# Self-dual Repeated Root Cyclic and Negacyclic Codes over Finite Fields 

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#### Abstract

In this paper we investigate repeated root cyclic and negacyclic codes of length $p^{r} m$ over $\mathbb{F}_{p^{s}}$ with $(m, p)=1$. In the case $p$ odd, we give necessary and sufficient conditions on the existence of negacyclic self-dual codes. When $m=2 m^{\prime}$ with $m^{\prime}$ odd, we characterize the codes in terms of their generator polynomials. This provides simple conditions on the existence of self-dual negacyclic codes, and generalizes the results of Dinh [6]. We also answer an open problem concerning the number of selfdual cyclic codes given by Jia et al. [11].


## I. Introduction

Let $p$ be a prime number and $\mathbb{F}_{p^{s}}$ the finite field with $p^{s}$ elements. An $[n, k]$ linear code $C$ over $\mathbb{F}_{p^{s}}$ is a $k$ dimensional subspace of $\mathbb{F}_{p^{s}}^{n}$. A linear code $C$ over $\mathbb{F}_{p^{s}}^{n}$ is said to be constacyclic if it is an ideal of the quotient ring $R_{n}=\mathbb{F}_{p^{s}}[x] /\left\langle x^{n}-a\right\rangle$. When $a=1$ the code is called cyclic, and when $a=-1$ the code is called negacyclic. The Euclidean dual code $C^{\perp}$ of $C$ is defined as $C^{\perp}=\{\mathrm{x} \in$ $\left.\mathbb{F}_{q}^{n}: \sum_{i=1}^{n} x_{i} y_{i}=0 \forall \mathrm{y} \in C\right\}$. An interesting class of codes is the so-called self-dual codes. A code is called Euclidean self-dual if it satisfies $C=C^{\perp}$. Note that the dual of a cyclic (respectively negacyclic) code is a cyclic (respectively negacyclic) code.

Cyclic codes are interesting from both theoretical and practical perspectives. For example, they can easily be encoded, and decoding algorithms exist in many cases. When $(n, p)=1$, these codes are called simple root codes, otherwise they are called repeated root codes. Castagnoli et al. and van Lint [4], [12] studied repeated root cyclic codes. They proved that these codes have a concatenated structure and are not asymptotically better than simple root codes. Negacyclic codes were introduced by Berlekamp [2]. Simple root self-dual negacyclic codes were studied by Blackford [3] and Guenda [8]. The algebraic structure of repeated root constacyclic codes of length $2 p^{r}$ over $\mathbb{F}_{p^{s}}$ as well as the self-duality of such codes has also been investigated by Dinh [6]. Conditions on the existence of cyclic self-dual codes of length $2^{r} m$ over $\mathbb{F}_{2^{s}}$ were studied independently by Kai and Zhu [10] and Jia et al. [11]. Jia et al. also determined the existence and the number of cyclic self-dual codes for $q=2^{m}$.

In this paper, we investigate repeated root cyclic and negacyclic codes of length $p^{r} m$ over $\mathbb{F}_{p^{s}}$ with $(m, p)=1$. When $p$ is odd, we give necessary and sufficient conditions on the
existence of negacyclic self-dual codes. When $m=2 m^{\prime}, m^{\prime}$ odd, we determine explicitly the generator polynomials using ring isomorphisms. This provides simple conditions on the existence of negacyclic self-dual codes. We also answer an open problem concerning the number of self-dual cyclic codes given by Jia et al. [11].

## II. Self-dual Negacyclic Codes of Length $m p^{r}$ OVER $\mathbb{F}_{p^{s}}$

Throughout this section, $p$ is an odd prime number and $n=$ $m p^{r}$, with $m$ an integer (odd or even) such that $(m, p)=1$. This section provides conditions on the existence of self-dual negacyclic codes of length $n=m p^{r}$ over $\mathbb{F}_{p^{s}}$. It is well known that negacyclic codes over $\mathbb{F}_{p^{s}}$ are principal ideals generated by the factors of $x^{m p^{r}}+1$. Since $\mathbb{F}_{p^{s}}$ has characteristic $p$, the polynomial $x^{m p^{r}}+1$ can be factored as

$$
\begin{equation*}
x^{m p^{r}}+1=\left(x^{m}+1\right)^{p^{r}} \tag{1}
\end{equation*}
$$

The polynomial $x^{m}+1$ is a monic square free polynomial, hence from [7] Proposition 2.7] it factors uniquely as a product of pairwise coprime monic irreducible polynomials $f_{1}(x), \ldots, f_{l}(x)$. Thus from (1) we obtain the following factorization of $x^{m p^{r}}+1$

$$
\begin{equation*}
x^{m p^{r}}+1=f_{1}(x)^{p^{r}} \ldots f_{l}(x)^{p^{r}} \tag{2}
\end{equation*}
$$

A negacyclic code of length $n=m p^{r}$ over $\mathbb{F}_{p^{s}}$ is then generated by a polynomial of the form

$$
\begin{equation*}
A(x)=\prod f_{i}^{k_{i}} \tag{3}
\end{equation*}
$$

where $f_{i}(x), i \leq l$, are the polynomials given in (2) and $0 \leq$ $k_{i} \leq p^{r}$.

For a polynomial $f(x)=a_{0}+a_{1} x \ldots+a_{r} x^{r}$, with $a_{0} \neq$ 0 and degree $r$ (hence $a_{r} \neq 0$ ), the reciprocal of $f$ is the polynomial denoted by $f^{*}$ and defined as

$$
\begin{equation*}
f^{*}(x)=x^{r} f\left(x^{-1}\right)=a_{r}+a_{r-1} x+\ldots+a_{0} x^{r} \tag{4}
\end{equation*}
$$

If a polynomial $f$ is equal to its reciprocal, then $f$ is called self-reciprocal. We can easily verify the following equalities

$$
\begin{equation*}
\left(f^{*}\right)^{*}=f \text { and }(f g)^{*}=f^{*} g^{*} \tag{5}
\end{equation*}
$$

It is well known (see [6, Proposition 2.4] or [9, Theorem 4.4.9]), that the dual of the negacyclic code generated by $A(x)$ is the negacyclic code generated by $B^{*}(x)$ where

$$
\begin{equation*}
B(x)=\frac{x^{n}+1}{A(x)} \tag{6}
\end{equation*}
$$

Hence we have the following lemma.
Lemma 2.1: A negacyclic code $C$ of length $n$ generated by a polynomial $A(x)$ is self-dual if and only if

$$
A(x)=B^{*}(x)
$$

Denote the factors $f_{i}$ in the factorization of $x^{m}+1$ which are self-reciprocal by $g_{1}, \ldots g_{s}$, and the remaining $f_{j}$ grouped in pairs by $h_{1}, h_{1}^{*}, \ldots, h_{t}, h_{t}^{*}$. Hence $l=s+2 t$, and the factorization given in (2) becomes

$$
\begin{align*}
x^{n}+1= & \left(x^{m}+1\right)^{p^{r}}=g_{1}^{p^{r}}(x) \ldots g_{s}^{p^{r}}(x) \\
& \times h_{1}^{p^{r}}(x) h_{1}^{* p^{r}}(x) \ldots h_{t}^{p^{r}}(x) h_{t}^{* p^{r}}(x) . \tag{7}
\end{align*}
$$

Theorem 2.2: There exists a self-dual negacyclic code of length $m p^{r}$ over $\mathbb{F}_{p^{s}}$ if and only if there is no $g_{i}$ (selfreciprocal polynomial) in the factorization of $x^{m p^{r}}+1$ given in (7). Furthermore, a self-dual negacyclic code $C$ is generated by a polynomial of the following form

$$
\begin{equation*}
h_{1}^{b_{1}}(x) h_{1}^{* p^{r}-b_{1}}(x) \ldots h_{t}^{b_{t}}(x) h_{t}^{* p^{r}-b_{t}}(x) \tag{8}
\end{equation*}
$$

Proof. Assume there exists a negacyclic self-dual code $C$ of length $n=m p^{r}$ over $\mathbb{F}_{p^{s}}$. Hence from (3) the code $C$ is generated by $A(x)=\prod f_{i}^{k_{i}}$, where the $f_{i}$ are factors of $x^{m}+1$. From (7), we can write

$$
A(x)=g_{1}^{a_{1}}(x) \ldots g_{s}^{a_{s}}(x) h_{1}^{b_{1}}(x) h_{1}^{* c_{1}}(x) \ldots h_{t}^{b_{t}}(x) h_{t}^{* c_{t}}(x)
$$

where $0 \leq a_{i} \leq p^{s}$ for $1 \leq i \leq s$, and $0 \leq b_{j} \leq p^{s}$ and $0 \leq c_{j} \leq p^{s}$ for $1 \leq j \leq t$. Let $B(x)=\frac{x^{m p^{s}}+1}{A(x)}$, and substituting $A(x)$ above gives

$$
\begin{aligned}
B(x)= & g_{1}^{p^{r}-a_{1}}(x) \ldots g_{s^{r}}^{p^{r}-a_{s}}(x) \\
& \times h_{1}^{p^{r}-b_{1}}(x) h_{1}^{* p^{r}-c_{1}}(x) \ldots h_{t}^{p^{r}-b_{t}}(x) h_{t}^{* p^{r}-c_{t}}(x)
\end{aligned}
$$

Using (5) repeatedly in the factorization of $H(x)$, we obtain

$$
\begin{aligned}
B^{*}(x)= & g_{1}^{p^{r}-a_{1}}(x) \ldots g_{s^{r}}^{p^{r}-a_{s}}(x) \\
& \times h_{1}^{* p^{r}-b_{1}}(x) h_{1}^{p^{r}-c_{1}}(x) \ldots h_{t}^{* p^{r}-b_{t}}(x) h_{t}^{p^{r}-c_{t}}(x) .
\end{aligned}
$$

Since $C$ is self-dual, from Lemma 2.1 we have that $A(x)=$ $B^{*}(x)$, and then by equating factors of $A(x)$ and $B^{*}(x)$, the powers of these factors must satisfy $a_{i}=p^{s}-a_{i}$ for $1 \leq i \leq s$, and $b_{j}=p^{s}-b_{j}$ for $1 \leq j \leq t$. Equivalently, $p^{s}=2 a_{i}$ for $1 \leq i \leq s$, and $c_{j}=p^{s}-b_{j}$ for $1 \leq j \leq t$. Since $p$ is odd, the last equalities are possible if and only if there is no $g_{i}$ in the factorization of $x^{m p^{s}}+1$ and $c_{j}=p^{s}-b_{j}$, for $1 \leq j \leq t$, i.e, $s=0$ in (7) and $c_{j}=p^{s}-b_{j}$, for $1 \leq j \leq t$. Hence a negacyclic self-dual code is generated by

$$
h_{1}^{b_{1}}(x) h_{1}^{* p^{r}-b_{1}}(x) \ldots h_{t}^{b_{t}}(x) h_{t}^{* p^{r}-b_{t}}(x)
$$

Lemma 2.3: Let $p^{s}$ be an odd prime. Then the following holds
(i) If $p \equiv 1 \bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even, then $x^{2}+1=0$ has a solution $\gamma \in \mathbb{F}_{p^{s}}$.
(ii) If $p \equiv 3 \bmod 4, s$ odd, then $x^{2}+1$ is irreducible in $\mathbb{F}_{p^{s}}$. Proof. Since $p \equiv 1(\bmod 4),-1$ is a quadratic residue in $\mathbb{F}_{p} \subset \mathbb{F}_{p^{s}}$ [9, Lemma 6.2.4]. Thus there exists $\gamma \in \mathbb{F}_{p^{s}}$ such that $\gamma^{2}=-1$. If $p \equiv 3(\bmod 4)$, then $p^{2} \equiv 1(\bmod 4)$, so that -1 is a quadratic residue in $\mathbb{F}_{p^{2}} \subset \mathbb{F}_{p^{s}}$. The proof of (ii) is in [6, Proposition 3.1 (ii)].

## III. Negacyclic Codes of Length $2 m p^{r}$ over $\mathbb{F}_{p^{s}}$

In this section, we consider the structure of negacyclic codes over $\mathbb{F}_{p^{s}}$ of length $2 m p^{r}$. We begin with the following lemma. When $(m, p)=1, m$ an odd integer, Dinh and López-Permouth [7, Proposition 5.1] proved that negacyclic codes of length $m$ are isomorphic to cyclic codes. Batoul et al. [1] proved that under some conditions, there also exists an isomorphism between constacyclic codes and cyclic codes of length $m$. In the following lemma, we prove that there is an isomorphism between cyclic codes and some constacyclic codes with conditions different from those in [1], [7].

Lemma 3.1: Let $p^{s}$ be an odd prime power such that $p \equiv 1$ $\bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even. Then there is a ring isomorphism between the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}-1}$ and the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}-\gamma}$ given by

$$
\mu(f(x))= \begin{cases}f(\gamma x) & \text { if } m \equiv 3 \quad \bmod 4 \\ f(-\gamma x) & \text { if } m \equiv 1 \quad \bmod 4\end{cases}
$$

Furthermore, there is a ring isomorphism between the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}-1}$ and the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}+\gamma}$ given by

$$
\mu(f(x))= \begin{cases}f(-\gamma x) & \text { if } m \equiv 3 \\ f(\gamma x) & \text { if } m \equiv 1 \quad \bmod 4 \\ f\end{cases}
$$

Proof. From the assumptions on $p$ and $s$ in Lemma 2.3, there exists a solution $\gamma$ to $x^{2}+1=0$. We only prove the ring isomorphism between the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}-1}$ and the ring $\frac{\mathbb{F}_{p^{s}}[x]}{x^{m}-\gamma}$. The other isomorphism can easily be obtained in a similar manner. Since $\gamma^{2}=-1$, we have that $\gamma^{m}=\gamma$ if $m \equiv 1 \bmod 4$, and $\gamma^{m}=-\gamma$ if $m \equiv 3 \bmod 4$. Assume that $m \equiv 3 \bmod 4$, so that $\mu f(x)=f(\gamma x)$ for $f(x) \in \mathbb{F}_{p^{s}}[x]$. It is obvious that $\mu$ is a ring homomorphism, hence we only need to prove that $\mu$ is a one-to-one map. For this, let $f(x)$ and $g(x)$ be polynomials in $\mathbb{F}_{p^{s}}[x]$ such that

$$
f(x) \equiv g(x) \quad\left(\bmod x^{m}-1\right)
$$

This is equivalent to the existence of $h(x) \in \mathbb{F}_{p^{s}}[x]$ such that $f(x)-g(x)=h(x)\left(x^{m}-1\right)$, and this equality is true if and only if $f(\gamma x)-g(\gamma x)=h(\gamma x)\left((\gamma x)^{m}-1\right)$ is true. The assumption on $m$ gives that $\gamma^{m}=-\gamma$. Then we have $f(\gamma x)-g(\gamma x)=-\gamma h(\gamma x)\left(x^{m}-\gamma\right)$. This equality is equivalent to $f(\gamma x)-g(\gamma x) \equiv 0 \bmod x^{n}-\gamma$. This means that for $f$ and $g$ in $\mathbb{F}_{p^{s}}[x] /\left\langle x^{m}-1\right\rangle$, we have $f(x)=g(x)$ if and only if $\mu(f(x))=\mu(g(x))$. Hence it follows that $\mu$ is an isomorphism. A similar argument holds with $m \equiv 1 \bmod 4$
for $\mu(f(x))=f(-\gamma x)$.

Theorem 3.2: Let $p^{s}$ be an odd prime power such that $p \equiv$ $1 \bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even, and $n=2 m p^{r}$ be an oddly even integer with $(m, p)=1$. Then a negacyclic code of length $n$ over $\mathbb{F}_{p^{s}}$ is a principal ideal of $\mathbb{F}_{p^{s}}[x] /\left\langle x^{n}+1\right\rangle$ generated by a polynomial of the following form

$$
\prod_{i \in I} f_{i}^{t_{i}}(\gamma x) \prod_{j \in J} f_{j}^{t_{j}}(-\gamma x)
$$

where $f_{i}(x), f_{j}(x)$ are monic irreducible factors of $x^{m}-1$, and $0 \leq t_{i}, t_{j} \leq p^{s}$.
Proof. It suffices to find the factors of $x^{2 m p^{r}}+1$. From Lemma 2.3, $x^{2}+1=0$ has a solution $\gamma \in \mathbb{F}_{p^{s}}$, so $x^{2 m p^{r}}+1$ can be decomposed as $\left(x^{2 m}+1\right)^{p^{r}}=\left(x^{m}+\right.$ $\gamma)^{p^{r}}\left(x^{m}-\gamma\right)^{p^{r}}$. The result then follows from the isomorphisms given in Lemma 3.1.

Example 3.3: In the case $p \equiv 1 \bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even, $n=2 p^{r}$, (i.e. $m=1$ ), there is a unique factor of $x-1$ which is $f(x)=x-1$. Hence from Theorem 3.2, negacyclic codes of length $2 p^{r}$ over $\mathbb{F}_{p^{s}}$ are generated by

$$
\begin{equation*}
C=\left\langle(x-\gamma)^{i}\left(x+\gamma^{j}\right)\right\rangle, \text { where } 0 \leq i, j \leq p^{r} \tag{9}
\end{equation*}
$$

The result given in (9) was also proven in [6, Theorem 3.2].

## A. Self-dual Negacyclic Codes of Length $2 m p^{r}$

The purpose of this section is to provide conditions on the existence of self-dual codes. This is done considering only the length and characteristic. This gives equivalent conditions to those in Theorem 2.2 which are much simpler to verify. We first present an example.

Example 3.4: For $m=1$, we have the following.
(i) If $p \equiv 1 \bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even, then from Theorem 2.2 there exist self-dual codes of length $2 p^{s}$ over $\mathbb{F}_{p^{s}}$ if and only if none of the irreducible factors of $x^{2}+1$ are self-reciprocal. From Lemma 2.3, there is a solution $\gamma$ of $x^{2}+1=0$ in $\mathbb{F}_{p^{s}}$. Hence the irreducible factors of $x^{2}+1$ are $x-\gamma$ and $x+\gamma$. Neither of these polynomials can be self-reciprocal, as we have $(x-\gamma)^{*}=-\gamma(x+\gamma)$ and $(x+\gamma)^{*}=\gamma(x-\gamma)$. Hence by Theorem 2.2 there exist negacyclic self-dual codes of the following form

$$
\left\langle(x-\gamma)^{i}(x+\gamma)^{p^{r}-i}, \text { where } 0 \leq i \leq p^{r}\right.
$$

(ii) If $p \equiv 3 \bmod 4, s$ odd, then from Lemma $2.3 x^{2}+1$ is irreducible in $\mathbb{F}_{p^{s}}$. Furthermore, we have $\left(x^{2}+1\right)^{*}=x^{2}+$ 1. Hence by Theorem 2.2 there are no self-dual negacyclic codes in this case.
The results in Example 3.4 are also given in [6, Corollary 3.3]. We now require the following Lemma.
Lemma 3.5: Let $m$ be an odd integer and $C l_{m}(i)$ the $p^{s}$ cyclotomic class of $i$ modulo $m$. The polynomial $f_{i}(x)$ is the
minimal polynomial associated with $C l_{m}(i)$, hence we have $C l_{m}(i)=C l_{m}(-i)$ if and only if $f_{i}(x)=f_{i}^{*}(x)$.
Proof. Let $\alpha$ be an $m$ th primitive root of unity. The elements of $C l(i)$ are such that $\alpha^{i}$ is a root of a monic irreducible polynomial $f_{i}(x)=a_{0}+a_{1} x+\ldots+x^{r}$. Hence $f_{i}^{*}(x)=x^{\operatorname{deg} f_{i}} f_{i}\left(x^{-1}\right)$ has $\alpha^{-i}$ as a root. Therefore $C l_{m}(i)=C l_{m}(-i)$ if and only if the polynomials $f_{i}(x)$ and $f_{i}^{*}(x)$ are monic with the same degree and the same roots, and hence are equal.

Lemma 3.6: Let $m$ be an odd integer and $p$ a prime number. Then $\operatorname{ord}_{m}\left(p^{s}\right)$ is even if and only if there exists a cyclotomic class $C l_{m}(i)$ which satisfies $C l_{m}(i)=C l_{m}(-i)$.
Proof. Assume that $\operatorname{or} d_{m}\left(p^{s}\right)$ is even. We start with the case where $m=q^{\alpha}$ is a prime power. We first prove the following implication

$$
\operatorname{ord}_{q^{\alpha}}\left(p^{r}\right) \text { is even } \Rightarrow \operatorname{ord}_{q}\left(p^{r}\right) \text { is even. }
$$

Assume that $\operatorname{ord}_{q^{\alpha}}\left(p^{r}\right)$ is even and $\operatorname{ord}_{q}\left(p^{r}\right)$ is odd. Then there exists odd $i>0$ such that $p^{r i} \equiv 1 \bmod q \Leftrightarrow p^{r i}=$ $1+k q$. Hence $p^{r i q^{\alpha-1}}=(1+k q)^{q^{\alpha-1}} \equiv 1 \bmod q^{\alpha}$, because $(1+k q)^{q^{\alpha-1}} \equiv 1+k q^{\alpha} \bmod q^{(\alpha+1)}$ (the proof of the last equality can be found in [5, Lemma 3.30]). Therefore we have that

$$
\begin{equation*}
p^{r i q^{\alpha-1}} \equiv 1 \quad \bmod q^{\alpha} \tag{10}
\end{equation*}
$$

If both $i$ and $q^{\alpha-1}$ are odd, then $\operatorname{ord}_{q^{\alpha}}\left(p^{r}\right)$ is odd, which is absurd. Then it must be that $\operatorname{ord}_{q}\left(p^{r}\right)$ is even, so there exists some integer $j$ such that $0<j<\operatorname{ord}_{q}\left(p^{r}\right)$ and $p^{j} \equiv-1 \bmod q$. Therefore we have $p^{r j q^{\alpha-1}} \equiv-1$ $\bmod q^{\alpha}$, which gives that $C l_{q^{\alpha}}(1)=C l_{q^{\alpha}}(-1)$. Then for all $i$ in the cyclotomic classes we have $C l_{q^{\alpha}}(i)=C l_{q^{\alpha}}(-i)$. Assume now that $m=p_{1} p_{2}$ such that $\left(p_{1}, p_{2}\right)=1$ and $\operatorname{ord}_{m}\left(p^{s}\right)$ is even. Since $m=p_{1} p_{2}$, we have that $\operatorname{ord}_{m}\left(p^{s}\right)=\operatorname{lcm}\left(\operatorname{ord}_{p_{1}}\left(p^{s}\right), \operatorname{ord}_{p_{2}}\left(p^{s}\right)\right)$ is even, and hence either $\operatorname{ord}_{p_{1}}\left(p^{s}\right)$ or $\operatorname{ord}_{p_{2}}\left(p^{s}\right)$ is even. Assume that $\operatorname{ord}_{p_{1}}\left(p^{s}\right)$ is even, then there exists $1 \leq k \leq \operatorname{ord}_{p_{1}}\left(p^{s}\right)$ such that $\left(p^{s}\right)^{k} \equiv-1 \bmod p_{1}$. Therefore $\left(p^{s}\right)^{k}\left(m-p_{2}\right) \equiv-\left(m-p_{2}\right) \bmod m$, with $k \leq \operatorname{ord}_{p_{1}}\left(p^{s}\right)$, and hence $C l_{m}\left(m-p_{2}\right)=C l_{m}\left(-\left(m-p_{2}\right)\right)$. The same result is obtained for $m=p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}}$. Conversely, assume there exists a class for which $C l_{m}(i)=C l_{m}(-i)$. Then the elements of $C l_{m}(i)$ are $\pm i q^{j}$ for some $j$, so $C l(i)$ contains an even number of elements. On the other hand the size of each $q$ cyclotomic class is a divisor of $\operatorname{ord}_{m}\left(p^{s}\right)$ [9, Theorem 4.1.4]. This gives that $\operatorname{ord}_{m}\left(p^{s}\right)$ is even.

Theorem 3.7: Let $p^{s}$ be an odd prime power such that $p \equiv$ $1 \bmod 4, s$ any integer or $p \equiv 3 \bmod 4$ and $s$ even, and $n=2 m p^{r}$ be an oddly even integer with $(m, p)=1$. Then there exists a negacyclic self-dual code of length $2 m p^{s}$ over $\mathbb{F}_{p^{s}}$ if and only if $\operatorname{ord}_{m}\left(p^{s}\right)$ is odd.
Proof. Under the hypothesis on $p, s$ and $m$ we have from Theorem 3.2 that the polynomial $x^{2 m p^{r}}+1=$ $\prod f_{i}(\gamma x)^{p^{r}}(x) \prod f_{j}^{p^{r}}(-\gamma x)$, where $f_{i}(x)$ and $f_{j}(x)$ are the
monic irreducible factors of $x^{m}-1$ in $\mathbb{F}_{p^{s}}$. By Lemma 3.6 $\operatorname{ord}_{m}\left(p^{s}\right)$ is odd if and only if there is no cyclotomic class such that $C l_{m}(i)=C l_{m}(-i)$. From Lemma 3.5, this is equivalent to saying that there are no irreducible factors of $x^{m}-1$ such that $f_{i}(x)=f_{i}^{*}(x)$. From the ring isomorphisms given in Lemma 3.1 we have that $f_{i}(x) \neq f_{i}^{*}(x)$ for all $i$ is true if and only if $f_{i}(\gamma x) \neq f_{i}^{*}(\gamma x)$ and $f_{j}(-\gamma x) \neq f_{j}^{*}(-\gamma x)$ are true. Then from Theorem 2.2 self-dual negacyclic codes exist.

Example 3.8: A self-dual negacyclic code of length 70 over $\mathbb{F}_{5}$ does not exist. There is no self-dual negacyclic code of length 30 over $\mathbb{F}_{9}$, but there is a self-dual code over $\mathbb{F}_{9}$ of length 126.

Lemma 3.9: Let $p$ and $q$ be distinct odd primes such that $p$ is not a quadratic residue modulo $q$. Then we have the following.
(i) If $q \equiv 1(\bmod 4)$, then $\operatorname{ord}_{q}(p) \equiv 0(\bmod 4)$.
(ii) If $q \equiv 3(\bmod 4)$, then $\operatorname{ord}_{q}(p) \equiv 0(\bmod 2)$.

Proof. Assume that $p$ is not a quadratic residue modulo $q$. Then from [9, Lemma 6.2.2] $\operatorname{ord}_{q}(p)$ is not a divisor of $\frac{p-1}{2}$, so from Fermat's Little Theorem $\operatorname{ord}_{q}(p)=q-1$. Hence $\operatorname{ord}_{q}(p) \equiv 0(\bmod 4)$ since $q \equiv 1(\bmod 4)$. If $q \equiv 3$ $(\bmod 4)$, then $\operatorname{ord}_{q}(p)=q-1 \equiv 0(\bmod 2)$.

Lemma 3.10: Let $n$ be a positive integer and $q$ a prime power such that $(q, n)=1$. Then we have the following.
(i) If $\operatorname{ord}_{n}(q)$ is even, then $\operatorname{ord}_{n}\left(q^{2}\right)=\frac{\operatorname{ord}_{n}(q)}{2}$.
(ii) If $\operatorname{ord}_{n}(q)$ is odd, then $\operatorname{ord}_{n}\left(q^{2}\right)=\operatorname{ord}_{n}(q)$.

Proof. Let $r=\operatorname{ord}_{n}(q)$ and $r^{\prime}=\operatorname{ord}_{n}\left(q^{2}\right)$. Then we have $q^{2 r^{\prime}} \equiv 1 \bmod n$, which implies that $r \mid 2 r^{\prime}$. Since $r$ is even, we have $\left(q^{2}\right)^{\frac{r}{2}}=q^{r} \equiv 1 \bmod n$, and then $r^{\prime} \left\lvert\, \frac{r}{2}\right.$. Hence we obtain that $r^{\prime}=\frac{r}{2}$. This proves part (i). For part (ii), assume again that $r=\operatorname{ord}_{n}(q)$ is odd and $r^{\prime}=\operatorname{ord}_{n}\left(q^{2}\right)$. We then have that $r \mid 2 r^{\prime}$, and since $r$ is odd it must be that $r \mid r^{\prime}$. On the other hand, we have $q^{2 r} \equiv n$, so that $r^{\prime} \mid r$, and therefore $r=r^{\prime}$.

Corollary 3.11: Let $p$ and $q$ be two distinct primes such that $p$ is not a quadratic residue modulo $q$. Then if $q \equiv 1$ $(\bmod 4)$ and $p \equiv 1(\bmod 4)$, there is no self-dual negacyclic code of length $2 p q^{\alpha}$ over $\mathbb{F}_{p}$ or $\mathbb{F}_{p^{2}}$.
Proof. From Lemma 3.9, if $q \equiv 1(\bmod 4)$ and $p$ is not a quadratic residue modulo $q$, then $\operatorname{ord}_{q}(p) \equiv 0(\bmod 4)$. Hence from Lemma 3.10 $\operatorname{ord}_{q}\left(p^{2}\right)$ is even. Then the proof of Lemma 3.6 implies that $\operatorname{ord}_{q^{\alpha}}\left(p^{2}\right)$ is even. Hence from Theorem 3.7 there are no self-dual negacyclic codes of length $2 q^{\alpha} p$ over $\mathbb{F}_{p}$ or $\mathbb{F}_{p^{2}}$.

Example 3.12: For $p=5, q=13$ and $q=17$ satisfy the hypothesis of Corollary 3.11 Hence there are no self-dual negacyclic codes over $\mathbb{F}_{5}$ and $\mathbb{F}_{25}$ with lengths 130 , 170 or 1690.

## IV. Repeated Root Cyclic Codes

It is well known that the cyclic codes of length $n$ over $\mathbb{F}_{p^{s}}$ are principal ideals of $\mathbb{F}_{p^{s}}[x] /\left(x^{n}-1\right)$, and these ideals are generated by the monic factors of $x^{n}-1$. Hence the importance of the decomposition of the polynomial $x^{n}-1$ over $\mathbb{F}_{p^{s}}$.

Let $n=2 m p^{r}$, with $m$ an odd integer such that $(m, p)=1$. Then we have the decomposition $x^{n}-1=\left(x^{2 m}-1\right)^{p^{r}}=$ $\left(x^{m}-1\right)^{p^{s}}\left(x^{m}+1\right)^{p^{r}}$. Since $(m, p)=1$, the polynomials $x^{m}-1$ and $x^{m}+1$ factor uniquely as the product of monic irreducible pairwise coprime polynomials given by $x^{m}-1=$ $\prod_{i=1}^{k} f_{i}$ and $x^{m}+1=\prod_{j=1}^{l} g_{j}$. This is due to the fact that $(m, p)=1$, so the roots are simple [7, Proposition 2.7]. Let $f_{i}(x)$ be a monic irreducible divisor of $x^{m}-1$. Then there exists $h(x) \in \mathbb{F}_{p^{s}}[x]$ such that $f_{i}(x) h(x)=x^{m}-1$, and hence $f_{i}(-x) h(-x)=(-x)^{m}-1=-\left(x^{m}+1\right)$. Therefore $f_{i}(-x)$ is a monic irreducible divisor of $x^{m}+1$. This gives that the factorization of $x^{n}-1$ is

$$
x^{2 m p^{r}}-1=\prod_{i=1}^{k}\left(f_{i}(x) f_{i}(-x)\right)^{p^{r}}
$$

Hence a cyclic code of length $n=2 m p^{r}$ over $\mathbb{F}_{p^{s}}$ is of the form

$$
\left.C=\left\langle\prod\left(f_{i}(x)\right)^{\alpha_{i}} \prod\left(f_{j}(-x)\right)^{\beta_{j}}\right)\right\rangle
$$

where $0 \leq \alpha_{i}, \beta_{j} \leq p^{r}, 1 \leq i, j \leq k$, and $f_{i}, i \leq k$ is an irreducible factor of $x^{m}-1$. This gives the following result.

Proposition 4.1: For $p$ an odd prime, the cyclic codes of length $n=2 m p^{r}, m$ an odd integer such that $(m, p)=1$, are generated by

$$
\left\langle\prod\left(f_{i}(x)\right)^{\alpha_{i}} \prod\left(f_{j}(-x)\right)^{\beta_{j}}\right\rangle
$$

where $0 \leq \alpha_{i}, \beta_{j} \leq p^{r}, 1 \leq i, j \leq k$, and $f_{i}, i \leq k$, is a monic irreducible factor of $x^{m}-1$.

## A. The Number of Cyclic Self-dual Codes

It has been proven [6], [10], [11] that cyclic self-dual codes exist if and only if the characteristic is 2 . Since a self-dual cyclic code must have even length and characteristic 2 , cyclic self-dual codes have repeated roots. In [11, Corollary 2], Jia et al. gave the number of self-dual cyclic codes in some cases. The remainder of this characterization was left as an open problem, namely the case when the length of the code contains a prime factor congruent to $1 \bmod 8$. The following proposition is used in answering this problem.

Proposition 4.2: Let $p \equiv 1 \bmod 8$ be an odd prime number, and $m$ be an odd number. Then we have the following implication

$$
\operatorname{ord}_{p}(2)=2^{k} e \Rightarrow \forall 0 \leq l \leq k, \operatorname{ord}_{p}\left(2^{2^{l}}\right)=2^{k-l} e
$$

Proof. Since $p \equiv 1 \bmod 8$, from [9, Lemma 6.2.5] 2 is a quadratic residue modulo $p$. Hence $\operatorname{ord}_{p}(2) \left\lvert\, \frac{p-1}{2}\right.$, i.e., $\operatorname{ord}_{p}(2)=2^{k} e$ for some $k>0$. Then from Lemma 3.10 (i) we have that $\operatorname{ord}_{p}\left(2^{2}\right)=2^{k-1} e$. Using the same argument $l$ times, the result follows.

Corollary 4.3: Let $n=2^{r} p^{\alpha}$. Then there is a unique cyclic self-dual code of length $n$ over $\mathbb{F}_{2^{s}}$ generated by $g(x)=\left(x^{p^{\alpha}}+1\right)^{2^{r-1}}$ in the following cases
(i) $p \equiv 3 \bmod 8, s$ odd,
(ii) $p \equiv 5 \bmod 8, s$ odd or $s \equiv 2 \bmod 4$,
(iii) $p \equiv 1 \bmod 8$ and $\operatorname{ord}_{p}(2)=2^{k} e$, and $s=2^{l}$,

$$
0<l<k .
$$

Proof. Parts (i) and (ii) follow from [11, Proposition 2]. When $p \equiv 1 \bmod 8$ and $\operatorname{ord}_{p}(2)=2^{k} e$, for $s=2^{l}$ with $0<l<k$, from Proposition 4.2 we have that $\operatorname{ord}_{p}\left(2^{s}\right)$ is an even integer. Hence from [11, Theorem 4] there is a unique self-dual code generated by $g(x)=\left(x^{p^{\alpha}}+1\right)^{2^{r-1}}$.

Example 4.4: Let $r$ and $\alpha$ be positive integers.
(i) For $p=3$ and $s=5$, the polynomial $g(x)=\left(x^{3^{\alpha}}+1\right)^{2^{r-1}}$ generates the unique self-dual cyclic code of length $2^{r} 3^{\alpha}$ over $\mathbb{F}_{2^{5}}$.
(ii) For $p=5$ and $s=6$, the polynomial $g(x)=\left(x^{5^{\alpha}}+1\right)^{2^{r-1}}$ generates the unique self-dual cyclic code of length $2^{r} 5^{\alpha}$ over $\mathbb{F}_{2^{6}}$.
(iii) For $p=17$, $\operatorname{ord}_{17}(2)=2^{3}$, so $l=2$. Then $g(x)=$ $\left(x^{17^{\alpha}}+1\right)^{2^{r-1}}$ generates the unique self-dual cyclic code of length $2^{r} 17^{\alpha}$ over $\mathbb{F}_{2^{2}}$.

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