The DoF of the K-user Interference Channel with a Cognitive Relay

Anas Chaaban and Aydin Sezgin
Chair of Communication Systems
RUB, 44780 Bochum, Germany
Email: anas.chaaban@rub.de, aydin.sezgin@rub.de

***Abstract*—It was shown recently that the 2-user interference channel with a cognitive relay (IC-CR) has full degrees of freedom (DoF) almost surely, that is, 2 DoF. The purpose of this work is to check whether the DoF of the K-user IC-CR, consisting of K user pairs and a cognitive relay, follow as a straight forward extension of the 2-user case. As it turns out, this is not the case. The K-user IC-CR is shown to have $2K/3$ DoF if $K > 2$ for the when the channel is time varying, achievable using interference alignment. Thus, while the basic K-user IC with time varying channel coefficients has $1/2$ DoF per user for all K, the K-user IC-CR with varying channels has 1 DoF per user if $K = 2$ and $2/3$ DoF per user if $K > 2$. Furthermore, the DoF region of the 3-user IC-CR with constant channels is characterized using interference neutralization, and a new upper bound on the sum-capacity of the 2-user IC-CR is given.**

I. Introduction

One of the approaches for approximating the capacity of interference networks is finding the multiplexing gain. The multiplexing gain, also known as the capacity pre-log or degrees of freedom (DoF), characterizes the capacity of the network at an asymptotically high signal-to-noise ratio. Recently, there has been an increasing interest in characterizing the DoF of interference networks, e.g, the K-user IC [1] and the X Channel [2], [3].

Besides the IC and the X channel, relaying setups have also been studied from DoF point of view. For instance, [4] studies the impact of relays on wireless networks and shows that causal relays do not increase the DoF of the network. Non-causal relays, on the other hand, can increase the DoF. In [5], achievable rate regions and upper bounds for the 2-user IC with a cognitive relay (IC-CR) were given, and it was shown the interference channel with a cognitive relay has full DoF, i.e., 2 DoF. The cognitive IC has also been studied in [6]–[8] where capacity results for some cases were given, in addition to new upper and lower bounds.

The question we try to answer in this paper is: How is the behavior of the DoF of the IC-CR with K users? A straight forward extension of the results of [5] suggest that the K-user IC-CR has K DoF. The goal of this paper is the characterization of the DoF for general K. Namely, we consider the effect of a cognitive relay on the DoF of the K-user IC. We study the K-user Gaussian IC-CR, and obtain the DoF of this channel under time varying channel coefficients assumption.

It turns out that the case with $K > 2$ users does not follow as a straight forward extension of the 2-user case. We show that while the sum-rate of the 2-user Gaussian IC-CR scales as $2\log(P)$ as the transmit power $P \to \infty$, the $K > 2$ user case scales as $\frac{2K}{3}\log(P)$. In other words, the 2-user case does not follow the same law as the $K > 2$ user case. This DoF is shown to be achievable using interference alignment as in a K-user 2×1 MISO IC [9]. Thus we give a characterization of the DoF of the K-user Gaussian IC-CR with time varying channel coefficients. It turns out that the per user DoF of the K-user Gaussian IC-CR drop from 1 to $2/3$ as we go from the $K = 2$ to $K > 2$. This is in contrast to the K-user IC, where the per-user DoF is $1/2$ for all $K \geq 2$. We also consider the constant channel case, for which we obtain the DoF region of the 3-user Gaussian IC-CR.

As a result, in contrast to [4], where it was shown that causal relays can not increase the DoF of the wireless network, a cognitive relay can increase the DoF of the K-user IC from $K/2$ to $2K/3$ with $K > 2$. Moreover, the results of this paper give an example where cognition/relaying can help in increasing the DoF of a wireless network.

The rest of the paper is organized as follows. In section II, we give the general model of the K-user IC-CR. In section III, we consider the time varying IC-CR and characterize its DoF, and in section IV we consider the IC-CR with constant channel coefficients, where we give a new sum-rate upper bound for the 2-user case and characterize the DoF region of the 3-user case. Finally, we conclude in section V.

II. System Model

The K-user Gaussian interference channel with a cognitive relay (IC-CR) is shown in Figure 1. It consists of K transmit-receive pairs and a cognitive relay, each with one antenna. For $k \in \{1, \dots, K\}$, source k has a message $m_k \in \mathcal{M}_k \triangleq \{1, \dots, 2^{nR_k}\}$ to be sent to destination k over n channel uses. The messages m_k are independent, uniformely distributed over the messages sets, and are made available non-causally at the relay. At each time instant (i), the output of the channel can be represented as follows

$$Y_k(i) = \sum_{j=1}^{K} h_{jk}(i)X_j(i) + h_{rk}(i)X_r(i) + Z_k(i),$$

where $X_r, X_k \in \mathbb{R}$, $k = 1, \dots, K$, are the channel inputs and $Y_k \in \mathbb{R}$ is the channel output, Z_k is an independent

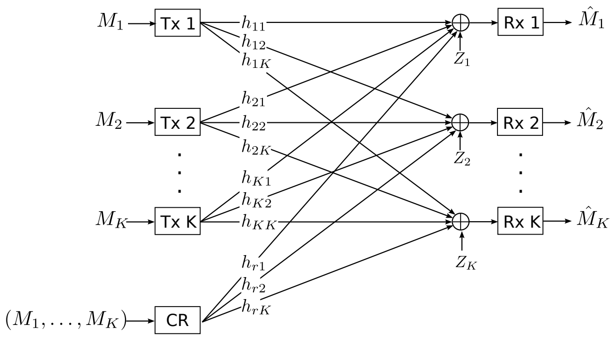

Fig. 1. The K-user Gaussian interference channel with a cognitive relay (CR) system model.

and identically distributed (i.i.d.) noise with zero mean and unit variance $Z_k \sim \mathcal{N}(0,1)$, and $h_{jk}(i)$ and $h_{rk}(i)$ represent time varying channel gains from source j and the relay to destination k, respectively. The channels are assumed to be known apriori at all nodes, and are i.i.d. and drawn from a continuous distribution. The IC-CR with constant channels is defined in the same way as above, with the exception that $h_{jk}(i_1) = h_{jk}(i_2)$ and $h_{rk}(i_1) = h_{rk}(i_2)$ for all $i_1, i_2 \in \mathbb{N}$.

The inputs satisfy the following power constraint

$$\mathbb{E}[X_j^2] \leq P, \quad \forall j \in \{1,\dots,K,r\}. \tag{1}$$

The transmitters and the relay use encoding functions to map the messages to codewords $X_k^n = (X_k(1),\dots,X_k(n))$ and $X_r^n = (X_r(1)\dots,X_r(n))$, respectively. The receivers want to decode their desired messages from their received signals Y_k^n which induces an error probability. A rate tuple $(R_1,\dots,R_K)$ is said to be achievable if the error probability can be made arbitrarily small by increasing the code length n. The closure of the set of all achievable rate tuples defines the capacity region $\mathcal{C}$.

An achievable sum-rate is defined as $R_\Sigma = \sum_{k=1}^K R_k$ with $(R_1,\dots,R_k) \in \mathcal{C}$ and the sum-capacity C_Σ is the maximum sum-rate. The sum DoF is defined as

$$d_\Sigma = \sum_{k=1}^K d_i = \lim_{P\to\infty} \frac{C_\Sigma(P)}{\frac{1}{2}\log(P)}.$$

The DoF region $\mathcal{D}$ is defined as in [1].

III. The IC-CR with Time Varying Channel Coefficients

In this section, we study the DoF of the K-user IC-CR. We state the main result in the following theorem, and describe it in more details afterwards.

Theorem 1. *The DoF of the K-user IC-CR with time varying channel coefficients is given by*

$$d_\Sigma = \begin{cases} 2 & \text{if } K=2 \\ \frac{2K}{3} & \text{if } K>2 \end{cases}$$

The proof of this theorem is given in the following subsections. We consider the 2-user case first, and then the K-user case, and derive upper bounds on the DoF. Then we provide the achievability of these upper bounds.

A. A Sum-capacity Upper Bound for the 2-User Gaussian IC-CR

The 2-User Gaussian IC-CR with constant channel coefficients was considered in [5], where achievable rate regions, upper bounds, and the DoF region were given. The same DoF upper bound as in [5] holds for the time varying case. That is

$$d_1 + d_2 \leq 2. \tag{2}$$

B. DoF Upper Bound for the K-User Gaussian IC-CR with $K \geq 3$:

We first consider the case $K = 3$. The DoF upper bound (2) yields

$$d_1 + d_2 + d_3 \leq 3, \tag{3}$$

when extended to the 3-user case. However, as we show next, this straight forward extension is not tight since the DoF of the 3-user IC-CR is upper bounded by 2. In the following lemma, we give a DoF upper bound for the 3-user IC-CR.

Lemma 1. *The DoF of the 3-user IC-CR is upper bounded by*

$$d_1 + d_2 + d_3 \leq 2.$$

Proof: See Appendix A. ∎

The 3-user Gaussian IC-CR DoF upper bound can be used to obtain the DoF upper bound for the K-user Gaussian IC-CR with $K \geq 3$ stated in the following theorem.

Theorem 2. *The DoF of the K-user IC-CR, $K \geq 3$, is upper bounded as follows*

$$d_\Sigma \leq \frac{2K}{3}. \tag{4}$$

Proof: Using Lemma 1, we have: $d_j + d_k + d_l \leq 2$, for all distinct $j,k,l \in \{1,\dots,K\}$. Adding all such inequalities, we obtain $\binom{K-1}{2} d_\Sigma \leq 2\binom{K}{3}$, and the result follows. ∎

C. Achievability of the K-User IC-CR DoF

Consider the following achievable scheme in a K-user Gaussian IC-CR. At time instant i, the message m_k, $k \in \{1,\dots,K\}$, is mapped to a vector $\mathbf{x}_k(i) = [x_k^{[1]}(i),\ x_k^{[2]}(i)]^T$, the first component of which is sent from Tx 1 and the second component is sent from the relay. The overall relay signal is $x_r(i) = \sum_{k=1}^K x_k^{[2]}(i)$ and the received signals at receiver j can be written as:

$$y_j(i) = \sum_{k=1}^K \mathbf{h}_{kj}^T(i)\mathbf{x}_k(i) + z_j(i), \tag{5}$$

$$\mathbf{h}_{kj}(i) = [h_{kj}(i),\ h_{rj}(i)]^T. \tag{6}$$

Therefore, we can model the IC-CR with this scheme as a K-user 2×1 MISO IC with time varying channel coefficients. Since the relay sends the sum of K signals, we guarantee that the power constraint at each node of the IC-CR is satisfied by defining the power constraint of the resulting MISO channel to be P/K at each node. Notice that this power scaling does not reduce the achievable DoF.

It was shown in [9] that using interference alignment in a K-user 2×1 MISO interference channel with time varying channel coefficients, 2 DoF are achievable if $K=2$, and $\frac{2K}{3}$ DoF are achievable if $K>2$. It is important to note that by the reciprocity of interference alignment [10], the same DoF is achievable in the SIMO IC (with the same physical channels).

Here, we use the same scheme as in [9] for our setup, i.e., we make use of reciprocity. We consider the reciprocal 1×2 K-user SIMO IC with the physical channels given by the 2×1 MISO IC interpretation of the IC-CR given in (5). In this SIMO IC, the channel from transmitter j to receiver k is $\mathbf{h}_{kj}(i)$. Notice here the special structure of the SIMO channel vectors: the second component of $\mathbf{h}_{jj}(i)$ is the same as $\mathbf{h}_{kj}(i)$ (see (6))

Now, as in [9], we consider μ_n symbol extensions of the channel. This makes the 1×2 SIMO IC and extended $\mu_n\times 2\mu_n$ SIMO IC, where the channel matrix from Tx j to Rx k is $2\mu_n\times\mu_n$ and has a block diagonal structure

$$H_{kj}=\begin{bmatrix}\mathbf{h}_{kj}(1) & \mathbf{0}_{2\times 1} & \dots\\ \mathbf{0}_{2\times 1} & \mathbf{h}_{kj}(2) & \dots\\ \vdots & \vdots & \ddots\end{bmatrix}, \tag{7}$$

where $\mathbf{0}_{2\times 1}$ is the all-zero vector of length 2. Notice that $\mathbf{h}_{kj}(i_2)$ and $\mathbf{h}_{kj}(i_2)$ are independent. User $j\in\mathcal{T}_1=\{1,2,3\}$ sends a data vector $\mathbf{w}_j=[\mathbf{x}_j^T(1),\ \mathbf{x}_j^T(2),\dots]^T$ of length[1] $\frac{2}{3}\mu_n$ using a pre-coding matrix V_j with dimension $\mu_n\times\frac{2}{3}\mu_n$. User $j\in\mathcal{T}_2=\{4,\dots,K\}$ sends a data vector $\mathbf{w}_j$ of length $\left(\frac{2}{3}-\epsilon_n\right)\mu_n$ using a pre-coding matrix V_j with dimension $\mu_n\times(\frac{2}{3}-\epsilon_n)\mu_n$, where $\epsilon_n\to 0$ as $n\to\infty$. Thus, Tx j sends

$$X_j=V_j\mathbf{w}_j. \tag{8}$$

As in [9], we choose $V_1=V_2=V_3$ and $V_4=V_5=\cdots=V_K$. The main idea is of alignment is to find pre-coding matrices V_k and post-coding matrices U_k such that

$$\text{rank}(U_kH_{kk}V_k)=d_k \tag{9}$$
$$U_kH_{kj}V_j=0\quad\forall k\neq j, \tag{10}$$

where $d_k=\frac{2}{3}\mu_n$ for $k\in\mathcal{T}_1$ and $d_k=(\frac{2}{3}-\epsilon_n)\mu_n$ for $k\in\mathcal{T}_2$. Here, d_k denotes the dimension of the subspace spanned by the desired signal at Rx k. Denote by $\bar{d}_k$ the dimension of the subspace spanned by all the interfering signals arriving at Rx k. Since user k needs to achieve d_k DoF, then the remaining dimensions of the overall $2\mu_n$-dimensional receive space to be occupied by interference should have $\bar{d}_k=\frac{4}{3}\mu_n$ for $k\in\mathcal{T}_1$ and $\bar{d}_k=(\frac{4}{3}+\epsilon_n)\mu_n$ for $k\in\mathcal{T}_2$. For example, at Rx 1 and K, we need to make sure that the following holds, respectively,

$$\text{span}([H_{12}V_2,\ H_{13}V_3,\dots,H_{1K}V_K])=\frac{4}{3}\mu_n$$
$$\text{span}([H_{K1}V_1,\ H_{K2}V_2,\dots,H_{K(K-1)}V_{K-1}])=(\frac{4}{3}+\epsilon_n)\mu_n.$$

This is guaranteed by using the same construction of V_k as in [9], where V_k is given as a function of all $H_{kj},\ j\neq k$. By choosing U_k to be the null space of the subspace spanned by the interference, we satisfy (10).

[1] μ_n is chosen so that all the relevant quantities are integer.

Now for the general SIMO IC, the construction of V_k given in [9] also satisfies (9) since their design of V_k is independent of the direct channels H_{kk} which are generated randomly and independently of all other channels. In our case, we should examine this more carefully, since we have some dependency in the channels given by

$$H_{kk}(2m,m)=H_{kj}(2m,m),\ m=1,2,\dots$$

where $H_{kj}(a,b)$ is the component in the a-th row and b-th column of H_{kj}. The design of V_k is not completely independent of H_{kk} in our case. However, let us write H_{kk} as follows

$$H_{kk}=\widehat{H}_{kk}+\widetilde{H}_{kk} \tag{11}$$

where

$$\widehat{H}_{kk}=\begin{bmatrix}h_{kk}(1) & 0 & \dots\\ 0 & 0 & \dots\\ 0 & h_{kk}(2) & \dots\\ 0 & 0 & \dots\\ \vdots & \vdots & \ddots\end{bmatrix}. \tag{12}$$

Then, the construction of V_k is clearly independent of $\widehat{H}_{kk}$ whose components are independent of all other channel matrices. Moreover, $\widehat{H}_{kk}$ has full rank. Therefore, rank$(U_k\widehat{H}_{kk}V_k)=d_k$ almost surely and hence condition (9) is satisfied. This achieves $3\left(\frac{2}{3}\right)\mu_n+(K-3)\left(\frac{2}{3}-\epsilon_n\right)\mu_n$ DoF almost surely which approaches $\frac{2K}{3}$ as $n\to\infty$. As a consequence (due to reciprocity), by using V_j and U_j as post-coding and pre-coding matrices at Rx j and Tx j in the original MISO IC, respectively, we achieve $2K/3$ DoF. Thus the DoF upper bounds (2) and (4) are achievable using interference alignment.

IV. The IC-CR with Constant Channel Coefficients

In this section, we focus on the IC-CR with constant channel coefficients. We give a new sum-rate upper bound for the 2-user IC-CR. The DoF upper bounds in section III are general and still hold in this case. However, what differs is that achievable scheme. In what follows, we give an upper bound on the sum-rate of the 2-user case, and we characterize the DoF region of the 3-user case.

A. The 2-User Gaussian IC-CR with constant channel coefficients

Theorem 3. *The sum-rate of the 2-user Gaussian IC-CR with constant channel coefficients is upper bounded by*

$$R_1+R_2\leq\max_{\mathbf{A}\succeq 0}\{I(X_1,X_2,X_r;Y_1)+I(X_2,X_r;Y_2|Y_1,X_1)\}$$

where (X_1,X_2,X_r) *are jointly Gaussian with covariance matrix*

$$\mathbf{A}=\begin{pmatrix}P_1 & 0 & \rho_1\sqrt{P_1P_r}\\ 0 & P_2 & \rho_2\sqrt{P_2P_r}\\ \rho_1\sqrt{P_1P_r} & \rho_2\sqrt{P_2P_r} & P_r\end{pmatrix},$$

and $P_j \leq P \; \forall j \in \{1,2,r\}$. This bounds gives the following DoF upper bound

$$d_1 + d_2 = \begin{cases} 1 & \textit{if} \quad h_{11}h_{r2} - h_{12}h_{r1} = 0 \\ & \textit{or} \quad h_{22}h_{r1} - h_{21}h_{r2} = 0 \\ 2 & \textit{otherwise} \end{cases} \tag{13}$$

The statement of this theorem is obtained by giving (Y_1^n, m_1) as side information to receiver 2 and using classical information theoretic approaches. In [5, Theorem 4], it was shown that $d_1 + d_2$ satisfies (13), and that this upper bound is indeed achievable using interference neutralization [11]. The sum-rate upper bound in Theorem 3 combines the two DoF cases in one expression. We notice a collapse of the DoF to 1 under the special conditions in (13). With random channel realizations, the condition under which $d_1+d_2 = 1$ constitutes a set of measure zero. Thus the 2-user IC-CR with constant channel coefficients has 2 DoF almost surely achievable using interference neutralization.

B. The 3-User Gaussian IC-CR with constant channel coefficients

The sum DoF upper bound in Lemma 1 still holds in this case. Thus

$$d_1 + d_2 + d_3 \leq 2.$$

In the following theorem, we give the DoF region of the 3-user IC-CR with constant channel coefficients.

Theorem 4. *The DoF region $\mathcal{D}$ of the 3-user Gaussian IC-CR is given by*

$$\mathcal{D} = \left\{ (d_1, d_2, d_3) : \begin{array}{l} d_k \leq 1, \forall k \in \{1,2,3\} \\ d_1 + d_2 + d_3 \leq 2 \end{array} \right\}. \tag{14}$$

Proof: We know that $d_1 + d_2 + d_3 \leq 2$. Together with the following trivial bounds

$$d_k \leq 1, \forall k \in \{1,2,3\},$$

it follows that the DoF region is outer bounded by $\mathcal{D}$. Since the corner points of this region, i.e. the points $(1,0,0)$,$(0,1,0)$, and $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ are all achievable, the former three corners by keeping two users silent, and the latter three corners by keeping one user silent and using interference neutralization as in the 2-user IC-CR, the whole region is achievable by time sharing, and the statement of the theorem follows. ■

Remark 1. *Interference neutralization can also be used as a DoF achieving scheme for the time varying 2 and 3 user Gaussian IC-CR.*

In some special cases, the 3-user Gaussian IC-CR has 3 DoF. However, these special cases occur under conditions that do not hold almost surely, i.e. constitute a set of measure 0. This is given in the following corollary.

Corollary 1. *If the 3-user Gaussian IC-CR satisfies the following conditions,*

$$\frac{h_{32}}{h_{31}} = \frac{h_{r2}}{h_{r1}}, \quad \frac{h_{23}}{h_{21}} = \frac{h_{r3}}{h_{r1}}, \quad \frac{h_{13}}{h_{12}} = \frac{h_{r3}}{h_{r2}},$$

and

$$\frac{h_{11}}{h_{12}} \neq \frac{h_{r1}}{h_{r2}}, \quad \frac{h_{22}}{h_{21}} \neq \frac{h_{r2}}{h_{r1}}, \quad \frac{h_{33}}{h_{31}} \neq \frac{h_{r3}}{h_{r1}},$$

then $d_1 + d_2 + d_3 = 3$.

Proof: See Appendix B. ■

V. Conclusion

We studied the K-user Gaussian interference channel with a cognitive relay. For the 2-user case, we have obtained a new upper bound on the sum-capacity. In the general K-user case with time varying channel coefficients, we characterized the DoF. We have shown that while for $K = 2$, the setup has 2 DoF, for $K > 2$ users the DoF are upper bounded by $2K/3$. Moreover $2K/3$ DoF are achievable using interference alignment when the channels are time varying. We notice that the DoF per user is more compared to that in the K-user IC, where we have $1/2$ DoF per user. Thus, a cognitive relay can increase the DoF of the IC. We notice also a decrease in the per-user DoF for the K-user case from 1 to 2/3 as we go from $K = 2$ to $K > 2$. We also considered the case with constant channel coefficients, where we gave the DoF region for the 3-user case and showed that it is achievable using interference neutralization.

Appendix A
Proof of Lemma 1

Let us give (Y_1^n, m_1) and $(Y_1^n, m_1, m_2, \tilde{Z}^n)$ as side information to receivers 2 and 3 respectively, where $\tilde{Z}^n = (\tilde{Z}(1), \ldots, \tilde{Z}(n))$ and

$$\tilde{Z}(i) = Z_2(i) - \frac{h_{r2}(i)}{h_{r1}(i)} Z_1(i) - \left(Z_3(i) - Z_1(i) \frac{h_{r3}(i)}{h_{r1}(i)} \right) \frac{h_{32}(i) - \frac{h_{31}(i)h_{r2}(i)}{h_{r1}(i)}}{h_{33}(i) - \frac{h_{31}(i)h_{r3}(i)}{h_{r1}(i)}}.$$

This random variable $\tilde{Z}$ is used to allow constructing Y_2^n from Y_3^n, Y_1^n, X_1^n, and X_2^n as we shall see next. Then, using Fano's inequality, with $\epsilon_n \to 0$ as $n \to \infty$, we write

$$\begin{aligned}
&n(R_1 + R_2 + R_3 - 3n\epsilon_n) \\
&\quad \leq I(m_1; Y_1^n) + I(m_2; Y_2^n, Y_1^n, m_1) \\
&\qquad + I(m_3; Y_3^n, Y_1^n, m_1, m_2, \tilde{Z}^n) && (15) \\
&\quad = I(m_1; Y_1^n) + I(m_2; Y_1^n | m_1) + I(m_2; Y_2^n | Y_1^n, m_1) \\
&\qquad + I(m_3; Y_1^n | m_1, m_2) + I(m_3; \tilde{Z}^n | Y_1^n, m_1, m_2) \\
&\qquad + I(m_3; Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) && (16) \\
&\quad \leq I(m_1, m_2, m_3; Y_1^n) + I(m_2; Y_2^n | m_1, Y_1^n) \\
&\qquad + I(m_3; \tilde{Z}^n | Y_1^n, m_1, m_2) \\
&\qquad + I(m_3; Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) && (17) \\
&\quad \leq I(m_1, m_2, m_3; Y_1^n) + h(Y_2^n | m_1, Y_1^n) \\
&\qquad - h(Y_2^n | m_1, m_2, Y_1^n) + I(m_3; \tilde{Z}^n | Y_1^n, m_1, m_2) \\
&\qquad + h(Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) - h(Z_3^n | \tilde{Z}^n). && (18)
\end{aligned}$$

where we have used the chain rule and the independence of m_1, m_2 and m_3. Consider now the first term in (18). This is bounded by

$$I(m_1, m_2, m_3; Y_1^n) \leq n \left(\frac{1}{2} \log(P) + o(\log(P)) \right). \quad (19)$$

Moreover,

$$h(Y_2^n | m_1, Y_1^n) - h(Z_3^n | \tilde{Z}^n) \leq n \left(\frac{1}{2} \log(P) + o(\log(P)) \right) \quad (20)$$

except if Y_2^n is a degraded version of Y_1^n given m_1, which is not the case almost surely due to the randomness of the channels. Consider then the fifth term in (18), $h(Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n)$. This can be bounded as follows

$$\begin{aligned} h(Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) &\overset{(a)}{=} h(Y_3^n | Y_1^n, m_1, m_2, X_1^n, X_2^n, \tilde{Z}^n) \\ &\overset{(b)}{\leq} h(\tilde{Y}_3^n | \tilde{Y}_1^n, m_1, m_2, \tilde{Z}^n) \\ &\overset{(c)}{=} h\left(\hat{Y}_3^n | \tilde{Y}_1^n, m_1, m_2, \tilde{Z}^n\right) \end{aligned}$$

where

- (a) follows since X_1^n and X_2^n can be constructed from m_1 and m_2,
- (b) follows by using the knowledge of X_1^n and X_2^n to cancel their contribution from Y_3^n and Y_1^n, where we defined $\tilde{Y}_3(i) \triangleq h_{33}(i)X_3(i) + h_{r3}(i)X_r(i) + Z_3(i)$ and $\tilde{Y}_1(i) \triangleq h_{31}(i)X_3(i) + h_{r1}(i)X_r(i) + Z_1(i)$, and we used the fact that conditioning reduces entropy, and
- (c) follows by the following operation

$$\hat{Y}_3(i) = \tilde{Y}_3(i) - \frac{h_{r3}(i)}{h_{r1}(i)} \tilde{Y}_1(i) \quad (21)$$

$$= \alpha(i)X_3(i) + Z_3(i) - \frac{h_{r3}(i)}{h_{r1}(i)} Z_1(i), \quad (22)$$

where $\alpha(i) = h_{33}(i) - h_{31}(i)\frac{h_{r3}(i)}{h_{r1}(i)} \neq 0$ almost surely.

We continue

$$\begin{aligned} &h(Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) \\ &\leq h\left(\hat{Y}_3^n | \tilde{Y}_1^n, m_1, m_2, \tilde{Z}^n\right) && (23) \\ &\overset{(d)}{=} h\left(\bar{Y}_3^n | \tilde{Y}_1^n, m_1, m_2, \tilde{Z}^n\right) - \frac{1}{2} \sum_{i=1}^n \log\left(\frac{\beta^2(i)}{\alpha^2(i)}\right) && (24) \\ &\overset{(e)}{=} h(Y_2^n | Y_1^n, m_1, m_2, \tilde{Z}^n) - \frac{1}{2} \sum_{i=1}^n \log\left(\frac{\beta^2(i)}{\alpha^2(i)}\right) && (25) \\ &\overset{(f)}{\leq} h(Y_2^n | Y_1^n, m_1, m_2) - \frac{1}{2} \sum_{i=1}^n \log\left(\frac{\beta^2(i)}{\alpha^2(i)}\right), && (26) \end{aligned}$$

where in

- (d) we defined $\bar{Y}_3(i) \triangleq \frac{\beta(i)}{\alpha(i)} \hat{Y}_3(i)$, with $\beta(i) = h_{32}(i) - h_{31}(i)\frac{h_{r2}(i)}{h_{r1}(i)} \neq 0$ almost surely, and we used $h(aX) = h(X) + \frac{1}{2}\log(a^2)$ [12],
- (e) follows by the constructing $Y_2(i) = \bar{Y}_3(i) + \frac{h_{r2}(i)}{h_{r1}(i)} \tilde{Y}_1(i) + \tilde{Z}(i)$ and reconstructing Y_1^n from $(\tilde{Y}_1^n, m_1, m_2)$, and
- (f) follows since conditioning reduces entropy.

As a result, if we consider the third and the fifth term in (18) together, and use (26), we get

$$h(Y_3^n | Y_1^n, m_1, m_2, \tilde{Z}^n) - h(Y_2^n | m_1, m_2, Y_1^n) \leq n\left(o(\log(P))\right). \quad (27)$$

Finally, the fourth term in (18) satisfies

$$I(m_3; \tilde{Z}^n | Y_1^n, m_1, m_2) \leq n\left(o(\log(P))\right). \quad (28)$$

Thus, by plugging (19), (20), (27), and (28) in (18), and letting $n \to \infty$, we obtain $R_1 + R_2 + R_3 \leq \log(P) + o(\log(P))$ and as a result, the degrees of freedom of the 3-user IC-CR is upper bounded by $d_1 + d_2 + d_3 \leq 2$.

APPENDIX B
PROOF OF THEOREM 1

If $h_{32}h_{r1} = h_{r2}h_{31}$, then the upper bound in Appendix A given by $d_1 + d_2 + d_3 \leq 2$ does not hold since $\beta = 0$. It can be similarly shown that, by giving similar side information as in Appendix A to receivers 1 and 3, and 1 and 2, the conditions $h_{23}h_{r1} = h_{r3}h_{21}$, and $h_{13}h_{r2} = h_{r3}h_{12}$, are required so that the DoF does not collapse to 2. Now, as long as

$$\frac{h_{11}}{h_{12}} \neq \frac{h_{r1}}{h_{r2}}, \quad \frac{h_{22}}{h_{21}} \neq \frac{h_{r2}}{h_{r1}}, \quad \frac{h_{33}}{h_{31}} \neq \frac{h_{r3}}{h_{r1}},$$

the relay can cancel the interference at all receivers simultaneously, and thus 3 DoF are achievable.

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