Degrees of Freedom of Sparsely Connected Wireless Networks

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***Abstract*—We investigate how the network connectivity can affect the degrees of freedom (DoF) of wireless networks. We consider a network of n source–destination (SD) pairs and assume that any two nodes are connected with a positive probability p, independent of other node pairs. We show that, for any arbitrarily small p, a constant DoF is achievable for every SD pair with probability approaching one as n tends to infinity. The achievability is based on the two-hop transmission with decode-and-forward relaying and over each-hop we adopt interference alignment. Considering that an achievable per-user DoF for direct or one-hop transmission can be arbitrarily small as the connectivity probability p decreases, our result shows that, somewhat surprisingly, two-hop transmission is enough to guarantee non-vanishing per-user DoF for any p showing that sparsely connected networks can still provide non-vanishing per-user DoF.**

I. Introduction

The degrees of freedom (DoF) of wireless networks has received a great deal of attention as a fundamental metric in the high signal-to-noise ratio (SNR) regime. In essence, DoF characterization provides the capacity of wireless networks within $o(\log(\text{SNR}))$. The key issue is how to efficiently manage inter-user interference in oder to achieve the optimal DoF and there has been remarkable progress in recent years.

- **Interference Networks:** For interference networks or interference channels, it appears that *interference alignment* is essentially required in order to achieve the optimal DoF for more than two users. Interference alignment for the K-user interference channel has been originally proposed by Cadambe and Jafar with signal space alignment [1] showing that each source–destination (SD) pair can achieve $1/2$ DoF regardless of the number of SD pairs. Other types of alignment have been proposed based on the number theory [2] and ergodic setting [3].
- **Partially Connected Interference Networks:** Obviously, network connectivities can affect the DoF and one of the important classes of partially connected networks are layered networks. For multi-hop networks, inter-user interference can be cancelled through multiple paths, which is referred to as *interference neutralization*. For 2-user 2-hop networks, aligned interference neutralization in [4] can achieve the optimal 2 DoF. Ergodic interference neutralization has been proposed in [5] for K-user K-hop networks showing that the optimal K DoF is achievable for isotropic fading. More recently, the result in [4] has been generalized to 2-user multi-hop networks [6], [7].

The main focus of interference networks and partially connected interference networks is on concrete understanding of small-scale networks and developing fundamental interference management techniques for basic communication blocks. In this paper, we address the question *how the network connectivity can affect the overall DoF of large-scale networks.* Before specifying this question, we will briefly introduce related works in large-scale networks.

- **Large-scale Networks:** The primary goal of a large-scale network with n SD pairs is to characterize the capacity within $O(\log n)$ in the limit of large n. In their famous random model, Gupta and Kumar showed that nearest-neighbor multi-hop routing is optimal for power-limited networks [8]. For interference-limited networks, on the other hand, cooperative transmission between nodes are essentially required [9]. Notice that for both cases *sources or destinations act as relays of other SD pairs*. Randomly connected networks have been studied in [10]. It was shown that for certain connectivity distributions the aggregate rate of $\frac{n}{(\log n)^d}$ is achievable for some $d > 0$.

From the large-scale network model perspective, any two nodes can be connected to each other through wireless channels. In this sense, modeling large-scale networks as interference networks or partially connected interference networks mentioned before cannot capture a potential benefit of using sources or destinations as relays of other SD pairs, which can be seen in [8], [9]. Motivated by this aspect, we consider a *network of n SD pairs in which any two nodes are connected to each other with a positive probability, independent of other node pairs.* For fully connected networks, i.e., connectivity probabilities are one for all possible node pairs, direct or one-hop transmission without relaying is sufficient both in terms of DoF and scaling law [11], [12]. We show that, on the other hand, as nodes are connected more sparsely to each other, relaying becomes a dominant strategy. We prove that, even for arbitrarily small connectivity probabilities, each SD pair can still achieve non-vanishing DoF in the limit of large n.

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This means that *very sparse connection is enough to provide a constant fraction of DoF for each SD pair.* Notice that direct transmission with interference alignment leads to negligible DoF as connectivity probabilities become arbitrarily small.

II. SYSTEM MODEL

For a vector A and a matrix B, A_i denotes the ith element of A and $B_{i,j}$ denotes the (i,j)th element of B. For a set S, $|S|$ denote the cardinality of S. Let $\mathcal{N}_{\mathbb{C}}(\mu,\sigma^2)$ denote the circularly symmetric complex Gaussian distribution with mean μ and variance σ^2.

A. Sparsely Connected Networks

We consider a *network of n SD pairs*. For simplicity, denote source i and destination i by node i and node $n+i$ respectively, where $i \in \{1,\cdots,n\}$.

1) Full-duplex: For full-duplex nodes, the $2n \times 1$ dimensional output vector at time t is given by

$$Y[t] = H[t]X[t] + Z[t], \tag{1}$$

where $H[t]$ is the $2n \times 2n$ dimensional complex channel matrix, $X[t]$ is the $2n \times 1$ dimensional input vector at time t, $Z[t]$ is the $2n \times 1$ dimensional noise vector at time t. The elements of $Z[t]$ are independent and identically distributed (i.i.d.) drawn from $\mathcal{N}_{\mathbb{C}}(0,1)$ and each node should satisfy the average power constraint P. That is, $\frac{1}{B}\sum_{t=1}^{B} \mathsf{E}[|X_i[t]|^2] \le P$ during B channel uses.

We assume that *channel coefficients vary independently over time*. Specifically, the (i,j)th element of $H[t]$ is given by

$$H_{i,j}[t] = e_{i,j}[t]h_{i,j}[t], \tag{2}$$

where $e_{i,j}[t]$'s are independently drawn having one with probability $p_{i,j} \in (0,1]$ and zero with probability $1-p_{i,j}$ and $h_{i,j}[t]$'s are again independently drawn from continuous distributions. Hence, as the connectivity probabilities $p_{i,j}$'s decrease, the considered network becomes more sparsely connected. We further assume that global channel state information $H[t]$ is available for each node at time t.

2) Half-duplex: The network model and assumptions are the same as those of full-duplex except that each node can either transmit or receive at a given time, but not simultaneously. Based on (1), we can define the input–output relation of half-duplex nodes. Assuming that, at given time t, the set of nodes $N(t) \subseteq \{1,\cdots,2n\}$ transmit and the set of the rest of nodes $N(t)^c = \{1,\cdots,2n\} \setminus N(t)$ receive, the input–output relation is given by

$$Y_{N(t)^c}[t] = H_{N(t)}[t]X_{N(t)}[t] + Z_{N(t)^c}[t],$$

where the $|N(t)^c| \times 1$ dimensional output vector at time t is given by $\{Y_i[t]\}_{i\in N(t)^c}$, the $|N(t)^c| \times |N(t)|$ dimensional complex channel matrix $H_{N(t)}[t]$ is given by $\{H_{i,j}[t]\}_{i\in N(t)^c, j\in N(t)}$, the $|N(t)|\times 1$ dimensional input vector at time t is given by $\{X_j[t]\}_{j\in N(t)}$, and the $|N(t)^c| \times 1$ dimensional noise vector at time t is given by $\{Z_i[t]\}_{i\in N(t)^c}$.

B. Setup

We consider a set of length-B block codes. Let W_i be the message of source i uniformly distributed over $\{1,\cdots,2^{BR_i}\}$, where R_i is the rate of source i. A rate tuple $(R_1,\cdots,R_n)$ is said to be *achievable* if there exists a sequence of $(2^{BR_1},\cdots,2^{BR_n};B)$ codes such that the probabilities of error for W_1 to W_n converge to zero as B increases. Then, for an achievable rate tuple $(R_1,\cdots,R_n)$, the sum DoF of

$$d_\Sigma = \lim_{P\to\infty} \frac{\sum_{i=1}^{n} R_i}{\log P}$$

is achievable. Throughout the paper, we will characterize an achievable per-user DoF $\frac{d_\Sigma}{n}$ in the limit of large n.

III. MAIN RESULTS

We first introduce our main results here and then explain the achievability in the next section.

A. Achievable DoF

We characterize an achievable DoF for both full-duplex and half-duplex in the limit of large n.

Theorem 1 (Time-varying connectivity): Consider a network of n SD pairs with time-varying connectivity. Let $p_d = \min_{j\in\{1,\cdots,n\}}\{p_{n+j,j}\} > 0$. For sufficiently large n,

$$\frac{d_\Sigma}{n} = \begin{cases} \frac{1}{3} + \frac{p_d}{6} - \delta_{1,n} & \text{for full-duplex,} \\ \frac{1}{4} + \frac{p_d}{4} - \delta_{2,n} & \text{for half-duplex} \end{cases}$$

is achievable with probability approaching one, where $\delta_{1,n}$ and $\delta_{2,n}$ converge to zero as n increases.

As shown in Theorem 1, *for any set of non-zero connectivity probabilities, a constant fraction of DoF is achievable for each SD pair (at least $1/3$ for full-duplex and $1/4$ for half-duplex).* This result shows that vary sparse connections between nodes is enough to guarantee non-vanishing DoF for each SD pair even as the number of SD pairs tend to infinity. For more details, we introduce the following example.

Example 1 (Comparision): Suppose that $p_{n+j,j} = p > 0$ for all $j \in \{1,\cdots,n\}$. Then, from Theorem 1,

$$\lim_{n\to\infty} \frac{d_\Sigma}{n} = \begin{cases} \frac{1}{3} + \frac{p}{6} & \text{for full-duplex,} \\ \frac{1}{4} + \frac{p}{4} & \text{for half-duplex} \end{cases} \tag{3}$$

is achievable with probability approaching one. Consider the direct transmission in which only a subset of sources having direct connections to their destinations send messages through their direct channels. Then we can apply interference alignment [2] to this one-hop transmission and each of the corresponding SD pairs achieves $1/2$ DoF. Since the number of SD pairs having direct connections is approximately given by np with probability approaching one as n increases,

$$\lim_{n\to\infty} \frac{d_\Sigma}{n} = \frac{p}{2} \tag{4}$$

is achievable with probability approaching one for both full-duplex and half-duplex in this case. Fig. 1 plots (3) and

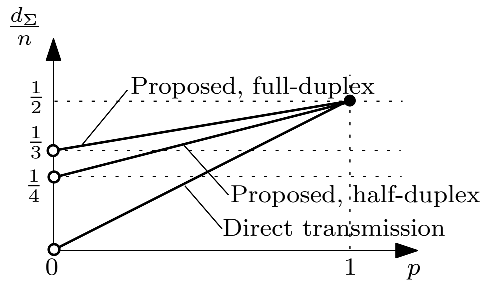

Fig. 1. Achievable $\frac{d_\Sigma}{n}$ in the limit of large n.

(4) with respect to $p \in (0,1]$. As p decreases, our result provides significant DoF improvement compared to the direct transmission.

B. Connectivity and DoF

The results presented in Section III-A dealt with connectivity probabilities in $(0,1]$, that are not scaled with n. Obviously, if $p_{i,j} = 0$, every node is isolated to each other and $\frac{d_\Sigma}{n} = 0$. In this subsection, we consider scalable connectivity probabilities, i.e., the connectivity probabilities $p_{i,j}(n)$'s are functions of n. The following theorem shows a sufficient condition on $p_{i,j}(n)$'s guaranteeing non-vanishing DoF for each SD pair.

Theorem 2 (Scalable $p_{i,j}(n)$): Consider a network of n SD pairs with time-varying connectivity. If there exist increasing sequences $a_{i,j}(n) > 0$ such that $p_{i,j}(n) = a_{i,j}(n)\frac{\log n}{n}$ for all $i, j \in \{1, \cdots, 2n\}$, then for sufficiently large n,

$$\frac{d_\Sigma}{n} = \begin{cases} \frac{1}{3} - \delta_{3,n} & \text{for full-duplex,} \\ \frac{1}{4} - \delta_{4,n} & \text{for half-duplex} \end{cases}$$

is achievable with probability approaching one, where $\delta_{3,n}$ and $\delta_{4,n}$ converge to zero as n increases.

This theorem shows that the connectivity probabilities can indeed decrease slower than the order of $\frac{\log n}{n}$ while preserving a constant fraction of DoF for each SD pair, independent of n.

C. Static Connectivity

For static connectivity, we show that similar results presented in Section III-A still hold. In this case, once $H[t]$ in (1) is drawn according to (2), they remain fixed during the entire transmission. The following theorem shows an achievable DoF for static connectivity.

Theorem 3 (Static conectivity): Consider a network of n SD pairs with static connectivity. Let n_d be the number of SD pairs having direct connections. For sufficiently large n,

$$\frac{d_\Sigma}{n} = \begin{cases} \frac{1}{3} + \frac{n_d}{6n} & \text{for full-duplex,} \\ \frac{1}{4} + \frac{n_d}{4n} & \text{for half-duplex} \end{cases}$$

is achievable with probability approaching one.

The same statement that, for any connectivity probabilities in $[0,1)$, each SD pair can achieve a constant DoF (at least $1/3$ for full-duplex and $1/4$ for half-duplex) is true for static connectivity. Furthermore, if $p_{n+j,j} = p$ for all $j \in \{1, \cdots, n\}$, then Theorem 3 coincides with the result in (3).

IV. Achievability

Due to the page limitation, we show the achievability of Theorem 1 for full-duplex nodes. For simplicity, 'with probability one' in this section means that the probability approaches to one as n increases.

The achievability is based on the *two-hop transmission with decode-and-forward (DF) relaying and over each-hop we adopt interference alignment in [2]*. As shown in Example 1, relaying is essentially required to improve DoF of sparsely connected networks. From the result of *matching in bipartite random graphs*, we can show that this two-hop transmission with DF relying is enough to utilize a constant fraction of nodes with probability one for each hop transmission and, as a result, a constant fraction of $\frac{d_\Sigma}{n}$ is achievable with probability one.

A. Matching in Bipartite Random Graphs

Before proving Theorem 1 for full-duplex, we introduce matching in bipartite random graphs. Denote a bipartite graph by $G = (U, V, E)$ in which U and V are sets of vertices and E is the set of edges. A bipartite graph $G = (U, V, E)$ with $|U| = |V| = m$ is said to contain a *perfect matching* if for each U_i, $i \in \{1, \cdots, m\}$, there exists a distinguishable V_j having an edge (U_i, V_j). The following theorem shows the existence of a perfect matching for a bipartite random graph in the limit of large m.

Theorem 4: Consider a bipartite random graph $G = (U, V, E)$ with $|U| = |V| = m$ in which there is an edge between U_j and V_i with probability $q_{i,j} \in (0,1]$ if $i \neq j$, independent of each other, and with probability zero otherwise. For sufficiently large m, G contains a perfect matching with probability approaching one.

Proof: The overall proof is based on Theorem 3.25 in [13]. The only differences are that $q_{i,j}$ are not the same and there is no edge between U_i and V_i. ■

B. Achievability: Time-varying Connectivity

In this subsection, we prove Theorem 1 assuming full-duplex nodes.

1) Construction of bipartite graphs: Denote $p_{\min} = \min_{i,j \in \{1,\cdots,2n\}}\{p_{i,j}\}$ and $n_1 = np_d - \sqrt{n \log n}$, where $p_d = \min_{i \in \{1,\cdots,n\}}\{p_{n+i,i}\}$. Define a random variable $e'_{i,j}[t]$ such that $\mathsf{P}[e'_{i,j}[t] = 1]$ is equal to $p_d/p_{i,j}$ if $i = n + j$ and $p_{\min}/p_{i,j}$ otherwise. Then define the $2n \times 2n$ dimensional connectivity matrix $\mathcal{E}[t]$ such that $\mathcal{E}_{i,j}[t] = e_{i,j}[t]e'_{i,j}[t]$. Hence

$$\mathsf{P}[\mathcal{E}_{i,j}[t] = 1] = \begin{cases} p_d & \text{if } i = n + j, \\ p_{\min} & \text{otherwise.} \end{cases}$$

Define $S^{[1]}[t] \subseteq \{1, \cdots, n\}$ as the set of sources such that $|S^{[1]}[t]| = n_1$ and $\mathcal{E}_{n+S^{[1]}_i[t], S^{[1]}_i[t]}[t] = 1$ for all

$i \in \{1, \cdots, n_1\}$. If there are multiple sets satisfying this condition, we choose one of them uniformly at random. Denote $D^{[1]}[t] = \{n+i | i \in S^{[1]}[t]\}$ as the set of the corresponding destinations. Let $S^{[2]}[t] = \{1, \cdots, n\} \setminus S^{[1]}[t]$ and $D^{[2]}[t] = \{n+1, \cdots, 2n\} \setminus D^{[1]}[t]$.

Now construct a bipartite graph $G^{[d2,s2]}[t] = (S^{[2]}[t], D^{[2]}[t], E^{[d2,s2]}[t])$ such that there is an edge between $S_j^{[2]}[t]$ and $D_i^{[2]}[t]$ if $\mathcal{E}_{D_i^{[2]}[t], S_j^{[2]}[t]}[t] = 1$ for $i, j \in \{1, \cdots, n-n_1\}, i \neq j$. Hence $G^{[d2,s2]}[t]$ has an edge between $S_j^{[2]}[t]$ and $D_i^{[2]}[t]$ with probability $p_{\min}$ if $i \neq j$, independent of each other, and with probability zero otherwise. Similarly, construct $G^{[s2,s2]}[t] = (S^{[2]}[t], S^{[2]}[t], E^{[s2,s2]}[t])$ such that there is an edge between $S_j^{[2]}[t]$ and $S_i^{[2]}[t]$ if $\mathcal{E}_{S_i^{[2]}[t], S_j^{[2]}[t]}[t] = 1$ for $i, j \in \{1, \cdots, n-n_1\}, i \neq j$ and $G^{[d2,d2]}[t] = (D^{[2]}[t], D^{[2]}[t], E^{[d2,d2]}[t])$ such that there is an edge between $D_j^{[2]}[t]$ and $D_i^{[2]}[t]$ if $\mathcal{E}_{D_i^{[2]}[t], D_j^{[2]}[t]}[t] = 1$ for $i, j \in \{1, \cdots, n-n_1\}, i \neq j$. These bipartite graphs will be used to find a set of relays for two-hop transmission.

Let $n_d[t] = \sum_{i \in \{1, \cdots, n\}} \mathcal{E}_{n+i,i}[t]$. From Chebyshev's inequality,

$$\mathsf{P}\left[\left|\frac{n_d[t]}{n} - p_d\right| \geq \epsilon_1\right] \leq \frac{\sigma^2}{n\epsilon_1^2},$$

where $\sigma^2 = p_d(1-p_d)$ is a variance of $\mathcal{E}_{n+i,i}[t]$. Then, by setting $\epsilon_1 = \sqrt{\frac{\log n}{n}}$, we have $np_d - \sqrt{n \log n} \leq n_d[t] \leq np_d + \sqrt{n \log n}$ with probability one. Hence $n_1 \leq n_d[t]$ with probability one and, therefore, we can find $S^{[1]}[t]$ with probability one. Also, $|S^{[2]}[t]| = n - n_1 \to \infty$ as $n \to \infty$ with probability one. Hence, from Theorem 4, each of $G^{[d2,s2]}[t]$, $G^{[s2,s2]}[t]$, and $G^{[d2,d2]}[t]$ contains a perfect matching with probability one. Let $M(G^{[d2,s2]}[t])$ as the resulting perfect matching set, which is the permutation set of $D^{[2]}[t]$ with respect to $S^{[2]}[t]$. If there are multiple matching sets, we choose one of them uniformly at random. Similarly, define $M(G^{[s2,s2]}[t])$, and $M(G^{[d2,d2]}[t])$.

For transmission, we will only use the time indices that the connectivities satisfy the above conditions, which holds with probability one. Hence the fractional loss on DoF becomes negligible as n increases. For simplicity, the connectivity at each time is assumed to satisfy the above conditions from now on.

2) Interference alignment based on two-hop relaying: The proposed scheme is operated over three blocks with length-B each. Let $n^{[1]}(l)$ be the total number of messages transmitted by each source when it belongs to $S^{[1]}[t]$ for $t \in \{(l-1)B+1, \cdots, lB\}$, where $l \in \{1, 2, 3\}$. Similarly, let $n^{[2]}(l)$ be the total number of messages transmitted by each source when it belongs to $S^{[2]}[t]$ for $t \in \{(l-1)B+1, \cdots, lB\}$, where $l \in \{1, 2\}$. Hence the total number of messages transmitted by all sources during three blocks is given by $n\left(\sum_{l=1}^{3} n^{[1]}(l) + \sum_{l=1}^{2} n^{[2]}(l)\right)$. The values of $n^{[1]}(l)$ and $n^{[2]}(l)$ will be specified later. The detailed transmission for each block is follows.

- (First block) For $t \in \{1, \cdots, B\}$, $S_i^{[1]}[t]$ transmits one of $n^{[1]}(1)$ messages to $D_i^{[1]}[t]$ and $S_j^{[2]}[t]$ transmits one of $n^{[2]}(1)$ messages to $M_j(G^{[d2,s2]}[t])$, which is not the final destination of $S_j^{[2]}[t]$, where $i \in \{1, \cdots, n_1\}$ and $j \in \{1, \cdots, n-n_1\}$.
- (Second block) For $t \in \{B+1, \cdots, 2B\}$, $S_i^{[1]}[t]$ transmits one of $n^{[1]}(2)$ messages to $D_i^{[1]}[t]$, $S_j^{[2]}[t]$ transmits one of $n^{[2]}(2)$ messages to $M_j(G^{[s2,s2]}[t])$, which is not the final destination of $S_j^{[2]}[t]$, and $D_j^{[2]}[t]$ transmits one of the received messages from the source of $M_j(G^{[d2,d2]}[t])$ to the final destination $M_j(G^{[d2,d2]}[t])$, where $i \in \{1, \cdots, n_1\}$ and $j \in \{1, \cdots, n-n_1\}$.
- (Third block) For $t \in \{2B+1, \cdots, 3B\}$, $S_i^{[1]}[t]$ transmits one of $n^{[1]}(3)$ messages to $D_i^{[1]}[t]$ and $S_j^{[2]}[t]$ transmits one of the received messages from the source of $M_j(G^{[d2,s2]}[t])$ to the final destination $M_j(G^{[d2,s2]}[t])$, where $i \in \{1, \cdots, n_1\}$ and $j \in \{1, \cdots, n-n_1\}$.

For the first and third blocks, i.e., $t \in \{1, \cdots, B, 2B+1, \cdots, 3B\}$, the overall transmission at given time t can be regarded as the n-user interference channel from $(S^{[1]}[t], S^{[2]}[t])$ to $(D^{[1]}[t], M(G^{[d2,s2]}[t]))$ having direct connections as shown in Fig. 2. Similarly, for the second block, i.e., $t \in \{B+1, \cdots, 2B\}$, it can be regarded as the $(2n-n_1)$-user interference channel from $(S^{[1]}[t], S^{[2]}[t], D^{[2]}[t])$ to $(D^{[1]}[t], M(G^{[s2,s2]}[t]), M(G^{[d2,d2]}[t]))$ having direct connections as shown in Fig. 2. Therefore, by applying interference alignment in [2] at each message transmission, $1/2$ DoF is achievable for each of message transmission.

3) Achievable DoF: For the proposed message transmission, each source or destination acts as a relay of other SD pairs. Therefore, $n^{[1]}(l)$ and $n^{[2]}(l)$ should be carefully chosen guaranteeing that all messages are delivered to their final destinations. Let $n_{i,j}(l)$ denote the maximum number of messages of source j that node i can receive during the l-th block, where $i \in \{1, \cdots, 2n\}$, $j \in \{1, \cdots, n\}$, and $l \in \{1, 2, 3\}$. Then all messages can be delivered to their final destinations if the following conditions are satisfied:

$$n_{n+j,j}(l) \geq n^{[1]}(l) \tag{5}$$

for $j \in \{1, \cdots, n\}, l \in \{1, 2, 3\}$ and

$$\sum_{i \in \{n+1, \cdots, 2n\}, i \neq n+j} \min\{n_{i,j}(l), n_{n+j,j}(l+1)\} \geq n^{[2]}(l) \tag{6}$$

for $j \in \{1, \cdots, n\}, l \in \{1, 2\}$. Let $q_{i,j}(l)$ denote the probability that node i can receive a message of the j-th source at a given time in the l-th block, where $i \in \{1, \cdots, 2n\}$, $j \in \{1, \cdots, n\}$, and $l \in \{1, 2, 3\}$. From the transmission scheme described before,

$$q_{i,j}(1) = \begin{cases} \frac{n_1}{n} & \text{if } i = n+j, \\ \frac{n-n_1}{n(n-1)} & \text{if } i \in \{n+1, \cdots, 2n\}, i \neq n+j, \\ 0 & \text{otherwise,} \end{cases}$$

$$
q_{i,j}(2) = \begin{cases} \frac{n_1}{n} & \text{if } i = n+j, \\ \frac{n-n_1}{n(n-1)} & \text{if } i \in \{1, \cdots, n\}, i \neq j, \\ \frac{n-n_1}{n(n-1)} & \text{if } i \in \{n+1, \cdots, 2n\}, i \neq n+j, \\ 0 & \text{otherwise,} \end{cases}
$$

and $q_{i,j}(3) = q_{i,j}(1)$. From the strong typicality lemma in [14], the probability that $\left|\frac{n_{i,j}(l)}{B} - q_{i,j}(l)\right| \leq \epsilon_2$ for all $i \in \{1, \cdots, 2n\}$, $j \in \{1, \cdots, n\}$, and $l \in \{1, 2, 3\}$ is greater than $1 - \frac{6n^2}{4B\epsilon_2^2}$. Then setting $\epsilon_2 = \frac{1}{n \log n}$ and $B = n^5$ gives that

$$
n^5 q_{i,j}(l) - n^4/\log n \leq n_{i,j}(l) \leq n^5 q_{i,j}(l) + n^4/\log n
$$

for all i, j, l with probability one. Hence,

$$
n_{n+j,j}(l) \geq n^4 n_1 - n^4/\log n
$$

for $j \in \{1, \cdots, n\}, l \in \{1, 2, 3\}$ and

$$
\sum_{i \in \{n+1, \cdots, 2n\}, i \neq n+j} \min\{n_{i,j}(l), n_{n+j,j}(l+1)\} \geq n^4(n - n_1) - n^5/\log n
$$

for $j \in \{1, \cdots, n\}, l \in \{1, 2\}$ with probability one. Therefore setting $n^{[1]}(l) = n^4 n_1 - n^4/\log n$ for all $l \in \{1, 2, 3\}$ and $n^{[2]}(l) = n^4(n - n_1) - n^5/\log n$ for all $l \in \{1, 2\}$ satisfies the conditions (5) and (6) with probability one. In conclusion,

$$
\begin{aligned}
\frac{d_\Sigma}{n} &= \frac{n\left(\sum_{l=1}^{3} n^{[1]}(l) + \sum_{l=1}^{2} n^{[2]}(l)\right)}{6bn} \\
&= \frac{1}{3} + \frac{p_d}{6} - \delta_{1,n}
\end{aligned}
$$

is achievable with probability one, where $\delta_{1,n} = \frac{1}{3\log n} + \frac{1}{6}\sqrt{\frac{\log n}{n}} + \frac{1}{2n\log n}$ converges to zero as n increases.

Remark 1: The underlying approach of the proposed two-hop relaying is similar to the scheme in [15]. For both cases, at the first phase, each node receives messages from multiple sources when appropriate connectivities or channel gains occur and then, at the second phase, it delivers the received messages to the final destinations when appropriate connectivities or channel gains occur.

Remark 2: The relaying approach presented in this paper is also beneficial for fully connected fading networks. Consider a fully connected network in which $H_{i,j}[t]$'s are i.i.d. drawn from $\mathcal{N}_{\mathbb{C}}(0, 1)$. If we directly apply ergodic interference alignment in [3] from the sources to the destinations (one-hop transmission), $\frac{\sum_{i=1}^{n} R_i}{n} = \frac{1}{2}\,\mathsf{E}\left[1 + 2|H_{n+1,1}[1]|^2 P\right]$ is achievable, where the expectation is over the channel coefficient. Hence the achievable rate for each SD pair is not scaled with n. Instead, we can construct a *hypothetical* connection between two nodes if their channel gain is greater than a certain threshold, and then apply the same two-hop relaying in Section IV with ergodic interference alignment for each hop transmission. This two-hop relaying can provide that $\frac{\sum_{i=1}^{n} R_i}{n}$ scales as $\log\log n$ as n increases. It was first shown in [12] that $\log\log n$ rate scaling is achievable by ergodic interference alignment using the channel gains greater than a threshold. We can prove the same $\log\log n$ gain with different relaying (or scheduling) presented in Section IV.

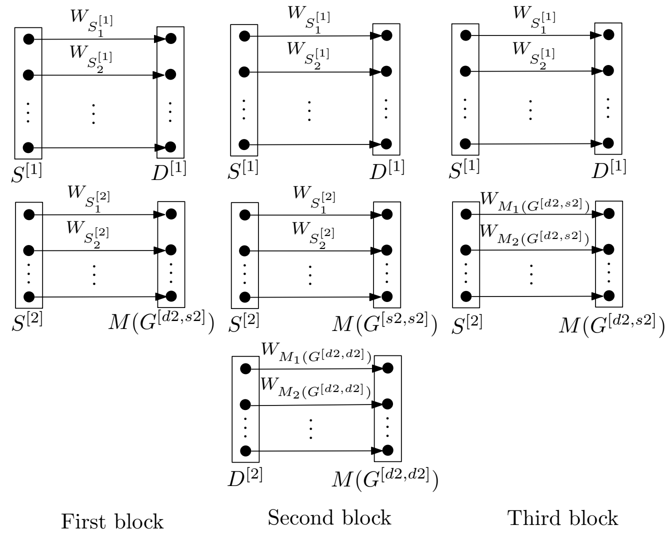

Fig. 2. Transmission of each sub-block for full-duplex nodes, where W_i denote the message of the ith SD pair. For simplicity, we omit the time index.

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