Simple schedules for half-duplex networks

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***Abstract*—We consider the diamond network where a source communicates with the destination through N non-interfering half-duplex relays. Deriving a simple approximation to the capacity of the network, we show that simple schedules having exactly two states and avoiding broadcast and multiple access communication can still achieve a significant constant fraction of the capacity of the 2 relay network, independent of the channel SNRs. The results are extended to the case of 3 relays for the special class of antisymmetric networks. We also study the structure of (approximately) optimal relaying strategies for such networks. Simulations show that these schedules have at most $N+1$ states, which we conjecture to be true in general. We prove the conjecture for $N=2$ and for special cases for $N=3$.[1]**

I. Introduction

Calculating the capacity of wireless relay networks is a hard problem; calculating the capacity when the relays are half-duplex is even harder. Indeed, in half duplex relay networks, an additional dimension of optimization comes into play: scheduling the relay states, i.e., whether each relay transmits (T) or listens (L) at any given time instance [5]. For example, for the N-relay diamond network in Fig. 1, there exist 2^N possible combinations of L, T states, and any capacity achieving strategy would need to optimize for how long each of these occurs.

In this paper, we consider half-duplex diamond networks [7]. Our position is the following: at least for small diamond networks, there might be no need for such an exponential size optimization. We base this claim on two observations. First, following the network simplification approach of [4], we show that even very simple schedules that use only two states and employ point-to-point connections (no broadcasting and no multiple access) can (approximately) achieve a rate that is a significant multiplicative fraction of the capacity of the whole network. This multiplicative fraction is independent of the strength of the links in the 2 and 3 relay diamond networks and the operating SNR. Second, the approximately optimal schedule may have at most $N+1$ active states, instead of the possible 2^N. For example, for 2 relays, although 4 states are possible, at most 3 are employed (this directly follows from the work in [2]), while for 3 relays, only 4 out of the 8 possible states are employed. This conjecture is supported by experimental results, as well as analytic proofs for some special cases.

The aim of this paper is to show that even with reduced schedule complexity, significant rates are achievable for small

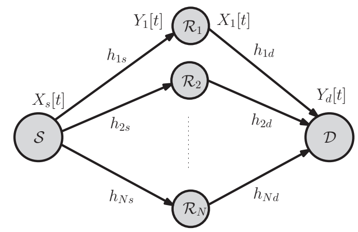

Fig. 1. The Gaussian N-relay half-duplex diamond network.

half-duplex diamond networks. In the rest of the paper: Section II provides the framework of our work, i.e., the network model, a simple approximation to the capacity of the half-duplex diamond network, the rate achieved by the simple strategies of interest and a Linear Programming (LP) problem formulation; Section III establishes lower bounds on the rates achieved by the simple strategies; Section IV presents our conjecture regarding the linear number of active states in the (approximately) optimal schedule.

II. Network Model and Problem Formulation

A. Network Model

We consider the Gaussian N-relay diamond network where a source $\mathcal{S}$ transmits information to a destination $\mathcal{D}$ with the help of half-duplex relays. At any given time t, each relay $\mathcal{R}_i$ can either listen (L) or transmit (T), but not both; we denote by $M_i[t] \in \{L, T\}$ its state. We denote by $M_s[t]$ and $M_d[t]$ the states of the source and the destination, respectively.

Let $X_s[t]$ be the signal transmitted by $\mathcal{S}$ at time t, $X_i[t]$ be the signal transmitted by relay $\mathcal{R}_i$, $Y_d[t]$ and $Y_i[t]$ the signals received by $\mathcal{D}$ and $\mathcal{R}_i$, respectively. Then

$$\begin{aligned} X_i[t] &= 0 \text{ when } M_i[t] = L \\ Y_i[t] &= h_{is}X_s[t] + Z_i[t] \text{ when } M_i[t] = L \\ &= 0 \text{ when } M_i[t] = T \\ Y_d[t] &= \sum_{i=1}^{N} h_{id}X_i[t] + Z[t] \text{ when } M_d[t] = L \\ &= 0 \text{ when } M_d[t] = T \end{aligned}$$

where h_{is}, h_{id} are the complex channel coefficients between $\mathcal{S}$ and $\mathcal{R}_i$ and $\mathcal{R}_i$ and $\mathcal{D}$, respectively. $Z_i[t]$ and $Z[t]$ are independent and identically distributed white Gaussian random processes of power spectral density $N_0/2$ Watts/Hz.

The power constraints for the source and all the relays are fixed to P. We can then calculate the individual link capacities from $\mathcal{S}$ to $\mathcal{R}_i$ as $R_{is} = \log(1 + |h_{is}|^2 P)$ and from $\mathcal{R}_i$ to $\mathcal{D}$ as $R_{id} = \log(1 + |h_{id}|^2 P)$. $[N]$ represents the set $\{1, 2, \cdots, N\}$ and the relays are ordered such that $R_{is} \geq R_{js}$ for $i < j$. Finally, unless otherwise stated, the term "constant" will mean

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a quantity that depends only on the number of relays and is independent of the channel SNRs.

B. An Approximation to the Capacity

For half-duplex relay networks, the capacity depends not only on the strength of the channel coefficients, but crucially also on the L-T scheduling. Let $m \in M = \{L, T\}^N$ denote a particular state of the relays and let $L(m)$ and $T(m)$ be the set of relays in listening and transmitting state in m, respectively. In a particular schedule, let $p(m)$ denote the fraction of time the relays are in state m. Let C_{hd}^N be the capacity of the N-relay half-duplex diamond network. [6, Section VI] derives an upper bound on the capacity of arbitrary half-duplex wireless networks, which combined with the simplification approach developed in [4] for diamond networks, yields the following upper bound on C_{hd}^N.

Lemma 2.1: We have $C_{hd}^N \leq C_{LP}^N + G(N)$ where $G(N) = N + 3\log N - 2.75$ and

$$C_{LP}^N = \max_{\substack{p(m)\\ m\in M}} \min_{\Lambda\subseteq[N]} \sum_{m\in M} p(m) \left(\max_{i\in\bar{\Lambda}\cap L(m)} R_{is} + \max_{i\in\Lambda\cap T(m)} R_{id} \right)$$

The minimization is over all the 2^N subsets Λ of the relay nodes $[N] = \{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ and the maximization is over all schedules $p(.)$ such that $\sum_{m\in M} p(m) = 1$. Starting with the cutset-upper bound to the capacity of the network similar to [4], the main idea in the proof is to show that the values of the broadcast and multiple-access cuts can be bounded by the maximum capacity of the individual constituent links within a certain constant gap. Following [1], [6], we can also show that C_{LP}^N is achievable by quantize-map-and-forward scheme within an additive constant gap. This is because C_{LP}^N is smaller than the cutset-upper bound to the capacity of the network and these works show that the cutset-upper bound is achievable within a certain gap. Therefore, we get the following approximation for the capacity C_{hd}^N

Theorem 2.2: For a N relay half-duplex diamond network, there exist constants $G(N)$ and $G'(N)$ such that

$$C_{LP}^N - G'(N) \leq C_{hd}^N \leq C_{LP}^N + G(N) \tag{1}$$

Thus C_{LP}^N, which only depends on the individual capacities of the links $\{R_{is}, R_{id}\}$, can approximate C_{hd}^N upto constant additive terms.

C. Simple Strategies

We define *simple strategies* to be relaying strategies that use exactly two states and avoid broadcast at the source and multiple access at the destination. Concretely, we look at the rates achievable under the *Decode-Forward* scheme:

1-relay simple strategy: Let $C_{s1,i}$ be the maximum achievable rate over the one hop network $\mathcal{S}$-$\mathcal{R}_i$-$\mathcal{D}$ when the relay node $\mathcal{R}_i$ decodes the source message, re-encodes and transmits it to $\mathcal{D}$. Let p_1, p_2 be the fraction of time $\mathcal{R}_i$ is in the L and T state, respectively. Then

$$C_{s1,i} = \max_{\substack{p_1,p_2\\ p_1+p_2=1}} \min(R_{is}p_1, R_{id}p_2)$$

Solving the maximization, we can easily conclude that $C_{s1,i} = R_{is}R_{id}/(R_{is} + R_{id})$. We define C_{s1} to be the maximum achievable rate by this strategy which uses decode and forward at a single relay, i.e.

$$C_{s1} = \max_{i\in[N]} C_{s1,i} \tag{2}$$

2-relay simple strategy: With two relays, we use the *Mutihop-Decode-Forward* (MDF) strategy as defined in [2], [8]. A pair of relays $\mathcal{R}_i$ and $\mathcal{R}_j$ $(i < j)$ are operated in a complementary fashion, using only the two states $\{L, T\}$ and $\{T, L\}$. Each of the relay performs decode-and-forward. Let p_1, p_2 be the fraction of time ($\mathcal{R}_i$, $\mathcal{R}_j$) are in the states (L, T) and (T, L) respectively. Then the maximum rate achieved by this strategy is given by

$$C_{s2,ij} = \max_{\substack{p_1,p_2\\ p_1+p_2=1}} \min(p_1R_{is}, p_2R_{id}) + \min(p_2R_{js}, p_1R_{jd})$$

Note that the first term is the rate carried by the first relay and the second term is the rate carried by the second relay. Assuming $R_{is} \geq R_{js}$, the maximization can be solved to obtain ([2], [8])

$$\begin{aligned} C_{s2,ij} &= \frac{R_{is}(R_{js} + R_{id})}{R_{is} + R_{id}} \text{ if } R_{is}R_{js} < R_{id}R_{jd} \\ &= \frac{R_{id}(R_{is} + R_{jd})}{R_{is} + R_{id}} \text{ if } R_{is}R_{js} \geq R_{id}R_{jd}, R_{jd} < R_{id} \\ &= \frac{R_{jd}(R_{js} + R_{id})}{R_{js} + R_{jd}} \text{ if } R_{is}R_{js} \geq R_{id}R_{jd}, R_{jd} \geq R_{id} \end{aligned}$$

The best achievable rate C_{s2} by this strategy is given by a maximization over all possible choices for the two relays, i.e.,

$$C_{s2} = \max_{i,j\in[N], i<j} C_{s2,ij} \tag{3}$$

Finally, suppose we can show that a particular relaying strategy achieves a rate C'. Then the next result, which follows easily from Theorem 2.2, can be used to prove bounds on the rates achievable by our simple strategies.

Lemma 2.3: If $C' \geq \alpha C_{LP}^N$, then $C' \geq \alpha C_{hd}^N - \alpha G(N)$.

D. Linear Programming Formulation

C_{LP}^N can be reformulated as a linear program as follows.

$\mathbf{LP1}$: Maximize C

$$\sum_{i=1}^{2^N} p_i \left(\max_{j\in\Lambda\cap L(m_i)} R_{js} + \max_{j\in\bar{\Lambda}\cap T(m_i)} R_{jd} \right) \geq C; \forall\Lambda \subseteq [N]$$

$$\sum_{i=1}^{2^N} p_i = 1;\ \forall i, p_i \geq 0, C \geq 0$$

The 2^N variables of type $p(m)$ have been numbered as p_i with m_i being the corresponding state. **LP1** can be visualized in a matrix form as follows. (All vectors are column vectors)

$$\text{Maximize } \mathbf{c}^T[\mathbf{p}\, C] \qquad \textbf{(LP1)}$$
$$\mathbf{A}[\mathbf{p}\, C] \geq \mathbf{b};\ [\mathbf{p}\, C] \geq \mathbf{0}$$

where the objective function vector $\mathbf{c}^T$ is of size $1 \times (2^N + 1)$, with all zero entries except the last one which is +1. $\mathbf{A}$ is a $(2^N + 1) \times (2^N + 1)$ matrix with

$$A_{k,i} = \max_{j\in\Lambda(k)\cap L(m_i)} R_{js} + \max_{j\in\overline{\Lambda(k)}\cap T(m_i)} R_{jd}$$
$$\text{for } 1 \leq k \leq 2^N; 1 \leq i \leq 2^N$$

$$= -1 \text{ for } 1 \le k \le 2^N; i = 2^N + 1$$
$$= -1 \text{ for } k = 2^N + 1; 1 \le i \le 2^N$$
$$= 0 \text{ for } k = 2^N + 1; i = 2^N + 1$$

where $\Lambda(k)$ is the k-th subset of $[N]$. $\mathbf{b}$ is a $(2^N + 1) \times 1$ vector with all zero entries except the last one which is -1. The variable vector $[\mathbf{p}\, C]$ consists of the 2^N variables $\{p_1, p_2, \cdots, p_{2^N}\}$ and the (approximate) capacity variable C. The dual of **LP1**, denoted by **DLP1**, is a minimization problem defined as follows.

$$\text{Minimize } \mathbf{c}^T[\mathbf{p}_d\, C_d] \qquad \textbf{(DLP1)}$$
$$\mathbf{A}[\mathbf{p}_d\, C_d] \le \mathbf{b};\ [\mathbf{p}_d\, C_d] \ge \mathbf{0}$$

The definitions of $\mathbf{A}, \mathbf{b}, \mathbf{c}$ are the same as above and $[\mathbf{p}_d\, C_d]$ is the corresponding variable vector in the dual program.

III. Performance of Simple Strategies

In [4], it was shown that for full-duplex N-relay diamond networks, we can always find a k-relay subnetwork that approximately achieves $\frac{k}{k+1}$ fraction of the full-duplex network capacity within an additive constant factor; for half-duplex, this implies the following lemma.

Lemma 3.1: For a N-relay half-duplex diamond network, there exist a k relay subnetwork that approximately achieves $\frac{k}{2(k+1)}$ of the capacity of the whole network within constant additive factors.

Therefore, a 1-relay subnetwork can approximately achieve 1/4 and a 2 relay subnetwork 1/3 of the network's capacity for any N. Network simplification [4] for half-duplex relays involves both using fewer relays and fewer number of states in the schedule. Therefore, what we show below can be thought of as improved simplification bounds for $N = 2$ and $N = 3$.

A. 2 Relay Networks

As shown in [2], the linear program for C_{LP}^2 can be solved exactly to obtain closed form expressions. Using them, we can prove the following result.

Lemma 3.2: For a 2-relay half-duplex diamond network, for some constants c_1, c_2,

$$C_{s1} \ge \frac{1}{2} C_{hd}^2 - c_1,\ C_{s2} \ge \frac{8}{9} C_{hd}^2 - c_2$$

Proof: We show that $C_{s1} \ge \frac{1}{2} C_{LP}^2$ and $C_{s2} \ge \frac{8}{9} C_{LP}^2$, whence the result follows from Lemma 2.3. For brevity, assume $\{R_{1s}, R_{2s}, R_{1d}, R_{2d}\} = \{a, b, c, d\}$. Note that $a \ge b$. We will show the proofs for the case ($ab < cd$). The other cases are similar. In this case, we have

$$\frac{C_{s1,1} + C_{s1,2}}{C_{LP}^2} - 1 = \frac{\frac{ac}{a+c} + \frac{bd}{b+d}}{\frac{ac(b+d)+bd(a-b)}{(b+d)(a+c-b)}} - 1$$
$$= \frac{bc(b+d)(cd - ab)}{(ac(b+d) + bd(a-b))(a+c)(b+d)} \ge 0$$

Hence, $C_{s1} = \max\{C_{s1,1} + C_{s1,2}\} \ge \frac{1}{2} C_{LP}^2$. For the other claim, since there are only two relays, $C_{s2} = C_{s2,12}$. For the case of ($ab < cd$), we have

$$\frac{9C_{s2}}{8C_{LP}^2} - 1 = \frac{9ab^2(a-b) + abc(a+c) + df_1(a,b,c)}{8(a+c)(ac(b+d) + bd(a-b))}$$

where $f_1(a,b,c) = a^2b - ab^2 + a^2c - 8abc + 8b^2c + ac^2$. Writing f_1 as a quadratic expression in c, we have

$$f_1(a,b,c) = ac^2 + (a^2 - 8ab + 8b^2)c + ab(a-b)$$

Clearly, if $a^2 - 8ab + 8b^2 \ge 0$, then $f_1(a,b,c) \ge 0$. Since the equation $x^2 - 8x + 8 = 0$ has two roots approximately equal to 1.17 and 6.82, as long as $a/b \in [1, 1.17] \cup [6.82, +\infty]$, $a^2 - 8ab + 8b^2 \ge 0$ and hence $f_1(a,b,c) \ge 0$. On the other hand, we can also look at f_1 as a quadratic function in c and look at its discriminant as a function of a, b. We have

$$\Delta_{a,b} = (a^2 - 8ab + 8b^2)^2 - 4a(ab(a-b))$$
$$= (a - 2b)^2(a^2 - 16ab + 16b^2)$$

Since the roots of $x^2 - 16x + 16 = 0$ are approximately 1.07 and 14.92, the discriminant $\Delta_{a,b} < 0$ if $1.07 \le a/b \le 14.92$, in which case f_1 as a function of c is non-negative. Since the interval $[1, 1.17] \cup [6.82, +\infty] \cup [1.07, 14.92]$ covers all possible values of a/b, we can conclude that $f_1(a,b,c) \ge 0$ in all cases. Hence

$$\frac{9C_{s2}}{8C_{LP}^2} - 1 \ge 0 \implies \frac{C_{s2}}{C_{LP}^2} \ge \frac{8}{9}$$

which proves the second claim of the lemma. ∎

The multiplicative ratios are essentially the best we can obtain.

Lemma 3.3: There exist 2-relay half-duplex diamond networks where

$$C_{s1} = \frac{1}{2} C_{LP}^2,\ C_{s2} \approx \frac{8}{9} C_{LP}^2$$

Proof: For the first claim, consider the network where $R_{1s} = a, R_{2s} = b, R_{1d} = b, R_{2d} = a$ for some $a, b \in \mathbb{R}^+$, ($a > b$). In this case, $C_{s1}/C_{LP}^2 = \frac{ab/(a+b)}{2ab/(a+b)} = 1/2$. For the second claim, consider the network with $R_{1s} = 2a, R_{2s} = a, R_{1d} = a, R_{2d} = ka$ for some $k > 2$. Then, plugging in the expressions for capacities, we have

$$\frac{C_{s1}}{C_{LP}^2} = \frac{4(2+2k)}{3(2+3k)} \to \frac{8}{9} \text{ as } k \to \infty$$

∎

To summarize, we have shown that for the 2-relay diamond network, we can universally achieve approximately 50% of the capacity using the *1-relay simple strategy* and 88% by using the *2-relay simple strategy*, independent of the channel SNRs.

B. 3 Relay Antisymmetric Networks

For the case of $N = 3$ relays, it is difficult to obtain closed form expressions for C_{LP}^3 involving the six terms $(R_{1s}, R_{2s}, R_{3s}, R_{1d}, R_{2d}, R_{3d})$. We distinguish the relay networks according to the order of the relative values of these capacities. Assuming that $R_{1s} \ge R_{2s} \ge R_{3s}$, the R_{id} values can occur in six possible permutations. Although bounds can be obtained for each of the cases separately, we present here the results for the special case of *antisymmetric networks* where $R_{1s} \ge R_{2s} \ge R_{3s}$ and $R_{1d} \le R_{2d} \le R_{3d}$.

Lemma 3.4: For the anti-symmetric 3-relay half-duplex diamond network, for some constants c_3, c_4,

$$C_{s1} \ge \frac{1}{3} C_{hd}^3 - c_3,\ C_{s2} \ge \frac{1}{2} C_{hd}^3 - c_4$$

Proof: To prove the result we show that $C_{s1} \ge \frac{1}{3} C_{LP}^3$ and $C_{s2} \ge \frac{1}{2} C_{LP}^3$ whence the result follows from Lemma 2.3.

Redefine $\{R_{1s}, R_{2s}, R_{3s}, R_{1d}, R_{2d}, R_{3d}\} = \{a, b, c, d, e, f\}$. and let $x = \max\{d, e\}$, $y = \max\{e, f\}$, $z = \max\{d, f\}, t = \max\{d, e, f\}$. For the anti-symmetric network, $a \geq b \geq c$ and $d \leq e \leq f$. Hence $x = e$ and $y, z, t = f$. The **LP1** matrix for the network is shown below.

$$\begin{pmatrix} a & a & a & b & a & b & c & 0 & -1 \\ a & a+f & a & b & a+f & b+f & 0 & f & -1 \\ a & a & a+e & c & a+e & 0 & c+e & e & -1 \\ b & b & c & b+d & 0 & b+d & c+d & d & -1 \\ a & a+f & a+e & 0 & a+y & f & e & y & -1 \\ b & b+f & 0 & b+d & f & b+z & d & z & -1 \\ c & 0 & c+e & c+d & e & d & c+x & x & -1 \\ 0 & f & e & d & y & z & x & t & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

We will construct three upper bounds to the optimum value of this program by picking three different dual feasible solutions. They are

$$\bar{\alpha_d} = [\frac{d}{d+a-b}, 0, 0, \frac{a-b}{d+a-b}, 0, 0, 0, 0, \frac{ad+ab-b^2}{d+a-b}]$$
$$\bar{\gamma_d} = [0, 0, 0, 0, 0, 0, \frac{f-e}{c+f-e}, \frac{c}{c+f-e}, \frac{fc+fe-e^2}{c+f-e}]$$

The third one $\bar{\beta_d}$ is defined as follows. When $e \neq d$ or $b \neq c$,

$$\bar{\beta_d} = [0, 0, 0, \frac{e-d}{e-d+b-c}, 0, 0, \frac{b-c}{e-d+b-c}, 0, \frac{(b+d)(e-d)+(c+d)(b-c)}{e-d+b-c}]$$

and when $e = d, b = c$, we define

$$\bar{\beta} = [0, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, b+d]$$

Let $\alpha_0 = \frac{ad+ab-b^2}{d+a-b}$, $\gamma_0 = \frac{fc+fe-e^2}{c+f-e}$ and $\beta_0 = \frac{(b+d)(e-d)+(c+d)(b-c)}{e-d+b-c}$ or $b+d$ depending on the parameter values. It can be verified that these three solutions are dual feasible and hence by weak duality [3] their objective values are upper bounds to C_{LP}^3. Hence, $\alpha_0, \beta_0, \gamma_0 \geq C_{LP}^3$, which implies $\min\{\alpha_0, \beta_0, \gamma_0\} \geq C_{LP}^3$.

We claim that the following holds,

$$\frac{\frac{ad}{a+d}}{\alpha_0} + 2\frac{\frac{be}{b+e}}{\beta_0} + \frac{\frac{cf}{c+f}}{\gamma_0} \geq \frac{4}{3}$$

This can be shown by expanding the terms and using the fact that $a \geq b \geq c$ and $d \leq e \leq f$. Therefore

$$\frac{4C_{s1}}{\min\{\alpha_0, \beta_0, \gamma_0\}} \geq \frac{\frac{ad}{a+d}}{\beta_0} + 2\frac{\frac{be}{b+e}}{\gamma_0} + \frac{\frac{cf}{c+f}}{\alpha_0} \geq \frac{4}{3}$$

which implies that $C_{s1} \geq \frac{1}{3}C_{LP}^3$. Now for the second claim, let us consider the pairs of relays $(\mathcal{R}_1, \mathcal{R}_2)$ and $(\mathcal{R}_2, \mathcal{R}_3)$. If $C' = C_{s2,12} + C_{s2,23}$, using the expressions above for the 2-relay simply strategy, we have

$$\begin{aligned} C' &= \frac{a(b+d)}{a+d} + \frac{b(e+c)}{b+e} \text{ if } \frac{e}{b} \geq \frac{a}{d} \geq \frac{c}{f} \\ &= \frac{e(b+d)}{b+e} + \frac{b(e+c)}{b+e} \text{ if } \frac{a}{d} \geq \frac{e}{b} \geq \frac{c}{f} \\ &= \frac{e(b+d)}{b+e} + \frac{f(e+c)}{f+c} \text{ if } \frac{a}{d} \geq \frac{c}{f} \geq \frac{e}{b} \end{aligned}$$

If $\left(\frac{e}{b} \geq \frac{a}{d} \geq \frac{c}{f}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12} + C_{s2,23}}{\alpha_0} = \frac{n_1(a,b,c,d,e,f)}{d_1(a,b,c,d,e,f)} \geq 1$$

If $\left(\frac{a}{d} \geq \frac{e}{b} \geq \frac{c}{f}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12}}{\alpha_0} + \frac{C_{s2,23}}{\gamma_0} = \frac{n_2(a,b,c,d,e,f)}{d_2(a,b,c,d,e,f)} \geq 1$$

If $\left(\frac{a}{d} \geq \frac{c}{f} \geq \frac{e}{b}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12} + C_{s2,23}}{\gamma_0} = \frac{n_3(a,b,c,d,e,f)}{d_3(a,b,c,d,e,f)} \geq 1$$

where $n_1, n_2, n_3, d_1, d_2, d_3$ are polynomials in (a, b, c, d, e, f) and the last inequalities in each of the three cases follows from substitution and expansion of terms and using the fact that $a \geq b \geq c$ and $d \leq e \leq f$. Therefore $C_{s2,12} + C_{s2,23} \geq C_{LP}^3$. Picking the maximum of the two pairs, we get

$$C_{s2} \geq \max\{C_{s2,12}, C_{s2,23}\} \geq \frac{1}{2}C_{LP}^3$$

■

The best lower bound multiplicative ratios we have been able to establish are the following.

Lemma 3.5: There exist 3-relay half-duplex diamond networks where

$$C_{s1} \approx 0.4C_{LP}^3, \; C_{s2} \approx 0.625C_{LP}^3$$

Proof: Consider the network $a = kr, b = 3r, c = 3r, d = 2r, e = 5r, f = 5r$ for some $k > 30, r > 0$. For this case, $C_{LP}^3 = \frac{(5k-9)r}{k-1}$, $C_{s1} = \frac{2kr}{k+2}$, $C_{s2} = \frac{25r}{8}$. Therefore, as $k \to \infty$,

$$\frac{C_{s1}}{C_{LP}^3} \to \frac{2}{5} = 0.4, \; \frac{C_{s2}}{C_{LP}^3} \to \frac{5}{8} = 0.625$$

■

To summarize, we have shown that for the 3-relay antisymmetric diamond network, we can universally achieve approximately 33% of the capacity using the *1-relay simple strategy* and 50% by using the *2-relay simple strategy*, independent of the channel SNRs.

IV. The Complexity of Optimal Schedules

In general, the optimal schedule in **LP1** corresponding to C_{LP}^N can have 2^N active states; we here present our conjecture that in fact, there always exists an optimal schedule with a linear number of active states. If true, this offers a significant reduction (from exponential to linear) to the number of states needed for optimal operation, making it more feasible to implement such schedules in practice.

Conjecture: For a N relay half-duplex diamond network, there exists a schedule that optimizes the value of C_{LP}^N and has at most $N + 1$ active states.

We support this conjecture in two ways:

Experimental results: Fig. 2 shows numerical evaluation results for **LP1**. We plot the average number of active states in the optimal schedule as a function of the number of relays N. The average is taken over several random instances of the networks, where the SNRs of the source to relay and relay to destination channels are chosen independently and uniformly

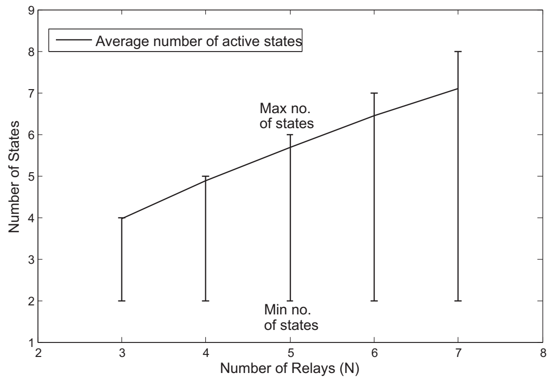

Fig. 2. Average, minimum and maximum number of active states for C_{LP}^N

at random from the interval $[1, 1000]$. For each value of N, the maximum and the minimum number of active states is found to be $N+1$ and 2, respectively.

Proof for special cases: For the case of $N=2$ relays, the claim follows easily by directly evaluating the optimal schedule [2] and checking that there are at most three states. We have not been able to come up with a general proof for $N>2$. In what follows, we prove the conjecture for a special case of $N=3$. Redefine $\{R_{1s}, R_{2s}, R_{3s}, R_{1d}, R_{2d}, R_{3d}\} = \{a, b, c, d, e, f\}$ Consider the case when the point to point capacities of all the relay to destination links dominates those of the source to relay links or vice-versa.

Lemma 4.1: For a 3-relay half-duplex diamond network where $\min\{d, e, f\} \geq \max\{a, b, c\}$ or $\min\{a, b, c\} \geq \max\{d, e, f\}$, the optimal solution for **LP1** has exactly 4 non-zero states.

Proof: Assume $\min\{d, e, f\} \geq \max\{a, b, c\}$. The matrix corresponding to **LP1** is the same as the one mentioned in the proof of Lemma 3.4. Name the rows of the matrix as $I_1, \cdots, I_9$ and columns as $J_1, \cdots, J_9$. Consider the sub-matrix $\mathbf{S}$ formed using rows I_1, I_4, I_7, I_8, I_9 and columns J_1, J_2, J_3, J_4, J_9 and the corresponding form of **LP1** with equality.

$$\mathbf{S}[p_1\, p_2\, p_3\, p_4\, C] = [0\,0\,0\,0\; -1]$$

This can be solved to get the following result.

$$\{p_1, p_2, p_3, p_4\} = \{\frac{\Delta_1}{(a-b+d)(b-c+e)(c+f)}, \frac{c}{c+f}, \frac{bc+(b-c)f}{(b-c+e)(c+f)}, \frac{e(a-b)(c+f)+(b-c)(ac+f(a-c))}{(a-b+d)(b-c+e)(c+f)}\}$$

and

$$C = \frac{(a((c+d)(e-d)+b(c+e))+d(b(c+e)+c(e-d)))fe}{(a+d)(b+e-d)(c+f-e)} - \frac{e\,(ad(e-d)+be(a+d))}{(a+d)(b+e-d)(c+f-e)} = a(p_1+p_2+p_3)+bp_4$$

where

$$\Delta_1 = b^2c - c^2f + def + bc(e+f-d) + a(c(c+f-e) - b(2c+f))$$

Since $a \geq b \geq c$, $p_2, p_3, p_4 \geq 0$. Further, since $\min\{d, e, f\} \geq \max\{a, b, c\} = a$, we have $f = a + l_1, e = a + l_2, d = a + l_3$, for some $l_1, l_2, l_3 \geq 0$. Therefore,

$$\Delta_1 = (a^2-bc)(a-b) + l_1(a(a-b)+c(b-c)+ac) + l_2(a(a-c)+bc) + l_3(a^2-bc) + a(l_1l_2+l_2l_3+l_3l_1) + l_1l_2l_3$$

Since $a \geq b \geq c$, $\Delta_1 \geq 0$ and $C \geq 0$. If we define $\mathbf{p} = \{p_1, p_2, p_3, p_4, 0, 0, 0, 0\}$ and C is the same as above, then

$$I_1[\mathbf{p}\, C] = I_4[\mathbf{p}\, C] = I_7[\mathbf{p}\, C] = I_8[\mathbf{p}\, C] = \mathbf{0}$$

It can be explicitly verified that this implies

$$I_2[\mathbf{p}\, C], I_3[\mathbf{p}\, C], I_5[\mathbf{p}\, C], I_6[\mathbf{p}\, C] \geq 0$$

In other words $[\mathbf{p}\, C]$ is a feasible solution for **LP1**. We will now consider the dual program and solve for the submatrix of the dual consisting of columns J_1, J_2, J_3, J_4, J_9 and rows I_1, I_4, I_7, I_8, I_9, which is the transpose of S considered above. Note that the dual variables in the **DP1** correspond to the rows in **LP1**. The corresponding form of **DLP1** with equality is

$$\mathbf{S}^T[p_1^d\, p_2^d\, p_3^d\, p_4^d\, C] = [0\,0\,0\,0\; -1]$$

On solving, we get

$$\{p_1^d, p_4^d, p_7^d, p_8^d\} = \{\frac{d}{a-b+d}, \frac{(a-b)e}{(a-b+d)(b-c+e)}, \frac{(a-b)(b-c)f}{(a-b+d)(b-c+e)(c+f)}, \frac{(a-b)(b-c)c}{(a-b+d)(b-c+e)(c+f)}\}$$

and where

$$C^d = ap_1^d + bp_4^d + cp_7^d = C$$

Clearly, $p_1^d, p_4^d, p_7^d, p_8^d \geq 0$. If we define $\mathbf{p}^d = \{p_1^d, 0, 0, p_4^d, 0, 0, p_7^d, p_8^d\}$, then

$$J_1^T[\mathbf{p}^d\, C^d] = J_2^T[\mathbf{p}^d\, C^d] = J_3^T[\mathbf{p}^d\, C^d] = J_4^T[\mathbf{p}^d\, C^d] = 0$$

Again, it can be explicitly verified that this implies

$$J_5^T[\mathbf{p}^d\, C^d], J_6^T[\mathbf{p}^d\, C^d], J_7^T[\mathbf{p}^d\, C^d], J_8^T[\mathbf{p}^d\, C^d] \leq 0$$

In other words, $[\mathbf{p}^d\, C^d]$ is feasible for **DLP1**. Thus, the objective value of $C = C^d$ corresponds to both a dual feasible and primal feasible solution, which means it is the optimum value of **LP1**. Since the optimal schedule given by $[\mathbf{p}\, C]$ has just 4 non-zero states and there are 3 relays, the conjecture is valid for this case. The case when $\min\{a, b, c\} \geq \max\{d, e, f\}$ can be proved in a similar manner by reordering the relays so that the relay to destination link capacities are in sorted order. ■

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