

Properties of Network Polynomials

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Abstract—It is well known that transfer polynomials play an important role in the network code design problem. In this paper we provide a graph theoretical description of the terms of such polynomials. We consider acyclic networks with arbitrary number of receivers and min-cut h between each source-receiver pair. We show that the associated polynomial can be described in terms of certain subgraphs of the network.¹

I. INTRODUCTION

It is well known in the network coding literature that the problem of designing a linear network code that allows to multicast information from a source to a set of receivers over a specific network, can be reduced to the problem of assigning values to variables so that a multivariate polynomial becomes nonzero [2], [4]. Thus, inherently, each linear network coding instance over a network is mapped to a polynomial, which we will call *network polynomial*.

In this paper we try to understand how the structure of these polynomials relates to the underlying network graph. We show that every monomial of the network polynomial is associated with a subgraph of the network with certain properties. For example, for networks with one receiver we show that there is, in fact, a bijection between the monomials of the network polynomial and sets of edge-disjoint paths.

Network polynomials play a significant role in network code design. In the seminal paper [2] it was shown that the existence of a network code over a graph relates to roots of such polynomials. The size of the network coding alphabet used also depends on algebraic properties of such polynomials [5], [6]. These polynomials arise not only in graphs, but also in deterministic networks [1], [3], [5]; In this paper as well, we provide a new method that relates alphabet size and code construction for special classes of networks to polynomial structure and properties. Thus we believe that studying properties of such polynomials is interesting, not only from a theoretical point of view, but also because of possible applications.

The rest of this paper is organized as follows. Section II reviews the algebraic framework, using a line-graph perspective; Section III looks at transfer polynomials of a single receiver; Section IV looks at multiple receivers; Section V presents a specific application and Section VI concludes the paper.

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II. MODEL AND BACKGROUND

In this section we describe the network model, and briefly review known results from [2], [4] from a line graph perspective; we use similar notation to [4].

a) Setup: We consider a directed acyclic graph $G = (V, E)$, where a source S would like to multicast information to N receivers R_1, \dots, R_N . We use the terms “graph” and “network” interchangeably. We are interested in scalar linear coding over a finite field \mathbb{F}_q , i.e., the source has h symbols $\{u_1, \dots, u_h\}$ that she would like to send to all receivers, and intermediate network nodes are allowed to linearly combine their incoming symbols using coefficients from the field \mathbb{F}_q . The min-cut from the source to each receiver is greater or equal to h , i.e., there exist h edge-disjoint paths from the source to each receiver.

b) Line Graph: Unless otherwise specified, in this paper we will work with the line graph of the original network. Given a graph $G = (V, E)$, the associated line graph is defined as the graph $H = (V_L, E_L)$ whose vertex set V_L is the same as the edge set of the graph G , i.e., $V_L = E$. Two vertices $e, e' \in V_L$ are connected by an arc if and only if the starting point(head) of e' is the same as the ending point(tail) of e in the graph G .

Without loss of generality, we can assume that H has h nodes, known as *source nodes* [4], each of which has a symbol u_i from a finite field \mathbb{F}_q to send to each receiver. Each receiver has also h associated *receiver nodes*, through which it receives information from the network. In the original graph G , the h source nodes in H can be thought of as h auxiliary edges, entering the source node and each bringing one of the symbols u_i ; the h receivers nodes in H correspond in G to h incoming edges each receiver has.

Note that in the graph H , for each receiver, there exist h vertex disjoint paths, where each path starts from one source node and ends at one of the receiver nodes; these correspond to the h edge-disjoint paths from the source to the receiver that exist in G . We will come back to these paths in Section IV. Also note that if G is directed and acyclic, so is H [4].

Definition 2.1 (h -minimal graph): A graph L is called h -minimal with respect to a source S and a set of receivers if the min-cut from S to each of the receivers is at least h and no proper subgraph of L has this property.

For further notation and terminology about graphs, see [8].

c) *Transfer and Network Polynomial*: In linear network coding over \mathbb{F}_q , intermediate nodes in the network G linearly combine their received information using coding coefficient $\{x_k\}$ from the field \mathbb{F}_q . These coefficients are the unknown variables in the algebraic formulation of the network code design problem. In the line graph notation, we have one variable x_i associated with each edge of the graph H ; thus we have $\nu \triangleq |E_L|$ such variables.

Let $X \in \mathbb{F}_q^h$ be a vector that collects the source symbols $\{u_1, \dots, u_h\}$, and $Y \in \mathbb{F}_q^h$ a vector that collects the symbols receiver i observes, then $Y = \mathbf{A}(R)X$ where $\mathbf{A}(R)$ is the $h \times h$ transfer matrix from the source to the receiver R [2], [4]. The transfer matrix can be efficiently calculated, and captures the linear transformation that the network operations impose on the sent source symbols.

Definition 2.2: The *transfer polynomial* p_i for a receiver R_i is defined as

$$p_i(x_1, \dots, x_\nu) \triangleq \det(\mathbf{A}(R_i))$$

Definition 2.3: The *network polynomial* p associated with a multicast network coding instance is the product of the transfer polynomials of all receivers, i.e.,

$$p(x_1, x_2, \dots, x_\nu) \triangleq p_1(x_1, \dots, x_\nu) \cdots p_N(x_1, \dots, x_\nu) \quad (1)$$

d) *Network Code Design*: In the framework we discuss, the network code design problem asks to find an assignment of values to the unknown variables $\{x_i\}$ so that the network polynomial evaluates to a nonzero value. Indeed, in this case, the transfer polynomial to each receiver evaluates to a nonzero value; the transfer matrix to each receiver is full rank; and thus each receiver can invert the transfer matrix and decode the source symbols. It is well known (see for example [2]) that such an assignment is always possible provided that the field size is larger than the number of receivers.

III. TRANSFER POLYNOMIAL

We now focus on a single receiver R_i . For simplicity, we will use \mathbf{A} and p (instead of $\mathbf{A}(R_i)$ and p_i) for the transfer matrix and the transfer polynomial, respectively.

We will work with the line graph of the original network; thus, as mentioned in Section II, we assume that we have a set $\mathcal{S} = \{s_1, s_2, \dots, s_h\}$ of h source nodes with in-degree 0 and a set $\mathcal{R} = \{r_1, r_2, \dots, r_h\}$ of h receiver nodes. We also assume that there are h vertex disjoint paths from the elements of \mathcal{S} to the elements of \mathcal{R} .

A. Monomials and Paths

As we discussed earlier, with every edge $e \in E$ of the line graph we have an associated variable x_e ; thus with every path $P = e_{i_1}, e_{i_2}, \dots, e_{i_k}$ we can associate the monomial

$$f(P) \triangleq x_{e_{i_1}} \cdot x_{e_{i_2}} \cdots x_{e_{i_k}}.$$

Let $\mathcal{P}_{(i,j)}$ denote the set of all (s_i, r_j) paths, i.e., all paths that connect source node s_i to receiver node r_j . We then define

$$f_{(i,j)} \triangleq \sum_{P \in \mathcal{P}_{(i,j)}} f(P).$$

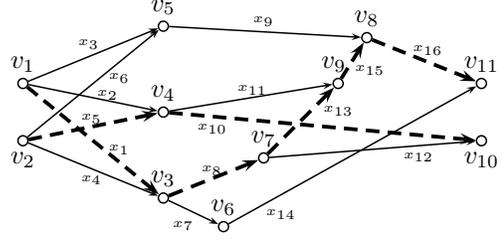


Fig. 1. An example of a network with $h = 2$. A DPQM is depicted with the figure using dashed edges. The term corresponding to this DPQM in the transfer polynomial is $x_1 x_8 x_{13} x_{15} x_{16} x_{18} x_{20}$.

It is well known (and straightforward) that the entry (i, j) of the transfer matrix \mathbf{A} is nothing but the polynomial $f_{(i,j)}$.

B. PQMs and DPQMs

We now define some new notation that will be useful in stating our results. Consider a permutation π of the set $\{1, 2, \dots, h\}$ and denote $\pi(i)$ the i th element in the particular permutation (recall there are $h!$ possible permutations).

Definition 3.1 (PQM): A *Perfect Quasi-Matching* (PQM) is a set of h paths in which each path starts from a different node s_i and ends at a different node $r_{\pi(i)}$, for some permutation π , so that no two paths have the same starting or ending node. The *(sgn)* of a PQM is defined as the sign of π .

Definition 3.2 (DPQM): A PQM is called *Disjoint PQM* (DPQM), if the h paths are vertex-disjoint. A DPQM corresponds to a set of edge-disjoint paths in the original graph.

C. Main Result

Our first result says that each monomial appearing in the transfer polynomial corresponds to a DPQM. In particular, each monomial is of the form $f(P_1) \cdots f(P_h)$ where P_1, \dots, P_h are paths corresponding to a DPQM (i.e., edge-disjoint paths in the original graph). More formally:

Theorem 3.1:

$$p(x_1, \dots, x_\nu) = \sum_{\pi} \sum_{\substack{P_i \in \mathcal{P}_{(i, \pi(i))} \\ P_i \text{'s form a DPQM}}} (-1)^{\text{sgn}(\pi)} \prod_{i=1}^h f(P_i)$$

in which the first summation is taken over all permutations π and the second summation is taken over all DPQMs.

Thus, one alternative way of finding the transfer polynomial, would be to find all DPQMs in the network, and sum the corresponding terms. Reversely, if we were given the transfer polynomial, simply by counting the monomials it has, we can learn how many DPQM's the network has towards this receiver; and we can identify for example intersection of DPQM's by identifying their common variables. Next we give an example, and then we prove Theorem 3.1.

Example 3.1: In the line network depicted in Fig. 1, the mincut from the source to the receiver is two, the nodes v_1, v_2 are the source nodes that bring the two source symbols, and the nodes v_{10}, v_{11} are the receiver nodes that the receiver

observes. Let x_i 's be the variables associated with the edges of the network of Figure 1, as shown in that Figure. The transfer polynomial is:

$$\begin{aligned} f = & x_3x_9x_{16}x_5x_{10} + x_3x_9x_{16}x_4x_8x_{12} \\ & + x_2x_{11}x_{15}x_{8,10}x_4x_8x_{12} + x_2x_{10}x_6x_9x_{16} \\ & + x_2x_{10}x_4x_8x_{13}x_{15}x_{16} + x_2x_{10}x_4x_7x_{14} \\ & + x_1x_8x_{13}x_{15}x_{16}x_5x_{10} + x_1x_7x_{14}x_5x_{10} \\ & + x_1x_8x_{12}x_6x_9x_{16} + x_1x_8x_{12}x_5x_{11}x_{15}x_{16} \end{aligned}$$

and each monomial corresponds to a DPQM.

Example 3.2: Let G be the network in Figure 2. For every receiver, there are exactly two different set of disjoint PQM's. The transfer polynomial of the first receiver (the lowest receiver in Figure 2) is equal to $x_1y_3a_1a_3p_1q_1 - x_3y_1a_1a_3p_1q_1 = a_1a_3p_1q_1(x_1y_3 - x_3y_1)$.

D. Steps in proving Theorem 3.1

We start from the following lemma, which states that the only terms that can possibly appear as monomials in the transfer polynomial are of the form $f(P_1)\dots f(P_h)$ where P_1, \dots, P_h are paths corresponding to a PQM.

Lemma 3.1:

$$p(x_1, \dots, x_\nu) = \sum_{\pi} \sum_{P_i \in \mathcal{P}_{(i, \pi(i))}, \text{ for all } i} (-1)^{\text{sgn}(\pi)} \prod_{i=1}^h f(P_i)$$

Proof: The proof is straightforward and follows from expanding the determinant of the transfer matrix $\mathbf{A}(G)$. ■

Next, we need to prove that in fact only the terms corresponding to disjoint paths (that form a DPQM) will appear in the transfer polynomial; all other terms will cancel out. For this proof, we need to introduce first some notation.

Partial Order

Let \prec_V be a partial order on the set of vertices of H such that $v \prec_V v'$ if and only if there exists a directed path from v to v' . This partial order can be extended to a total order on the set V . For simplicity, we use the same notation \prec_V for the total order. Similarly, we can define the total order \prec_E for the set of edges of G .

We can also define a partial order \prec_P on the set of source-receiver simple paths as follows. $P_1 \prec_P P_2$ if $s_1 \prec_V s_2$ in which s_i is the starting point of the path P_i for $i = 1, 2$.

Let P_1, P_2 be two source-receiver paths with different end points. We say that P_1, P_2 are crossing paths if they share a common vertex. If P_1, P_2 are crossing path and v is a common vertex of P_1, P_2 , we say $(v, \{P_1, P_2\})$ is a crossing pattern. Suppose that $(v, \{P_1, P_2\})$ is a crossing pattern and assume that $P_i = Q_iQ'_i$ for $i = 1, 2$ in which P_i is an (s_i, r_i) path, Q_i is an (s_i, v) path and Q'_i is an (v, r_i) path. By the dual of $(v, \{P_1, P_2\})$ pattern we refer to the crossing pattern $(v, \{Q_1Q'_2, Q_2Q'_1\})$. It is easy to observe that $Q_1Q'_2$ and $Q_2Q'_1$ are source-receiver paths that intersect at v and also it can be easily checked that the dual of the pattern $(v, \{Q_1Q'_2, Q_2Q'_1\})$ is $(v, \{P_1, P_2\})$. Furthermore, it is easy to see that the dual of each pattern can not be identical as the pattern.

Conclusion of the Proof

From Lemma 3.1, it suffices to show that the terms $(-1)^{\text{sgn}(\pi)} \prod_{i=1}^h f(P_i)$ cancel each other when the paths P_i are not pairwise vertex disjoint. We will show that we can pair up all the crossing PQM's into pairs so that both PQM's in a pair use the same set of edges but have opposite sgn 's. As a result, their corresponding terms in the expansion of $P(G)$ will cancel each other.

We define the dual of a crossing PQM $\mathcal{P} = \{P_1, P_2, \dots, P_h\}$ as follows. Let $C = \{v_1, v_2, \dots, v_k\} \subset V$ be the set of all the vertices of the network that belong to more than one of the paths $P_i, i = 1, \dots, h$. Let v_1 be the minimum of the elements of C with respect to the order \prec_V . Let $P_1, P_2, \dots, P_l, l \geq 2$ be all the elements of \mathcal{P} which pass through v_1 . Also, assume that P_1, P_2 are the smallest elements of P_1, P_2, \dots, P_l with respect to the order \prec_P . Clearly $(v_1, \{P_1, P_2\})$ is a crossing pattern. Let $v_1, \{Q_1, A_2\}$ be the dual of this pattern. Now, we define the dual of \mathcal{P} to be the following PQM:

$$\mathcal{P}' = \{Q_1, Q_2, P_3, P_4, \dots, P_h\}.$$

In Figure 2, Let $P_1 = v_1v_4v_{10}$ and $P_2 = v_2v_4v_9v_8v_{11}$. Then $(v_4, \{P_1, p_2\})$ is a crossing pattern.

Notice that:

- 1- \mathcal{P}' is also a crossing PQM.
- 2- \mathcal{P}' uses the same edges as of \mathcal{P} .
- 3- $\mathcal{P}' \neq \mathcal{P}$.
- 4- $\text{sgn}(\mathcal{P}') = -\text{sgn}(\mathcal{P})$.
- 5- The dual of \mathcal{P}' is \mathcal{P} .

The only nontrivial parts of the above is the last two parts. For the part (4), notice that the end points of the paths of \mathcal{P}' are matched the same way as the endpoints of the paths in \mathcal{P} with one exception for the endpoints of the paths P_1, P_2 and Q_1, Q_2 which are matched differently.

To see the last part, notice that if v_1 is the smallest crossing point of \mathcal{P} , it is also the smallest crossing point of \mathcal{P}' . Also, since P_1, P_2 are the two smallest elements of \mathcal{P} and Q_1, Q_2 have the same set of the starting points as of P_1, P_2 , by definition of \prec_P , Q_1, Q_2 are the two smallest elements of \mathcal{P}' . This means that the dual of the dual of a crossing pattern is the original pattern. Therefore dual of \mathcal{P}' is \mathcal{P} .

IV. NETWORK POLYNOMIAL

In the case of a single receiver the terms in the transfer polynomial corresponded to h disjoint paths, i.e., a subgraph of the network with some special properties. Similarly, in the case of N receivers, each term of the network polynomial now also corresponds to a subgraph with some special properties.

A. Terms in the network polynomial

For simplicity we describe for the case of two receivers, R_1 and R_2 . Consider an acyclic line network with one source set $\mathcal{S} = \{s_1, s_2, \dots, s_h\} \subset V$ and two receiver sets $\mathcal{R}_1 = \{r_1, r_2, \dots, r_h\}$ and $\mathcal{R}_2 = \{r'_1, r'_2, \dots, r'_h\}$. As discussed in Section II, the network polynomial can be calculated as

$$p(x_1, \dots, x_\nu) = \det(A(R_1)A(R_2)) = p_1(x_1, \dots)p_2(x_1, \dots).$$

As we already showed, each monomial of p_1 (and p_2) corresponds to a DPQM with respect to the set \mathcal{S} and the set \mathcal{R}_1 (\mathcal{R}_2). Therefore, each monomial of p corresponds to a subgraph which is a union of two DPQM's, one with respect to the sets S, R and the other one with respect to the set S, R' . Notice that the converse of this statement is not necessarily true. This is due to the fact that one subgraph of the network can be decomposed as the union of two DPQM's in two different ways and therefore, in the network polynomial some terms might appear several time and they can possibly cancel each other. Thus it is important to classify those subgraphs of the network that correspond to a monomial in the network polynomial.

We next attempt to extract properties that these subgraphs have; the following lemma summarizes some such easy properties.

Lemma 4.1 (Properties): Consider a subgraph L that corresponds to a term appearing in the network polynomial of a network with two receivers.

- 1) The edges of L can be decomposed into two DPQM's; one for each receiver.
- 2) Each vertex of L has in-degree 0,1 or 2. If it has in-degree 0, then it is a source node. If it has in-degree 1 and the its out-degree is 2, then its incoming edge must appear in both DPQM's.
- 3) Each vertex of L has out-degree 0,1 or 2. If it has out-degree 0, then it is a receiver node. If it has out-degree 1 and the its in-degree is 2, then its outgoing edge must appear in both DPQM's.
- 4) The mincut of each receiver on L is at least h .
- 5) The power of each variable in a monomial indicates whether the corresponding edge (in the original graph) appears in one of the DPQM's or both.

Proof: Before we prove these properties, notice that each term of the network polynomial is product of two terms of transfer polynomials of the receivers.

- 1) This property is a direct implication of the previous sentence.
- 2) Each DPQM is a subgraph of the graph for which the in-degree and out-degree of each vertex is 0 or one. For the union of two DPQM's, the in-degree and out-degree of each vertex is 0,1 or 2. The second part of this property is also clear.
- 3) Similar to the previous property.
- 4) This property is a direct consequence of the first property.
- 5) Trivial. ■

Notice that these properties can be naturally extended for arbitrary number of receivers.

B. The case of two receivers

Throughout this section we assume that the field \mathbb{F}_q is of characteristic 2. For the case of two receivers, we have a more concise characterization of these subgraphs.

Theorem 4.1: In the expansion of the product of the transfer polynomials of the two receivers, each monomial appears either only once or even number of times. In particular, if the field \mathbb{F}_q has characteristic 2, then the subgraphs corresponding to the monomials of the network polynomial can be uniquely decomposable into two DPQM's.

The proof of this theorem follows from Theorem 4.2.

Theorem 4.2: Suppose that H is an acyclic line-network with the source set S and the receiver sets $\mathcal{R}_1, \mathcal{R}_2$ each of which of min-cut h . Also, assume that the edges of H can be decomposed into two DPQM's, one for each receiver. The following statements are equivalent:

- i) There exist at least two different ways for decomposing the edges of H into two DPQM's.
- ii) There are even number of ways that H can be decomposed into two DPQM's.
- iii) If the characteristic of the field is 2, then there is no term in the network polynomial corresponding to the edges of H .

The proof is provided in [7], along with the proof of the following lemma.

Lemma 4.2: If H is an h -minimal subgraph of G then the network polynomial has a unique monomial corresponding to the edges of H .

Corollary 4.1: An h -minimal subgraph H of G is uniquely decomposable into DPQM's.

As a direct application of 4.2, we get an alternative proof for the following known result.

Corollary 4.2: For every network with two receivers, there is always a network code for the multicast network coding problem over the binary field.

V. A CODE-DESIGN APPLICATION

We now give an example of why studying the structure of network polynomials is useful. We look at a special case of combination networks, and using a simple combinatorial argument, we provide an alternative code construction as well as an associated bound on the field size this construction uses, that matches the best known such bound.

Network polynomial of combination network: The line graph of a combination network with $h = 2$ is a 4-layered network. The first layer has two source nodes s_1, s_2 . The second layer consists of m nodes v_1, v_2, \dots, v_m . For each $i = 1, 2, j = 1, 2, \dots, n$, s_i is connected to v_j . Let $x_i(y_i)$ be the variable associated with the edge $s_1, v_i(s_2, v_i)$. The third layer has also n nodes w_1, w_2, \dots, w_m . Each v_i is connected to w_i . Let a_i be the variable associated with that edge. The last layer contains N pairs of receiver nodes. The i -the pair has 2 nodes r_i, t_i . Each pair has two in-neighbors from w_j 's where one is connected to r_i and one is connected to t_i . Suppose that r_i is connected to $w_{f(i)}$ and t_i is connected to $w_{g(i)}$ in which f, g are two functions from the set $[N]$ to the set $[m]$, in which $[j] = \{1, 2, \dots, j\}$. In Figure 2 and Example 3.2, we used p_i and q_i instead of $w_{f(i)}$ and $w_{g(i)}$ for simplicity.

Suppose that the variable associated to the edge $r_i, w_{f(i)}$ is p_i and the one associated to the edge $t_i, w_{g(i)}$ is q_i . As we

saw in Example 3.2, the transfer polynomial of each receiver can be computed. Therefore, the network polynomial of G is:

$$p = \prod_{i=1}^N a_{f(i)} a_{g(i)} p_i q_i (x_{f(i)} y_{g(i)} - y_{f(i)} x_{g(i)}) \quad (2)$$

Figure 2 shows the line graph of a combination network with $h = 2, m = 4, N = 5$.

Alphabet Size: We use the results of the previous sections to prove the following theorem.

Theorem 5.1: For every combination network with N receivers and min-cut 2 to every receiver, there exists a network code over any field of size larger than $\sqrt{2N}$.

Proof: Let G be a combination network. The network polynomial of G is expressed in 2. We must find an assignment of the values to the variables so that I evaluates to a non-zero value. Set $a_i = p_i = q_1 = x_{g(i)} = x_{f(i)} = 1$ for all $i = 1, 2, \dots, m$.

The network polynomial then becomes:

$$I = \prod_{i=1}^N (y_{f(i)} - y_{g(i)})$$

Thus, we only need to show that if the field size is larger than $\sqrt{2N}$, we can always assign values to y_i 's such that $y_{f(i)} \neq y_{g(i)}$, for $i \in [m]$. Let \mathbb{F} be a finite field of size larger than $\sqrt{2N}$. Each variable y_i appears in certain number of parenthesis. Without loss of generality suppose that y_1 is a variable that appears in the minimum number of parenthesis. Let's assume that y_1 appears in l_1 parenthesis.

We remove all the parenthesis containing y_1 from the product and again without loss of generality, we assume that y_2 is the least appeared variable among the remaining terms. Let's assume that y_2 appears in l_2 of the remaining parenthesis. We exclude all the terms with y_1, y_2 from the product and we repeat the procedure. What we end up is an ordering of the variables and N numbers l_1, \dots, l_N . Let $l_k = \max \{l_1, \dots, l_N\}$. We show that we can always find an assignment to the variables y_i from any field of size larger than l_k such that I is not zero. We assign values in the variables based on the ordering we defined above, in the opposite direction. Namely, we first assign arbitrary value to y_N , then we chose an appropriate value for y_{N-1} and at the end we find a right value for y_1 . At each step i we must make sure that we select a value for the variable i such that it is different from the value of every other variable that appears with y_i in some parenthesis. Clearly, if the field size is larger than l_i , we have enough element in the field to select an appropriate value for y_i . Since l_k is the largest l_i , we can find an appropriate value for all the variables. Thus, it is enough to show that $l_k \leq \sqrt{2N}$. We prove this inequality using two inequalities.

- i $l_k \leq m - k$
- ii $l_k \leq 2N / (m - k)$

The first inequality holds because when we select the k -th variable, there are $m - k$ other variables left. Even if y_k appears with all the left variables, it will be appeared $m - k$ times. The

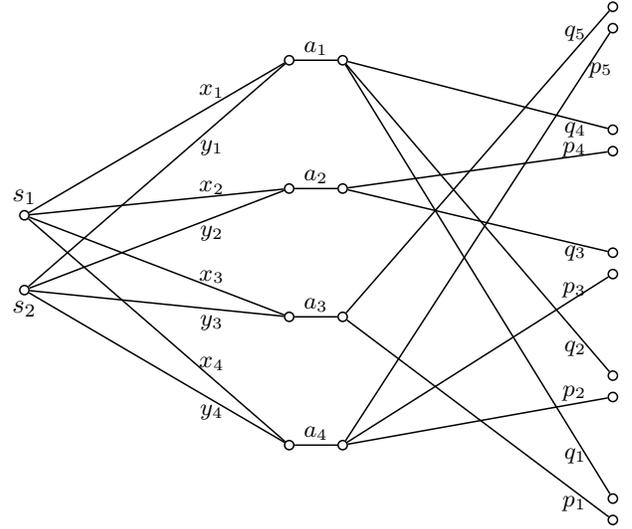


Fig. 2. An example of the line graph of a combination network.

second inequality holds because in the k -th step, each of the $m - k + 1$ variables appear at least l_k times in the parenthesis. There are at most N parenthesis and each parenthesis has exactly two elements. Therefore, $l_k(m - k + 1) \leq 2N$ and therefrom, we deduce the desired inequality. If we multiply both sides of the two inequalities, we can deduce that $l_k \leq \sqrt{2N}$. ■

VI. CONCLUSIONS

In this paper, we established relationships between the monomials that appear in the transfer and network polynomials to graph theoretical properties of the underlying network configuration. Several questions remain open, with most prominent a more exact characterization of the terms of the network polynomial for an arbitrary number of receivers.

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