

# Delay–Rate Tradeoff in Ergodic Interference Alignment

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**Abstract**—Ergodic interference alignment, as introduced by Nazer et al (NGJV), is a technique that allows high-rate communication in  $n$ -user interference networks with fast fading. It works by splitting communication across a pair of fading matrices. However, it comes with the overhead of a long time delay until matchable matrices occur: the delay is  $q^{n^2}$  for field size  $q$ .

In this paper, we outline two new families of schemes, called JAP and JAP-B, that reduce the expected delay, sometimes at the cost of a reduction in rate from the NGJV scheme. In particular, we give examples of good schemes for networks with few users, and show that in large  $n$ -user networks, the delay scales like  $q^T$ , where  $T$  is quadratic in  $n$  for a constant per-user rate and  $T$  is constant for a constant sum-rate. We also show that half the single-user rate can be achieved while reducing NGJV’s delay from  $q^{n^2}$  to  $q^{(n-1)(n-2)}$ .

This extended version includes complete proofs and more details of good schemes for small  $n$ .

## I. INTRODUCTION

Interference alignment [1], [2] describes a set of techniques that allow communication in multiuser networks at much higher rates than standard resource division schemes such as TDMA. Interference alignment schemes work by ‘aligning’ all interfering signals so they can be cancelled together.

Given a fast-fading channel – that is a channel with independent and identically distributed (IID) fading coefficients in each time slot – Nazer, Gastpar, Jafar and Vishwanath [3] proposed a scheme, which we call the NGJV scheme. In brief, the NGJV scheme pairs together communications using fading matrix  $H$  with those using fading matrix  $I - H$ , providing a situation with no interference between receivers. (We outline the NGJV scheme in more detail in Section III.)

The NGJV scheme can be regarded as optimal for an  $n$ -user interference network, since half the single-user rate is achievable for each user, no matter how large  $n$  is [3], [4], [5], [6]. However, this is achieved at the cost of a significant delay in communications.

For definiteness, we consider a model of communication over a finite field  $\mathbb{F}_q$  of size  $q$ . Since the NGJV scheme requires a particular  $n \times n$  channel matrix with entries in  $\mathbb{F}_q \setminus \{0\}$  to occur, the expected delay for a given message is  $(q-1)^{n^2}$ , which is roughly  $q^{n^2}$  for large  $q$ . It is clear that even for  $n$  and  $q$  relatively small, this is not a practical delay. For  $n = 6$  and  $q = 3$ , for example, the delay is  $2^{36} \approx 7 \times 10^{10}$ .

There are five questions we would like to try to answer:

- 1) Can we find a scheme that, like NGJV, achieves half the single-user rate, but has a shorter delay?
- 2) Can we find schemes that have shorter delays than NGJV, even at some cost to the rate achieved?
- 3) Specifically, which schemes from Question 2 perform well for situations where we have few users ( $n$  small)?
- 4) Specifically, which schemes from Question 2 perform well for situations where we have many users ( $n \rightarrow \infty$ )?
- 5) What is the shortest delay possible for any scheme achieving a given rate for a given number of users?

In Section IV, we define a new family of schemes, called JAP (Subsection IV-A), a beamforming extension JAP-B (Subsection IV-B), and child schemes derived from them (Section IV-D) that have lower time delays than the NGJV scheme for a variety of different rates, answering Question 2. As a special case, the JAP-B( $n$ ) schemes (Subsection IV-C) achieve half the single-user rate, like NGJV, while reducing the delay from  $q^{n^2}$  to  $q^{(n-1)(n-2)}$ , answering Question 1. In Section V, we answer Questions 3 and 4, by finding and analysing the JAP-B schemes that perform the best for small and large  $n$ ; Table 1 and Figure 1 illustrate the best schemes for small  $n$ , and Theorems 6 and 7 give the asymptotic behaviour. Question 5 remains an open problem, although we do give bounds on the delay achievable for the best schemes listed above.

Koo, Wu and Gill [7] have attempted to answer Questions 2 and 3. We briefly outline their work in Section III.

This is a slightly longer version of a paper with the same title [8] presented at the 2011 IEEE Symposium on Information Theory. This version includes complete proofs, and extends the results in Table 1 and Figure 1.

## II. MODEL: THE FINITE FIELD CHANNEL

Since ergodic interference alignment relies on matrices being exactly aligned, Nazer et al [3] give their main results in the context of the finite field channel, where there are only finitely-many possible fading matrices. They then use a quantisation argument to apply their results to the Gaussian case. In order to allow comparison of our results, we use the same finite field model.

Each transmitter  $i$  has an independent message  $\mathbf{w}_i \in \mathbb{F}_q^{m_i}$ , which it encodes as a codeword  $(x_i[1], \dots, x_i[N])^\top$  of block

length  $N$ , giving rate  $R_i = (\log q^{m_i})/N = \frac{m_i}{N} \log q$ .

At time  $t$ , receiver  $j$  sees channel output

$$\mathbf{Y}_j[t] = \sum_{i=1}^n H_{ji}[t]x_i[t] + Z_j[t], \quad (1)$$

and needs to decode the message  $\mathbf{w}_j$ . We can rewrite (1) in matrix form as  $\mathbf{Y}[t] = \mathbf{H}[t]\mathbf{x}[t] + \mathbf{Z}[t]$ . (Throughout, matrices are in sans-serif and vectors in bold.) We call  $\mathbf{H}[t]$  the *channel matrix* or *fading matrix*. As in [3], the noise terms  $Z_j[t]$  are IID sequences from a distribution on  $\mathbb{F}_q$  that is a mixture of a uniform distribution and a point mass at zero:

$$Z_j = \begin{cases} 0 & \text{with prob. } 1 - \rho, \\ z \in \{1, \dots, q-1\} & \text{with prob. } \rho/(q-1), \end{cases}$$

Like [3], we use an ‘ergodic’ model, where the channel coefficients  $H_{ji}[t]$  are drawn IID and uniformly from the field  $\mathbb{F}_q \setminus \{0\}$  and are redrawn for each time slot.

We assume all transmitters and receivers have full causal channel state information for all transmitter–receiver pairs.

By a simple mutual information maximisation, it is easy to show that the capacity of the single-user finite field channel  $Y = Hx + Z$  for a constant  $H \neq 0$  is  $D(Z) := \log q - H(Z)$ , where  $H(Z)$  is the entropy of  $Z$ .

Schemes for the finite field interference channel often allow each user pair to achieve a fixed fraction of the single-user rate. We refer to the ‘pre- $D(Z)$  term’ as the *degrees of freedom*.

*Definition 1:* Given an achievable symmetric rate point  $(R, \dots, R)$ , we define the *degrees of freedom* to be  $\text{DOF} = R/D(Z)$ .

In particular, a single user can achieve 1 degree of freedom.

### III. THREE EXISTING SCHEMES: NGJV, KWG AND TDMA

The NGJV scheme [3] is based on an idea from previous work of Nazer and Gastpar [9] involving the performance of finite-field multiple-access channels.

Each receiver’s problem takes the form of (1). Rather than reconstructing a single message  $\mathbf{w}_j$  (or all messages  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ ) as receiver  $j$  would normally wish to do, it can be shown [9, Theorem 1], [3, Lemma 3] that receiver  $j$  can actually recover the ‘pseudomessage’  $\sum_{i=1}^n H_{ji}\mathbf{w}_i$  at rate  $D(Z)$ . This is done by all transmitters using the same linear code, so for all  $i$ ,  $(x_i[1], \dots, x_i[N])^\top = \mathbf{G}\mathbf{w}_i$ , for an appropriate  $m \times N$  generator matrix  $\mathbf{G}$ .

The NGJV scheme works as follows. Each transmitter  $i$  sends two signals encoding the same message  $\mathbf{w}_i$ ; first when the channel matrix is  $\mathbf{H}$ , and second when the channel matrix is  $\mathbf{H}' := \mathbf{I} - \mathbf{H}$ . In the first time slot, each receiver  $j$  can decode the pseudomessage  $\sum_{i=1}^n H_{ji}\mathbf{w}_i$ , using the argument above. The receiver stores this pseudomessage in its memory. In the second time slot it can decode the pseudomessage  $\sum_{i=1}^n H'_{ji}\mathbf{w}_i$ , which it also stores in its memory. The receiver then adds together these two estimates of pseudomessages, to

get its own message

$$\sum_{i=1}^n H_{ji}\mathbf{w}_i + \sum_{i=1}^n H'_{ji}\mathbf{w}_i = \sum_{i=1}^n (H_{ji} + \delta_{ji} - H_{ji})\mathbf{w}_i = \mathbf{w}_j.$$

Since both pseudomessages can be communicated at rate  $D(Z)$ , we conclude that each transmitter can send its message to its receiver at rate  $D(Z)/2$ , since two channel uses are needed per message. This corresponds to  $\text{DOF} = 1/2$ .

We define the expected time delay for the NGJV scheme to be the average number of time slots we must wait after seeing a channel matrix  $\mathbf{H}$  until we see the corresponding matrix  $\mathbf{I} - \mathbf{H}$ . The time delay is geometrically distributed with mean  $1/p$ , where  $p$  is the probability that the random channel matrix takes the value  $\mathbf{I} - \mathbf{H}$ . Hence finding the average time delay is reduced to finding the probability that the desired matrix appears in the next time slot. Since a channel matrix has  $n^2$  entries, each of which needs to take the correct one value of  $q-1$  possible values, the average time delay is

$$D = \frac{1}{\left(\frac{1}{q-1}\right)^{n^2}} = (q-1)^{n^2} \sim q^{n^2}. \quad (2)$$

(Here and elsewhere, we write  $f(q) \sim g(q)$  if  $f(q)/g(q) \rightarrow 1$  as  $q \rightarrow \infty$ .)

Since this expected delay will be quite large even for modest values of  $q$  and  $n$ , we will concentrate on the *delay exponent*.

*Definition 2:* An interference alignment scheme with expected delay  $D \sim Cq^T$  for some  $C$  and  $T$  has *delay exponent*  $T$  and *delay constant*  $C$ . Formally,  $T := \lim_{q \rightarrow \infty} \log D / \log q$ .

Reduction of the delay exponent is the key aim, with the delay coefficient playing a secondary role. Since the finite field model is in some sense an a quantisation of the continuous case,  $q$  should be large enough for accurate quantisation. Alternatively, for fixed quantisation quality, SNR scales like  $q^2$ , so the high-SNR region corresponds to large  $q$ . When  $q$  is large, the delay exponent  $T$  dominates the delay constant  $C$  in determining size of the expected delay  $D$ .

From (2), we see that the NGJV scheme, which achieves  $\text{DOF} = 1/2$ , has a delay exponent of  $n^2$ .

For comparison, time-division multiple access (TDMA), where each transmitter–receiver pair has sole access to the channel for an  $n$ th of the total time, achieves  $\text{DOF} = 1/n$  for an expected delay  $D = n = nq^0$ , and hence a delay exponent of  $T = 0$ . To an extent, our new schemes can be seen as ‘interpolating’ between the extremes of NGJV (high rate, high delay) and TDMA (low rate, low delay).

We also mention some new schemes due to Koo, Wu and Gill [7] (KWG). They attempted to answer Questions 2 and 3, by finding schemes with lower delay than NGJV. The KWG schemes suggest matching a larger class of matrices than simply  $\mathbf{H}$  and  $\mathbf{I} - \mathbf{H}$ . By analysing the hitting probability of an associated Markov chain, they were able to reduce the expected delay, at the cost of a reduction in rate and hence degrees of freedom. However, their schemes only affect the delay by a constant multiple, with the most successful scheme only reducing the delay to  $0.64(q-1)^{n^2} \sim 0.64q^{n^2}$  for a sum-rate of  $0.79D(Z)$ . That is, they only reduce the delay constant

$C$ , leaving the delay exponent as  $T = n^2$ . For modest  $q$  and  $n$ , say  $q = 3$ ,  $n = 6$ , again, we regard this delay as still impractical. Since the KWG schemes achieve a lower rate than the NGJV scheme for the same delay exponent, we need only compare our results with the NGJV scheme.

#### IV. NEW ALIGNMENT SCHEMES: JAP AND JAP-B

##### A. The basic scheme: JAP

Write  $\mathbf{H}_j^{\text{int}}$  for the interference vector

$$\mathbf{H}_j^{\text{int}} := (H_{j1}, \dots, H_{jj-1}, H_{jj+1}, \dots, H_{jn}).$$

The NGJV scheme has a long delay as after seeing the matrix  $\mathbf{H}[t_0]$ , we have to wait for a single matrix  $\mathbf{H}[t_1] = \mathbf{I} - \mathbf{H}[t_0]$  to appear to simultaneously complete the linear dependences

$$H_{jj}[t_0] + H_{jj}[t_1] = 1, \quad \mathbf{H}_j^{\text{int}}[t_0] + \mathbf{H}_j^{\text{int}}[t_1] = \mathbf{0} \quad \text{for all } j.$$

We can widen the range of acceptable matrices – and so reduce the expected delay – by

- Building the dependences with more than two matrices;
- Forming any linear combination rather than just a sum;
- Allowing transmitters to complete their dependences at different times, rather than simultaneously;
- Recovering a multiple of the desired message, rather than the message itself.

In other words, if there exist scalars  $\lambda, \lambda_1, \dots, \lambda_K$  such that

$$\lambda_0 H_{jj}[t_0] + \lambda_1 H_{jj}[t_1] + \dots + \lambda_K H_{jj}[t_K] = \lambda \neq 0 \quad (3)$$

$$\lambda_0 \mathbf{H}_j^{\text{int}}[t_0] + \lambda_1 \mathbf{H}_j^{\text{int}}[t_1] + \dots + \lambda_K \mathbf{H}_j^{\text{int}}[t_K] = \mathbf{0}, \quad (4)$$

then receiver  $j$  can recover its message from  $\mathbf{H}[t_0], \mathbf{H}[t_1], \dots, \mathbf{H}[t_K]$  by forming the linear combination of pseudomessages

$$\lambda_0 \sum_{i=1}^n H_{ji}[t_0] \mathbf{w}_i + \dots + \lambda_K \sum_{i=1}^n H_{ji}[t_K] \mathbf{w}_i = \lambda \mathbf{w}_j.$$

The time delay for user  $j$  is  $t_K - t_0$ , and they have communicated at rate  $D(Z)/(K+1)$  for  $\text{DOF} = 1/(K+1)$ .

We will need to analyse the probability of (3) and (4) holding, in order to analyse the expected time delay of our schemes. Hence, it will be useful to note the following lemma.

*Lemma 2 $\frac{1}{2}$ :* Conditional on the interference vectors  $\mathbf{H}_j^{\text{int}}[t_0], \mathbf{H}_j^{\text{int}}[t_1], \dots, \mathbf{H}_j^{\text{int}}[t_K]$  being linearly dependent, the probability that (3) holds is  $1 - O(q^{-1})$ , as  $q \rightarrow \infty$ .

*Proof:* Assume the interference vectors are linearly dependent. That is, assume there exist scalars  $\lambda_1, \lambda_2, \dots, \lambda_K$  such that

$$\lambda_0 \mathbf{H}_j^{\text{int}}[t_0] + \lambda_1 \mathbf{H}_j^{\text{int}}[t_1] + \dots + \lambda_K \mathbf{H}_j^{\text{int}}[t_K] = \mathbf{0}$$

where  $L > 0$  of the  $\lambda_k$  are nonzero. Then, (3) also holds provided that the corresponding linear combination

$$\lambda_0 H_{jj}[t_0] + \lambda_1 H_{jj}[t_1] + \dots + \lambda_K H_{jj}[t_K] \quad (5)$$

is nonzero; call the probability that this happens  $p$ .

If  $\lambda_k \neq 0$ , then  $\lambda_k H_{jj}[t_k] =: V_k$  is uniform on  $\mathbb{F}_q \setminus \{0\}$ ; and if  $\lambda_k = 0$ , then  $\lambda_k H_{jj}[t_k]$  is always 0. So (5) is the sum of  $L$  random variables  $V_k$  IID uniform on  $\mathbb{F}_q \setminus \{0\}$ . We can

write the mass function of each  $V_k$  as  $(1+\rho)U - \rho\delta_0$ , where  $U$  is uniform on  $\mathbb{F}_q$ ,  $\delta_0$  is a point mass on 0, and  $\rho = 1/(q-1)$ . Then the mass function  $L$  IID copies is

$$(1 - (-\rho)^L)U + (-\rho)^L \delta_0.$$

Hence, the probability that (5) is zero is

$$1-p = (1 - (-\rho)^L) \frac{1}{q} + (-\rho)^L = \frac{1}{q} + \frac{1}{q(q-1)^{L-1}} = O(q^{-1}),$$

as desired.  $\blacksquare$

We now define our first new scheme.

Start by fixing  $K \leq n$  and a sequence  $\mathbf{a} = (a_1, a_2, \dots, a_K)$  of length  $K$  and weight  $n$ , so in the set

$$\mathcal{A}(n, K) := \left\{ \mathbf{a} \in \mathbb{Z}_+^K : \sum_{k=1}^K a_k = n \right\},$$

and write  $A_k$  for the partial sums  $A_k := a_1 + \dots + a_k$ . Then we define the scheme  $\text{JAP}(\mathbf{a})$  as consisting of the following  $K+1$  steps:

- **Step 0:** Start with a matrix  $\mathbf{H}[t_0]$ .
- **Step 1:** Set  $t_1$  to be the first time slot that allows the first  $a_1$  receivers  $1, 2, \dots, A_1$  to recover their message from  $\mathbf{H}[t_0], \mathbf{H}[t_1]$ .
- **Step  $k$ :** Set  $t_k$  to be the first time slot that allows the next  $a_k$  receivers  $A_{k-1} + 1, A_{k-1} + 2, \dots, A_k$  to recover their message from  $\mathbf{H}[t_0], \mathbf{H}[t_1], \dots, \mathbf{H}[t_k]$ .
- **Step  $K$ :** Set  $t_K$  to be the first time slot that allows the final  $a_K$  receivers  $A_{K-1} + 1, A_{K-1} + 2, \dots, A_K$  to recover their message from  $\mathbf{H}[t_0], \mathbf{H}[t_1], \dots, \mathbf{H}[t_K]$ .

By the end of this process, all  $n = A_K$  receivers have recovered their message.

Since the message was split over  $K+1$  time slots, the common rate of communication is  $D(Z)/(K+1)$ , which corresponds to  $\text{DOF} = 1/(K+1)$ .

We now examine the delay exponent for our new schemes.

*Theorem 3:* Consider the  $n$ -user finite field interference network. Fix  $K$  and  $\mathbf{a} \in \mathcal{A}(n, K)$ . We use the scheme  $\text{JAP}(\mathbf{a})$  as outlined above. Then

- 1) the expected time for the  $k$ th round to take place is  $D \sim q^{T_k(\mathbf{a})}$ , where  $T_k(\mathbf{a}) = a_k(n - k - 1)$ ;
- 2) the delay exponent for the whole scheme is

$$T(\mathbf{a}) := \max_k T_k(\mathbf{a}) = \max_k a_k(n - k - 1).$$

*Proof:* Recall that the expected delay is the reciprocal of the probability the desired match can be made.

Suppose we are about to begin stage  $k$  of a scheme  $\text{JAP}(\mathbf{a})$ . By Lemma 2 $\frac{1}{2}$ , provided that for the next  $a_k$  receivers  $\mathbf{H}_j^{\text{int}}[t_0], \dots, \mathbf{H}_j^{\text{int}}[t_k]$  are linearly dependent – ensuring that (4) holds – then (3) holds as well with probability  $1 - O(q^{-1})$ . Since we are only interested in the leading order in  $q$ , we may assume that (3) will hold.

Also by Lemma 2 $\frac{1}{2}$ , the probability that the first  $k$  interference vectors are already linearly dependent, is only  $O(q^{-1})$ . So again, we may assume they are not.

Write  $\mathcal{S}$  for the span of the first  $k$  interference vectors of one of the desired  $a_k$  receivers  $j$ ,

$$\mathcal{S} := \text{span}\{\mathbf{H}_j^{\text{int}}[t_0], \dots, \mathbf{H}_j^{\text{int}}[t_{k-1}]\}.$$

The idea is that  $\mathcal{S}$  has size roughly  $q^k$ , whereas the space of all possible interfering has size roughly  $q^{n-1}$ , giving a probability  $q^{-(n-k-1)}$  of completing a dependence. But we have to be a little more careful, as the  $H_{ji}$  cannot take the value 0.

Specifically, since all possible interference vectors in  $(\mathbb{F}_q \setminus \{0\})^{n-1}$  are equally likely, the probability that the next matrix completes a linear dependence is indeed

$$\frac{|\mathcal{S} \cap (\mathbb{F}_q \setminus \{0\})^{n-1}|}{|(\mathbb{F}_q \setminus \{0\})^{n-1}|} = \frac{q^k s}{(q-1)^{n-1}},$$

where  $s$  is the proportion of vectors in  $\mathcal{S}$  with no zero entries. By counting the possible coefficients in  $\mathbb{F}_q$  used in the span, the inclusion–exclusion formula gives us

$$s = 1 - (K-1)\frac{1}{q} + O\left(\frac{1}{q^2}\right) = 1 - O(q^{-1}).$$

Hence, the probability that the interference vectors are linearly dependent is

$$\frac{q^k s}{(q-1)^{n-1}} = \frac{q^k (1 - O(q^{-1}))}{(q-1)^{n-1}} \sim q^{-(n-k-1)}.$$

This must hold for all  $a_k$  receivers, which happens with probability  $(q^{-(n-k-1)})^{a_k} = q^{-a_k(n-k-1)}$ , hence the first result.

For the second result, note that, as  $q \rightarrow \infty$ , the delay is dominated by the delay for the slowest round. ■

### B. Improving delay with beamforming: JAP-B

Beamforming slightly improves the performance of JAP(a) schemes, combining ideas from the original Cadambe–Jafar interference alignment [1] with the JAP scheme.

In round  $k$  we can guarantee that the interference matches up for receiver  $l := A_{k-1} + 1$ . Each transmitter  $i$ , instead of repeating their message  $\mathbf{w}_i$ , rather encodes  $(H_{li}[t_k])^{-1} H_{li}[t_0] \mathbf{w}_i$ . (Since the coefficient  $H_{li}$  cannot be 0, the inverse term certainly exists.) The total received interferences at receiver  $l$  at times  $t_0$  and  $t_k$  are both equal to  $\sum_{i \neq l} H_{li}[t_0] \mathbf{w}_i$ , so can be cancelled.

We refer to such schemes that take advantage of beamforming as JAP-B(a) schemes.

*Theorem 4:* The delay exponent of a JAP-B(a) scheme with parameter sequence  $\mathbf{a}$  is

$$T_B(\mathbf{a}) := \max_k (a_k - 1)(n - k - 1).$$

*Proof:* At each round, receiver  $A_{k-1} + 1$  will automatically recover its message, leaving the JAP scheme to align interference for the other  $a_k - 1$  users. (Independence of the coefficients  $H_{ji}$  ensures that the other users still have the same problem to solve.) ■

In particular, the JAP-B scheme will always outperform the JAP scheme with the same sequence  $\mathbf{a}$ .

### C. An interesting special case: JAP-B(n)

An interesting special case of the JAP-B schemes is the case when  $K = 1$  and  $a_1 = n$ ; we call this scheme JAP-B(n).

In this case, we have  $1/(K+1) = 1/2$  degrees of freedom for a rate of  $D(Z)/2$ . From Theorem 4, we see that the delay exponent is

$$T_B(n) = (a_1 - 1)(n - 1 - 1) = (n - 1)(n - 2).$$

Effectively, the JAP-B(n) scheme works by using beamforming to automatically cancel user 1’s interference, then for users 2, 3,  $\dots$ ,  $n$  requiring the existence of diagonal matrices  $D_0, D_1$  such that  $D_0 \mathbf{H}[t_0] + D_1 \mathbf{H}[t_1] = \mathbf{I}$ .

Note that this is the same rate as is achieved by the original NGJV scheme, but the delay exponent has been reduced from NGJV’s  $n^2$  to  $(n-1)(n-2) = n^2 - (3n-2)$ . For small  $n$  in particular this is a worthwhile improvement (see Figure 1). For  $n = 3$  users, where experiments have shown the feasibility of interference alignment [10], the delay exponent is reduced from  $n^2 = 9$  to  $(n-1)(n-2) = 2$ .

### D. Using time-sharing: child schemes

Another way to generate new alignment schemes is by time-sharing schemes designed for a smaller number of users.

Call the NGJV, KWG, JAP and JAP-B schemes ‘parent schemes’. Given a parent scheme for an  $m$ -user network, we can modify the scheme for use in an  $n$ -user network for  $n > m$ , giving what we call a ‘child scheme’.

We use TDMA to split an  $n$ -user network into  $\binom{n}{m}$  subnetworks, each of which contains a unique collection of just  $m < n$  of the users. Within each of these  $m$ -user subnetworks, an  $m$ -user parent scheme is used, while the other  $n - m$  transmitters remain silent.

An  $m$ -user child scheme has the same delay exponent as the  $n$ -user parent scheme, with the rate, and thus the degrees of freedom, reduced by a factor of  $m/n$ . So an  $m$ -user JAP-B scheme shared between  $n$  users gives  $\text{DOF} = m/n(K+1)$ .

In particular, a child scheme from parent NGJV schemes has a lower delay exponent  $m^2 < n^2$  than the main NGJV scheme, reducing the degrees of freedom from  $1/2$  to  $m/2n$ . As even this outperforms the KWG schemes, we regard this as the benchmark against which to compare our new JAP-B parent and child schemes.

Child schemes derived from the JAP-B(n) parent scheme are particularly effective, and usually perform better than other JAP-B schemes, as we discuss in the next section.

## V. BEST SCHEMES

### A. General case

Given a number of users  $n$  and a desired number of degrees of freedom  $\text{DOF} = 1/(K+1)$ , we wish to find a scheme with the lowest delay exponent.

For  $K = n - 1$  or  $n$ , when  $\text{DOF} = 1/n$  or  $1/(n+1)$ , the best JAP-B schemes have delay exponent 0. This is the same delay exponent as TDMA, which has  $\text{DOF} = 1/n$  also.

TABLE I  
BEST JAP-B( $\mathbf{a}$ ) SCHEMES FOR SMALL VALUES OF  $n$  AND  $K$ : DELAY EXPONENTS (ABOVE) AND OPTIMAL  $\mathbf{a}$  (BELOW). (\* = NON-UNIQUE)

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$K = 1$ DOF = 1/2	2 (3)	6 (4)	12 (5)	20 (6)	30 (7)	42 (8)
$K = 2$ DOF = 1/3	0 TDMA	2 (1, 3)	4 (2, 3)	8 (3, 3)	12 (3, 4)	18 (4, 4)
$K = 3$ DOF = 1/4		0 TDMA	2 (1, 1, 3)*	4 (1, 2, 3)*	6 (2, 2, 3)	10 (2, 3, 3)
$K = 4$ DOF = 1/6			0 TDMA	2 (1, 1, 1, 3)*	4 (1, 1, 2, 3)*	6 (1, 2, 2, 3)*
$K = 5$ DOF = 1/6				0 TDMA	2 (1, 1, 1, 1, 3)*	4 (1, 1, 1, 2, 3)*
$K = 6$ DOF = 1/7					0 TDMA	2 (1, 1, 1, 1, 1, 3)*
$K = 7$ DOF = 1/8						0 TDMA

For  $K \leq n - 2$  the best parent scheme will be a JAP-B scheme with parameter sequence  $\mathbf{a} \in \mathcal{A}(n, K)$ . We write  $T(n, K)$  for this best delay exponent, that is

$$T(n, K) := \min_{\mathbf{a} \in \mathcal{A}(n, K)} T_B(\mathbf{a}) \\ = \min_{\mathbf{a} \in \mathcal{A}(n, K)} \max_{k \in \{1, 2, \dots, K\}} (a_k - 1)(n - k - 1).$$

We can bound  $T(n, K)$  as follows.

*Theorem 5:* Fix  $n$  and  $K \leq n - 2$ . For  $T(n, K)$  as defined above, we have the following bounds:

$$\frac{n}{K}(n - 1) - (2n - K - 1) \leq T(n, K) \leq \frac{n}{K}(n - 2).$$

In particular, for fixed  $K$ , we have  $T(n, K) = n^2/K + o(n^2)$ .

The gap between the bounds grows linearly with  $n$ .

The following lemma on partial harmonic sums will be useful.

*Lemma 5 $\frac{1}{2}$ :* Let  $S(n, K)$  be the partial harmonic sum

$$S(n, K) := \sum_{k=1}^K \frac{1}{n - k - 1} = \frac{1}{n - 2} + \dots + \frac{1}{n - K - 1}.$$

Then we have the bounds

$$\frac{K}{n - 2} \leq S(n, K) \leq \frac{K}{n - K - 1}.$$

*Proof:* There are  $K$  terms in the sum. The largest term is  $1/(n - K - 1)$ ; the smallest term is  $1/(n - 2)$ . ■

We can now prove Theorem 5.

*Proof of Theorem 5:* The value of  $T(n, K)$  is lower-bounded by the value of the same minimisation problem relaxed to allow the  $a_k$  to be real. That is,

$$T(n, K) = \min_{\mathbf{a} \in \mathbb{Z}_+^K: \sum_k a_k = n} \max_{k \in \{1, 2, \dots, K\}} (a_k - 1)(n - k - 1) \\ \geq \min_{\mathbf{a} \in \mathbb{R}_+^K: \sum_k a_k = n} \max_{k \in \{1, 2, \dots, K\}} (a_k - 1)(n - k - 1).$$

The relaxed problem is solved by waterfilling, setting  $a_k - 1 = c/(n - k - 1)$ . Requiring the weight of  $\mathbf{a}$  to be  $n$  forces

$$T(n, K) \geq c = \frac{n - K}{S(n, K)} \geq \frac{(n - K)(n - K - 1)}{K},$$

where we have used Lemma 5 $\frac{1}{2}$ . Rearrangement gives the lower bound.

An upper bound is obtained by using the same  $c$  and taking

$$a_k - 1 = \left\lceil \frac{c}{n - k - 1} \right\rceil \leq \frac{c}{n - k - 1} + 1.$$

This gives

$$T(n, K) \leq c + \max_k (n - k - 1) \\ = \frac{n - K}{S(n, K)} + (n - 1 - 1) \\ \leq \frac{(n - K)(n - 2)}{K} + (n - 2),$$

where we have used Lemma 5 $\frac{1}{2}$ . Rearrangement gives the upper bound.

The dominant term in the upper and lower bounds is easily seen to be  $n^2/K$ . ■

### B. Few users: Small $n$

For small values of  $n$ , we can find the best parent JAP-B schemes by hand. (The task is simplified by noting that the optimal  $a_k$  will be nonzero and increasing in  $k$ .) Table I gives the delay exponents of the best JAP-B schemes for  $n = 3, \dots, 8$  and  $K \leq n - 2$ .

We also consider child schemes based on parent JAP-B schemes. Figure 1 plots the performance of NGJV and all JAP-B schemes, as well as child schemes derived from them, for  $n = 3, \dots, 8$ . For many values of  $n$  and DOF, the scheme with the lowest delay exponent is JAP-B( $n$ ) or a child scheme

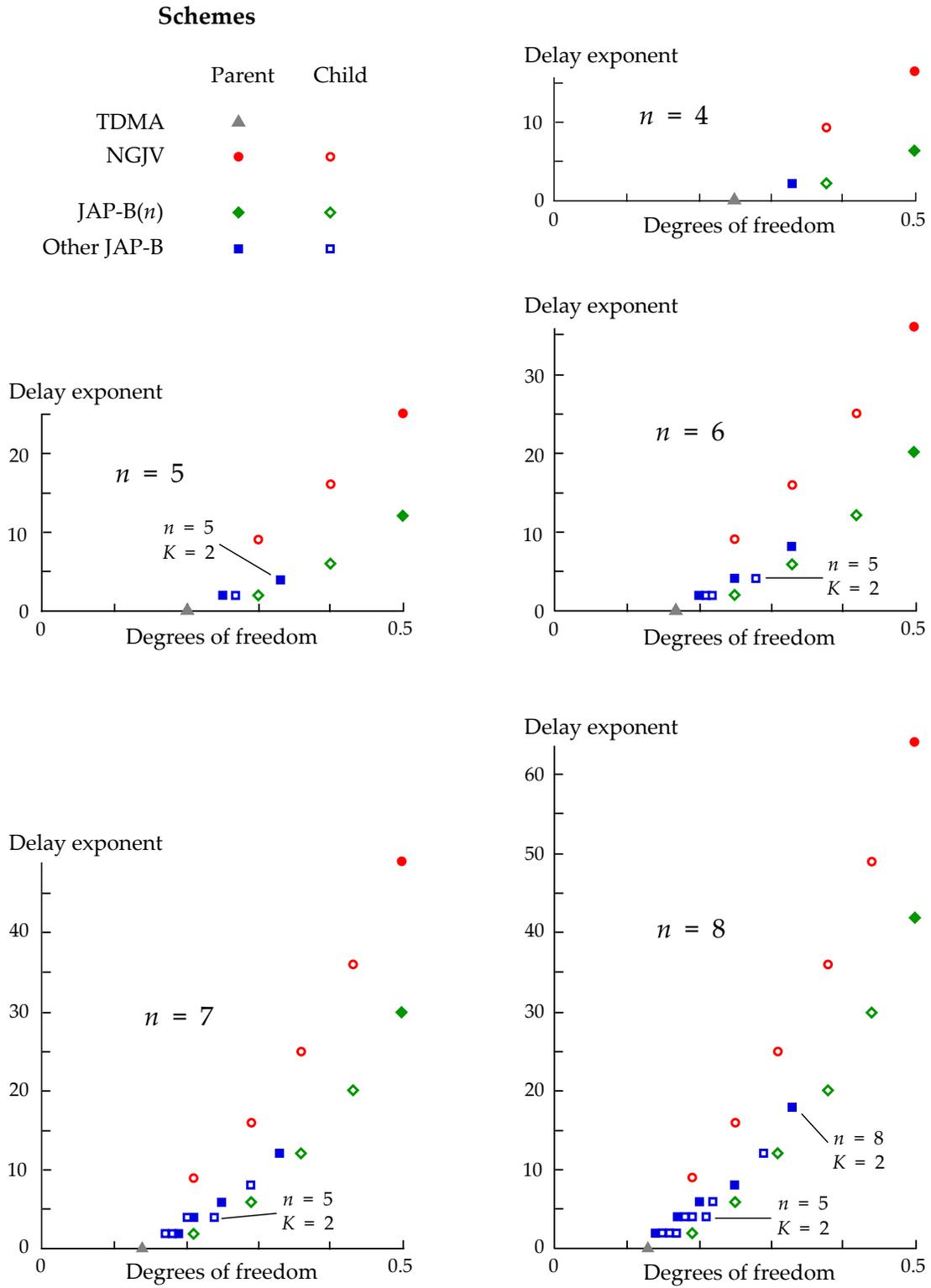


Fig. 1. Graphs of delay exponent against degrees of freedom NGJV and best JAP-B parent schemes and child schemes derived from them and TDMA.

derived from it, although the JAP-B parent schemes with  $(n, K) = (5, 2)$  or  $(8, 2)$ , for example, and their child schemes always outperform JAP( $n$ ) for some degrees of freedom.

C. Many users:  $n \rightarrow \infty$

We now consider the performance of schemes in the many-user limit  $n \rightarrow \infty$ . We consider two limiting regimes, depending on how the degrees of freedom  $\text{DOF}(n)$  should scale with the number of users  $n$ .

- **Regime I**, where we hold  $\text{DOF}(n) = \alpha$  constant, for  $\alpha \in (0, 1/2]$ . That is, we want to communicate at fixed fraction of the single-user rate, as in the NGJV scheme. NGJV itself corresponds to  $\alpha = 1/2$ .
- **Regime II**, where we hold the sum-rate constant, so the  $\text{DOF}(n) = \beta/n$ , for  $\beta \geq 1$ . That is, we want to communicate at a multiple of the rate allowed by resource division schemes like TDMA. TDMA itself corresponds to  $\beta = 1$ .

First, we consider parent JAP-B(a) schemes.

*Theorem 6:* Let  $T(n)$  be the delay exponent of the best parent JAP-B(a) scheme for  $n$ -users with at least  $\text{DOF}(n)$  degrees of freedom. Then:

- **Regime I:** Fix  $\alpha \in (0, 1/2]$ . Then the delay exponent for  $\text{DOF}(n) = \alpha$  scales quadratically, in that

$$T(n) = \frac{1}{\lfloor 1/\alpha \rfloor - 1} n^2 + o(n^2) = O(n^2).$$

- **Regime II:** Fix  $\beta > 1$ . Then the delay exponent for  $\text{DOF}(n) = \beta/n$  scales linearly, in that  $T(n) = O(n)$ , or more specifically

$$\left( \beta + \frac{1}{\beta} - 2 \right) n - o(n) \leq T(n) \leq \beta n + o(n).$$

*Proof:* For regime I, we have  $1/(K+1) \geq \text{DOF} = \alpha$ , so we need to take

$$K = \left\lfloor \frac{1}{\alpha} - 1 \right\rfloor = \left\lfloor \frac{1}{\alpha} \right\rfloor - 1.$$

But Theorem 5 tells us that for fixed  $K$  we have  $T(n, K) = n^2/K + o(n^2)$ .

For regime II, we have  $1/(K+1) \geq \text{DOF} = \beta/n$ , so we need to take

$$K = \left\lfloor \frac{n}{\beta} - 1 \right\rfloor = \frac{n}{\beta} - O(1).$$

The lower bound from Theorem 5 then gives us

$$\begin{aligned} T(n) &\geq \frac{n}{n/\beta - O(1)} (n-1) - \left( 2n - \frac{n}{\beta} - O(1) \right) \\ &= \left( \beta + \frac{1}{\beta} - 2 \right) n - O(1), \end{aligned}$$

and the upper bound gives us

$$T(n) \leq \frac{n}{n/\beta - O(1)} (n-2) = \beta n + O(1).$$

This gives the result.  $\blacksquare$

In regime I with  $\alpha = 1/2$ , we get  $T(n) \approx n^2$ , like NGJV.

Figure 1 shows that sharing the parent scheme JAP-B( $m$ ) is particularly effective. The following theorem shows this.

*Theorem 7:* Let  $T(n)$  be the delay exponent of child schemes based on JAP-B( $m$ ) parent schemes for  $n$ -users with at least  $\text{DOF}(n)$  degrees of freedom. Then:

- **Regime I:** Fix  $\alpha \in (0, 1/2]$ . Then the delay exponent for  $\text{DOF}(n) = \alpha$  scales quadratically, in that

$$T(n) = 4\alpha^2 n^2 + o(n^2).$$

- **Regime II:** Fix  $\beta > 1$ . Then the delay exponent for  $\text{DOF}(n) = \beta/n$  is constant, in that

$$T(n) = (\lfloor 2\beta \rfloor - 1)(\lfloor 2\beta \rfloor - 2).$$

*Proof:* Recall from Subsection IV-D that sharing the scheme JAP-B( $m$ ) amongst  $n$  users gives  $\text{DOF} = m/2n$  for delay exponent  $T = (m-1)(m-2)$ .

For regime I, we have  $m/2n \geq \text{DOF}(n) = \alpha$ , so we need to take  $m = \lfloor 2\alpha n \rfloor$ , giving  $T(n) = (\lfloor 2\alpha n \rfloor - 1)(\lfloor 2\alpha n \rfloor - 2)$ . The result follows.

For regime II, we have  $m/2n \geq \text{DOF}(n) = \beta/n$ , so we need to take  $m = \lfloor 2\beta \rfloor$ , giving  $T(n) = (\lfloor 2\beta \rfloor - 1)(\lfloor 2\beta \rfloor - 2)$ .  $\blacksquare$

Note that asymptotically this means that in both regimes child schemes from JAP-B( $m$ ) parent schemes are asymptotically more effective than any other parent scheme.

Note also that by the same argument as above, sharing the NGJV parent scheme gives  $T(n) = 4\alpha^2 n^2$  in regime I – less good than sharing JAP-B( $m$ ), but the same to first-order terms.

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