Constructions of Rank Modulation Codes

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Abstract—Rank modulation is a way of encoding information to correct errors in flash memory devices as well as impulse noise in transmission lines. Modeling rank modulation involves construction of packings of the space of permutations equipped with the Kendall tau distance.

We present several general constructions of codes in permutations that cover a broad range of code parameters. In particular, we show a number of ways in which conventional error-correcting codes can be modified to correct errors in the Kendall space. Codes that we construct afford simple encoding and decoding algorithms of essentially the same complexity as required to correct errors in the Hamming metric. For instance, from binary BCH codes we obtain codes correcting t Kendall errors in nmemory cells that support the order of $n!/(\log_2 n!)^t$ messages, for any constant $t = 1, 2, \ldots$ We also construct families of codes that correct a number of errors that grows with n at varying rates, from $\Theta(n)$ to $\Theta(n^2)$. One of our constructions gives rise to a family of rank modulation codes for which the trade-off between the number of messages and the number of correctable Kendall errors approaches the optimal scaling rate. Finally, we list a number of possibilities for constructing codes of finite length, and give examples of rank modulation codes with specific parameters.

Index Terms—Flash memory, codes in permutations, rank modulation, transpositions, Kendall tau distance, Gray map

I. INTRODUCTION

Recently considerable attention in the literature was devoted to coding problems for non-volatile memory devices, including error correction in various models as well as data management in memories [3], [5], [13]–[15]. Non-volatile memories, in particular flash memory devices, store data by injecting charges of varying levels in memory cells that form the device. The current technology supports multi-level cells with two or more charge levels. The write procedure into the memory is asymmetric in that it is possible to increase the charge of an individual cell, while to decrease the charge one must erase and overwrite a large block of cells using a mechanism called block erasure. This raises the issue of data management in memory, requiring data encoding for efficient rewriting of the data [12]. A related issue concerns the reliability of the stored information which is affected by the drift of the charge of the cells caused by ageing devices or other reasons. Since

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the drift in different cells may occur at different speed, errors introduced in the data are adequately accounted for by tracking the relative value of adjacent cells rather than the absolute values of cell charges. Storing information in relative values of the charges also simplifies the rewriting of the data because we do not need to reach any particular value of the charge as long as we have the desired ranking, thereby reducing the risk of overprogramming. Based on these ideas, Jiang et al. [14], [15] suggested to use the rank modulation scheme for errorcorrecting coding of data in flash memories. A similar noise model arises in transmission over channels subject to impulse noise that changes the value of the signal substantially but has less effect on the relative magnitude of the neighboring signals. In an earlier work devoted to modeling impulse noise, Chadwick and Kurz [6] introduced the same error model and considered coding problems for rank modulation. Drift of resistance in memory cells is also the main source of errors in multilevel-cell phase-change memories [22].

Motivated by the application to flash memories, we consider reliable storage of information in the rank modulation scheme. Relative ranks of cell charges in a block of n cells define a permutation on the set of n elements. Our problem therefore can be formulated as encoding of data into permutations so that it can be recovered from errors introduced by the drift (decrease) of the cell charges.

To define the error process formally, let $[n] = \{1, 2, ..., n\}$ be a set of n elements and consider the set S_n of permutations of [n]. In this paper we use a one-line notation for permutations: for instance (2,1,3) refers to the permutation $\binom{123}{213}$. Referring to the discussion of charge levels of cells, permutation (2,1,3) means that the highest-charged cell is the second one followed by the first and then the third cell. Permutations can be multiplied by applying them successively to the set [n], namely the action of the permutation $\pi\sigma$, where $\pi, \sigma \in S_n$, results in $i \mapsto \sigma(\pi(i)), i = 1, ..., n$. (Here and elsewhere we assume that permutations act on the right). Every permutation has an inverse, denoted σ^{-1} , and e denotes the identity permutation.

Let $\sigma=(\sigma(1),\ldots,\sigma(n))$ be a permutation of [n]. An elementary error occurs when the charge of cell j passes the level of the charge of the cell with rank one smaller than the rank of j. If the n-block is encoded into a permutation, σ , this error corresponds to the exchanging of the locations of the elements $\sigma(j)$ and $\sigma(j+1)$ in the permutation. For instance, let $\sigma=(3,1,4,2)$ then the effect of the error $\pi=(2,1,3,4)$ is to exchange the locations of the two highest-ranked elements, i.e., $\pi\sigma=(1,3,4,2)$.

Accordingly, define the *Kendall tau distance* $d_{\tau}(\sigma,\pi)$ from σ to another permutation π as the minimum number of transpositions of pairwise adjacent elements required to change σ into π . Denote by $\mathcal{X}_n = (S_n, d_{\tau})$ the metric space of

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permutations on n elements equipped with the distance d_{τ} . The Kendall metric was studied in statistics [16] where it was introduced as a measure of proximity of data samples, as well as in combinatorics and coding theory [3], [10]. The Kendall metric also arises naturally as a Cayley metric on the group S_n if one takes the adjacent transpositions as its generators.

The Kendall distance is one of many metrics on permutations considered in the literature; see the survey [9]. Coding for the Hamming metric was considered recently in [7] following the observation in [25] that permutation arrays are useful for error correction in powerline communication. Papers [20], [23], [24] considered coding for the ℓ_{∞} distance on permutations from the perspective of the rank modulation scheme. Generalizations of Gray codes for rank modulation are considered in [26], while an application of LDPC codes to this scheme is proposed in [27].

An (n,d) code $\mathcal{C} \subset \mathcal{X}_n$ is a set of permutations in S_n such that the minimum distance $d_{ au}$ separating any two of them is at least d. The main questions associated with the coding problem for the Kendall space \mathcal{X}_n are to establish the size of optimal codes that correct a given number of errors and, more importantly, to construct explicit coding schemes. In our previous work [3] we addressed the first of these problems, analyzing both the finite-length and the asymptotic bounds on codes. Since the maximum value of the distance in \mathcal{X}_n is $\binom{n}{2}$, this leaves a number of possibilities for the scaling rate of the distance for asymptotic analysis, ranging from d = O(n) to $d = \Theta(n^2)$. Define the rate of the code

$$R(\mathcal{C}) = \log |\mathcal{C}| / \log(n!) \tag{1}$$

(all logarithms are base 2 unless otherwise mentioned) and let

$$R(n,d) = \max_{\mathcal{C} \subset \mathcal{X}_n} R(\mathcal{C})$$

$$\mathscr{C}(d) = \lim_{n \to \infty} R(n,d)$$
(3)

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where the maximum in (2) is over all codes with distance > d. We have the following result.

Theorem 1: [3] The limit in (3) exists, and

$$\mathscr{C}(d) = \begin{cases} 1 & \text{if } d = O(n) \\ 1 - \epsilon & \text{if } d = \Theta(n^{1+\epsilon}), \ 0 < \epsilon < 1 \\ 0 & \text{if } d = \Theta(n^2). \end{cases}$$
 (4)

Moreover.

$$R(n,d) = \begin{cases} O(\log^{-1} n) & \text{if } d = \Theta(n^2) \\ 1 - O(\log^{-1} n) & \text{if } d = O(n). \end{cases}$$

We remark [3] that the equality $\mathscr{C}(d) = 1 - \epsilon$ holds under a slightly weaker condition, namely, $d = n^{1+\epsilon}\alpha(n)$, where $\alpha(n)$ grows slower than any positive power of n.

Equation (4) suggests the following definition. Let us say that an infinite family of codes scales optimally if there exists $\epsilon \in (0,1)$ such that, for any positive α, β , all codes of the family of length n larger than some n_0 , have rate at least $1 - \epsilon - \beta$ and minimum distance $\Omega(n^{1+\epsilon-\alpha})$.

The proof of Theorem 1 relied on near-isometric embeddings of \mathcal{X}_n into other metric spaces that provide insights into the asymptotic size of codes. We also showed [3] that there exists a family of rank modulation codes that correct a constant number of errors and have size within a constant factor of the upper (sphere packing) bound.

Regarding the problem of explicit constructions, apart from a construction in [15] of codes that correct one Kendall error, no other code families for the Kendall distance are presently known. Addressing this issue, we provide several general constructions of codes that cover a broad range of parameters in terms of the code length n and the number of correctable errors. We present constructions of rank modulation codes that correct a given number of errors as well as several asymptotic results that cover the entire range of possibilities for the scaling of the number of errors with the code's length. Sect. II we present a construction of low-rate rank modulation codes that form subcodes of Reed-Solomon codes, and can be decoded using their decoding algorithms. In Sect. III we present another construction that gives rank modulation codes capable of correcting errors whose multiplicity can be anywhere from a constant to $O(n^{1+\epsilon})$, $0 < \epsilon < 1/2$, although the code rate is below the optimal rate of (4). Relying on this construction, we also show that there exist sequences of rank modulation codes derived from binary codes whose parameters exhibit the same scaling rate as (4) for any $0 < \epsilon < 1$. Moreover, we show that almost all linear binary codes can be used to construct rank modulation codes with this optimal trade-off. Finally, we present a third construction of rank modulation codes from codes in the Hamming space that correct a large number of errors. If the number of errors grows as $\Theta(n^2)$, then the rate of the codes obtained from binary codes using this construction attains the optimal scaling of $O(\log^{-1} n)$. Generalizing this construction to start from nonbinary codes, we design families of rank modulation codes that scale optimally (in the sense of the above definition) for all values of ϵ , $0 < \epsilon < 1$.

Finally, Sect. IV contains some examples of codes obtained using the new constructions proposed here.

Our constructions rely on codes that correct conventional (Hamming) errors, converting them into Kendall-errorcorrecting codes. For this reason, the proposed methods can be applied to most families of codes designed for the Hamming distance, thereby drawing on the rich variety of available constructions with their simple encoding and decoding algorithms.

II. CONSTRUCTION I: RANK MODULATION CODES FROM PERMUTATION POLYNOMIALS

Our first construction of rank modulation codes is algebraic in nature. Let $q = p^m$ for some prime p and let $\mathbb{F}_q = \{\alpha_0, \alpha_1, \dots, \alpha_{q-1}\}$ be the finite field of q elements. A polynomial $g(x) \in \mathbb{F}_q[x]$ is called a permutation polynomial if the values g(a) are distinct for distinct values of $a \in \mathbb{F}_q$

Consider the evaluation map $f \mapsto (f(\alpha_0), \dots, f(\alpha_{q-1}))$ which sends permutation polynomials to permutations of nelements. Evaluations of permutation polynomials of degree $\leq k$ form a subset of a q-ary Reed-Solomon code of dimension k+1. Reed-Solomon codes are a family of error-correcting codes in the Hamming space with a number of desirable properties including efficient decoding. For an introduction to them see [21, Ch. 10].

At the same time, evaluating the size of a rank modulation code constructed in this way is a difficult problem because it is hard to compute the number of permutation polynomials of a given degree. In this section we formalize a strategy for constructing codes along these lines. This does not result in very good rank modulation codes; in fact, our later combinatorial constructions will be better in terms of the size of the codes with given error-correcting capabilities. Nonetheless, the construction involves some interesting observations which is why we decided to include it.

A polynomial over \mathbb{F}_q is called *linearized of degree* ν if it has the form

$$\mathcal{L}(x) = \sum_{i=0}^{\nu} a_i x^{p^i}$$

Note that a linearized polynomial of degree ν has degree p^{ν} when viewed as a standard polynomial.

Lemma 2: The number of linearized polynomials over \mathbb{F}_q of degree less than or equal to ν that are permutation polynomials in \mathbb{F}_q is at least

$$\left(1 - \frac{1}{p-1} + \frac{1}{q(p-1)}\right)q^{\nu+1} \ge q^{\nu}.$$

Proof: The polynomial $\mathcal{L}(x)$ acts on \mathbb{F}_q as a linear homomorphism. It is injective if and only if it has a trivial kernel, in other words if the only root of $\mathcal{L}(x)$ in \mathbb{F}_q is 0. Hence, $\mathcal{L}(x)$ is a permutation polynomial if and only if the only root of $\mathcal{L}(x)$ in \mathbb{F}_q is 0.

The total number of linearized polynomials of degree up to ν is $q^{\nu+1}$. We are going to prove that at least a $(1-\frac{1}{p-1}+\frac{1}{q(p-1)})$ proportion of them are permutation polynomials. To show this, choose the coefficients $a_i, 0 \leq i \leq \nu$ of $\mathcal{L}(x) = \sum_{i=0}^{\nu} a_i x^{p^i}$ uniformly and randomly from \mathbb{F}_q . For a fixed $\alpha \in \mathbb{F}_q^*$, the probability that $\mathcal{L}(\alpha) = 0$ is 1/q. Furthermore, the set of roots of a linearized polynomial is an \mathbb{F}_p -vector space [21, p.119], hence the set of non-zero roots is a multiple of p-1. The number of 1-dimensional subspaces of \mathbb{F}_q over \mathbb{F}_p is $\frac{q-1}{p-1}$. The probability that one of these sets is included in the set of roots of $\mathcal{L}(x)$ is, from the union bound,

$$\Pr(\exists \alpha \in \mathbb{F}_q^* : \mathcal{L}(\alpha) = 0) \le \frac{q-1}{p-1} \cdot \frac{1}{q}.$$

Hence, the probability that $\mathcal{L}(x)$ is a permutation polynomial is greater than or equal to $1 - \frac{q-1}{q(p-1)}$.

A. Code construction

We use linearized permutation polynomials of \mathbb{F}_q to construct codes in the space \mathcal{X}_n . Note that a linearized polynomial $\mathcal{L}(x)$ always maps zero to zero, so that when it is a permutation polynomial it can be considered to be a permutation of the elements of \mathbb{F}_q and also of the elements of \mathbb{F}_q^* . Let t be a positive integer and let $\nu = \lfloor \log_p(n-2t-1) \rfloor$. Let \mathcal{P}_t be the set of all linearized polynomials of degree $\leq \nu$ that permute \mathbb{F}_q . Set n=q-1 and define the set $A \subset \mathbb{F}_q^n$

$$A = \{ (\mathcal{L}(a), a \in \mathbb{F}_q^*), \ \mathcal{L} \in \mathcal{P}_t \}$$

to be the set of vectors obtained by evaluating the polynomials in \mathcal{P}_t at the points of \mathbb{F}_q^* . Form a code \mathcal{C}_τ by writing the

vectors in A as permutations (for that, we fix some bijection between [n] and \mathbb{F}_q^* , which will be implicit in the subsequent discussion). We can have n=q rather than n=q-1 if desired: for that we add the zero field element in the first position of the (q-1)-tuples of A, and the construction below readily extends.

The idea behind the construction is quite simple: the set A is a subset of a Reed-Solomon code that corrects t Hamming errors. Every Kendall error is a transposition, and as such, affects at most two coordinates of the codeword of \mathcal{C}_{τ} . Therefore the code \mathcal{C}_{τ} can correct up to t/2 errors. By handling Kendall errors more carefully, we can actually correct up to t errors. The main result of this part of our work is given by the following statement.

Theorem 3: The code \mathcal{C}_{τ} has length n=q-1 and size at least $q^{\lfloor \log_p(n-2t-1) \rfloor}$. It corrects all patterns of up to t Kendall errors in the rank modulation scheme under a decoding algorithm of complexity polynomial in n.

Proof: It is clear that $|\mathcal{C}_{\tau}| = |A|$, and from Lemma 2 $|A| \geq q^{\lfloor \log_p(n-2t-1) \rfloor}$.

Let $\sigma=(a_1,a_2,\ldots,a_i,a_{i+1},\ldots,a_n)$, where $a_j\in\mathbb{F}_q^*,1\leq j\leq n$, be a permutation in \mathcal{X}_n (with the implied bijection between [n] and \mathbb{F}_q^*) and let $\sigma'=(a_1,a_2,\ldots,a_{i+1},a_i,\ldots,a_n)$ be a permutation obtained from σ by one Kendall step (an adjacent transposition). We have

$$\sigma - \sigma' = (0, \dots, 0, \theta, -\theta, \dots, 0)$$

where $\theta = a_i - a_{i+1} \in \mathbb{F}_q^*$.

Let

be an $n \times n$ matrix. Note that

$$P(\sigma - \sigma')^T = (0, \dots, 0, \theta, 0, \dots, 0)^T.$$

This means that multiplication by the accumulator matrix P converts one adjacent transposition error into one Hamming error. Extending this observation, we claim that if $d_{\tau}(\sigma,\pi) \leq t$ with π being some permutation, and any $t \leq \frac{n}{2}$, then the Hamming weight of the vector $P(\sigma-\pi)^T$ is not more than t. Here we again take σ and π to be vectors with elements from \mathbb{F}_q^* with the implied bijection between [n] and \mathbb{F}_q^* .

Now let $\mathcal{L}(x)$ be a linearized permutation polynomial and let $1, \alpha, \alpha^2, \ldots, \alpha^{q-2}$ be the elements of \mathbb{F}_q^* for some choice of the primitive element α . Let

$$\sigma = (\mathcal{L}(1), \mathcal{L}(\alpha), \mathcal{L}(\alpha^2), \dots, \mathcal{L}(\alpha^{q-2})).$$

Since $\mathcal{L}(a+b) = \mathcal{L}(a) + \mathcal{L}(b)$, we have

$$P\sigma^T = (\mathcal{L}(\beta_0), \mathcal{L}(\beta_1), \mathcal{L}(\beta_2), \dots, \mathcal{L}(\beta_{n-2}))^T$$

where

$$\beta_i = \sum_{j=0}^{i} \alpha^j, \quad i = 0, 1, \dots, q - 2.$$

It is clear that $\beta_i \neq 0$, $0 \leq i \leq n-1$ and also $\beta_{i_1} \neq \beta_{i_2}$ for $0 \leq i_1 < i_2 \leq n-1$, Therefore, the vector $P\sigma^T$ is a permutation of the elements of \mathbb{F}_q^* . At the same time, it is the evaluation vector of a polynomial of degree $\leq n-2t-1$. We conclude that the set $\{P\sigma^T, \sigma \in A\}$ is a subset of vectors of an (extended) Reed-Solomon code of length n, dimension n-2t and distance 2t+1. Any t errors in a codeword of such a code can be corrected by standard RS decoding algorithms in polynomial time.

The following decoding algorithm of the code C_{τ} corrects any t Kendall errors. Suppose $\sigma \in A$ is read off from memory as σ_1 .

Decoding algorithm (Construction I):

- Evaluate $z = P\sigma_1^T$.
- Use a Reed-Solomon decoding algorithm to correct up to t Hamming errors in the vector z, obtaining a vector y (if the Reed-Solomon decoder returns no results, the algorithm detects more than t errors).
- Compute $\sigma = P^{-1} \mathbf{y}^T$, i.e.,

$$\sigma_i = y_{i+1} - y_i, \ 1 \le i \le n-1; \ \sigma_n = y_n.$$

The correctness of the algorithm follows from the construction. Namely, if $d_{\tau}(\sigma, \sigma_1) \leq t$, then \boldsymbol{y} corresponds to a transformed version of σ , i.e., $\boldsymbol{y} = P\sigma^T$. Then the last step of the decoder correctly identifies the permutation σ .

Some examples of code parameters arising from this theorem are given in Sect. IV.

We note an earlier use of permutation polynomials for constructing permutation codes in [7]. At the same time, since the coding problem considered in that paper relies on the Hamming metric rather than the Kendall tau distance, its results have no immediate link to the above construction.

III. CONSTRUCTION II: RANK MODULATION CODES FROM THE GRAY MAP

In this section we present constructions of rank modulation codes using a weight-preserving embedding of the Kendall space \mathcal{X}_n into a subset of integer vectors. To evaluate the error-correcting capability of the resulting codes, we further link codes over integers with codes correcting Hamming errors.

A. From permutations to inversion vectors

We begin with a description of basic properties of the distance d_{τ} such as its relation to the number of inversions in the permutation, and weight-preserving embeddings of S_n into other metric spaces. Their proofs and a detailed discussion are found for instance in the books by Comtet [8] or Knuth [17, Sect. 5.1.1].

The distance d_{τ} is a right-invariant metric which means that $d_{\tau}(\sigma_1, \sigma_2) = d_{\tau}(\sigma_1 \sigma, \sigma_2 \sigma)$ for any $\sigma, \sigma_1, \sigma_2 \in S_n$ where the operation is the usual multiplication of permutations. Therefore, we can define the weight of the permutation σ as its distance to the identity permutation $e = (1, 2, \ldots, n)$.

Because of the invariance, the Cayley graph of S_n (i.e., the graph whose vertices are indexed by the permutations and whose edges connect permutations one Kendall step apart) is

regular of degree n-1. At the same time it is not distance-regular, and so the machinery of algebraic combinatorics does not apply to the analysis of the code structure. The diameter of the space \mathcal{X}_n equals $N \triangleq \binom{n}{2}$ and is realized by pairs of opposite permutations such as (1,2,3,4) and (4,3,2,1).

The main tool to study properties of d_{τ} is provided by the inversion vector of the permutation. An *inversion* in a permutation $\sigma \in S_n$ is a pair (i,j) such that i>j and $\sigma^{-1}(j)>\sigma^{-1}(i)$. It is easy to see that $d_{\tau}(\sigma,e)=I(\sigma)$, the total number of inversions in σ . Therefore, for any two permutations σ_1,σ_2 we have $d_{\tau}(\sigma_1,\sigma_2)=I(\sigma_2\sigma_1^{-1})=I(\sigma_1\sigma_2^{-1})$. In other words,

$$d_{\tau}(\sigma, \pi) = |\{(i, j) \in [n]^2 : i \neq j, \pi^{-1}(i) > \pi^{-1}(j),$$

$$\sigma^{-1}(i) < \sigma^{-1}(j)\}|.$$

To a permutation $\sigma \in S_n$ we associate an inversion vector $\boldsymbol{x}_{\sigma} \in \mathcal{G}_n \triangleq [0,1] \times [0,2] \times \cdots \times [0,n-1],$ where $\boldsymbol{x}_{\sigma}(i) = |\{j \in [n]: j < i+1, \sigma^{-1}(j) > \sigma^{-1}(i+1)\}|, i=1,\ldots,n-1.$ In words, $\boldsymbol{x}_{\sigma}(i), i=1,\ldots,n-1$ is the number of inversions in σ in which i+1 is the first element. For instance, we have

It is well known that the mapping from permutations to the space of inversion vectors is bijective, and any permutation can be easily reconstructed from its inversion vector¹. Clearly,

$$I(\sigma) = \sum_{i=1}^{n-1} \boldsymbol{x}_{\sigma}(i). \tag{5}$$

Denote by $J: \mathcal{G}_n \to S_n$ the inverse map from \mathcal{G}_n to S_n , so that $J(\boldsymbol{x}_{\sigma}) = \sigma$. The correspondence between inversion vectors and permutations was used in [15] to construct rank modulation codes that correct one error.

For the type of errors that we consider below we introduce the following ℓ_1 distance function on \mathcal{G}_n :

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n-1} |\boldsymbol{x}(i) - \boldsymbol{y}(i)|, \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}_n)$$
 (6)

where the computations are performed over the integers, and write $\|\boldsymbol{x}\|$ for the corresponding weight function (this is not a properly defined norm because \mathcal{G}_n is not a linear space). Recall that $d_{\tau}(\sigma,\pi)=I(\pi\sigma^{-1})$; hence the relevance of the ℓ_1 distance for our problem. For instance, let $\sigma_1=(2,1,4,3),\sigma_2=(2,3,4,1)$, then $\boldsymbol{x}_{\sigma_1}=(1,0,1),\boldsymbol{x}_{\sigma_2}=(1,1,1)$. To compute the distance $d_{\tau}(\sigma_1,\sigma_2)$ we note that $\sigma_1^{-1}=\sigma_1$ and so

$$I(\sigma_2\sigma_1^{-1}) = I((1,4,3,2)) = ||(0,1,2)|| = 3.$$

Observe that the mapping $\sigma \to x_\sigma$ is a weight-preserving bijection between \mathcal{X}_n and the set \mathcal{G}_n . At the same time, the above example shows that this mapping is not distance preserving. Indeed, $d_\tau(\sigma_1,\sigma_2)=3$ while $d_1(x_{\sigma_1},x_{\sigma_2})=1$. However, a weaker property pointed out in [15] is true, namely:

¹There is more than one way to count inversions and to define the inversion vector: for instance, one can define $\boldsymbol{x}_{\sigma}(i) = |\{j: j \leq i, \sigma(j) > \sigma(i+1)\}|, i=1,\ldots,n-1$. In this case, given $\sigma=(2,1,6,4,3,7,5,9,8)$ we would have $\boldsymbol{x}_{\sigma}=(1,0,2,1,2,0,0,1)$. The definition in the main text is better suited to our needs in that it supports Lemma 4 below.

Lemma 4: Let $\sigma_1, \sigma_2 \in S_n$, then

$$d_{\tau}(\sigma_1, \sigma_2) \ge d_1(\boldsymbol{x}_{\sigma_1}, \boldsymbol{x}_{\sigma_2}). \tag{7}$$

Proof: Let $\sigma(m)$, $\sigma(m+1)$ be two adjacent elements in a permutation σ . Let $i=\sigma(m)$, $j=\sigma(m+1)$ and suppose that i< j. Form a permutation σ' which is the same as σ except that $\sigma'(m)=j$, $\sigma'(m+1)=i$, so that $d_{\tau}(\sigma,\sigma')=1$. The count of inversions for which i is the first element is unchanged, while the same for j has increased by one. We then have $\boldsymbol{x}_{\sigma'}(k)=\boldsymbol{x}_{\sigma}(k), k\neq j$ and $\boldsymbol{x}_{\sigma'}(j)=\boldsymbol{x}_{\sigma}(j)+1$. Thus, $d_1(\boldsymbol{x}_{\sigma'},\boldsymbol{x}_{\sigma})=1$, and the same conclusion is clearly true if i>j.

Hence, if the Kendall distance between σ_1 and σ_2 is 1 then the ℓ_1 distance between the corresponding inversion vectors is also 1. Now consider two graphs G and G' with the same vertex set S_n . In G there will be an edge between two vertices if and only if the Kendall distance between them is 1. On the other hand there will an edge between two vertices in G' if and only if the ℓ_1 distance between corresponding inversion vectors is 1. We have just shown that the set of edges of G is a subset of the set of edges of G'. The Kendall distance between two permutations is the minimum distance between them in the graph G. A similar statement is true for the ℓ_1 distance with the graph G'.

This proves the lemma.

We conclude as follows.

Proposition 5: If there exists a code $\mathcal C$ in $\mathcal G_n$ with ℓ_1 distance d then the set $\mathcal C_\tau:=\{J(\boldsymbol x): \boldsymbol x\in \mathcal C\}$ forms a rank modulation code in S_n of cardinality $|\mathcal C|$ with Kendall distance at least d.

B. From inversion vectors to the Hamming space via Gray Map

We will need the *Gray map* which is a mapping ϕ_s from the ordered set of integers $[0,2^s-1]$ to $\{0,1\}^s$ with the property that the images of two successive integers differ in exactly one bit. Suppose that $b_{s-1}b_{s-2}\dots b_0$, $b_i\in\{0,1\}, 0\leq i< s$, is the binary representation of an integer $u\in[0,2^s-1]$. Set by definition $b_s=0$ and define $\phi_s(u)=(g_{s-1},g_{s-2},\dots,g_0)$, where

$$g_i = (b_i + b_{i+1}) \pmod{2}$$
 $(j = 0, 1, \dots s - 1)$ (8)

(note that for $s \ge 4$ there are several ways of defining maps from integers to binary vectors with the required property).

Example: The Gray map for the first 10 integers looks as follows:

Note the "reflective" nature of the map: the last 2 bits of the second block of four are a reflection of the last 2 digits of the first block with respect to the horizontal line; the last 3 bits of the second block of eight follow a similar rule, and so on. This property, easy to prove from (8), will be put to use below (see Prop. 9).

Now, for $i = 2, \ldots, n$, let

$$m_i = \lfloor \log i \rfloor,$$

and let

$$\psi_i: \{0,1\}^{m_i} \to [0,i-1]$$

be the inverse Gray map $\psi_i = \phi_i^{-1}$. Clearly ψ_i is well defined; it is injective but not surjective for most *i*'s since the size of its domain is only 2^{m_i} .

Proposition 6: Suppose that $x, y \in \{0, 1\}^{m_i}$. Then

$$|\psi_i(\boldsymbol{x}) - \psi_i(\boldsymbol{y})| \ge d_H(\boldsymbol{x}, \boldsymbol{y}), \tag{9}$$

where d_H denotes the Hamming distance.

Proof: This follows from the fact that if u, v are two integers such that |u - v| = 1, then their Gray images satisfy $d_H(\phi(u), \phi(v)) = 1$. If the number are such that u < v and |u - v| = d, then by the triangle inequality,

$$d_H(\phi(u), \phi(v)) \le d_H(\phi(u), \phi(u+1)) + \dots + d_H(\phi(v-1), \phi(v))$$

$$= d$$

Consider a vector $\boldsymbol{x}=(\boldsymbol{x}_2|\boldsymbol{x}_3|\dots|\boldsymbol{x}_n)$, where $\boldsymbol{x}_i\in\{0,1\}^{m_i},\ i=2,\dots,n$. The dimension of \boldsymbol{x} equals $m=\sum_{i=2}^n m_i \approx \log n!$, or more precisely

$$m = \sum_{j=1}^{m_n-1} (2^{j+1} - 2^j)j + m_n(n+1-2^{m_n})$$

$$= \sum_{j=1}^{m_n-1} j2^j + m_n(n+1-2^{m_n})$$

$$= (m_n - 2)2^{m_n} + 2 + m_n(n+1-2^{m_n})$$

$$= (n+1)m_n - 2^{m_n+1} + 2.$$

On the first line of this calculation we used the fact that among the numbers m_i there are exactly $2^{j+1}-2^j$ numbers equal to j for all $j \leq n-1$, namely those with $i=2^j, 2^j+1, \ldots, 2^{j+1}-1$. The remaining $(n+1)-2^{m_n}$ numbers equal m_n .

For a vector $x \in \{0,1\}^m$ let

$$\Psi(x) = \Psi(x_2|x_3|\dots|x_n) = (\psi_2(x_2),\dots,\psi_n(x_n)).$$

Proposition 7: Let $x, y \in \{0, 1\}^m$. Then

$$d_1(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})) \geq d_H(\boldsymbol{x}, \boldsymbol{y}),$$

where the distance d_1 is the ℓ_1 distance defined in (6).

Proof: Using (9), we obtain

$$d_1(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})) = \sum_{i=2}^n |\psi_i(\boldsymbol{x}_i) - \psi_i(\boldsymbol{y}_i)|$$

$$\geq \sum_{i=2}^n d_H(\boldsymbol{x}_i, \boldsymbol{y}_i)$$

$$= d(\boldsymbol{x}, \boldsymbol{y}).$$

C. The code construction: correcting up to $O(n \log n)$ number of errors

Now we can formulate a general method to construct rank modulation codes. We begin with a binary code \mathcal{A} of length m and cardinality M in the Hamming space.

Encoding algorithm (Construction II):

- Given a message m encode it with the code A. We obtain a vector $x \in \{0,1\}^m$.
- Write $x = (x_2|x_3|...|x_n)$, where $x_i \in \{0,1\}^{m_i}$.
- Evaluate $\pi = J(\Psi(x))$

This algorithm is of essentially the same complexity as the encoding of the code \mathcal{A} , and if this latter code is linear, is easy to implement, Both J and Ψ are injective, so the cardinality of the resulting code is M. Moreover, each of the two mappings can only increase the distance (namely, see (7) and the previous Proposition). Summarizing, we have the following statement.

Theorem 8: Let A be a binary code of length

$$m = (n+1)\lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2,$$

cardinality M and Hamming distance d. Then the set of permutations

$$C_{\tau} = \{ \pi \in S_n : \pi = J(\Psi(\boldsymbol{x})), \boldsymbol{x} \in \mathcal{A} \}$$

forms a rank modulation code on n elements of cardinality M with distance at least d in the Kendall space \mathcal{X}_n .

The resulting rank modulation code \mathcal{C}_{τ} can be decoded to correct any $t = \lfloor (d-1)/2 \rfloor$ Kendall errors if t Hamming errors are correctable with a decoding algorithm of the binary code \mathcal{A} . Namely, suppose that σ' is the permutation that represents a corrupted memory state. To recover the data we perform the following steps.

Decoding algorithm (Construction II):

- Construct the inversion vector $\boldsymbol{x}_{\sigma'}$. Form a new inversion vector \boldsymbol{y} as follows. For $i=2,\ldots,n$, if $\boldsymbol{x}_{\sigma'}(i-1)\in[0,i-1]$ is greater than $2^{m_i}-1$ then put $\boldsymbol{y}_{\sigma'}(i)=2^{m_i}-1$, else put $\boldsymbol{y}_{\sigma'}(i)=\boldsymbol{x}_{\sigma'}(i)$.
- Form a vector $\boldsymbol{y} \in \{0,1\}^m$, $\boldsymbol{y} = (\boldsymbol{y}_2|\boldsymbol{y}_3|\dots|\boldsymbol{y}_n)$ where $\boldsymbol{y}_i \in \{0,1\}^{m_i}$ is given by $\phi_i(\boldsymbol{y}_{\sigma'}(i))$.
- Apply the t-error-correcting decoding algorithm of the code \mathcal{A} to y. If the decoder returns no result, the algorithm detects more than t errors. Otherwise suppose that y is decoded as x.
- $\bullet \ \ \text{Output} \ \sigma = J(\Psi(\boldsymbol{x})).$

The correctness of this algorithm is justified as follows. Suppose $\sigma \in \mathcal{C}_{\tau}$ is the original permutation written into the memory, and $d_{\tau}(\sigma,\sigma') \leq t$. Let \boldsymbol{x}_{σ} be its inversion vector and let \boldsymbol{x} be its Gray image, i.e., a vector such that $\Psi(\boldsymbol{x}) = \boldsymbol{x}_{\sigma}$. By Lemma 4 and Prop. 7 we conclude that $d_H(\boldsymbol{x},\boldsymbol{y}) \leq t$, and therefore the decoder of the code $\mathcal A$ correctly recovers \boldsymbol{x} from \boldsymbol{y} . Therefore σ' will be decoded to σ as desired.

Example: Consider a t-error-correcting primitive BCH code \mathcal{A} in the binary Hamming space of length $m=(n+1)\lfloor \log n\rfloor - 2^{\lfloor \log n\rfloor +1} + 2$ and designed distance 2t+1 (generally, we will need to shorten the code to get to the desired length m). The cardinality of the code satisfies

$$M \ge \frac{2^m}{(m+1)^t}.$$

The previous theorem shows that we can construct a set of (n, M) rank modulation codes that correct t Kendall errors. Note that, by the sphere packing bound, the size of any code $C \in \mathcal{X}_n$ that corrects t Kendall errors satisfies $|C| = O(n!/n^t)$. The rank modulation codes constructed from binary BCH codes have size $M = \Omega(n!/(\log n!)^t) = \Omega(n!/(n^t \log^t n))$.

Specific examples of code parameters that can be obtained from the above construction are given in Sect. IV.

Remark (Encoding into permutations): Suppose that the construction in this section is used to encode binary messages into permutations (i.e., the code $\mathcal A$ in the above encoding algorithm is an identity map). We obtain an encoding procedure of binary m-bit messages into permutations of n symbols. This redundancy of this encoding equals $1-m/\log(n!)$. Using the Stirling formula, we have for $n \geq 1$

$$\log n! \le \log(\sqrt{2\pi n}) + n\log n - \left(n - \frac{1}{12n}\right)\log e$$

([1], Eq. 6.1.38). Writing $m \ge (n+1)(\log n - 1) - 2n + 2$, we can estimate the redundancy as

$$1 - \frac{m}{\log n!} \le \frac{(3 - \log e)n}{\log n!}, \quad n \ge 2.$$

Thus the encoding is asymptotically nonredundant. The redundancy is the largest when n is a power of 2. It is less than 10% for all $n \ge 69$, less than 7% for all $n \ge 527$, etc.

D. Correcting $O(n^{1+\epsilon})$ number of errors, $0 < \epsilon < 1/2$

Consider now the case when the number of errors t grows with n. Since the binary codes constructed above are of length about $n \log n$, we can obtain rank modulation codes in \mathcal{X}_n that correct error patterns of Kendall weight $t = \Omega(n \log n)$. But in fact more is true. We need the following proposition.

Proposition 9: Let $x, y \in \{0, 1\}^m$. Then

$$d_1(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})) \ge \frac{n-1}{2} \left(2^{\frac{d_H(\boldsymbol{x}, \boldsymbol{y})}{n-1}} - 1 \right).$$

Proof: Assume without loss of generality that $x \neq y$. We first claim that, for any such $x, y \in \{0, 1\}^{m_i}$, the inequality $d_H(x, y) \geq w_i \geq 1$ implies that $|\psi_i(x) - \psi_i(y)| \geq 2^{w_i - 1}$. This is true because of the reflection property of the standard Gray map as exemplified above.

Now consider vectors $\boldsymbol{x}=(\boldsymbol{x}_2|\boldsymbol{x}_3|\dots|\boldsymbol{x}_n), \boldsymbol{y}=(\boldsymbol{y}_2|\boldsymbol{y}_3|\dots|\boldsymbol{y}_n)$ in $\{0,1\}^m$ where $\boldsymbol{x}_i,\boldsymbol{y}_i\in\{0,1\}^{m_i},2\leq i\leq n$. Suppose that $d_H(\boldsymbol{x}_i,\boldsymbol{y}_i)=w_i$ for all i, and $\sum_{i=2}^n w_i=w$ where $w=d_H(\boldsymbol{x},\boldsymbol{y})$.

Hence,

$$\begin{split} d_1(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})) &= \sum_{i=2}^n |\psi_i(\boldsymbol{x}_i) - \psi_i(\boldsymbol{y}_i)| \\ &\geq \sum_{i: w_i > 0} 2^{w_i - 1} \\ &= \sum_{i=2}^n 2^{w_i - 1} - \sum_{i: w_i = 0} \frac{1}{2} \end{split}$$

We do not have control over the number of nonzero w_i 's, so let us take the worst case. We have

$$\sum_{i=2}^{n} \frac{1}{n-1} 2^{w_i} \ge 2^{\sum_{i=2}^{n} \frac{w_i}{n-1}} = 2^{\frac{w}{n-1}}.$$

As for $\sum_{i: w_i=0} \frac{1}{2}$, use the trivial upper bound (n-1)/2. Together the last two results conclude the proof. \blacksquare We have the following theorem as a result.

Theorem 10: Let \mathcal{C} and \mathcal{C}_{τ} be the binary and rank modulation codes defined in Theorem 8. Suppose furthermore that the minimum Hamming distance d of the code \mathcal{C} satisfies $d=\epsilon m$, where m is the blocklength of \mathcal{C} . Then the minimum Kendall distance of the code \mathcal{C}_{τ} is $\Omega(n^{1+\epsilon})$.

Proof: We have $\log n - 1 \le \lfloor \log n \rfloor \le \log n$. Use this in the definition of m to obtain that $m \ge n(\log n - 3)$. Therefore, $d = \epsilon m \ge \epsilon n(\log n - 3)$. From the previous proposition the minimum Kendall distance of \mathcal{C}_{τ} is at least

$$\frac{n-1}{2} \left(2^{\epsilon n(\log n - 3)/(n-1)} - 1 \right) = \Omega(n^{1+\epsilon}).$$

Examples of specific codes that can be constructed from this theorem are again deferred to Sect. IV.

Let us analyze the asymptotic trade-off between the rate and the distance of the codes. We begin with an asymptotically good family of binary codes, i.e., a sequence of codes $C_i, i=1,2\ldots$, of increasing length m for which the rate $\log |C_i|/m$ converges to a positive number R, and the relative Hamming distance behaves as ϵm , where $0<\epsilon<1/2$. Such families of codes can be efficiently constructed by means of concatenating several short codes into a longer binary code (e.g., [21, Ch. 10]) Using this family in the previous theorem, we obtain a family of rank modulation codes in S_n of Kendall distance that behaves as $\Omega(n^{1+\epsilon})$, and of rate R (see (1)). The upper limit of 1/2 on ϵ is due to the fact [21, p. 565] that no binary codes of large size (of positive rate) are capable of correcting a higher proportion of errors.

E. Correcting even more, $O(n^{1+\epsilon})$, errors, $1/2 \le \epsilon < 1$

It is nevertheless possible to extend the above theorem to the case of $\epsilon \geq 1/2$, obtaining rank modulation codes of distance $\Omega(n^{1+\epsilon}), \ 1/2 \leq \epsilon < 1$ and positive rate. However, this extension is not direct, and results in an existential claim as opposed to the constructive results above. To be precise, one

can show that for any $0 \le \varepsilon < 1$, there exist infinite families of binary (m,M,d) codes \mathcal{C} , with rate $R=1-\epsilon$, such that the associated rank modulation code \mathcal{C}_{τ} for permutations of [n] in Theorem 8 has minimum Kendall distance $\Omega(n^{1+\varepsilon})$.

Theorem 11: For any $0 < \epsilon < 1$, there exist infinite families of binary (m,M) codes $\mathcal C$ such that $(1/m)\log M \to 1-\epsilon > 0$, and the associated rank modulation code $\mathcal C_\tau$ constructed in Theorem 8 has minimum Kendall distance that scales as $\Omega(n^{1+\epsilon})$. Moreover all but an exponentially decaying fraction of the binary linear codes are such.

The rank modulation codes described above have asymptotically optimal trade-off between the rate and the distance. Therefore, this family of codes achieves the capacity of rank modulation codes (see [3, Thm. 3.1]).

To prove the above theorem we need the help of the following lemma.

Lemma 12: Let $0 \le \alpha \le 1$ and let $T \subset [m], |T| \ge \alpha m$ be a coordinate subset. There exists a binary code $\mathcal C$ of length m and any rate $R < \alpha$ such that the projections of any two codewords $x,y \in \mathcal C, x \ne y$ on T are distinct. Moreover all but an exponentially decaying fraction of binary linear codes of any rate less than α are such.

Proof: The proof is a standard application of the probabilistic method. Construct a random binary code $\mathcal C$ of length m and size $M=2^{mR}$ randomly and independently selecting M vectors from $\{0,1\}^m$ with uniform probability. Denote by $\mathcal E_{\boldsymbol x,\boldsymbol y}$ the event that two different vectors $\boldsymbol x,\boldsymbol y\in\mathcal C$ agree on T. Clearly $\Pr(\mathcal E_{\boldsymbol x,\boldsymbol y})=2^{-\alpha m}$, for all $\boldsymbol x,\boldsymbol y\in\mathcal C$. The event $\mathcal E_{\boldsymbol x,\boldsymbol y}$ is dependent on at most 2(M-1) other such events. Using the Lovász Local Lemma [2], all such events can be avoided, i.e.,

$$\Pr\left(\bigcap_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{C}}\bar{\mathcal{E}}_{\boldsymbol{x},\boldsymbol{y}}\right)>0,$$

if

$$e2^{-\alpha m}(2M-1) \le 1$$

or

$$M \le 2^{\alpha m - 1}/e + 1/2.$$

Hence as long as $R < \alpha$, there exists a code of rate R that contains no pairs of vectors $\boldsymbol{x}, \boldsymbol{y}$ that agree on T. This proves the first part of the lemma.

To prove the claim regarding random linear codes chose a linear code $\mathcal C$ spanned by the rows of an $mR\times m$ binary matrix G each entry of which is chosen independently with $P(0)=P(1)={}^1\!/2$. The code $\mathcal C$ will not contain two codewords that project identically on T if the $mR\times |T|$ submatrix of G with columns indexed by T has full rank. If mR<|T| then a given $mR\times |T|$ sub-matrix of G has full rank with probability at least $1-5\cdot 2^{-(|T|-mR)^2}$ [11]. Thus if |T| grows at least as $T=mR+\sqrt{m}$, the proportion of matrices G in which the $(mR\times T)$ submatrix is singular falls exponentially with m. Even if each of these matrices generates a different code, the proportion of undesirable codes will decline exponentially with m.

Proof of Thm. 11: Suppose that $x, y \in \{0, 1\}^m$ where $m = \sum_{i=2}^{n} m_i$ and $m_i = \lfloor \log i \rfloor$ as above in this section. Let

$$d_1(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})) = \sum_{i=2}^n |\psi_i(\boldsymbol{x}_i) - \psi_i(\boldsymbol{y}_i)| \le n^{1+\epsilon}$$

for some $0 \le \epsilon \le 1$. Let $0 < \beta < 1$. For at least a $1 - \beta$ proportion of indices i we can claim that

$$|\psi_i(\boldsymbol{x}_i) - \psi_i(\boldsymbol{y}_i)| \le \frac{n^{1+\epsilon}}{\beta(n-1)}.$$

On the other hand, if x_i and y_i have the same value in the first t_i of the m_i coordinates, then the construction of the Gray map implies that $|\psi_i(\boldsymbol{x}_i) - \psi_i(\boldsymbol{y}_i)| \ge 2^{m_i - t_i}$. Hence for at least a $1 - \beta$ fraction of the *i*'s,

$$2^{m_i - t_i} \le \frac{n^{1 + \epsilon}}{\beta(n - 1)},$$

i.e., $t_i \ge m_i - \epsilon \log n - \log \frac{n}{\beta(n-1)}$. Therefore, \boldsymbol{x} and \boldsymbol{y} must coincide in a well-defined subset of coordinates of size

$$\sum_{i=2}^{\lceil (1-\beta)(n-1) \rceil} t_i \ge \sum_{i=2}^{\lceil (1-\beta)(n-1) \rceil} \left(m_i - \epsilon \log n - \log \frac{n}{\beta(n-1)} \right)$$

$$= \sum_{i=2}^{\lceil (1-\beta)(n-1) \rceil} \lfloor \log i \rfloor$$

$$- \epsilon (1-\beta)(n-1) \log n - O(n)$$

$$= m(1 - \epsilon - O(1/\log n)).$$

Invoking Lemma 12 now concludes the proof: indeed, it implies that there exists a binary code of rate at least $1-\epsilon$ where no such pair of vectors x and y exists. The claim about linear codes also follows immediately.

F. Construction III: A quantization map

In this section we describe another construction of rank modulation codes from codes in the Hamming space over an alphabet of size $q \geq 2$. The focus of this construction is on the case when the number of errors is large, for instance, forms a proportion of n^2 .

The first result in this section serves as a warm-up for a more involved construction given later. In the first construction we use binary codes in a rather simple manner to obtain codes in permutations. This nevertheless gives codes in \mathcal{X}_n that correct a large number of errors. Then we generalize the construction by using codes over larger alphabets.

1) Construction IIIA: Rank modulation codes from binary base codes: Recall our notation \mathcal{G}_n for the space of inversion vectors and the map $J: \mathcal{G}_n \to S_n$ that sends them to permutations. Let $C \in \{0,1\}^{n-1}$ be a binary code that encodes k bits into n-1 bits.

Encoding algorithm (Construction IIIA):

• Let $m \in \{0,1\}^k$ be a message. Find its encoding **b** with the code C.

• Compute the vector $x = \vartheta(b)$, where $\vartheta : \{0,1\}^{n-1} \to$ \mathcal{G}_n is as follows:

$$\mathbf{b} = (b_1, b_2, \dots, b_{n-1}) \stackrel{\vartheta}{\mapsto} \mathbf{x} = (x_1, \dots, x_{n-1})$$
$$x_i = \begin{cases} 0 & \text{if } b_i = 0 \\ i & \text{if } b_i = 1 \end{cases}, \quad i = 1, \dots, n-1.$$

• Find the encoding of m as $\sigma = J(x)$.

Theorem 13: Let $C(n-1, M, d \ge 2t+1)$ be a code in the binary Hamming space and let $\mathcal{C}_{\tau} \subset S_n$ be the set of permutations obtained from it using the above encoding algorithm. Then the code $\mathcal{C}_{ au} \subset S_n$ has cardinality M and corrects any r Kendall errors where $r = t^2/4$ if $t \ge 2$ is even and $r = (t^2 - 1)/4$ if t > 3 is odd.

Proof: To prove the claim about error correction, consider the following decoding procedure of the code \mathcal{C}_{τ} . Let π be a permutation read off from memory.

Decoding algorithm (Construction IIIA):

- Find the inversion vector $\boldsymbol{x}_{\pi} = (x_1, \dots, x_{n-1}).$
- Form a vector $\mathbf{y} \in \{0,1\}^{n-1}$ by putting

$$y_i = \begin{cases} 0 & \text{if } x_i \le \lfloor i/2 \rfloor \\ 1 & \text{if } x_i > \lfloor i/2 \rfloor. \end{cases}$$

- Decode y with the code C to obtain a codevector c. If the decoder returns no result, the algorithm detects more than t errors.
- Compute the overall decoding result as $J(\vartheta(c))$.

Let σ be the original permutation, let x_{σ} be its inversion vector, and let $c(\sigma)$ be the corresponding codeword of C. The above decoding can go wrong only if the Hamming distance $d_H(\boldsymbol{c}(\sigma), \boldsymbol{y}) > t$. For this to happen the ℓ_1 distance between $m{x}_{\pi}$ and $m{x}_{\sigma}$ must be large, in the worst case satisfying the condition $d_1(m{x}_{\pi}, m{x}_{\sigma}) > \sum_{i=1}^t \lfloor i/2 \rfloor$. This gives the claimed result.

From a binary code in Hamming space of rate R that corrects any τn errors, the above construction produces a rank modulation code C_{τ} of size 2^{Rn} that is able to correct $\Omega(n^2)$ errors. The rate of the obtained code equals $\approx R(\log n)^{-1}$. According to Theorem 1 this scaling is optimal for the multiplicity of errors considered. Some numerical examples are given in Sect. IV.

2) Construction IIIB: Rank modulation codes from nonbinary codes: This construction can be further generalized to obtain codes that are able to correct a wide range of Kendall errors by observing that the quantization map employed above is a rather coarse tool which can be refined if we rely on codes in the q-ary Hamming space for q > 2. As a result, for any $\epsilon < 1$ we will be able to construct families of rank modulation codes of rate $R = R(\epsilon) > 0$ that correct $\Omega(n^{1+\epsilon})$ Kendall

Let l > 1 be an integer. Let $Q = \{a_1, a_2, \dots, a_q\}$ be the code alphabet. Consider a code \mathcal{C} of length n' = 2(l-1)(q-1)over Q and assume that it corrects any t Hamming errors (i.e., its minimum Hamming distance is at least 2t + 1). Let n=(2l+1)(q-1). Consider the mapping $\Theta_q:Q^{n-1}\to\mathcal{G}_n$, defined as $\Theta_q(\mathbf{b}) = (\vartheta_1(b_1), \vartheta_2(b_2), \dots, \vartheta_{n-1}(b_{n-1})), \mathbf{b} =$ $(b_1, \ldots, b_{n-1}) \in Q^{n-1}$, where

$$\vartheta_i(a_j) = \begin{cases} 0 & \text{if } i < 3(q-1) \\ (2k-1)(j-1) & \text{if } (2k-1)(q-1) \leq i \\ & < (2k+1)(q-1) \\ & k = 2, 3, \dots, l, \end{cases}$$
 vectors equals
$$\sum_{j=1}^{l-1} jt_j \geq \min_{\substack{t_j \leq 2(q-1) \\ \sum_j t_j \geq t+1}} \sum_{j=1}^{l-1} jt_j$$

$$= 2(q-1)(1+2+\cdots + (t+1-2(q-1)s)(s+1) + (t+1-2(q$$

To construct a rank modulation code C_{τ} from the code Cwe perform the following steps.

Encoding algorithm (Construction IIIB):

- ullet Encode the message m into a codeword $c \in \mathcal{C}$
- Prepend the vector c with 3(q-1)-1 symbols a_1 .
- Map the obtained (n-1)-dimensional vector to S_n using the map $J \circ \Theta_q$.

The properties of this construction are summarized in the following statement.

Theorem 14: Let n' = 2(l-1)(q-1), n = (2l+1)(q-1)1), $l \geq 2$. Let C(n', M, d = 2t + 1) be a code in the qary Hamming space. Then the code $\mathcal{C}_{\tau} \subset S_n$ described by the above construction has cardinality M and corrects any rKendall errors, where

$$r = (t+1-(q-1)s)(s+1)-1$$

and
$$s = |(t+1)/(2(q-1))|, s \ge 0.$$

Proof: We generalize the proof of the previous theorem. Let π be the permutation read off from the memory.

Decoding algorithm (Construction IIIB):

- Find the inversion vector $\mathbf{x}_{\pi} = (x_1, \dots, x_{n-1}).$
- Form a q-ary vector y by putting

$$y_i = \begin{cases} a_1 & \text{if } i < 3(q-1) \\ a_j & \text{if } (2k-1)(q-1) \le i < (2k+1)(q-1) \\ & \text{and } (2k-1)(j-1) - (k-1) \le x_i \\ & \le (2k-1)(j-1) + k, \\ & k = 2, 3 \dots, l \end{cases}$$

for i = 1, ..., n - 1.

- Decode $y' = (y_{3(q-1)}, \ldots, y_{n-1})$ with the code C to obtain a codevector c. If the decoder returns no results, the algorithm detects more than t errors.
- Find the decoded permutation as $\sigma = J(\Theta_a(\mathbf{c}))$.

There will be an error in decoding only when y' contains at least t+1 Hamming errors. y' contains coordinates 3(q-1)to n-1 of y. Suppose that $t_j, 1 \le j \le l-1$ is the number of Hamming errors in coordinates between (2j+1)(q-1) and (2j+3)(q-1). We have $\sum_{j=1}^{l-1} t_j \geq t+1$ and $t_j \leq 2(q-1)$. The ℓ_1 distance between the received and original inversion vectors equals

$$\sum_{j=1}^{l-1} jt_j \ge \min_{\substack{t_j \le 2(q-1) \\ \sum_j t_j \ge t+1}} \sum_{j=1}^{l-1} jt_j$$

$$= 2(q-1)(1+2+\cdots+s)$$

$$+ (t+1-2(q-1)s)(s+1)$$

$$= (q-1)s(s+1)+(t+1-2(q-1)s)(s+1)$$

$$= (t+1-(q-1)s)(s+1).$$

In estimating the minimum in the above calculation we have used the fact that the smaller-indexed t_j 's should be given the maximum value before the higher-indexed ones are used.

Therefore if the ℓ_1 distance between the received and original inversion vectors is less than or equal to r then decoding y' with the code C will recover x_{σ} . Using (7) we complete the proof.

Asymptotic analysis: For large values of the parameters we obtain that the number of errors correctable by \mathcal{C}_{τ} is

$$r \approx \frac{t^2}{4q}$$

or, in other words, $d(\mathcal{C}_{\tau}) \approx d^2/8q$. In particular, if $d = n'\delta$ and $q = O(n^{1-\epsilon}), 0 < \epsilon < 1$, then we get $d(\mathcal{C}_{\tau}) = \Omega(n^{1+\epsilon})$. If the code \mathcal{C} has cardinality $q^{Rn'}$ then $|\mathcal{C}_{\tau}| = q^{Rn'} = q^{R(n-3(q-1))}$. Using (1) yields the value $(1 - \epsilon)R$ for the rate of the code \mathcal{C}_{τ} . This is only by a factor of R less than the optimal scaling rate of (4). To achieve the optimal asymptotic rate-distance trade-off one need to use a q-ary code of rate very close to one and non-vanishing relative distance; moreover q needs to grow with code length n as $n^{1-\epsilon}$.

To show an example, let us take the family of linear codes on Hermitian curves (see e.g., [4, Ch. 10]). The codes can be constructed over any alphabet of size $q = b^2$, where b is a prime power. Let u be an integer, $b+1 \le u < b^2-b+1$. The length n', dimension k and Hamming distance d of the codes are as follows:

$$n' = b^3 + 1, \ k = (b+1)u - (1/2)b(b-1) + 1, \ d \ge n' - (b+1)u.$$

In the next section we will give a few examples of codes with specific parameters. For the moment, let us look at the scaling order of R and r as functions of the length of the codes \mathcal{C}_{τ} obtained from the above arguments. We have $n \approx qb$, so $q \approx n^{2/3}$, and

$$R = \frac{k}{n'} = \frac{(b+1)u - (1/2)b(b-1) + 1}{b^3 + 1},$$
$$\frac{d}{n'} \ge \frac{b^3 - (b+1)u}{b^3 + 1}.$$

Let us choose $u=b^2/2$, which gives $R\approx \frac{1}{2}\alpha$ and $\delta\approx \frac{1}{2}\alpha$, where $\alpha = 1 - O(1/b)$. Finally, we obtain that the rate of the codes C_{τ} behaves as

$$\frac{\log q^{Rn'}}{\log n!} = \frac{2}{3}R(1 - o(1))$$

and the number of correctable Kendall errors is $r \approx$ $(1/64)n^{4/3}$, which gives the scaling order mentioned in the previous paragraph for $\epsilon = 1/3$.

By taking $u=b^{1+\gamma}$, for $0<\gamma<1$, and by shortening the Hermitian code to the length $\lambda(b+1)u$, for $\lambda>1$ arbitrarily close to 1 we obtain a code with rate arbitrarily close to 1 with relative minimum distance equal to $1-1/\lambda$. This yields asymptotically optimal scaling, in the sense defined in section I, for values of ϵ that range in the interval (0,1/3). For values of ϵ in the range (1/3,1), families of codes with optimal scaling can similarly be constructed by starting from Algebraic Geometry codes with lengths that exceed larger powers of q than $q^{3/2}$, for instance, codes from the Garcia-Stichtenoth curves or other curves with a large number of rational points.

Another general example can be derived from the family of quadratic residue (QR) codes [21]. Let p be a prime, then there exist QR codes over \mathbb{F}_{ℓ} of length n' = p, cardinality $M=\ell^{(p+1)/2}$ and distance $\geq \sqrt{p}$, where ℓ is a prime that is a quadratic residue modulo p. Using them in Theorem 14 (after an appropriate shortening), we obtain rank modulation codes in S_n , where $n = p + 3(\ell - 1)$, with cardinality M and distance $d(\mathcal{C}_{\tau}) = \Omega(p/\ell)$. Let us take a sufficiently large prime p and let ℓ be a prime and a quadratic residue modulo p. Suppose that $\ell = \Theta(p^{\frac{1}{2}-\alpha})$ for some small $\alpha > 0$. Pairs of primes with the needed properties can be shown to exist under the assumption that the generalized Riemann hypothesis is true (see e.g. [18]). Using the corresponding QR code \mathcal{C} in Theorem 14, we obtain $n = p + 3(\ell - 1) = \Theta(p), d(\mathcal{C}_{\tau}) = \Theta(n^{\frac{1}{2} + \alpha})$ and $\log M = \Theta(\frac{n}{2}(\frac{1}{2} - \alpha) \log n)$, giving the rate $\frac{1}{2}(\frac{1}{2} - \alpha)$. Although this trade-off does not achieve the scaling order of (4), it still accounts for a good asymptotic family of codes.

IV. EXAMPLES

Below \mathcal{C}_{τ} refers to the rank modulation code that we are constructing, $M=|\mathcal{C}_{\tau}|$, and t is the number of Kendall errors that it corrects. We write the code parameters as a triple $(n, \log M, d)$ where d=2t+1. In the examples we do not attempt to optimize the parameters of rank modulation codes; rather, our goal is to show that there is a large variety of constructions that can be adapted to the needs of concrete applications. More codes can be constructed from the codes obtained below by using standard operations such as shortening or lengthening of codes [3], [15]. Note also that the design distance of rank modulation codes constructed below may be smaller than their true distance, so all the values of the distance given below are lower estimates of the actual values.

From Theorem 3 we obtain codes with the following parameters. Let $q=2^l$, then n=q-1 and $\log M \geq l \lfloor \log(q-2t-2) \rfloor$. For instance, let l=6, then we obtain the triples (63,30,31),(63,24,47), etc. Taking l=8, we obtain for instance the following sets of parameters: (255,56,127),(255,48,191).

Better codes are constructed using Theorem 8. Let us take n=62, then m=253. Taking twice shortened BCH codes \mathcal{B}_t of length m, we obtain a range of rank modulation codes according to the designed distance of \mathcal{B}_t . In particular, there are rank modulation codes in \mathcal{X}_{62} with the parameters

$$(62, 253 - 8t, 2t + 1), t = 1, 2, 3, \dots$$

Similarly, taking n=105, we can construct a suite of rank modulation codes from shortened BCH codes of length m=100

510, obtaining codes C_{τ} with the parameters

$$(105, 510 - 9t, 2t + 1), t = 1, 2, 3, \dots$$

We remark that for the case of t=1 better codes were constructed in [15]. Namely, there exist single-error-correcting codes in S_n of size $M \ge n!/(2n)$. For instance, for n=62 this gives $M=2^{277.064}$ as opposed to our $M=2^{245}$. A Hamming-type upper bound on M has the form

$$M(t) \le \frac{n!}{\sum_{i=0}^{t} K_n(i)}$$

where

$$K_n(0) = 1$$

 $K_n(1) = n - 1$
 $K_n(2) = (n^2 - n - 2)/2$
 $K_n(3) = {n+1 \choose 3} - n$

(see e.g., [17, p.15] which also gives a general formula for $K_n(i)$ for $i \le n$). The codes constructed above are not close to this bound (note however that, except for small t, Hamming-type bounds are usually loose).

Now let us use binary BCH codes in Theorem 13. Starting with codes of length n'=63,255 we obtain rank modulation codes with the parameters (64,36,13),(64,30,19),(64,24,25),(64,18,51),(64,16,61), (64,10,85), (256,215,13), (256,207,19), (256,199,25), (256,191,33), etc. These codes are not so good for a small number of errors, but become better as their distance increases.

Finally consider examples of codes constructed from Theorem 14. As our seed codes we consider the following possibilities: products of Reed-Solomon codes and codes on Hermitian curves.

Let us take $\mathcal{C}=\mathcal{A}\otimes\mathcal{B}$, where $\mathcal{A}[15,9,7]$ and $\mathcal{B}[14,3,12]$ are Reed-Solomon codes over \mathbb{F}_{16} . Then the code \mathcal{C} has length $n'=14\cdot 15=210$, (so l=8), cardinality $16^{27}=2^{108}$ and distance 84, so t=41. From Theorem 14 we obtain a rank modulation code \mathcal{C}_{τ} with the parameters $(n=255,\log M=108,d\geq 107)$. Some further sets of parameters for codes of length n=255 obtained as we vary $\dim(\mathcal{B})$ are as follows:

The code parameters obtained for n=255 are better than the parameters obtained for the same length in the above examples with binary BCH codes, although decoding product RS codes is somewhat more difficult than decoding BCH codes On the other hand, relying on product RS codes offers a great deal of flexibility in terms of the resulting parameters of rank modulation codes.

We have seen above that Hermitian codes account for some of the best asymptotic code families when used in Theorem 14. They can also be used to obtain good finite-length rank modulation codes. To give an example, let \mathcal{C} be a projective Hermitian code of length 4097 over \mathbb{F}_{2^8} . We have $\dim(\mathcal{C}) =$

17a - 119, $d(\mathcal{C}) \ge 4097 - 17a$ for any integer a such that $17 \le a \le 240$; see [4, p. 441]. Let us delete any 17 coordinates (puncture the code) to get a code \mathcal{C}' with

$$n' = 4080 = 16(q - 1),$$

$$\dim(\mathcal{C}') = \dim(\mathcal{C}),$$

$$d_H(\mathcal{C}') \ge n' - 17a.$$

We have n = n' + 3(q - 1) = 4845. For $a \in \{60, ..., 100\}$ we obtain a suite of rank modulation codes with the parameters (n, 7208, 6119), (n, 7344, 6071), ..., (n, 12648, 4079).

As a final remark, note that most existing coding schemes for the Hamming space, binary or not, can be used in one or more of our constructions to produce rank modulation codes. The decoding complexity of the obtained codes essentially equals the decoding complexity of decoding the original codes for correcting Hamming errors or for low error probability. This includes codes for which the Hamming distance is not known or not relevant for the decoding performance, such as LDPC and polar coding schemes. In this case, the performance of rank modulation schemes should be studied by computer simulations, similarly to the analysis of the codes used as building elements in the constructions.

V. CONCLUSION

We have constructed a number of large classes of rank modulation codes, associating them with binary and q-ary codes in the Hamming space. If the latter codes possess efficient decoding algorithms, then the methods discussed above translate these algorithms to decoding algorithms of rank modulation codes of essentially the same complexity. Our constructions also afford simple encoding of the data into permutations which essentially reduces to the encoding of linear error-correcting codes in the Hamming space. Thus, the existing theory of error-correcting codes can be used to design practical error-correcting codes and procedures for the rank modulation scheme.

A direction of research that has not been addressed in the literature including the present work, is to construct an adequate model of a probabilistic communication channel that is associated with the Kendall tau distance. We believe that the underpinnings of the channel model should be related to the process of charge dissipation of cells in flash memory devices. Once a reasonably simple probabilistic description of the error process is formally modelled, the next task will be to examine the performance on that channel of code families constructed in this work.

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