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Channel Modelling of MU-MIMO Systems by Quaternionic Free Probability

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***Abstract*—This paper studies the asymptotic eigenvalue distribution (AED) and the mutual information of a multiuser (MU) multiple-input multiple output (MIMO) channel with a certain fraction of users experiencing line-of-sight. It shows that the AED of the channel matrix decomposes into two separate bulks for practically relevant parameter choices and differs very much from the common assumption of independent identically distributed (iid) entries which induces the quarter circle law. This happens even without antenna correlation at either side of the channel. In order to tackle this problem the paper makes use of recent developments in free probability theory which allow to deal with complex-valued eigenvalue distributions of non-Hermitian matrices by means of quaternions.**

***Index Terms*—antenna arrays, channel modeling, eigenvalue distribution, free probability, line of sight, multiple-input multiple output, quaternions, random matrices, R-diagonal elements, Rician fading, Stieltjes transform**

I. Introduction

MU-MIMO systems have received a great deal of attention recently as they also serve as a models to describe the propagation of virtual MIMO systems were the multiple antennas are not co-located but belong to different cooperating users. The capacity region of a MU-MIMO system depends on the singular values of the channel matrix that governs the propagation from all (virtual) transmitting antennas to all (virtual) receiving antennas.

In multi-antenna systems, signal propagation is dominated by two mechanism: line-of-sight and scattering. In single-user MIMO systems, the line-of-sight component, if present, has high power, but rank one while the scattered component has lower power but high rank. In MU-MIMO systems, the line-of-sight component of the channel matrix is not limited to rank one, as the antennas need not be co-located. Several earlier works have addressed this scenario, see e.g. [1]–[4].

The rank of the line-of-sight component is typically lower than the rank of the scattered component as the existence of a direct path is less probable than the existence of an indirect path. With the scattered component having higher rank, but lower power, the question which of the two components is more important is non-trivial. Furthermore, it is expected that the interplay of both components is important to understand the properties of MU-MIMO systems.

In the large system limit, i.e. the number of antennas grows to infinity with a fixed ratio between the number of transmit to receive antennas, the properties of MIMO systems can be studied by means of free probability theory, see e.g. [5]–[11]. Free probability, as proposed in [12], [13] allows to infer the AEDs of sums or products of Hermitian random matrices with known eigenvalue distribution provided that these Hermitian random matrices are free. This allowed to deal with a great number of channel models in wireless communications and put the basis for the success of free probability in information theory of wireless channels, see e.g. [8], [10], [14].

Still there are channel models with great practical importance which are not simply sums or products of Hermitian random matrices with known eigenvalues distributions. The MU-MIMO system described by

$$\boldsymbol{H} = \sigma \boldsymbol{H}_2 \boldsymbol{H}_1 + \boldsymbol{H}_0 \tag{1}$$

with $\boldsymbol{H}_0$, $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, σ denoting the line-of-sight path, the propagation from the transmit antennas to the scatterers, the propagation from the scatterers to the receive antennas, and the attenuation of the scattered paths relative to the line-of sight paths, respectively, is one of them. Note that the terms to be summed in

$$\boldsymbol{H}\boldsymbol{H}^\dagger = \sigma^2 \boldsymbol{H}_2 \boldsymbol{H}_1 \boldsymbol{H}_1^\dagger \boldsymbol{H}_2^\dagger + \boldsymbol{H}_0 \boldsymbol{H}_0^\dagger + \tag{2}$$

$$+ \sigma \boldsymbol{H}_2 \boldsymbol{H}_1 \boldsymbol{H}_0^\dagger + \sigma \boldsymbol{H}_0 \boldsymbol{H}_1^\dagger \boldsymbol{H}_2^\dagger \tag{3}$$

are not free, while the terms in (1) are not Hermitian.

In this paper, we will make use of a free probability calculus for non-Hermitian random matrices recently discovered in [15][1] to analyze the AEDs of MU-MIMO systems. First, we will introduce the system model in Section II. Then, analyze it in Section III making use of the quaternionic extension of free probability theory discussed in Section IV. Conclusions are outlined in Section V.

II. System Model

Consider the MU-MIMO communication link introduced in (1). This model neglects potential antenna correlation at transmitter and receiver side. Such correlations can be easily taken into account by the modified channel model [9]

$$\sqrt{\boldsymbol{C}_{\mathrm{R}}}\boldsymbol{H}\sqrt{\boldsymbol{C}_{\mathrm{T}}} \tag{4}$$

by means of classical multiplicative free convolution, once the AED of $\boldsymbol{H}\boldsymbol{H}^\dagger$ is known. For sake of space limitations, we

[1] A less explicit calculus for non-Hermitian random matrices was already proposed in [16], [17].

will thus ignore antenna correlation in this paper and solely focus on the more challenging problem of finding the AED of $\boldsymbol{H}\boldsymbol{H}^\dagger$.

Assuming users with equal powers[2] and following [7], the random matrices $\boldsymbol{H}_1 \in \mathbb{C}^{S\times T}$ and $\boldsymbol{H}_2 \in \mathbb{C}^{R\times S}$ are assumed to have iid complex Gaussian entries with zero mean and variances $1/S$ and $1/R$, respectively, where T, S and R denote the number of transmit antennas, of scatterers, and of receive antennas, respectively. The line-of-sight matrix $\boldsymbol{H}_0$ is not iid, since some users may not experience line of sight, in general. Assuming users with equal powers again, it is well modeled as

$$\boldsymbol{H}_0 = \boldsymbol{G}_0\boldsymbol{P}_0 \tag{5}$$

where $\boldsymbol{G}_0$ is iid complex Gaussian with zero-mean and variance $1/R$ and $\boldsymbol{P}_0$ is a diagonal matrix with L ones and $T-L$ zeros on the diagonal.

Quaternionic free probability allows to find complex-valued asymptotic eigenvalue distributions and is thus well-suited for the analysis of a channel model like (1). However, $\boldsymbol{H}$ is not square, in general. Nevertheless, assuming $T \leq R$,[3] we can write

$$\boldsymbol{H} = \tilde{\boldsymbol{H}}\boldsymbol{P} \tag{6}$$

with $\tilde{\boldsymbol{H}} \in \mathbb{C}^{R\times R}$ and $\boldsymbol{P} \in \{0,1\}^{R\times T}$ with ones on the diagonal and zeros elsewhere. Obtaining the AED of $\boldsymbol{H}\boldsymbol{H}^\dagger$ given the AED of $\tilde{\boldsymbol{H}}\tilde{\boldsymbol{H}}^\dagger$ is a straightforward exercise in classical multiplicative free convolution and omitted here due to space limitations. Similarly, the rectangular matrices $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ can be represented by equivalent square matrices. Thus, we assume $T = R$ in the sequel for sake of readability and space limitations and leave the generalization to non-square matrices to the reader. We are, thus, left with only three parameters in the large system limit. One is the relative scattering attenuation σ introduced in (1). The second one is

$$\phi \equiv \frac{L}{T} \tag{7}$$

which we will call *line-of-sight fraction* in the following. It specifies the relative rank of the light-of-sight component of the MU-MIMO system. The third one is

$$\rho \equiv \frac{S}{R} \tag{8}$$

which is called the *scattering richness* or richness for short.

In the following, we will address the wireless MIMO system described by

$$\boldsymbol{y} = \sqrt{\frac{\gamma}{\phi+\sigma^2}}\boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \tag{9}$$

where $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{n}$, γ are the channel input, the channel output, additive white Gaussian noise (AWGN), and the signal-to-noise ratio, respectively. The entries of $\boldsymbol{x}$ and $\boldsymbol{n}$ are assumed to be iid with zero mean and unit variance.

[2]The generalization to unequal powers is a straightforward, though tedious exercise and omitted for sake of readability and space limitations.

[3]The case $T > R$ works accordingly by left multiplication of a projector.

III. MU-MIMO Systems

Using the results outlined in Section IV, we easily find:

COROLLARY 1 *Let the entries of the independent $T \times T$ matrices $\boldsymbol{H}_n$ be iid*[4] *with zero mean and variance $1/T$ for all n. Then, the empirical eigenvalue distributions of $\prod_{n=1}^N \boldsymbol{H}_n$ and $\boldsymbol{H}_n^N$ converge almost surely to the same limit given by*

$$\mathrm{p}_{\boldsymbol{H}_n^N}(z) = \begin{cases} \frac{1}{\pi N}|z|^{\frac{2}{N}-2} & |z| \leq 1 \\ 0 & \textit{elsewhere} \end{cases} \tag{10}$$

$\forall n$ as $T \to \infty$.

In other words, independent square random matrices with iid entries behave with respect to multiplication asymptotically as if they were identical. This means, that running through the same i.i.d. random channel twice or running consecutively through two random channels with the same statistics makes no difference in the large-system limit. By contrast, this does not even hold approximately if the channel is a diagonal matrix. A more general form of Corollary 1 is found in Theorem 4 which explains this surprising equivalence in distribution by means of the bi-unitary invariance of the measure of $\boldsymbol{H}_n$.

COROLLARY 2 *Let the entries of the $T \times T$ matrix $\boldsymbol{G}$ be iid with variance $1/T$ and the matrix $\boldsymbol{P} \in \{0,1\}^{T\times T}$ be diagonal with L non-zero entries. Then, the empirical eigenvalue distribution of $\boldsymbol{GP}$ converges almost surely to*

$$\mathrm{p}_{\boldsymbol{GP}}(z) = (1-\phi)\delta(z) + \begin{cases} \frac{1}{\pi} & |z| \leq \sqrt{\phi} \\ 0 & \textit{elsewhere} \end{cases} \tag{11}$$

as $T, L \to \infty$ with $\phi = \frac{L}{T}$ fixed.

In other words, the projection of iid square random matrices from T to L dimensions replaces the $T-L$ eigenvalues with greatest modulus by zero eigenvalues.

COROLLARY 3 *Let the entries of the $R \times S$ matrix $\boldsymbol{X}$ and the $S \times T$ matrix $\boldsymbol{Y}$ be iid with zero mean and variance $1/\sqrt{RS}$. Then, the empirical eigenvalue distribution of the $R\times R$ matrix $\boldsymbol{Z} = [\boldsymbol{XY}|\boldsymbol{0}]$ converges almost surely to the limit*

$$\mathrm{p}_{\boldsymbol{Z}}(z) = (1-\beta)\delta(z) + \begin{cases} \frac{1}{\pi}\frac{\rho}{\sqrt{(\rho-\beta)^2+4\rho|z|^2}} & |z| \leq \sqrt{\beta} \\ 0 & \textit{elsewhere} \end{cases} \tag{12}$$

as $T, S, R \to \infty$ with $\rho = S/R \geq 1$ and $\beta = T/R \leq 1$ fixed.

Setting $\beta = 1$, the AEDs of Corollary 1 with $N = 2$ and $N = 1$ are recovered for $\rho = 1$ and $\rho \to \infty$, respectively.

More involved calculations based on the results in Section IV lead to:

THEOREM 1 *Let the entries of the $R \times S$ matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2^\dagger$ be iid with zero mean, variance $1/\sqrt{RS}$ and m^{th} moments upper bounded by $\alpha_m R^{-m/2}$ for some α_m and all $m \geq 1$. Let*

[4]For matrices with independent Gaussian entries, the result is stated in [18].

$\boldsymbol{B}$ be an arbitrary matrix free of $\boldsymbol{A}_1\boldsymbol{A}_2$ such that the empirical distribution of eigenvalues of $\boldsymbol{B}\boldsymbol{B}^\dagger$ converges, as $R\to\infty$, to a limit distribution with Stieltjes transform $\mathrm{G}_{\boldsymbol{B}\boldsymbol{B}^\dagger}(s)$ defined in (22)*. Furthermore, let*

$$\boldsymbol{C} = \sigma\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{B} \tag{13}$$

with $\sigma\in\mathbb{C}$ and define

$$\tilde{\mathrm{G}}_{\boldsymbol{B}}(s) \equiv s\,\mathrm{G}_{\boldsymbol{B}\boldsymbol{B}^\dagger}\left(s^2\right). \tag{14}$$

Then, the empirical distribution of eigenvalues of $\boldsymbol{C}\boldsymbol{C}^\dagger$ converges almost surely to a limit distribution whose Stieltjes transform satisfies

$$\mathrm{G}_{\boldsymbol{C}\boldsymbol{C}^\dagger}(s) = \frac{1}{\sqrt{s}}\,\tilde{\mathrm{G}}_{\boldsymbol{B}}\left[\sqrt{s} - \frac{|\sigma|^2\rho\sqrt{s}\,\mathrm{G}_{\boldsymbol{C}\boldsymbol{C}^\dagger}(s)}{\rho-|\sigma|^2 s\,\mathrm{G}^2_{\boldsymbol{C}\boldsymbol{C}^\dagger}(s)}\right] \tag{15}$$

as $R,S\to\infty$ with $\rho=S/R$ fixed.

The proof is omitted due to space limitations.

The AED of the line-of-sight component, i.e.

$$\boldsymbol{H}_0\boldsymbol{H}_0^\dagger = \boldsymbol{G}\boldsymbol{P}\boldsymbol{G}^\dagger \tag{16}$$

is the well-known Marchenko-Pastur distribution with parameter ϕ [10]. Its Stieltjes transform is a solution to a quadratic equation which has the two solutions [19, Table I][5]

$$\mathrm{G}_{\boldsymbol{H}_0\boldsymbol{H}_0^\dagger}(s) = \frac{1}{2} + \frac{1-\phi}{2s} \pm \sqrt{\frac{(1-\phi)^2}{4s^2} - \frac{1+\phi}{2s} + \frac{1}{4}}. \tag{17}$$

With the help of (17), Theorem 1 allows to calculate the asymptotic singular value distribution of the channel (1). The respective density function is shown in Fig. 1 for $4\phi = 4\sigma = \rho = 1$. This examples was chosen, since the relative scattering attenuation σ and line-of-sight fraction ϕ are small, in practice. In that case, the asymptotic singular value distribution of $\boldsymbol{H}$ decomposes into two bulks with each bulk being shaped very similar to the cases of pure scattering and pure line-of-sight when scaled or shifted appropriately. This deviates strongly from the quarter circle law that would be obtained, if $\boldsymbol{H}$ were composed of iid entries.

The mutual information of the channel defined in (9) and measured in nats is given by

$$\lim_{T\to\infty}\frac{\mathrm{I}(X;Y)}{T} = \int \log\left(1+\frac{x}{s}\right)\mathrm{dP}_{\boldsymbol{H}\boldsymbol{H}^\dagger}(x)\bigg|_{s=\frac{\phi+\sigma^2}{\gamma}} \tag{18}$$

$$= \int_{\frac{\phi+\sigma^2}{\gamma}}^{\infty} \mathrm{G}_{\boldsymbol{H}\boldsymbol{H}^\dagger}(-s) + \frac{1}{s}\,\mathrm{d}s \tag{19}$$

$$= 2\int_{\sqrt{\frac{\phi+\sigma^2}{\gamma}}}^{\infty} \frac{1}{s} - \tilde{\mathrm{G}}_{\boldsymbol{H}}(-s)\,\mathrm{d}s \tag{20}$$

where (19) and (20) follow from [7, Sec. IV.C][5] and (14), respectively. It is shown in Fig. 2 for a fixed signal-to-noise ratio of 9 dB. One can observe that small values of the line-of-sight fraction ϕ and the relative scattering attenuation σ that are typical in many practical scenarios are quite deleterious for the mutual information of the channel. Furthermore, the figure seems to suggest that blocking the line of sight is better than a small, but non-zero value of the line-of-sight fraction ϕ. However, Fig. 2 is plotted for constant SNR and blocking the line of sight will surely decrease the SNR.

The hit in mutual information for small line-of-sight fraction and relative scattering attenuation is exacerbated in practice by the fact that analog-to-digital conversion and precise estimation of the scattered paths is challenging in the presence of much stronger direct paths.

[5]Note that [7], [19] use a different definition of the Stieltjes transform.

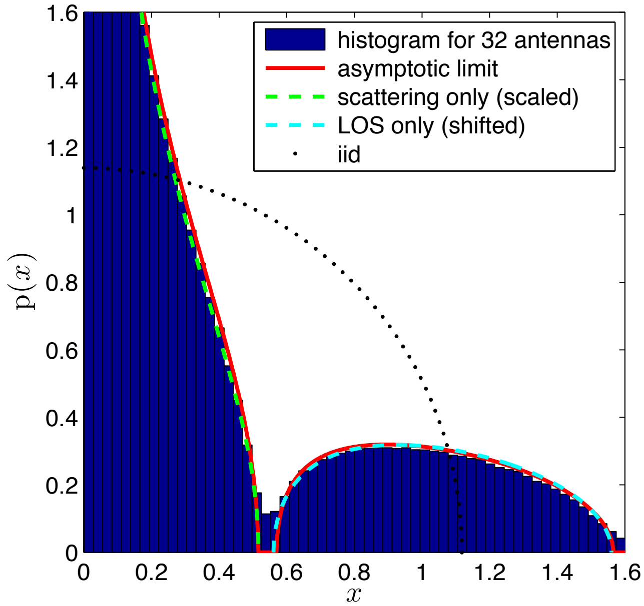

Fig. 1. Probability density function of the singular values of the matrix $\boldsymbol{H}$ in (1) for $4\sigma = 4\phi = \rho = 1$. The dashed lines show scaled and shifted versions of pure scattering ($\phi = 0$) and pure line-of-sight ($\sigma = 0$), respectively. The dotted line refers to the iid case.

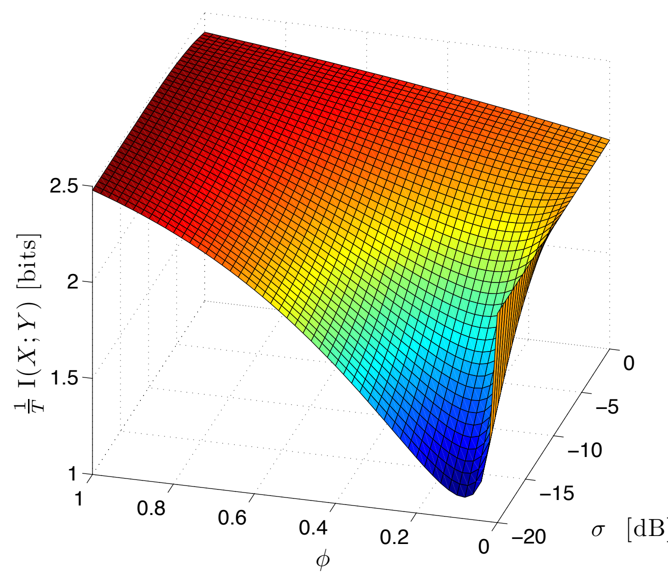

Fig. 2. Mutual information for $\gamma = 9$ dB and $\rho = 1$ versus ϕ and σ.

IV. Quaternionic Free Probability Theory

Hermitian matrices have real eigenvalues. The method of choice to deal with real-valued eigenvalue distributions in free probability is to utilize complex analysis, i.e. to represent a real-valued eigenvalue distribution

$$\mathrm{p}(x) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \Re \left\{ \mathrm{j} \mathrm{G}(x + \mathrm{j}\epsilon) \right\} \tag{21}$$

as a limit of a complex-valued holomorphic function $\mathrm{G}(s)$, which is known as the *Stieltjes transform* and defined by

$$\mathrm{G}(s) \equiv \int \frac{\mathrm{dP}(x)}{s - x}. \tag{22}$$

Complex-valued eigenvalue distributions are often not holomorphic. They can be represented by a pair of holomorphic functions representing real and imaginary part. Instead of real and imaginary part of a complex variable z, one can also consider z, its complex conjugate z^*, and apply the Wirtinger rule [20] for differentiation, i.e.

$$\frac{\partial z^*}{\partial z} \equiv 0 \equiv \frac{\partial z}{\partial z^*}. \tag{23}$$

A. Stieltjes Transform

In order to generalize the Stieltjes transform to two complex variables z and z^*, we first rewrite (22) by

$$\mathrm{G}(s) = \frac{\mathrm{d}}{\mathrm{d}s} \int \log(s - x) \mathrm{dP}(x) \,. \tag{24}$$

Further, note that the Dirac function of complex argument can be represented as the limit

$$\delta(z) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2}. \tag{25}$$

Thus, we have

$$\mathrm{p}(z) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\partial^2}{\partial z \partial z^*} \int \log \left[|z - z'|^2 + \epsilon^2 \right] \mathrm{dP}(z') \,. \tag{26}$$

We define the bivariate Stieltjes transform by

$$\mathrm{G}(s, \epsilon) \equiv \frac{\partial}{\partial s} \int \log \left[|s - z|^2 + \epsilon^2 \right] \mathrm{dP}(z) \tag{27}$$

$$= \int \frac{(s - z)^*}{|s - z|^2 + \epsilon^2} \mathrm{dP}(z) \tag{28}$$

and get the bivariate Stieltjes inversion formula to read

$$\mathrm{p}(z) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\partial}{\partial z^*} \mathrm{G}(z, \epsilon) \,. \tag{29}$$

At first sight, the bivariate Stieltjes transform looks quite different from (22). However, we can rewrite (27) as

$$\mathrm{G}(s, \epsilon) = \int \left[\left(\begin{array}{cc} s - z & \mathrm{j}\epsilon \\ \mathrm{j}\epsilon & s^* - z^* \end{array} \right)^{-1} \right]_{11} \mathrm{dP}(z) \tag{30}$$

which clearly resembles the form of (22). To get an even more striking analogy with (22), we can introduce the Stieltjes transform with *quaternionic* argument $q \equiv v + \mathrm{i}w, (v, w) \in \mathbb{C}^2, \mathrm{i}^2 \equiv -1, \mathrm{ij} \equiv -\mathrm{ji}$

$$\mathrm{G}(q) \equiv \int \frac{\mathrm{dP}(z)}{q - z} \tag{31}$$

with the respective inversion formula

$$\mathrm{p}(z) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\partial}{\partial z^*} \Re \, \mathrm{G}(z + \mathrm{i}\epsilon) \tag{32}$$

and the definition $\Re(v + \mathrm{i}w) \equiv v \in \mathbb{C}$.[6] Quaternions are inconvenient to deal with since multiplication of quaternions does not commute, in general. However, any quaternion $q = v + \mathrm{i}w$ can be conveniently represented by the complex-valued 2×2 matrix

$$\left(\begin{array}{cc} v & w \\ -w^* & v^* \end{array} \right). \tag{33}$$

This matrix representation directly connects (30) with (31) via

$$\mathrm{G}(s, \epsilon) = \Re \, \mathrm{G}(s + \mathrm{i}\epsilon) \,. \tag{34}$$

B. Free Convolution

We define the R-transform of quaternion argument p in complete analogy to the complex case in [12] as

$$\mathrm{R}(p) \equiv \mathrm{G}^{-1}(p) - \frac{1}{p} \tag{35}$$

and obtain for free random matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, with $\mathrm{R}_{\boldsymbol{A}}(p)$ and $\mathrm{R}_{\boldsymbol{B}}(p)$ denoting the R-transforms of the respective AEDs,

$$\mathrm{R}_{\boldsymbol{A}+\boldsymbol{B}}(p) = \mathrm{R}_{\boldsymbol{A}}(p) + \mathrm{R}_{\boldsymbol{B}}(p) \,. \tag{36}$$

The scaling law of the R-transform generalizes as follows

$$\mathrm{R}_{z\boldsymbol{H}}(p) = z \mathrm{R}_{\boldsymbol{H}}(pz) \tag{37}$$

for $z \in \mathbb{C}$. Note that the order of factors does matter here, since $pz = z^* p$.

While additive free convolution generalizes straightforwardly, this is very different for multiplicative free convolution. Due to space limitations, we refer the reader to [21].

C. R-Diagonal Random Matrices

In practice, non-Hermitian random matrices are often, though not always, R-diagonal.

Definition 1 (Definition 2.3 in [16]) . *A random matrix $\boldsymbol{X}$ is called R-diagonal, if it can be decomposed as $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{Y}$ where $\boldsymbol{Y} = \sqrt{\boldsymbol{X}\boldsymbol{X}^\dagger}$ and $\boldsymbol{U}$ is Haar distributed and free of $\boldsymbol{Y}$.*

R-diagonal random matrices are a superset of bi-unitarily invariant matrices. Their additive free convolution can be performed without quaternionic free calculus as follows:

Theorem 2 (Proposition 3.5 in [22]) *Let the asymptotically free random matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ be R-diagonal and denote the respective asymptotic singular value distributions by $\mathrm{P}_{\boldsymbol{A}}(x)$ and $\mathrm{P}_{\boldsymbol{B}}(x)$. Define the symmetrization of a density by*

$$\tilde{\mathrm{p}}(x) = \frac{\mathrm{p}(x) + \mathrm{p}(-x)}{2}. \tag{38}$$

[6] Note that the real and imaginary part of a quaternion are its first and second complex component, respectively.

Then, we have for the respective R-transforms of the symmetrized singular value distributions

$$\tilde{\mathrm{R}}_{\boldsymbol{A}+\boldsymbol{B}}(w) = \tilde{\mathrm{R}}_{\boldsymbol{A}}(w) + \tilde{\mathrm{R}}_{\boldsymbol{B}}(w). \tag{39}$$

The AED relates to the respective asymptotic singular value distribution as follows:

THEOREM 3 (COROLLARY 4.5 IN [22]) *Let random matrix $\boldsymbol{H}$ be R-diagonal. Let $\mathrm{S}_{\boldsymbol{H}\boldsymbol{H}^\dagger}(s)$ denote the S-transform of the AED of $\boldsymbol{H}\boldsymbol{H}^\dagger$ and define the function*

$$f(s) = \frac{1}{\sqrt{\mathrm{S}_{\boldsymbol{H}\boldsymbol{H}^\dagger}(s-1)}}. \tag{40}$$

Then, the AED of $\boldsymbol{H}$ is circularly symmetric and given by

$$\mathrm{p}_{\boldsymbol{H}}(z) = \frac{1}{2\pi z f'\left[f^{-1}(z)\right]} \tag{41}$$

with $f'(s) = \mathrm{d}f(s)/\mathrm{d}s$ wherever the density is positive and continuous.

Furthermore, the following lemma relating the Stieltjes transform of the symmetrized singular value distribution of a square matrix $\boldsymbol{X}$ to the Stieltjes transform of the eigenvalue distribution of $\boldsymbol{X}\boldsymbol{X}^\dagger$ turns out quite helpful:

LEMMA 1 (TABLE II IN [19]) *Let*

$$\tilde{\mathrm{G}}_\lambda(s) = \frac{1}{2}\int \frac{\mathrm{p}_\lambda(x) + \mathrm{p}_\lambda(-x)}{s-x}\,\mathrm{d}x. \tag{42}$$

Then

$$\tilde{\mathrm{G}}_\lambda(s) = s\,\mathrm{G}_{\lambda^2}\left(s^2\right). \tag{43}$$

R-diagonal random matrices have the property to behave as if they are identical with respect to multiplication:

THEOREM 4 (PROPOSITION 3.10 [22]) *Let the random matrices $\boldsymbol{A}_n$ have the same AED, be R-diagonal, and asymptotically free of each other for all n. Then, the AEDs of $\prod_{n=1}^N \boldsymbol{A}_n$ and $\boldsymbol{A}_1^N$ are identical.*

V. CONCLUSIONS AND OUTLOOK

Line of sight strongly influences the eigenvalue distribution of multi-user MIMO channels. If the line-of-sight component is significantly stronger than the scattered paths and/or the fraction of users who experience line of sight is small, the eigenvalue distribution is composed of two separate bulks, one corresponding to the scattered paths and one corresponding to the direct paths. In that case, the AED can be accurately approximated by a scaled version of pure scattering and a shifted version of pure line of sight which, in contrast to the exact solution, can be given in closed explicit form.

While Stieltjes transforms in one complex variable cannot uniquely define a complex-valued eigenvalue distribution, a generalized Stieltjes transform in two complex-valued or one quaternion-valued variable can do so. Quaternion-valued Stieltjes transforms allow to deal with a much larger class of vector-valued communication systems than complex-valued univariate Stieltjes transforms do.

In a subsequent journal version of this paper, we will include the more general case of rectangular channel matrices, i.e. $R \neq T$, as well as the detailed derivation of Theorem 1.

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