_{28} = a_{25} = 0$, (12) to (25) do not have a solution, $c_{j5} = 0$, $j = 2, 3$. Now, $c_{j1} = 0$ or $c_{j4} = c_{j6} = c_{j7} = 0$ has to be satisfied. Let $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$. Then, we have (for $i, j \in \{2, 3\}$)

$$\frac{c_{j2}}{pc_{j8}} = \frac{a_{i6}}{a_{i4}} = y_4; \quad \frac{c_{j3}}{pc_{j8}} = \frac{a_{i7}}{a_{i4}} = y_5; \quad \frac{c_{j2}}{c_{j3}} = \frac{a_{i6}}{a_{i7}} = y_6;$$

$$a_{i2}c_{j2} + a_{i3}c_{j3} - pa_{i8}c_{j8} = 0, \quad (26)$$

for some constants y_4, y_5, y_6 . For $\mathbf{c}_js$ to be linearly independent, we need $a_{i4} = a_{i6} = a_{i7} = 0 = a_{35}$. Since $\mathbf{c}_js$ and $\mathbf{a}_is$ have to satisfy linearly independence conditions, we need 4 vectors $[a_{22} \ a_{23} \ a_{28}]$, $[a_{32} \ a_{33} \ a_{38}]$, $[c_{22} \ c_{23} \ c_{28}]$, $[c_{32} \ c_{33} \ c_{38}]$ such that $a_{i2}c_{j2} + a_{i3}c_{j3} - pa_{i8}c_{j8} = 0$ ($i, j \in \{2, 3\}$).

Since, $a_{i4} = a_{i5} = a_{i6} = a_{i7} = 0 = c_{j4} = c_{j5} = c_{j6} = c_{j7}$ ($i, j \in \{2, 3\}$) and A_{1is}, A_{2js} ($i, j \in \{1, 2, 3\}$) (from 11) are to be linearly independent, $[a_{22} \ a_{23} \ a_{28}]$, $[a_{32} \ a_{33} \ a_{38}]$, $[c_{22} \ c_{23} \ c_{28}]$ and $[c_{32} \ c_{33} \ c_{38}]$ have to be linearly independent over $\mathbb{R}$, which is not possible. Hence, $c_{j4} = c_{j6} = c_{j7} = 0$, $j = 2, 3$ is not a valid option.

Now, let $c_{24} = c_{26} = c_{27} = 0$ and $c_{31} = 0$. No solution if $a_{i2} = a_{i3} = a_{i8} = 0$ for some $i \in \{2, 3\}$. Let $a_{i2} \neq 0$, $a_{i3} \neq 0$ and $a_{i8} \neq 0$, $i \in \{2, 3\}$. Then $c_{34} \neq 0$, $c_{36} \neq 0$ and $c_{37} \neq 0$. If $a_{i4} = a_{i6} = a_{i7} = 0$ for both $i \in \{2, 3\}$, $\mathbf{a}_is$ won't be linearly independent. So, let $a_{k4} \neq 0$, $a_{k6} \neq 0$ and $a_{k7} \neq 0$ for some $k \in \{2, 3\}$. Then from (26) $c_{j2} = c_{j3} = c_{j8} = 0$, $j = 2, 3$, so not valid. Now, let $a_{28} = 0$. Then $c_{34} = 0$, $a_{i4} = 0$, $c_{i8} = 0$, $i \in \{2, 3\}$ and $a_{38} = 0$. If $a_{22} \neq 0$ and $a_{23} \neq 0$, $c_{j2} = c_{j3} = 0$, $j = 2, 3$, so not a solution. So, let $a_{23} = 0$. Then $c_{37} = 0$, $a_{i7} = 0$ and $c_{j2} = c_{j3} = 0$, $j = 2, 3$, so not a solution.

Now, let $c_{21} = c_{31} = 0$. By proceeding as in above assumption, we get $c_{i4} = 0$, $a_{i4} = 0$, $c_{i8} = 0$, $i \in \{2, 3\}$ and $a_{38} = 0$, when $a_{28} = 0$. If $a_{22} \neq 0$ and $a_{23} \neq 0$, $c_{j2} = c_{j3} = 0$, $j = 2, 3$, so not a solution. So, let $a_{23} = 0$. Then $c_{i7} = 0$, $a_{i7} = 0$ and $c_{j2} = c_{j3} = 0$, $j = 2, 3$, so $a_{21} = a_{31} = 0$ not valid.

Case 2: Let $a_{25} = 0$. Then, $a_{21} = 0$ or $a_{24} = a_{26} = a_{27} = 0$. Since $a_{21} = 0$ is not possible, $a_{24} = a_{25} = a_{26} = a_{27} = 0$. Now, we get $a_{23}c_{j5} = a_{28}c_{j5} = a_{22}c_{j5} = 0$, $j \in \{2, 3\}$. Since $a_{22} = a_{23} = a_{24} = a_{25} = a_{26} = a_{27} = a_{28} = 0$ is not a valid solution, $c_{j5} = 0$, $j \in \{2, 3\}$. Then, $c_{j1} = 0$ or $c_{j4} = c_{j6} = c_{j7} = 0$. Since $c_{j1} = 0$ is not valid (from Case 1, because of similarity between $\mathbf{a}_i$ and $\mathbf{c}_j$), $c_{j4} = c_{j5} = c_{j6} = c_{j7} = 0$.

Similarly, since $c_{j5} = 0$, $a_{35} = 0$ and $a_{34} = a_{36} = a_{37} = 0$. So, from Case 1, we cannot get a_i s and c_i s satisfying linear independence conditions. So, $a_{25} = 0$ is not possible.

Case 3: Let $a_{22} = 0$. Then, $a_{26} = 0$ or $a_{21} = a_{23} = a_{28} = 0$. From Case 1, $a_{21} = 0$ is not valid. So, $a_{26} = 0$. Now, $c_{j6}a_{28} = c_{j6}a_{23} = c_{j2}a_{25} = 0$, $j \in \{2, 3\}$. Since $a_{25} = 0$ is not valid, $c_{j2} = 0$ ($j \in \{2, 3\}$). Since $c_{j1} = c_{j3} = c_{j8} = 0$ is not valid, $c_{j6} = 0$ ($j \in \{2, 3\}$). Now, $a_{i2} = a_{i6} = c_{j2} = c_{j6} = 0$, $i, j \in \{2, 3\}$. If we denote a_i and c_j using a_{i3} and c_{j3} , instead of a_{i2} and c_{j2} , and follow the lines of (24) and (25), we get a_i s and c_j s, satisfying the linear independence conditions.

Case 4: Let $a_{23} = 0$. Then, $a_{27} = 0$ or $a_{21} = a_{22} = a_{28} = 0$. From Case 1, $a_{21} = 0$ is not valid. So, $a_{27} = 0$. Then, we get $a_{22}c_{j7} = a_{28}c_{j7} = a_{25}c_{j3} = a_{24}c_{j3} = a_{24}c_{j7} = a_{26}c_{j3} = a_{26}c_{j7} = 0$, $j \in \{2, 3\}$. If $c_{j3} \neq 0$, then $a_{25} = a_{24} = a_{26} = a_{27} = 0$, and this is not possible from Case 2. So, $c_{j3} = 0$. Since $c_{j1} = c_{j2} = c_{j8} = 0$ is not valid, $c_{j7} = 0$ ($j \in \{2, 3\}$). Then, $a_{i3} = a_{i7} = c_{j3} = c_{j7} = 0$. By following the lines of (24) and (25), we get a_i s and c_j s, satisfying the linear independence conditions.

Case 5: Let $a_{24} = 0$. Then, $a_{28} = 0$, because $a_{24} = a_{25} = a_{26} = a_{27} = 0$ is not valid from Case 2. Now, we get $a_{22}c_{j4} = a_{23}c_{j4} = a_{26}c_{j8} = a_{27}c_{j8} = a_{26}c_{j4} = a_{27}c_{j4} = a_{25}c_{j8} = 0$, $j \in \{2, 3\}$. If $c_{j8} \neq 0$, $a_{25} = a_{26} = a_{27} = a_{24} = a_{28} = 0$, and this is not possible from Case 2. So, $c_{j8} = 0$. Since $c_{j1} = c_{j2} = c_{j3} = 0$ is not valid, $c_{j4} = 0$ ($j \in \{2, 3\}$). Now, $a_{i4} = a_{i8} = c_{j4} = c_{j8} = 0$. By following the lines of (24) and (25), we get a_i s and c_j s, satisfying the linear independence conditions.

Case 6 : Let $a_{26} = 0$. Since, $a_{24} = a_{25} = a_{26} = a_{27} = 0$ is not valid from Case 2, $a_{22} = 0$. So, Case 3 follows.

Case 7 : Let $a_{27} = 0$. Since, $a_{24} = a_{25} = a_{26} = a_{27} = 0$ is not valid from Case 2, $a_{23} = 0$. So Case 4 follows.

Case 8 : Let $a_{28} = 0$. Then, $a_{24} = 0$ or $a_{21} = a_{22} = a_{23} = 0$. From Case 1, $a_{21} = a_{23} = a_{28} = a_{22} = 0$ is not valid. So, $a_{24} = 0$ and Case 5 follows.

From the above cases, it follows that a_i s, and c_i s ($i \in \{2, 3\}$), satisfying the linear independence conditions are possible only if

- 1) None of the coefficients in a_i or c_i are 0.
- 2) $a_{i2} = a_{i6} = c_{i2} = c_{i6} = 0$ and other coefficients are non-zero.
- 3) $a_{i3} = a_{i7} = c_{i3} = c_{i7} = 0$ and other coefficients are non-zero.
- 4) $a_{i4} = a_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.
- 5) $a_{i2} = a_{i6} = c_{i2} = c_{i6} = a_{i3} = a_{i7} = c_{i3} = c_{i7} = 0$ and other coefficients are non-zero.
- 6) $a_{i2} = a_{i6} = c_{i2} = c_{i6} = a_{i4} = a_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.
- 7) $a_{i3} = a_{i7} = c_{i3} = c_{i7} = a_{i4} = a_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.

Here, $a_{i2} = a_{i6} = c_{i2} = c_{i6} = a_{i3} = a_{i7} = c_{i3} = c_{i7} = a_{i4} = a_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero, possibility is not taken into account, because, we are left with

only a_{i1} , a_{i5} , c_{i1} , c_{i5} as non-zeros. From which we cannot get a_i s, and c_i s ($i \in \{2, 3\}$) satisfying the linear independence conditions.

Step 4:

As in the case of solving for (a_i and c_j), for solving (a_i and b_k) or (b_k and c_j) we get similar conditions on their coefficients. Therefore, we may be able to get solutions to a_i s, b_k s and c_j s, which satisfy the linear independence conditions, if they satisfy one of the following conditions: (for $i \in \{2, 3\}$)

- 1) None of the coefficients in a_i or b_i or c_i are 0
- 2) $a_{i2} = a_{i6} = b_{i2} = b_{i6} = c_{i2} = c_{i6} = 0$ and other coefficients are non-zero.
- 3) $a_{i3} = a_{i7} = b_{i3} = b_{i7} = c_{i3} = c_{i7} = 0$ and other coefficients are non-zero.
- 4) $a_{i4} = a_{i8} = b_{i4} = b_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.
- 5) $a_{i2} = a_{i6} = b_{i2} = b_{i6} = c_{i2} = c_{i6} = a_{i3} = a_{i7} = b_{i3} = b_{i7} = c_{i3} = c_{i7} = 0$ and other coefficients are non-zero.
- 6) $a_{i2} = a_{i6} = b_{i2} = b_{i6} = c_{i2} = c_{i6} = a_{i4} = a_{i8} = b_{i4} = b_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.
- 7) $a_{i3} = a_{i7} = b_{i3} = b_{i7} = c_{i3} = c_{i7} = a_{i4} = a_{i8} = b_{i4} = b_{i8} = c_{i4} = c_{i8} = 0$ and other coefficients are non-zero.

Step 5:

From (19), (20) and (21), since, a_{i5} , b_{k5} , c_{j5} cannot be zero, we have (for $i, j, k \in \{2, 3\}$)

$$\begin{aligned} a_{i2} = a_{i6} = b_{k2} = b_{k6} = c_{j2} = c_{j6} = 0 & \text{ from } z_4, z_{11}, z_{17}, \\ a_{i3} = a_{i7} = b_{k3} = b_{k7} = c_{j3} = c_{j7} = 0 & \text{ from } z_7, z_{14}, z_{20}, \\ a_{i4} = a_{i8} = b_{k4} = b_{k8} = c_{j4} = c_{j8} = 0 & \text{ from } z_8, z_{15}, z_{21}. \end{aligned}$$

Now, we are left with only a_{i1} , a_{i5} , b_{k1} , b_{k5} , c_{j1} , c_{j5} , which are non-zero, from which we cannot get a_i s, b_k s and c_j s which satisfy the linear independence conditions. Hence, $g = 2a - 1$ is not possible.

From Corollary 1, since we are able to generate $g = 2a - 2$ group 3-real symbol decodable UWDs, the maximum achievable rate is $\frac{3(2a-2)}{2^{a+1}} = \frac{3(a-1)}{2^a}$ cspcu. Since this rate is achievable, this upper bound is tight. This completes the proof. ■

APPENDIX C

Proof of Theorem 4

Proof: Let g be the number of groups. Then, the rate is given by $\frac{4g}{2^{a+1}} = \frac{g}{2^{a-1}}$ cspcu. From Theorem 3, the maximum number of groups possible for 3-real symbol decodable UWDs is $2a - 2$. So, for 4-real symbol decodable UWDs the maximum number of groups possible is $\leq 2a - 2$, but in Theorem 2, we constructed $2a - 2$ group 4-real symbol decodable UWDs. So, the maximum number of groups possible (achievable) for 4-real symbol decodable UWDs is $2a - 2$. Hence, the tight upper bound on the rate of 4-real symbol decodable $2^a \times 2^a$ ($a \geq 2$) UWD is $\frac{a-1}{2^{a-2}}$ cspcu. ■

APPENDIX D

Proof of Theorem 5

Let S and S' be two distinct codewords of the code as in Theorem 2. Let

$$S = \sum_{i=0}^{2a-3} \sum_{j=1}^4 x_{ij} A_{ij}, \quad S' = \sum_{i=0}^{2a-3} \sum_{j=1}^4 x'_{ij} A_{ij},$$

where $A_{01} = I_n$. For $0 \leq i \leq 2a-3$ and $1 \leq j_1, j_2 \leq 4$,

$$A_{ij_1}^H A_{ij_2} + A_{ij_2}^H A_{ij_1} = 2A_{0j_1} A_{0j_2}. \quad (27)$$

Let $\Delta S \triangleq S - S'$, $(\Delta \mathbf{x}_i \mathbf{x}'_i)_j \triangleq x_{ij} - x'_{ij}$ and $\Delta \mathbf{x}_i \mathbf{x}'_i \triangleq [(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 \ (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 \ (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 \ (\Delta \mathbf{x}_i \mathbf{x}'_i)_4]^T$. Then, $(\Delta S)^H (\Delta S)$ is given by

$$\begin{aligned} & \left[\sum_{i=0}^{2a-3} \sum_{j=1}^4 (\Delta \mathbf{x}_i \mathbf{x}'_i)_j A_{ij} \right]^H \left[\sum_{m=0}^{2a-3} \sum_{j=1}^4 (\Delta \mathbf{x}_m \mathbf{x}'_m)_j A_{mj} \right] \\ &= \sum_{i=0}^{2a-3} \left(\sum_{j=1}^4 (\Delta \mathbf{x}_i \mathbf{x}'_i)_j A_{ij} \right)^H \left(\sum_{j=1}^4 (\Delta \mathbf{x}_i \mathbf{x}'_i)_j A_{ij} \right) \quad (28) \\ &= \sum_{i=0}^{2a-3} [((\Delta \mathbf{x}_i \mathbf{x}'_i)_1)^2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2^2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3^2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4^2] I_n + \\ & \quad 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 A_{02} + 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 A_{03} + \\ & \quad 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_1 (\Delta \mathbf{x}_i \mathbf{x}'_i)_4 A_{04} + 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_2 (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 A_{02} A_{03} + \\ & \quad 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_2 (\Delta \mathbf{x}_i \mathbf{x}'_i)_4 A_{02} A_{04} + 2(\Delta \mathbf{x}_i \mathbf{x}'_i)_3 (\Delta \mathbf{x}_i \mathbf{x}'_i)_4 A_{03} A_{04}], \quad (29) \end{aligned}$$

where, (28) and (29) occurs because of (4), (5) and (27).

Now, we calculate the determinant of $(\Delta S)^H (\Delta S)$ using A_{02} , A_{03} , A_{04} as given in Theorem 2.

$$\begin{aligned} & \det[(\Delta S)^H \Delta S] = \\ & \left[\sum_{i=0}^{2a-3} ((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[\sum_{i=0}^{2a-3} ((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[\sum_{i=0}^{2a-3} ((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[\sum_{i=0}^{2a-3} ((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \quad (30) \end{aligned}$$

The minimum of the determinant, denoted by Δ_{min} , of $(\Delta S)^H (\Delta S)$ for all possible non-zero ΔS is given as

$$\Delta_{min} = \min_{\Delta S \neq 0} \det[(\Delta S)^H (\Delta S)]. \quad (31)$$

Since the expression in the right hand side of equation (30) is a product of sum of squares of real numbers, we can write (30), (31) as

$$\begin{aligned} & \det[(\Delta S)^H (\Delta S)] \geq \\ & \left[((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \\ & \left[((\Delta \mathbf{x}_i \mathbf{x}'_i)_1 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_2 - (\Delta \mathbf{x}_i \mathbf{x}'_i)_3 + (\Delta \mathbf{x}_i \mathbf{x}'_i)_4)^2 \right]^{\frac{n}{4}} \end{aligned}$$

for some $0 \leq i \leq 2a-3$. And, Δ_{min} occurs when all but one among $\Delta \mathbf{x}_i \mathbf{x}'_i$, $0 \leq i \leq 2a-3$ are zeros.

Therefore, $\Delta_{min} = \min_{\Delta \mathbf{x}_i \mathbf{x}'_i \neq 0} (\Psi(\Delta \mathbf{x}_i \mathbf{x}'_i))^n$, for $\Delta \mathbf{x}_i \mathbf{x}'_i \in \Delta \mathcal{B}$ (i.e. $\mathbf{x}_i \in \mathcal{B}$) $\forall 0 \leq i \leq 2a-3$. To achieve full diversity we need $\Delta_{min} > 0$, which can be guaranteed if (9) is satisfied. This completes the proof.