Construction of MIMO MAC Codes Achieving the Pigeon Hole Bound

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Abstract—This paper provides a general construction method for multiple-input multiple-output multiple access channel codes (MIMO MAC codes) that have so called generalized full rank property. The achieved constructions give a positive answer to the question whether it is generally possible to reach the so called pigeon hole bound, that is an upper bound for the decay of determinants of MIMO-MAC channel codes.

I. INTRODUCTION

In MIMO MAC the DMT optimality criteria can be given by splitting the whole error probability space to separate error events and then giving a criteria for each event separately. In the case where each user is in error, the criteria given in [2] closely resembles that of the classical NVD condition used in single user case. This criteria would give us a natural goal if we would like to build DMT achieving codes. Unfortunately in [5] it was proved that this criteria is too tight in the case where the codes of the single users are lattice space-time codes. The *pigeon hole bound* in [5] proves that, irrespective of the code design, the determinants of the overall code matrices will decay with a polynomial speed.

In [3] it was proved that the two user single antenna code (BB-code) given in [9] do achieve the pigeon hole bound. In this paper we are giving a wast generalization of BB-code to general MAC-MIMO. The codes we will build fulfill the so called *generalized rank criteria* and do reach the pigeon hole bound.

The single user codes we are using are based on the *multi-block* codes from division algebras [6]. This approach has been taken in several recent papers on MIMO-MAC. However, in these papers the full rank criteria has been achieved by using either transcendental elements [9] or algebraic elements with high degree [4]. Both of these methods make it extremely difficult to measure the decay of the codes and likely lead to bad decay.

Instead of the usual algebraic independence strategy we will use *valuation theory* to achieve the full rank condition. This technical tool allows us to use algebraic elements with low degree. By applying Galois theoretic method of Lu et al. [5] and methods from Diophantine approximation, originally introduced by Lahtonen et al. in [3], we will prove that our codes achieve good decay and in particular reach the pigeon hole bound.

As a warm up we discuss, when it is possible to achieve the generalized full rank condition for two users, each having a single transmit antenna. In particular these results clarify the construction of BB-code in [9]. Then in Section IV we will give our general construction and state that such an construction always exists and that each code does fulfill the generalized rank criteria. In Section V we analyze the decay of the constructed codes. In particular Theorem 5.4 gives a general lower bound and Corollary 5.5 shows that our codes achieves the pigeon hole bound. In the last section we give few explicit examples of our codes.

II. DECAY FUNCTION, MULTIUSER CODE, AND OTHER DEFINITIONS

In this paper we are considering MIMO MAC code design in the Rayleigh flat fading channel. We suppose that we have U users, each having n_t antennas and that the receiver has n_r antennas and complete channels state information. We also suppose that the fading for each user stays stable for Un_t time units. Let us refer to the channel matrix of the ith user with $H_i \in M_{n_r \times n_t}$ and let us suppose that each of these have i.i.d complex Gaussian random variables with zero mean and unit variance as coefficients. In this scenario the base station receives

$$Y = \sum_{i=1}^{U} H_i X_i + N,$$

where, $X_i \in M_{n_t \times Un_t}(\mathbf{C})$, is the transmitted codeword from the *i*th user, and $N \in \mathcal{M}_{n_r \times n_t U}(\mathbf{C})$ presents the noise having i.i.d complex Gaussian random variables as coefficients.

In this scenario multiuser MIMO signal is a $Un_t \times Un_t$ matrix where the rows $(j-1)n_t+1, (j-1)n_t+2, \ldots, (j-1)n_t+n_t$ represent jth user's signal $(j=1,\ldots,U)$.

We suppose that each user applies a lattice space-time code $\mathbf{L}_j \subseteq \mathcal{M}_{n_t \times Un_t}, j = 1, \dots, U$. We also assume that each user's lattice is of full rank $r = 2Un_t^2$, and denote the basis of the lattice \mathbf{L}_j by $B_{j,1}, \dots, B_{j,r}$. Now the code associated with the jth user is a restriction of lattice \mathbf{L}_j

$$\mathbf{L}_{j}(N_{j}) = \left\{ \sum_{i=1}^{r} b_{i} B_{j,i} | b_{i} \in \mathbf{Z}, -N_{j} \le b_{i} \le N_{j} \right\},$$

where N_j is a given positive number.

Using these definitions the *U-user MIMO MAC code* is $(\mathbf{L}_1(N_1), \mathbf{L}_2(N_2), \dots, \mathbf{L}_U(N_U))$.

For a matrix $M(X_1, ..., X_U) = (X_1^T, X_2^T, ..., X_U^T)^T$ where $X_j \in \mathbf{L}_j(N_j)$ that jth user has sent, we define

$$D(N_1,\ldots,N_U) = \min_{X_j \in \mathbf{L}_j(N_j) \setminus \{0\}} |\det M(X_1,\ldots,X_U)|.$$

This is called the *decay function*. In the special case $N_1 = \ldots = N_U = N$ we write

$$D(N) = D(N_1 = N, \dots, N_U = N).$$

If we have a U-user code with decay function $D(N_1, \ldots, N_n)$, such that $D(N_1, \ldots, N_n) \neq 0$ for all $N_1, \ldots, N_U \in \mathbf{Z}_+$, we say that the code satisfies (generalized) rank criterion.

III. 2-USER CODE

In [9] Badr and Belfiore introduced a 2-user single antenna MAC code where the matrix coefficients were from the field $\mathbf{Q}(i,\sqrt{5})$. In the following we give a complete characterization of when their construction method leads to codes with generalized full rank property.

Theorem 3.1: Let K/\mathbf{Q} and L/K be two field extensions of degree 2 and $a, b, c, d \in L$. Let also σ be the non-trivial element in the Galois group Gal(L/K). Define

$$C = \left\{ \left(\begin{array}{cc} ax & b\sigma(x) \\ cy & d\sigma(y) \end{array} \right) | x, y \in \mathcal{O}_L^* \right\}.$$

There exists a matrix in C with zero determinant if and only if

$$\left|\begin{array}{cc} N(a) & N(b) \\ N(c) & N(d) \end{array}\right| = 0, \tag{1}$$

where the function ${\cal N}={\cal N}_{L/K}$ denotes the norm of extension L/K .

Proof: Assume first that we have a matrix

$$\left(\begin{array}{cc} ax & b\sigma(x) \\ cy & d\sigma(y) \end{array}\right)$$

with zero determinant. This means that $adx\sigma(y)=bc\sigma(x)y$, which gives $N(a)N(d)N(x)N(\sigma(y))=N(b)N(c)N(\sigma(x))N(y)$. Continuing we get N(a)N(d)=N(b)N(c) i.e. N(a)N(d)-N(b)N(c)=0.

Assume then that N(a)N(d)-N(b)N(c)=0. If b or c is zero then N(a)N(d)=0 i.e. a or d is zero and then for all $x,y\in\mathcal{O}_L$ we have

$$\left| \begin{array}{cc} ax & b\sigma(x) \\ cy & d\sigma(y) \end{array} \right| = 0.$$

Otherwise $N(\frac{ad}{bc})=1$. Then by Hilbert 90 we have some $z\in L$ such that $\frac{ad}{bc}=\frac{\sigma(z)}{z}$. Then write $z=\frac{w}{n}$ with $w\in \mathcal{O}_L^*$ and $n\in \mathbf{Z}$. This gives $\frac{ad}{bc}=\frac{\sigma(w)}{w}$ i.e. $adw-bc\sigma(w)=0$. This means that the determinant of

$$\left(\begin{array}{cc} aw & b\sigma(w) \\ c1 & d\sigma(1) \end{array}\right)$$

is zero.

IV. MULTI ACCESS CODE WITH SEVERAL TRANSMISSION ANTENNAS

A. Construction of U-user code for n_t transmission antennas

From now on we concentrate on the scenario where we have $U \in \mathbf{Z}_+$ users and each user has $n_t \in \mathbf{Z}_+$ transmission antennas. Throughout this section we assume K to be an imaginary quadratic extension of \mathbf{Q} with class number 1, L is a cyclic Galois extension of K of degree Un_t , such that $L = K(\alpha)$ with $\alpha \in \mathbf{R}$, σ a generating element in Gal(L/K) and $p \in \mathcal{O}_K$ an inert prime in L/K. We also define $\tau = \sigma^U$ and F to be the fixed field of τ . So we have $[L:F] = n_t$, [F:K] = U, $Gal(L/F) = <\tau>$, and $Gal(F/K) = <\sigma_F>$ where σ_F is a restriction of σ in F. Let $v = v_p$ be the p-adic valuation of the field L. In this section, when we say that L/K, p, and σ are suitable we mean that they are as above.

Due to the space limitations, we skip the proof of the following proposition.

Proposition 4.1: For every complex quadratic field K, having class number 1, and for any U and for any n_t we have a suitable degree n_tU extension L/K, prime $p \in \mathcal{O}_K$ and automorphism $\sigma \in Gal(L/K)$.

Now we are ready for our construction. Let $x_j \in \mathcal{O}_L$ for all $j = 1, \ldots, n_t$. Define $M = M(x_1, x_2, \ldots, x_{n_t})$ to be

$$\begin{pmatrix} x_1 & p\tau(x_{n_t}) & p\tau^2(x_{n_t-1}) & \dots & p\tau^{n_t-1}(x_2) \\ x_2 & \tau(x_1) & p\tau^2(x_{n_t}) & \dots & p\tau^{n_t-1}(x_3) \\ x_3 & \tau(x_2) & \tau^2(x_1) & \dots & p\tau^{n-1}(x_4) \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n_t-1} & \tau(x_{n_t-2}) & \tau^2(x_{n_t-3}) & \dots & p\tau^{n_t-1}(x_{n_t}) \\ x_{n_t} & \tau(x_{n_t-1}) & \tau^2(x_{n_t-2}) & \dots & \tau^{n_t-1}(x_1) \end{pmatrix}$$

i.e. a matrix representation of an element in the cyclic algebra $(L/F, \tau, p)$. The following lemma proofs that $(L/F, \tau, p)$ is a division algebra.

Lemma 4.2: Let $x_j \in \mathcal{O}_L$ for all $j=1,\ldots,n_t$ such that $x_l \neq 0$ for some l and $\min(v(x_1),\ldots,v(x_{n_t}))=0$. Then we have

$$\det(M(x_1, x_2, \dots, x_{n_t})) \neq 0$$

and

$$v(\det(M(x_1, x_2, \dots, x_{n_t}))) \le n_t - 1.$$

Proof: Write $M=M(x_1,x_2,\ldots,x_{n_t})$ and $N=N_{L/F}$. Assume first that $v(x_1)=0$. Then the determinant is $N(x_1)+py$ for some $y\in\mathcal{O}_L$ and hence we have $v(\det(M))=\min(v(N(x_1)),v(py))=0$. Assume then that $v(x_1),v(x_2),\ldots,v(x_{l-1})>0$ and $v(x_l)=0$ with $1< l\leq n_t$. Notice that in this case all the other elements a of matrix M than those in the left lower corner block of side length n_t-l+1 have v(a)>0. Either they have coefficient p or they are automorphic images of elements x_1,x_2,\ldots,x_{l-1} . Now $\det(M)=\pm p^{l-1}N(x_l)+p^lz$ for some $z\in\mathcal{O}_L$ since all the other terms except $\pm p^{l-1}N(x_l)$ have at most n_t-l factors from this left lower corner and hence at least $n_t-(n_t-l)=l$ terms have factor p. This gives that $v(\det(M))=\min(v(p^{l-1}N(x_l)),v(p^lz))=l-1\leq n_t-1$.

Definition 4.1: Define $M_j = M(x_{j,1}, x_{j,2}, \dots, x_{j,n_t})$ for all j = 1, ..., U. In our multi access system the code C_i of jth user consists of $n_t \times U n_t$ matrices $B_i =$

$$(M_j, \sigma(M_j), \sigma^2(M_j), \dots, p^{-k}\sigma^{j-1}(M_j), \dots, \sigma^{U-1}(M_j))$$

where k is any rational integer strictly greater than $\frac{U(n_t-1)}{2}$ and $x_{i,l} \neq 0$ for some l. Here k is same for all users. The the whole code C_{U,n_t} consists of matrices

$$A = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{II} \end{pmatrix},$$

where $B_j \in C_j$ for all i = 1, ..., U. This means that the matrices $A \in C_{U,n_t}$ have form

$$\begin{pmatrix} p^{-k}M_1 & \sigma(M_1) & \dots & \sigma^{U-1}(M_1) \\ M_2 & p^{-k}\sigma(M_2) & \dots & \sigma^{U-1}(M_2) \\ \vdots & \vdots & & \vdots \\ M_U & \sigma(M_U) & \dots & p^{-k}\sigma^{U-1}(M_U) \end{pmatrix}.$$

We will skip the proof of the following proposition, stating that each of the single user codes satisfies the NVD condition and are therefore DMT optimal.

Proposition 4.3: Let $C_{U,n_t} \in \mathfrak{C}_{U,n_t}$ and C_j be the jth users code in the system C_{U,n_t} for some $j \in 1, ..., U$. Then the code C_j is a $2Un_t^2$ -dimensional lattice code with the NVD property.

The code depends on how we did choose L/K, p, σ , and k, so to be precise, we can also refer to C_{U,n_t} with $C_{U,n_t}(L/K, p, \sigma, k)$. Let us call the family of all such codes $C_{U,n_t}(L/K, p, \sigma, k)$ (i.e. codes constructed with any suitable L/K, p, σ , and k) by \mathfrak{C}_{U,n_t} . That is

$$\mathfrak{C}_{U,n_t} = \bigcup_{L/K,p,\sigma,k} \{C_{U,n_t}(L/K,p,\sigma,k)\}$$

where L/K, p, σ , and k are any suitable ones.

According to Proposition 4.1 we can always find suitable L/K, p, σ , and k for any $U \in \mathbf{Z}_+$ and $n_t \in \mathbf{Z}_+$. We therefore have the following theorem.

Theorem 4.4: For any choice of $U \in \mathbf{Z}_+$ and $n_t \in \mathbf{Z}_+$ we have $\mathfrak{C}_{U,n} \neq \emptyset$.

Note that the code $C_{U,1} = C_{U,1}(L/K, p, \sigma, 1) \in \mathfrak{C}_{U,1}$, a code for U users each having one transmission antenna, consists of matrices of form

$$\begin{pmatrix} p^{-1}x_1 & \sigma(x_1) & \sigma^2(x_1) & \dots & \sigma^{U-1}(x_1) \\ x_2 & p^{-1}\sigma(x_2) & \sigma^2(x_2) & \dots & \sigma^{U-1}(x_2) \\ x_3 & \sigma(x_3) & p^{-1}\sigma^2(x_3) & \dots & \sigma^{U-1}(x_3) \\ \vdots & \vdots & \vdots & & \vdots \\ x_U & \sigma(x_U) & \sigma^2(x_U) & \dots & p^{-1}\sigma^{U-1}(x_U) \end{pmatrix}$$

Theorem 4.5: Let $C_{U,n_t} \in \mathfrak{C}_{U,n_t}$. The code C_{U,n_t} is a full rate code and satisfies the generalized rank criterion.

Proof: Let $A \in C_{U,n_t} = C_{U,n_t}(L/K, p, \sigma, k)$. We may assume that $\min(v(x_{j,1}), \dots, v(x_{j,n_t})) = 0$, for all $j = 1, \dots, U$, because otherwise we can divide extra p's off. That does not have any impact on whether det(A) = 0 or not. The determinant of A is

$$p^{-kUn_t} \prod_{l=1}^{U} \det(\sigma^{l-1}(M_l)) + y$$

where $v(y) \ge -k(Un_t-2)$. We know that $v(\sigma^{l-1}(\det(M_l))) = v(\det(M_l))$ because p is from K, i.e. from the fixed field of σ , and $\det(M_l) \neq 0$ for all l.

$$v(p^{-kUn_t} \prod_{l=1}^{U} \det(\sigma^{l-1}(M_l))) = -kUn + \sum_{l=1}^{U} v(\det(M_l))$$

that is less or equal than $-kUn_t+U(n_t-1)=U(n_t-1-kn_t)$ by 4.2. But if we would have det(A) = 0 then

$$v(y) = v(p^{-kUn_t} \prod_{l=1}^{U} \det(\sigma^{l-1}(M_l)))$$

and hence $v(y) \leq U(n_t-1-kn_t)$ implying $-k(Un_t-2) \leq U(n_t-1-kn_t)$. This gives $2k \leq U(n_t-1)$ i.e. $k \leq \frac{U(n_t-1)}{2}$ a contradiction.

Remark 4.1: Using multiblock codes from division algebras as single user codes in the MIMO MAC scenario has been used before for example in [9], [4] and [5]. In [9] the full rank condition for codes with $n_t > 1$ is achieved by using transcendental elements. In [4] the same effect is achieved with algebraic elements of high degree.

V. On the decay function of codes in \mathfrak{C}_{U,n_t}

In this chapter we will prove an asymptotic lower bound for the decay function of codes from \mathfrak{C}_{U,n_t} . In [3] the authors give a general asymptotic upper bound for a decay function in the case that only one user is properly using the code i.e. N_1 can be anything but $N_2 = \cdots = N_U = 1$ are restricted. We will see that in this special case our codes have asymptotically the best possible decay.

Lemma 5.1: Let $C_{U,n_t} = C_{U,n_t}(L/K, p, \sigma, k) \in \mathfrak{C}_{U,n_t}$, $A \in C_{U,n_t}$, and let F be the fixed field of $\tau = \sigma^U$. Then $det(A) \in F$.

Proof: As F is the fixed field of τ it is enough

consists of matrices of form
$$Proof: \text{ As } F \text{ is the fixed field of } \tau \text{ it is enough}$$
 to prove that $\tau(\det(A)) = \det(A)$. Write again $M_j = (D_j - D_j) = (D_j - D_$

Here we did an even number of row and column changes so when calling the above matrix M'_i we see that $\tau(\det(A))$ is

$$\begin{vmatrix} p^{-k}M_1' & \sigma(M_1') & \sigma^2(M_1') & \dots & \sigma^{n-1}(M_1') \\ M_2' & p^{-k}\sigma(M_2') & \sigma^2(M_2') & \dots & \sigma^{n-1}(M_2') \\ M_3' & \sigma(M_3') & p^{-k}\sigma^2(M_3') & \dots & \sigma^{n-1}(M_3') \\ \vdots & \vdots & \vdots & & \vdots \\ M_U' & \sigma(M_U') & \sigma^2(M_U') & \dots & p^{-k}\sigma^{n-1}(M_U') \end{vmatrix} .$$
 where $s_{l,a} \in \mathbf{Q}$ for all $l = 1, \dots, 2Un_t$ and $a \in S_1 \cup S_2$. Let $A \in C_{U,n}$ be
$$\begin{pmatrix} p^{-k}M_1 & \sigma(M_1) & \dots & \sigma^{U-1}(M_1) \\ M_1 & m^{-k}\sigma^{-k}M_1 & \sigma(M_1) & \dots & \sigma^{U-1}(M_1) \\ M_1 & m^{-k}\sigma^{-k}M_1 & \sigma(M_1) & \dots & \sigma^{U-1}(M_1) \end{pmatrix}$$

Let us call the above matrix A'. We notice that matrices Aand A' are exactly similar apart from the fact that in M_i elements in places $(1,2),(1,3),\ldots,(1,n_t)$ have coefficient p^{-k} and elements in places $(2,1),(3,1),\ldots,(n_t,1)$ have not coefficient p^{-k} and in M'_i these are vice versa. But this does not change the value of the determinant because in the expansion of the determinant each summand include a product of same number of elements from columns that are congruent to 1 modulo n_t and from rows that are congruent to 1 modulo n_t .

Theorem 5.2: [7] For any full-rate U-user lattice code, each user transmitting with n_t antennas there exists a constant k > 10 such that

$$D(N_1 = N, N_2 = N_3 = \dots = N_U = 1) \le \frac{k}{N^{(U-1)n_t}}.$$

For the next theorem we need few definitions. Let p(x) = $p_0 + p_1 x + \cdots + p_m x^m \in \mathbf{Z}[x]$ be a polynomial. Then we say that $H(p(x)) = \max\{|p_i|\}$ is the height of the polynomial p(x) and for an algebraic number α we define $H(\alpha) = H(\phi_{\alpha})$ where ϕ_{α} is the minimal polynomial of α . The next generalization of Liouville's theorem can be found from [8, p. 31].

Theorem 5.3: Let $\alpha \in \mathbf{R}$ be an algebraic number of degree κ , $H(\alpha) < h$, H(P) < H and $\deg(P(x)) = m \in \mathbb{Z}^+$. Then either $P(\alpha) = 0$ or

$$|P(\alpha)| \ge \frac{c^m}{H^{\kappa - 1}}$$

with $c = \frac{1}{3^{\kappa-1}h^{\kappa}}$.

Now we are ready to give a lower bound for the decay function of our codes. The proof can be seen as an extension to the analysis given for BB-code in [3].

Theorem 5.4: For a code $C_{U,n_t} \in \mathfrak{C}_{U,n_t}$ there exists constant K > 0 such that

$$D(N_1, N_2, \dots, N_U) \ge \frac{K}{(N_1 N_2 \dots N_U)^{(U-1)n_t}}.$$

Especially

$$D(N) \ge \frac{K}{N^{U(U-1)n_t}}.$$

Proof: Let $C_{U,n_t}=C_{U,n_t}(L/K,p,\sigma,k)$. Field extension $L/{\bf Q}$ has a basis $S_1\cup S_2$ where $S_1=$ $\{1, \delta, \delta^2, \dots, \delta^{U-1}, \beta, \beta\delta, \beta\delta^2, \dots, \beta\delta^{U-1}\}$ is a basis of F/\mathbf{Q} with $\delta \in \mathbf{R}$, $K = \mathbf{Q}(\beta)$ and $\beta = \sqrt{-w}$ for some positive integer w. Notice that if L = F then $S_2 = \emptyset$.

The ring \mathcal{O}_L has a **Z**-basis $\{\gamma_1, \ldots, \gamma_{2Un_t}\}$. Each of these basis elements can be presented as

$$\gamma_l = \sum_{a \in S_1 \cup S_2} s_{l,a} a,$$

$$\begin{pmatrix} p^{-k}M_1 & \sigma(M_1) & \dots & \sigma^{U-1}(M_1) \\ M_2 & p^{-k}\sigma(M_2) & \dots & \sigma^{U-1}(M_2) \\ \vdots & \vdots & & \vdots \\ M_U & \sigma(M_U) & \dots & p^{-k}\sigma^{U-1}(M_U) \end{pmatrix}$$

and $M_j = M(x_{j,1}, x_{j,2}, \dots, x_{j,n_t})$ be

$$\begin{pmatrix} x_{j,1} & p\tau(x_{j,n_t}) & p\tau^2(x_{j,n_t-1}) & \dots & p\tau^{n_t-1}(x_{j,2}) \\ x_{j,2} & \tau(x_{j,1}) & p\tau^2(x_{j,n_t}) & \dots & p\tau^{n_t-1}(x_{j,3}) \\ x_{j,3} & \tau(x_{j,2}) & \tau^2(x_{j,1}) & \dots & p\tau^{n_t-1}(x_{j,4}) \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{j,n_t} & \tau(x_{j,n_t-1}) & \tau^2(x_{j,n_t-2}) & \dots & \tau^{n_t-1}(x_{j,1}) \end{pmatrix}$$

as usual.

Now for any $j = 0, ..., Un_t - 1$ we have

$$\sigma^{j}(x_{m,h}) = \sum_{l=1}^{2Un_{t}} u_{m,h,l} \sigma^{j}(\gamma_{l})$$

where $u_{m,h,l} \in \mathbf{Z}$ and $|u_{m,h,l}| \leq N_m$ for all m, h and l.

Then the determinant det(A) is a sum consisting of $Un_t!$ elements of form

$$p^{-f} \prod_{j=0}^{Un_t} \sigma^j(x_{m_j,h_j}) = p^{-f} \prod_{j=0}^{Un_t} (\sum_{l=1}^{2Un_t} u_{m_j,h_j,l} \sigma^j(\gamma_l))$$

where $f \leq kUn_t$ and m_j gets exactly n_t times all the values $1, \ldots, U$ and h_j gets values from $\{1, \ldots, n_t\}$.

Now substituting $\gamma_l=\sum_{a\in S_1\cup S_2}s_{l,a}a$ gives that the determinant is a sum consisting of elements of form

$$p^{-f} \prod_{j=0}^{Un_t} (\sum_{l=1}^{2Un_t} u_{m_j,h_j,l} \sum_{a \in S_1 \cup S_2} s_{l,a} \sigma^j(a)).$$

We also write

$$\sigma^j(a) = \sum_{a \in S_1 \cup S_2} t_{j,a} a$$

where $t_{j,a} \in \mathbf{Q}$ for all j,a and $p^{-f}\prod_{j=0}^{Un_t}(\sum_{l=1}^{2Un_t}u_{m_j,h_j,l}\sum_{a\in S_1\cup S_2}s_{l,a}\sigma^j(a))$ written as a sum of elements of form that be

$$K_1 p^{-f} \sum_{a \in S_1 \cup S_2} u_a a$$

where $K_1 \in \mathbf{Q}$ is some constant, $u_a \in \mathbf{Z}$, and $u_a =$ $\mathcal{O}((N_1 \dots N_U)^{n_t}).$

Writing also p using basis $S_1 \cup S_2$ we see that the whole determinant det(A) can be written as a sum of elements of form

$$\sum_{a \in S_1 \cup S_2} u_a' a$$

multiplied by some constant K_2 and here we have $u'_a \in \mathbf{Z}$, and $u'_a = \mathcal{O}((N_1 \dots N_U)^{n_t})$.

On the other hand we know that $\det(A) \in F$ so by uniqueness of basis representation we know that $\det(A)$ is a sum consisting of elements of form

$$\sum_{a \in S_1} u_a' a = \sum_{m=0}^{U-1} u_{\delta^m}' \delta^m + \beta \sum_{m=0}^{U-1} u_{\delta^m \beta}' \delta^m$$

and hence

$$|\det(A)| = |K_2||\sum_{l=0}^{U-1} H_l \delta^l + \beta \sum_{l=0}^{U-1} J_l \delta^l|,$$

where $H_l, J_l \in \mathbf{Z}$ and $|H_l|, |J_l|$ are of size $\mathcal{O}((N_1 \cdots N_U)^{n_t})$ for all $l = 0, \dots, U - 1$.

Using the fact that δ is real we get

$$|\det(A)| \ge \frac{K_2}{2} (|\sum_{l=0}^{U-1} H_l \delta^l| + |\sum_{l=0}^{U-1} J_l \delta^l|).$$

Now using 5.3 and noticing that $deg(\delta) = U$ we have

$$|\det(A)| \ge \frac{K}{(N_1 \cdots N_U)^{(U-1)n_t}},$$

where K is some positive constant.

Corollary 5.5: For a code $C_{U,n_t} \in \mathfrak{C}_{U,n_t}$ there exists constants k>0 and K>0 such that

$$\frac{k}{N^{(U-1)n_t}} \le D(N_1 = N, N_2 = \dots = N_U = 1) \le \frac{K}{N^{(U-1)n_t}}$$

VI. EXAMPLES

Let us now give few examples of our general code constructions. In table Table 1 we have collected some examples of suitable fields K and L and inert primes p, fulfilling the conditions of Proposition 4.1. If $K = \mathbf{Q}(i)$ then $p_i \mathcal{O}_K$ refers to the inert prime and if $K = \mathbf{Q}(\sqrt{-3})$ then $p_{\sqrt{-3}} \mathcal{O}_K$ is inert. The inert primes and fields L are found by looking at totally real subfields of $\mathbf{Q}(\zeta_m)/\mathbf{Q}$ and then composing them with the field K.

TABLE I

[L:K]	L	p_i	$p_{\sqrt{-3}}$
3	$K(\zeta_7 + \zeta_7^{-1})$	2+i	$\sqrt{-3}$
4	$K(\zeta_{17} + \zeta_{17}^4 + \zeta_{17}^{-4} + \zeta_{17}^{-1})$	2+i	$\sqrt{-3}$
5	$K(\zeta_{11} + \zeta_{11}^{-1})$	1+i	$2 + \sqrt{-3}$
6	$K(\zeta_{13} + \zeta_{13}^{-1})$	1+i	$2 + \sqrt{-3}$
7	$K(\zeta_{29} + \zeta_{29}^{12} + \zeta_{29}^{-12} + \zeta_{29}^{-1})$	1+i	$\sqrt{-3}$

A. Actual examples

We get a code $C_{3,1} = C_{3,1}(\mathbf{Q}(i,\zeta_7+\zeta_7^{-1}),2+i,\sigma,1)$ *i.e.* 3-user code with each user having 1 antenna by setting L=

 $K(\zeta_7 + \zeta_7^{-1})$, $K = \mathbf{Q}(i)$, p = 2 + i, and $Gal(L/K) = <\sigma>$. Now the actual code consists of matrices

$$\left(\begin{array}{ccc}
p^{-1}x & \sigma(x) & \sigma^2(x) \\
y & p^{-1}\sigma(y) & \sigma^2(y) \\
z & \sigma(z) & p^{-1}\sigma^2(z)
\end{array}\right)$$

where $x, y, z \in \mathcal{O}_L^*$.

We get a code $C_{2,2} = C_{2,2}(\mathbf{Q}(\sqrt{-3},\zeta_{17} + \zeta_{17}^4 + \zeta_{17}^{-4} + \zeta_{17}^{-1}), \sqrt{-3}, \sigma, 2)$ i.e. 2-user code with each user having 2 antennas by setting $L = K(\zeta_{17} + \zeta_{17}^4 + \zeta_{17}^{-4} + \zeta_{17}^{-1}), K = \mathbf{Q}(\sqrt{-3}), \ p = \sqrt{-3}, \ k = 2 > \frac{U(n_t - 1)}{2}, \ \text{and} \ Gal(L/K) = < \sigma >$. Now the actual code consists of matrices

$$\begin{pmatrix} p^{-2}x_1 & p^{-1}\sigma^2(x_2) & \sigma(x_1) & p\sigma^3(x_2) \\ p^{-2}x_2 & p^{-2}\sigma^2(x_1) & \sigma(x_2) & \sigma^3(x_1) \\ y_1 & p\sigma^2(y_2) & p^{-2}\sigma(y_1) & p^{-1}\sigma^3(y_2) \\ y_2 & \sigma^2(y_1) & p^{-2}\sigma(y_2) & p^{-2}\sigma^3(y_1) \end{pmatrix}$$

where $x_1, x_2, y_1, y_2 \in \mathcal{O}_L$ and $x_1 \neq 0$ or $x_2 \neq 0$ and $y_1 \neq 0$ or $y_2 \neq 0$.

VII. CONCLUSION

We gave a general construction for MIMO MAC codes satisfying the generalized rank criteria and gave a lower bound for their decay. As a corollary we got that all our codes do achieve the pigeon hole bound.

ACKNOWLEDGEMENT

The research of T. Ernvall is supported in part by the Academy of Finland grant 131745. The research of R. Vehkalahti is supported by the Academy of Finland grants 131745 and 252457. The authors would like to thank Jyrki Lahtonen for suggesting this problem.

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