Linear Programming Upper Bounds on Permutation Code Sizes From Coherent Configurations Related to the Kendall Tau Distance Metric

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Abstract—Recent interest on permutation rank modulation shows the Kendall tau metric as an important distance metric. This note documents our first efforts to obtain upper bounds on optimal code sizes (for said metric) ala Delsarte's approach. For the Hamming metric, Delsarte's seminal work on powerful linear programming (LP) bounds have been extended to permutation codes, via association scheme theory. For the Kendall tau metric, the same extension needs the more general theory of coherent configurations, whereby the optimal code size problem can be formulated as an extremely huge semidefinite programming (SDP) problem. Inspired by recent algebraic techniques for solving SDP's, we consider the dual problem, and propose an LP to search over a subset of dual feasible solutions. We obtain modest improvement over a recent Singleton bound due to Barg and Mazumdar. We regard this work as a starting point, towards fully exploiting the power of Delsarte's method, which are known to give some of the best bounds in the context of binary codes.

Index Terms—association schemes, coherent configurations, permutations, linear programming, semidefinite programming

I. INTRODUCTION

A permutation code is designed to only allow certain pairwise distances between any two codewords. These codes have been studied in various contexts, *e.g.*, group codes [1], signal modulation [2], [3], vector quantization [4], rank modulation [5], [6], cost-constrained transpositions [7], etc. This work is motivated by a recent study on a fundamental coding problem. In [6] they looked at optimal code sizes with respect to the Kendall tau distance metric. This metric is important to rank modulation and its applications, *e.g.*, flash memories.

For binary codes, Delsarte's optimization-based methods [8] give some of the best known bounds [9]. For permutation codes, we observe during initial experiments (for very small lengths) that Delsarte-like methods outperform Hamming (sphere packing) bounds [6], [10]. Our interest is to investigate, if this improvement carries over for larger codes. Tarnanen extended Delsarte's work over to permutation codes [11], however only for the Hamming metric (and other metrics with similar symmetries). The novelty here is to consider the Kendall tau metric, and as pointed out in [6], lacks required symmetry to straight-forwardly apply Tarnanen's techniques.

Delsarte's (and Tarnanen) techniques are based on association schemes, from which linear programming (LP) formulations (of the optimal code size problem) are obtained. For the Kendall tau metric, one needs to consider the more general

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theory of coherent configurations (CC), which instead deliver semidefinite programming (SDP) formulations. The matrices in these SDP's turn out to be of unwieldy size, but recent work [12], [13], [14] suggest possible approaches. One may exploit the algebraic structure of the CC's, to only work with block-diagonalized (and possibly much smaller) versions of these matrices. To our knowledge, such recent techniques are new in the area of permutation codes. However, the solution is not straight-forward. As code lengths increase, the CC's (related to the Kendall tau metric) become huge quickly, motivating the techniques presented in this preliminary report.

While we believe to be presently unable to fully exploit the power of SDP bounds, we show some initial success. We consider the dual problem (also a SDP), and use an LP to search over a subset of feasible solutions. We obtained modest improvement over a recently published Singleton bound in [6]. The reduced complexity allows us to compute up to permutation codes of length 11 (where the matrices were previously of order 11 factorial). Certain bottlenecks, if solved, could allow computation for longer codes. As it stands, our proposed LP bounds perform poorer than known Hamming bounds [6], and it remains to see how far sophisticated SDP-based approaches can ultimately bring us. This note aims to motivate new research to resolve this open question.

II. BACKGROUND

A. Optimal Code Size Problem and Two Metrics

Let S_n denote the *symmetric group* on a set $\{1, 2, ..., n\}$ and $\operatorname{dist}(,)$ be a distance metric on S_n . A subset \mathcal{V} of S_n is an (n, δ_{\min}) **permutation code** (with respect to $\operatorname{dist}(,)$), if for any $g, h \in \mathcal{V}$ such that $g \neq h$, we have $\operatorname{dist}(g, h) \geq \delta_{\min}$.

Definition 1 (Optimal code size problem). Let dist(,) be a distance metric on the symmetric group S_n . Let $\delta_{min} \geq 1$. The following problem is the optimal code size problem.

$$\max_{\mathcal{V} \subset \mathcal{S}_n} \# \mathcal{V} \tag{1}$$

s.t. $\operatorname{dist}(g,h) \geq \delta_{\min}$ for all $g,h \in \mathcal{V}$ where $g \neq h$,

and #V denotes the cardinality of the set V. Denote $\mu(n, \delta_{min})$ to be the maximal cardinality achieved by (n, δ_{min}) codes, i.e., $\mu(n, \delta_{min})$ equals the optimal value of the above problem.

The *image* of i by g is denoted g(i). The *inverse* of g is denoted g^{-1} . The *product* of permutations g and h is

denoted gh, whereby (gh)(i) = g(h(i)). Most literature (e.g., Tarnanen [11]) consider the **Hamming metric**

$$\operatorname{dist}(g,h) \stackrel{\Delta}{=} \#\{1 \le x \le n : (g^{-1}h)(i) \ne i\}, \tag{2}$$

i.e., the Hamming distance $\operatorname{dist}(g,h)$ equals the number of *moved* points of $g^{-1}h$. For the *direct product* group $\mathcal{S}_n \times \mathcal{S}_n$, define its action on \mathcal{S}_n , as $(\alpha,\beta) \cdot g \stackrel{\Delta}{=} \alpha g \beta^{-1}$, where $(\alpha,\beta) \in \mathcal{S}_n \times \mathcal{S}_n$ and $g \in \mathcal{S}_n$. For any subgroup \mathcal{G} of $\mathcal{S}_n \times \mathcal{S}_n$, a metric $\operatorname{dist}(,)$ on \mathcal{S}_n is \mathcal{G} -invariant if for any $g,h \in \mathcal{S}_n$, we have $\operatorname{dist}(g,h) = \operatorname{dist}((\alpha,\beta) \cdot g,(\alpha,\beta) \cdot h)$ for all $(\alpha,\beta) \in \mathcal{G}$. The Hamming metric (2) can be verified to be $(\mathcal{S}_n \times \mathcal{S}_n)$ -invariant.

The length of a permutation g, denoted length(g), equals the minimum integer r satisfying $g = \alpha_1 \alpha_2 \dots \alpha_r$ whereby α_i are *adjacent transpositions* in \mathcal{S}_n . For rank modulation [5], [6] we consider the **Kendall tau metric**, given as

$$\operatorname{dist}(g,h) \stackrel{\Delta}{=} \operatorname{length}(g^{-1}h).$$
 (3)

There exists a unique element w_0 , termed the **longest element**, that satisfies $\operatorname{length}(w_0) = n(n-1)/2$. Then w_0 is an involution, *i.e.*, $w_0^{-1} = w_0$, and $\operatorname{dist}(g,h) = \operatorname{dist}(gw_0,hw_0)$, see [15], p. 119. Denote a subgroup $\{e,w_0\}$ of \mathcal{S}_n by Ψ_n , where e is the identity element of \mathcal{S}_n . In general, the Kendall tau metric is $(\mathcal{S}_n \times \Psi_n)$ -invariant.

A permutation g written as g=(123) means g(1)=2, g(2)=3 and g(3)=1. Note (12),(23),(13) are transpositions, in particular the first two are adjacent transpositions.

Example 1. Consider S_3 with elements e, (12), (23), (132), (13), and the Hamming metric. The minimum distance between any two non-equal permutations is 2. For $\delta_{min} = 1$ and 2 we have $\mu(n, \delta_{min}) = \#S_3$. For $\delta_{min} = 3$ the code \mathcal{V} with the optimal size satisfies $\mathcal{V} = \{e, (123), (132)\}$. Check $\operatorname{dist}(e, (123)) = \operatorname{dist}(e, (132)) = 3$, and $\operatorname{dist}((123), (132)) = \operatorname{dist}(e, (123)^{-1}(132)) = \operatorname{dist}(e, (123)) = 3$.

The minimum possible non-zero Kendall tau pairwise distance is 1. For $\delta_{min}=1$, we have $\mu(n,\delta_{min})=\#\mathcal{S}_3$ as before. For $\delta_{min}=2$ the optimal code satisfies $\mathcal{V}=\{e,(123),(132)\}$. Check $\mathrm{dist}(e,(123))=\mathrm{length}((123))=2$, where (123)=(12)(23). For $\delta_{min}=3$ the optimal code satisfies $\mathcal{V}=\{e,(13)\}$, where (13) is the longest element w_0 in \mathcal{S}_3 and $\mathrm{length}((13))=3$ (here (13)=(12)(23)(12)).

B. Coherent Configurations (CC)

We now describe objects used to formulate relaxations of (1). For a subgroup $\mathcal G$ of $\mathcal S_n \times \mathcal S_n$, define an **induced action** of $\mathcal G$ on $\mathcal S_n \times \mathcal S_n$, as $g \cdot (x,y) \stackrel{\Delta}{=} (g(x),g(y))$ where $g \in \mathcal G$ and $x,y \in \mathcal S_n$. An orbit of this induced action is termed an **orbital**. These orbitals $\Delta_1,\Delta_2,\cdots,\Delta_d$ of the induced action partition $\{(x,y):x,y \in \mathcal S_n\}=\cup_{i=1}^d \Delta_i$. If the action of $\mathcal G$ on $\mathcal S_n$ is *transitive*, we use the convention $\Delta_1=\{(x,x):x \in \mathcal S_n\}$. For each orbital Δ_i , we correspond an **adjacency matrix** A_i as follows. Here A_i is a 0-1 matrix, whose rows/columns are indexed by $\mathcal S_n$, and we have $(A_i)_{x,y}=1$ if and only if $(x,y)\in \Delta_i$. Let A_i^T denote the transposed matrix of A_i .

Theorem 1 (c.f. [16], p. 52). Let \mathcal{G} be a group which acts on \mathcal{S}_n transitively. For the induced action of \mathcal{G} on $\mathcal{S}_n \times \mathcal{S}_n$, the adjacency matrices A_i corresponding to the d orbitals Δ_i , satisfy

i) A_1 equals the identity matrix.

- ii) the sum $\sum_{i=1}^{d} A_i$ equals the all ones matrix.
- iii) for any $\overline{A_i}$, there exists some $A_{i'}$ that satisfies $A_i^T = A_{i'}$.
- iv) for any $i, j \in \{1, 2, \dots, d\}$, there exists numbers p_{ij}^k that satisfy $A_i A_j = \sum_{k=1}^d p_{ij}^k A_k$.

A **coherent configuration** (CC) denoted $(\mathcal{G}, \mathcal{S}_n)$, refers to the set $\{A_1, A_2, \cdots, A_d\}$ of corresponding adjacency matrices. A CC with the additional property $p_{ij}^k = p_{ji}^k$ is an association scheme; in this special case, Delsarte showed how combinatorial properties can deliver linear programming (LP) bounds [8]. Construct two CC's related to the \mathcal{G} -invariances of the Hamming and Kendall tau metrics. For the former metric, set $\mathcal{G} = \mathcal{S}_n \times \mathcal{S}_n$ and call $(\mathcal{S}_n \times \mathcal{S}_n, \mathcal{S}_n)$ the **conjugacy** CC - the name comes from [11]. For the latter metric, set $\mathcal{G} = \mathcal{S}_n \times \Psi_n$ and term $(\mathcal{S}_n \times \Psi_n, \mathcal{S}_n)$ the **length** CC. Let $\mathbb{R}^{\mathcal{S}_n \times \mathcal{S}_n}$ denote the set of real matrices and index set \mathcal{S}_n . Write $A_{\mathcal{S}_n,i}$ and $A_{\Psi_n,i}$ for adjacency matrices of conjugacy, and length CC.

Example 2. The matrices in $\mathbb{R}^{S_3 \times S_3}$ corresponding to the conjugacy and length CC (the indexing on S_3 is done in the same order that appears in Eg. 1), are written as follows. First $A_{S_n,1} = A_{\Psi_n,1} = I$, where I is the identity matrix. Next

$$A_{\mathcal{S}_{n},2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \ A_{\Psi_{n},2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

By Theorem 1, $A_{S_n,3} = J - I - A_{S_n,2}$, here J has all ones. Finally, it so happens that we get $A_{\Psi_n,3} = A_{S_n,3}$ and $A_{\Psi_n,4} = A_{S_n,2} - A_{\Psi_n,2}$. Matrices $A_{S_n,1}$ to $A_{S_n,3}$ corresponding to Hamming distances 0,2,3, and $A_{\Psi_n,1}$ to $A_{\Psi_n,4}$ to Kendall tau distances 0,1,2,3.

The focus here is on the length CC, related to the Kendall tau metric. The conjugacy CC (related to the Hamming metric) is actually an association scheme, and is treated in [11]; the recollection is because of connections exploited later.

III. SEMIDEFINITE PROGRAMMING (SDP) BOUNDS

A symmetric matrix M in $\mathbb{R}^{S_n \times S_n}$ is positive semidefinite, if all its eigenvalues are non-negative. We now use CC's to formulate the relaxation of the optimal code size problem. By iii), Theorem 1, a set $\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_{\tilde{d}}\}$ of **symmetrized adjacency matrices** are obtained, whereby $\tilde{d} \leq d$. If A_i is not symmetric, then find $A_{i'}$ such that $A_i^T = A_{i'}$, and set $\tilde{A}_{i} = A_{i} + A_{i'}$. Similarly the **symmetrized orbitals** $\tilde{\Delta}_{i}$ are obtained by setting $\hat{\Delta}_j = \Delta_i \cup \Delta_{i'}$ if $\hat{A}_j = A_i + A_{i'}$. Note both (g,h) and (h,g) belong to the same $\hat{\Delta}_j$, and $\operatorname{dist}(g,h) =$ $\operatorname{dist}(h, g)$. Thus by \mathcal{G} -invariance of $\operatorname{dist}(h, g)$ set $\delta_i = \operatorname{dist}(g, g)$ for any $(g,h) \in \tilde{\Delta}_j$, since $\operatorname{dist}(g,h) = \operatorname{dist}(g',h')$ for any $(g,h),(g',h')\in \tilde{\Delta}_j$. The values δ_j are called **orbit-distances** (with respect to a \mathcal{G} -invariant metric dist(,)). If \mathcal{G} acts transitively on S_n , then by convention $\Delta_1 = \{(g,g) : g \in S_n\},\$ thus $\delta_i \geq 1$ for all $j \geq 2$. The properties of the CC's can simplify the following optimizations.

Definition 2 (Primal SDP, $(\mathcal{G}, \mathcal{S}_n)$ and δ_{\min}). Let \mathcal{G} be a group which acts on \mathcal{S}_n transitively and dist(,) a \mathcal{G} -invariant distance metric on \mathcal{S}_n . Let δ_j be the orbit-distances w.r.t.

 $\mbox{TABLE I} \\ \mbox{[Initial Experiments] SDP Bounds for } 3 \leq n \leq 5$

		n =	= 3		n = 5						
$\delta_{ ext{min}}$	b_1^*	SB	НВ	Search	$\delta_{ ext{min}}$	b_1^*	SB	НВ	Search		
1	6	6	6	6	1	120	120	120	120		
2 3	3 2	6	6	3	2	60	120	120	-		
3	2	2	2	2	3	22	120	24	-		
	n=4					14	120	24	-		
$\delta_{ ext{min}}$	b_1^*	SB	$^{\rm HB}$	Search	5	7	24	8	-		
1	24	24	24	24	6	5	24	8	3		
2 3	12	24	24	12	7	3	24	4	2		
3	5	24	6	5	8	2	6	4	2		
4	3	6	6	3	9	2	6	2	2		
5	2	6	2	2	10	2	2	2	2		
6	2	2	2	2							

[†] Singleton bound (SB), published in [6], equation (5).

[‡] Hamming bound (HB) from ball-size estimates, see [10], [6]. *Note: Above table created by taking numerical floor.*

 $(\mathcal{G}, \mathcal{S}_n)$ and $\operatorname{dist}(,)$. Define the semidefinite programming (SDP) problem correp. to $(\mathcal{G}, \mathcal{S}_n)$ and some $\delta_{min} \geq 1$, as

$$\max_{M \in \mathbb{R}^{S_n \times S_n}} \operatorname{Tr}(JM) \tag{4}$$

s.t. M is positive semidefinite, and $\operatorname{Tr}(M)=1$,

$$\operatorname{Tr}(\tilde{A}_{j}M) \geq 0$$
, for $2 \leq j \leq \tilde{d}$,
 $\operatorname{Tr}(\tilde{A}_{i}M) = 0$, for $2 \leq j \leq \tilde{d}$ with $\delta_{i} < \delta_{min}$,

where \tilde{A}_j is a corresponding symmetrized adjacency matrix, J is the all-one matrix, and Tr is the trace function.

Proposition 1. Let \mathcal{G} be a group which acts on \mathcal{S}_n transitively and $\operatorname{dist}(,)$ a \mathcal{G} -invariant distance metric on \mathcal{S}_n . Let $\delta_{\min} \geq 1$. Then, the optimal objective value of (4) upper bounds the optimal objective value of (1) for $\operatorname{dist}(,)$ and δ_{\min} .

The SDP (4) is a relaxation of the optimal code size problem (1), see appendix for proof. The optimal value of the SDP (4) is at most $\#S_n$, as for any feasible M, we have $\text{Tr}(JM) \leq \text{Tr}(J) = \#S_n$. Software like SeDuMi [17] can solve SDP's.

Example 3. Consider $\mathcal{G} = \mathcal{S}_3 \times \Psi_3$, whereby the Kendall tau metric is \mathcal{G} -invariant. Let $\tilde{\Delta}_1$ to $\tilde{\Delta}_4$ correspond to $A_{\Psi_n,1}$ to $A_{\Psi_n,4}$ (all symmetric). Using SeDuMi we solve for $\delta_{\min} = 1$, 2 and 3, and get the optimal solutions

$$\frac{1}{6} \cdot J, \quad \frac{1}{6} \cdot (A_{\Psi_n,1} + A_{\Psi_n,3}), \quad \frac{1}{6} \cdot (A_{\Psi_n,1} + A_{\Psi_n,4}).$$
 which correspond to optimal objective values 6, 3 and 2.

We need to work with the dual problem to (4).

Definition 3 (Dual problem, $(\mathcal{G}, \mathcal{S}_n)$ and δ_{\min}). Let \mathcal{G} be a group which acts on \mathcal{S}_n transitively and $\mathrm{dist}(,)$ a \mathcal{G} -invariant distance metric on \mathcal{S}_n . Let δ_j be the orbit-distances w.r.t. $(\mathcal{G}, \mathcal{S}_n)$ and $\mathrm{dist}(,)$. Let \tilde{A}_j be a corresponding symmetrized adjacency matrix to $(\mathcal{G}, \mathcal{S}_n)$. Let $\delta_{\min} \geq 1$. Define the following

$$\min_{\substack{(b_1,b_2,...,b_{\tilde{d}}) \in \mathbb{R}^{\tilde{d}} \\ s.t. \ b_j \leq 0 \ for \ 2 \leq j \leq \tilde{d} \ with \ \delta_j \geq \delta_{min},}$$

$$\sum_{j=1}^{\tilde{d}} b_j \tilde{A}_j - J \ is \ positive \ semidefinite,$$
(5)

to be the dual problem of the SDP in Definition 4.

Any feasible solution b in $\mathbb{R}^{\tilde{d}}$ to the dual program (5), provides an upper bound to the optimal objective value of the SDP (4), see [14]; we have the following chain of inequalities

$$Tr(JM^*) \le b_1^* \le b_1,\tag{6}$$

n	Len.	Conj.	\tilde{d}_{Θ_n}	n	Len.	Conj.	\tilde{d}_{Θ_n}
4	13	5	8	8	10558	22	171
5	45	7	21	9	92126	30	860
6	230	11	34	10	912908	42	1052
7	1388	15	122	11	9998008	56	7578

where M^* and b^* are optimal solns. of (4) and (5), resp.

Our interest in SDP bounds is motivated by initial experimentation. Table I shows optimal objective values of (5) obtained using SeDuMi, for (small) n=3 to 5. We compare with two other bounds, i) a *Singleton bound* (SB) recently published in [6], and ii) a *Hamming bound* (HB) obtained by sphere packing, see [6]. Ball-sizes for HB were obtained from *exact* numbers of permutations with k inversions [10]. For cases shown, SDP bounds always perform the best, with some tightness verified by limited *exhaustive searches*. Given that optimization-based bounds are (some of) the best-known for binary codes, e.g. see discussion in [9], it is not unusual to ask: for permutation codes, are SDP bounds always better for all n?

To seek an answer we should compute for larger n, thus motivating the proposed method in the next section. When \mathcal{S}_n gets large, problems (4) and (5) become increasingly difficult to solve, as the matrices \tilde{A}_j have order $\#\mathcal{S}_n$. Our method is inspired by recent work [12], [13], [14], which show that if \tilde{A}_j come from a CC, then the \tilde{A}_j can be replaced (in (4) and (5)) by block-diagonalized versions - exact details omitted here. This may result in huge complexity reduction, e.g., [14] shows how SDP's related to the conjugacy CC reduces to simpler LP problems. The caveat is that number of matrix blocks (obtained from diagonalization) is at least d, the number of adjacency matrices A_i , see [12]. Unfortunately for the length CC, this number quickly becomes large for increasing n, see Table II. Thus in our case it becomes difficult to directly apply the techniques in [12], and modifications of the ideas are needed.

IV. LENGTH CC: LINEAR PROGRAMMING (LP) BOUNDS

Using "duality" we consider the feasible solutions b to (5) (for some \mathcal{G} -invariant $\operatorname{dist}(,)$ and $\delta_{\min} \geq 1$) that furnish upper estimates b_1 to $\mu(n, \delta_{\min})$, see (6) and Proposition 1. While "duality" ideas are not new, the novelty here is to "guess a good subset" of feasible solutions (in the dual program) described by a manageable number of linear equations, and use an LP to optimize over them. For a CC $(\mathcal{G}, \mathcal{S}_n)$, a feasible solution b corresponds to a positive semidefinite matrix in the following set¹

$$\mathcal{A}_{\mathcal{G},\mathcal{S}_n} \stackrel{\Delta}{=} \left\{ \sum_{i=1}^{\tilde{d}} b_j \tilde{A}_j : b_j \in \mathbb{R}, \text{ for all } 1 \leq j \leq \tilde{d} \right\}.$$
 (7)

Recall that the all-ones matrix J is also in $\mathcal{A}_{\mathcal{G},\mathcal{S}_n}$.

To build an intuition on how such a strategy is possible, we first connect with the LP bound of the conjugacy CC $(S_n \times S_n, S_n)$ described in [11]. To clarify between conjugacy and length CC's, we respectively denote $A_{S_n,i}$ and $A_{\Psi_n,i}$ for adjacency matrices, and d_{S_n} and d_{Ψ_n} for their numbers.

¹The set $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$ is usually known as the adjacency algebra (over ℝ) of the CC ($\mathcal{G}, \mathcal{S}_n$), which has the properties of a matrix-* algebra [16].

We claim that the set $A_{S_n \times S_n, S_n}$ is a subset of $A_{S_n \times \Psi_n, S_n}$, seen by showing each A_i to lie in $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$. Observe that $S_n \times \Psi_n$ is a subgroup of $S_n \times S_n$, hence the orbitals of the length CC, lie within those of the conjugacy CC. In other words, there exists index subsets $\mathcal{I}_{\mathcal{S}_n,i}$, where $\bigcup_{i=1}^{a_{\mathcal{S}_n}} \mathcal{I}_{\mathcal{S}_n,i} =$ $\{1,2,\cdots,d_{\Psi_n}\}$, such that $A_{\mathcal{S}_n,i}=\sum_{j\in\mathcal{I}_{\mathcal{S}_n,i}}A_{\Psi_n,j}$ hold (for all i). The claim $A_{\mathcal{S}_n,i}\in\mathcal{A}_{\mathcal{S}_n\times\Psi_n,\mathcal{S}_n}$ follows if $A_{\mathcal{S}_n,i}$ is a symmetric matrix, see property i) of the following theorem from [11].

Theorem 2 (c.f. [11]). Let $(S_n \times S_n, S_n)$ denote the conjugacy CC, where $(S_n \times S_n, S_n) = \{A_{S_n,i} : 1 \leq i \leq d_{S_n}\}$, and $A_{S_n,1} = I$. Then all of the following hold for $A_{S_n,i}$:

- i) symmetry, i.e., $A_{\mathcal{S}_n,i}^T = A_{\mathcal{S}_n,i}$ (or $\tilde{A}_{\mathcal{S}_n,i} = A_{\mathcal{S}_n,i}$).
- ii) commutativity, i.e., $A_{S_n,i}A_{S_n,j} = A_{S_n,j}A_{S_n,i}$ for all i, j.
- iii) diagonalization by an orthonormal matrix U in $\mathbb{R}^{S_n \times S_n}$. i.e., $U^T A_{\mathcal{S}_n,i} U = \sum_{j=1}^{d_{\mathcal{S}_n}} p_{i,j} \cdot I_j$ for some $p_{i,j} \in \mathbb{R}$ and 0-1 diagonal matrix \check{I}_j .

 - $I = A_{S_n,1} = \sum_{j=1}^{d_{S_n}} U I_j U^T$, therefore $\sum_{j=1}^{d_{S_n}} I_j = I$. $\sum_{i=1}^{d_{S_n}} A_{S_n,i} = J$, so $U^T J U = \sum_{j=1}^{d} c_j \cdot I_j$ where $c_j = \sum_{i=1}^d p_{i,j}$. By convention $c_1 = \#S_n$ (the only non-zero eigenvalue of J) and $c_j = 0$ for $j \ge 2$.

The numbers $d_{\mathcal{S}_n}$, tabulated in Table II, equal the partition number of n, see [11]. Consider a matrix $\sum_{j=1}^{d_{\Psi_n}} b_j \tilde{A}_{\Psi_n,j}$ in $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$, that for some $a \in \mathbb{R}^{d_{\mathcal{S}_n}}$, can be expressed as $\sum_{i=1}^{d} a_i A_{S_n,i}$. Theorem 2 allows us to further express $\sum_{j=1}^{d_{\Psi_n}} b_j \tilde{A}_{\Psi_n,j} = \sum_{j=1}^{d_{S_n}} z_j \cdot (UI_jU^T)$ where $z_j = \sum_{i=1}^{d_{S_n}} p_{i,j} a_i$. Then $\sum_{j=1}^{d_{\Psi_n}} b_j \tilde{A}_{\Psi_n,j} - J$ is positive semidefinite (see (5)) if the linear constraints $\sum_{i=1}^{d_{S_n}} p_{i,j} a_i \geq c_j$ hold for all j, for constants c_j in iii). Intuitively, Theorem 2 is an explicit "diagonalization" of all matrices in the subset $\mathcal{A}_{\mathcal{S}_n \times \mathcal{S}_n, \mathcal{S}_n}$ of $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$, and facilitates checking of positive semidef.

A simple extension of the "diagonalization" idea to the following *larger* subset of matrices, works reasonably well. Property ii) of Theorem 2 implies iii), as symmetric matrices that commute share common eigenspaces. As such, we desire² a subset \mathcal{B} of $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$, with the property that any $M \in \mathcal{B}$, commutes with any $M' \in \mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$. Thus any two matrices in $\ensuremath{\mathcal{B}}$ commute. Such a subset $\ensuremath{\mathcal{B}}$ may be obtained

$$\mathcal{B} = \left\{ \sum_{i=1}^{d_{\mathcal{S}_n}} (a_i A_{\mathcal{S}_n, i}) + \sum_{i=1}^{d_{\mathcal{S}_n}} (a_{d_{\mathcal{S}_n} + i} A_{\mathcal{S}_n, i} W) : a \in \mathbb{R}^{2d_{\mathcal{S}_n}} \right\}, (8)$$

where W is an orthonormal, 0-1 matrix in $\mathbb{R}^{S_n \times S_n}$, that satisfies $(W)_{x,y} = 1$ if and only if $yw_0^{-1} = x$ for any $x, y \in \mathcal{S}_n$. From (8) we see \mathcal{B} contains the set $\mathcal{A}_{\mathcal{S}_n \times \mathcal{S}_n, \mathcal{S}_n}$ considered in Theorem 2. Also by the previous correspondence between B_i and the orbital Δ_i , one can check (see appendix) W commutes with all of $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$ (and each $A_{\mathcal{S}_n, i}$). Because the longest element satisfies $w_0^{-1} = w_0$, thus $W^T = W^{-1} = W$. So $A_{S_n,i}W$ are symmetric, and \mathcal{B} is a set of symmetric matrices.

One technical lemma, that connects (8) with the dual problem (5), stands in way of finally describing our LP bound. This lemma involves a special subgroup Θ_n of \mathcal{S}_n , where Θ_n is also involved in a few final definitions. Let $\Theta_n = \{\alpha \in \mathcal{S}_n : (\alpha, \alpha) \cdot w_0 = w_0\}, \text{ where } (\alpha, \alpha) \cdot w_0 \text{ is }$

²Try to show, see [16], pp. 50-51., that $A_{S_n,i}$ and $A_{S_n,i}W$ are *conjugacy*sums, and \mathcal{B} in (8) is the *center* of the adjacency algebra (7) for $\mathcal{G} = \mathcal{S}_n \times \Psi_n$. computed using the action of $S_n \times S_n$ on S_n . Let $A_{\Theta_n,\ell}$ denote the symmetrized adjacency matrices belonging to the CC $(S_n \times \Theta_n, S_n)$, where there are d_{Θ_n} of them. Note $d_{\Theta_n} \leq d_{\Psi_n}$.

Lemma 1. Let $A_{S_n,i}$ and $\tilde{A}_{\Theta_n,\ell}$ be the symmetrized adjacency matrices belonging to the conjugacy CC and $(S_n \times \Theta_n, S_n)$, respectively. Let W be defined as before. For $1 \leq \ell \leq d_{\Theta_n}$ and $1 \le i \le 2d_{S_n}$ there exists 0-1 coefficients $t_{\ell,i}$ that satisfy

$$A_{\mathcal{S}_n,i} = \sum_{\ell=1}^{\tilde{d}_{\Theta_n}} t_{\ell,i} \tilde{A}_{\Theta_n,\ell}, \quad A_{\mathcal{S}_n,i} W = \sum_{\ell=1}^{\tilde{d}_{\Theta_n}} t_{\ell,d_{\mathcal{S}_n}+i} \tilde{A}_{\Theta_n,\ell}.$$
(9)

See appendix for the proof of Lemma 1. The coefficients $t_{\ell,i}$ satisfying (9) are used to state the following main theorem. For $\Theta_n \subseteq \mathcal{S}_n$, let index subsets $\mathcal{I}_{\Theta_n,\ell}$ satisfy $\mathcal{A}_{\Theta_n,\ell} =$ $\sum_{j\in \tilde{\mathcal{I}}_{\Theta_n,\ell}} \tilde{A}_{\Psi_n,i}$. Using orbit-distances δ_j w.r.t. $(\mathcal{S}_n \times \Psi_n, \mathcal{S}_n)$ and the Kendall tau metric dist(,), define constants γ_{ℓ} that satisfy $\gamma_{\ell} = \max\{\delta_j : j \in \mathcal{I}_{\Theta_n,\ell}\}.$

Theorem 3 (LP Bound on $(S_n \times \Psi_n, S_n)$ and δ_{\min}). Let W be the 0-1 orthornormal matrix defined as before.

For $1 \leq i, j \leq d_{S_n}$, let constants $p_{i,j}, c_j$ and matrices U, I_j be obtained from Theorem 2. Let matrices $M_{1,j}$ and $M_{2,j}$ satisfy $M_{1,j} = \frac{1}{2}(UI_jU^T)(I+W)$ and $M_{2,j} =$ $\frac{1}{2}(UI_jU^T)(I-W).$

For $1 \le \ell \le d_{\Theta_n}$, let the constants γ_ℓ be defined as above. For $1 \leq i \leq 2d_{\mathcal{S}_n}$, let the coefficients $t_{\ell,i}$ satisfy (9). Let a^* in $\mathbb{R}^{2d_{S_n}}$ solve the following LP problem

$$\min_{\substack{(a_1, a_2, \dots, a_{2d_S}) \in \mathbb{R}^{2d_{S_n}}}} \sum_{i=1}^{2d_{S_n}} t_{1,i} \cdot a_i \tag{10}$$

$$\min_{\substack{(a_1,a_2,\ldots,a_{2d_{\mathcal{S}_n}})\in\mathbb{R}^{2d_{\mathcal{S}_n}}\\ s.t.}} \sum_{i=1}^{2d_{\mathcal{S}_n}} t_{1,i}\cdot a_i$$

$$s.t. \sum_{i=1}^{2d_{\mathcal{S}_n}} t_{\ell,i}\cdot a_i \leq 0 \quad \textit{for } 2\leq \ell \leq \tilde{d}_{\Theta_n} \textit{ with } \gamma_\ell \geq \delta_{\textit{min}},$$

$$\sum_{i=1}^{d_{\mathcal{S}_n}} (a_i + a_{d_{\mathcal{S}_n}+i}) \cdot p_{i,j} \ge c_j \text{ for } 1 \le j \le d_{\mathcal{S}_n} \text{ with } M_{1,j} \ne 0$$

$$\sum_{i=1}^{d_{\mathcal{S}_n}} (a_i - a_{d_{\mathcal{S}_n}+i}) \cdot p_{i,j} \ge c_j \text{ for } 1 \le j \le d_{\mathcal{S}_n} \text{ with } M_{2,j} \ne 0$$

$$\sum_{i=1}^{d_{\mathcal{S}_n}} (a_i - a_{d_{\mathcal{S}_n} + i}) \cdot p_{i,j} \ge c_j \text{ for } 1 \le j \le d_{\mathcal{S}_n} \text{ with } M_{2,j} \ne 0$$

Let b_1^* and $\mu(n, \delta_{min})$ respectively denote the optimal objective values of the dual problem (5), and the optimal code size problem (1), for $\mathcal{G} = \mathcal{S}_n \times \Psi_n$ and the Kendall tau metric dist(,) and δ_{min} . Then we have the following inequalities

$$\mu(n, \delta_{min}) \le b_1^* \le \sum_{i=1}^{2d_{S_n}} t_{1,i} \cdot a_i^*.$$

As promised our main result Theorem 3 furnishes an LP bound on the optimal code size $\mu(n, \delta_{\min})$. See appendix for proof. The number \tilde{d}_{Θ_n} of matrices $\tilde{A}_{\Theta_n,\ell}$ is given in the previous Table II, where observe $d_{\Theta_n} > d_{S_n}$, but d_{Θ_n} is much reduced from d_{Ψ_n} . Table III shows our computed LP bounds whereby n is between 3 and 11. Our proposed LP bound fails to completely answer the question posed (at the end) of Section III, but some initial success is obtained. Observe that our LP bound is at least as tight as the SB in the places highlighted in bold font. Improvements are mainly obtained when δ_{\min} is close to n(n-1)/2. Interestingly, these two bounds are useful for similar ranges of δ_{\min} (the SB is known to be non-trivial only when $\delta_{\min} \geq n$, see [6]). For the case n=3 the LP and SDP bounds are equal, though unfortunately for n > 4, our LP

 $\mbox{TABLE III} \\ \mbox{Bounds computed for Various } 3 \leq n \leq 11. \\$

n=3			n = 7				n = 9			n = 10			
δ_{\min} 1 2 3	LP 6 3 2	SB [†] 6 6 2	HB [‡] 6 6 2	$\begin{array}{c} \delta_{\min} \\ 10 \\ 11 \\ 12 \\ 15 \end{array}$	LP 5040 630 543 140	SB 720 720 120 120	HB 28 14 14 5	δ_{\min} 14 15 16 23	LP 362880 45360 32989 7560	SB 40320 40320 5040 720	δ _{min} 42 43 44 45	LP 6 3 3 2	SB 24 6 6 2
δ_{\min} 3 4 5	n = LP 24 12 4 2	= 4 SB 24 6 6 2	HB 6 6 2 2	16 17 18 19 20 21	75 14 7 3 2 2	24 24 24 6 6 2	5 5 3 2 2 2	25 27 29 30 31 32 33	2016 1008 186 93 15 9	720 120 120 120 24 24 24 24	δ_{\min} 18 19 31 33	n = 11 LP 39916800 3326400 359611 193458	SB 3628800 3628800 40320 40320
δ_{\min} 6 7 8 9 10 δ_{\min} 8 9 11 12 13 14 15	n = LP 120 10 5 2 2 2 n = LP 720 120 27 13 6 4 2	SB 24 24 6 6 2	HB 8 4 4 2 2 2 2 HB 14 7 7 4 4 4 2 2 2 2 2	δ _{min} 12 13 14 19 21 22 23 24 25 26 27 28	$n = \frac{\text{LP}}{40320}$ 5040 4135 896 384 192 41 21 8 5 2 2	8 SB 5040 5040 720 120 120 120 24 24 24 6 6 2	HB 64 32 32 7 5 5 3 3 2 2 2 2 2	δ _{min} 16 17 18 27 29 31 35 36 37 38 39	$\begin{array}{c} \textbf{4} \\ \textbf{3} \\ \textbf{2} \\ \textbf{2} \\ \textbf{2} \\ \\ \\ n = 10 \\ \text{LP} \\ 3628800 \\ \textbf{329891} \\ 302400 \\ 49371 \\ 21098 \\ 9735 \\ 4995 \\ 3900 \\ 446 \\ 230 \\ \textbf{55} \\ \end{array}$	6 6 2	33 34 35 37 41 42 43 44 45 46 47 48 49 50 51 52 53 54	193458 177678 94924 66176 33662 26050 11152 8700 6349 3541 222 111 17 11 5 4 3 2 2	40320 5040 5040 720 720 720 720 720 120 120 120 120 4 24 24 24
	-	-						40 41	30 11	24 24	55	$\bar{2}$	6 2

[†], [‡] See footnotes on previous Table I.

bound does worse than the HB, and the performance gap gets bigger for smaller δ_{\min} . Inspired by [6] (which points out three regions with different asymptotics), it is tempting to conjecture that different strategies work for cases $\delta_{\min} < n$ and $\delta_{\min} \geq n$. The subset searched here works reasonably well for the latter case, for the former what are the "good" dual-feasible subsets?

One issue: no known efficient method to compute "max. distances" γ_{ℓ} , where $\gamma_{\ell} = \max\{\delta_j: j \in \tilde{\mathcal{I}}_{\Theta_n,\ell}\}$. If one replaces Θ_n by \mathcal{S}_n in the expression for γ_{ℓ} , (where $A_{\mathcal{S}_n,i} = \sum_{j \in \tilde{\mathcal{I}}_{\mathcal{S}_n,i}} A_{\Psi_n,j}$), then [18] has closed-forms for γ_{ℓ} . Also its is unclear how large the number $\#\{1 \leq \delta_{\ell} \leq \tilde{d}_{\Theta_n}: \delta_{\ell} \geq \delta_{\min}\}$ of non-positive constraints could be. No rigorous analysis is done here, but see [19] for a characterization of Θ_n .

V. CONCLUSION & FUTURE DIRECTIONS

Motivated by recent work on solving SDP's with algebraic structure, we formulated the optimal code size problem w.r.t. Kendall tau metric as a SDP, and propose using LP to search for solutions. The problem seems difficult, but we report modest improvement over a recent Singleton bound.

The interest is to progress toward (possibly) beating known Hamming bounds, for the cases $n \geq 6$ (other than those shown here). We offer some future directions. As previously mentioned, it would be nice to analyze the subsets that should be searched (for $\delta_{\min} < n$). Next, one might generalize to larger subsets where a manageable SDP (not a LP as here) is used for searching. Finally, one might seek a similar Fourier-type analysis as [9], using *representation-theoretic* techniques.

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APPENDIX

1) SDP relaxation of optimal code size problem

First we prove (4) is a relaxation of (1). In the following for a subset \mathcal{V} of \mathcal{S}_n , let \mathcal{V}^2 denote the product set $\mathcal{V} \times \mathcal{V}$. Let $\mathbb{R}^{\mathcal{S}_n}$ denote the set of vectors with real number entries with index set \mathcal{S}_n .

Proof of Proposition 1: Let $\mathcal V$ denote a solution of (1), i.e., let $\#\mathcal V=\mu(n,\delta_{\min})$. Identify the subset $\mathcal V$ of $\mathcal S_n$ with an 0-1 vector b in $\mathbb R^{\mathcal S_n}$, where $b_g=1$ if and only if $g\in\mathcal V$. We construct a matrix M whose objective value in (4) equals $\#\mathcal V$, i.e., $\mathrm{Tr}(JM)=\#\mathcal V$. Let $M=\frac{1}{b^Tb}bb^T$, i.e., $M=\frac{1}{\#\mathcal V}bb^T$, and let 1 denote the all-ones vector. Observe that $\mathrm{Tr}(JM)=\mathrm{Tr}(\mathbf 1^TM)=\mathrm{Tr}(\mathbf 1^TM)=\frac{1}{\#\mathcal V}\mathrm{Tr}((\mathbf 1^Tb)(b^T\mathbf 1))=\frac{1}{\#\mathcal V}\mathrm{Tr}(\#\mathcal V\#\mathcal V)=\#\mathcal V$. Next we show that matrix M is a feasible solution to (4).

Because $M=\frac{1}{\#\mathcal{V}}bb^T$, therefore M is positive semidefinite and $\operatorname{Tr}(M)=1$ is satisfied. Next observe $\operatorname{Tr}(\tilde{A}_jM)=\#(\tilde{\Delta}_j\cap\mathcal{V}^2)/\#\mathcal{V}$, so $\operatorname{Tr}(\tilde{A}_jM)\geq 0$ is satisfied. Now consider any $x,y\in\mathcal{S}_n$ where $x\neq y$. If $x,y\in\mathcal{V}$ then $\operatorname{dist}(x,y)\geq \delta_{\min}$. By contraposition, if $\operatorname{dist}(x,y)<\delta_{\min}$ then $(x,y)\notin\mathcal{V}^2$. Let $(x,y)\in\tilde{\Delta}_j$ for some $j\geq 2$, then $\delta_j<\delta_{\min}$ also implies $(x,y)\notin\mathcal{V}^2$, which in turn implies $\operatorname{Tr}(\tilde{A}_jM)=0$.

2) Matrix W and set $A_{S_n \times \Psi_n, S_n}$

Next we prove the orthonormal, 0-1 matrix W commutes with all matrices in the set $\mathcal{A}_{\mathcal{S}_n \times \Psi_n, \mathcal{S}_n}$. Recall W is related to the longest element w_0 , where for any $x,y \in \mathcal{S}_n$, we have $(W)_{x,y}=1$ if and only if $yw_0^{-1}=x$.

Proof: It suffices to show that W commutes with any adjacency matrix $A_{\Psi_n,j}$ of the length CC. For any $A_{\Psi_n,j}$, observe that $(W^T A_{\Psi_n,j} W)_{x,y} = (A_{\Psi_n,j})_{xw_0^{-1},yw_0^{-1}}$. Recall $(A_{\Psi_n,j})_{x,y} = 1$ if and only if $(x,y) \in \Delta_j$, whereby Δ_j is an orbital of the induced action of $\mathcal{S}_n \times \Psi_n$ on $\mathcal{S}_n \times \mathcal{S}_n$. Hence $(A_{\Psi_n,j})_{xw_0^{-1},yw_0^{-1}} = (A_{\Psi_n,j})_{x,y}$ because (x,y) and (xw_0^{-1},yw_0^{-1}) both belong to same orbital. Hence for any j we have $A_{\Psi_n,j}W = WA_{\Psi_n,j}$, which implies W commutes with all of $\mathcal{A}_{\mathcal{S}_n \times \Psi_n,\mathcal{S}_n}$.

3) Technical Lemma 1

To show Lemma 1 we need to first establish a relationship between the adjacency matrices $A_{\mathcal{Z},i}$ of the $\operatorname{CC}(\mathcal{S}_n \times \mathcal{Z}, \mathcal{S}_n)$, where \mathcal{Z} is a subgroup of \mathcal{S}_n , with orbits on \mathcal{S}_n , of a subgroup of $\mathcal{S}_n \times \mathcal{S}_n$ that is related to \mathcal{Z} . Recall our definition of the action of any (α,β) in $\mathcal{S}_n \times \mathcal{S}_n$ on any $x \in \mathcal{S}_n$, given as $(\alpha,\beta)x = \alpha x \beta^{-1}$. For any subgroup \mathcal{Z} of \mathcal{S}_n , denote the subgroup $\{(\beta,\beta):\beta\in\mathcal{Z}\}$ of $\mathcal{S}_n \times \mathcal{S}_n$ by $\mathcal{H}_{\mathcal{Z}}$. Let $\mathcal{C}_{\mathcal{Z},1},\mathcal{C}_{\mathcal{Z},1},\cdots,\mathcal{C}_{\mathcal{Z},r}$ denote the r orbits, obtained from the action of $\mathcal{H}_{\mathcal{Z}}$ on \mathcal{S}_n . Each orbit $\mathcal{C}_{\mathcal{Z},i}$ is called a **conjugacy class** (of the action of $\mathcal{H}_{\mathcal{Z}}$ on \mathcal{S}_n). Let $\rho(\beta)$ denote the 0-1 matrix where $(\rho(\beta))_{x,y}=1$ if and only if $y\beta^{-1}=x$ (i.e., by our previous definition, $W=\rho(w_0)$). We claim a one-to-one correspondence between some conjugacy class $\mathcal{C}_{\mathcal{Z},i}$ and some adjacency matrix $A_{\mathcal{Z},i}$ of the CC $(\mathcal{S}_n \times \mathcal{Z}, \mathcal{S}_n)$, given as

$$A_{\mathcal{Z},i} = \sum_{\beta \in \mathcal{C}_{\mathcal{Z},i}} \rho(\beta). \tag{11}$$

By this claim the number r of conjugacy classes $\mathcal{C}_{\mathcal{Z},i}$, equals the number d of adjacency matrices $A_{\mathcal{Z},i}$. To show (11), consider the following.

First, we establish the one-to-one correspondence. By the definition of the orbital Δ_i , for any $(x,y), (\tilde{x},\tilde{y}) \in \Delta_i$ there exists some $\alpha \in \mathcal{S}_n$ and $\beta \in \mathcal{Z}$ such that $\tilde{x} = \alpha x \beta^{-1}$ and $\tilde{y} = \alpha y \beta^{-1}$. Equivalently for any $(x,y), (\tilde{x},\tilde{y}) \in \Delta_i$, there exists some $\beta \in \mathcal{Z}$ that satisfies $\tilde{x}^{-1}\tilde{y} = \beta x^{-1}y\beta^{-1}$, which means that $\tilde{x}^{-1}\tilde{y}$ and $x^{-1}y$ are both in $\mathcal{C}_{\mathcal{Z},i}$. Note that $\sum_{\beta \in \mathcal{C}_{\mathcal{Z},i}} \rho(\beta)$ is a 0-1 matrix, and by the definition of $\rho(\beta)$ we conclude

$$\left(\sum_{\beta \in \mathcal{C}_{\mathcal{Z},i}} \rho(\beta)\right)_{x,y} = 1, \quad \text{if and only if} \quad x^{-1}y \in \mathcal{C}_{\mathcal{Z},i},$$

if and only if $(x, y) \in \Delta_i$. This establishes (11) by by referring to the original definition of $A_{\mathcal{Z},i}$ from Δ_i .

Proof of Lemma 1: Denote a set $\{\beta w_0 : \beta \in \mathcal{C}_{\mathcal{S}_n,i}\}$ of elements in \mathcal{S}_n by $\mathcal{P}_{\mathcal{S}_n,i}$. Hence $\mathcal{P}_{\mathcal{S}_n,i}$ is obtained using the conjugacy class $\mathcal{C}_{\mathcal{S}_n,i}$ and the longest element w_0 . Let $A_{\mathcal{S}_n,i}$ be an adjacency matrix of the conjugacy CC, and $W = \rho(w_0)$. It follows $A_{\mathcal{S}_n,i}W = \sum_{\beta \in \mathcal{C}_{\mathcal{S}_n,i}} \rho(\beta w_0) = \sum_{\beta \in \mathcal{P}_{\mathcal{S}_n,i}} \rho(\beta)$. Because Ψ_n is a subgroup of Θ_n , so for each conjugacy class $\mathcal{C}_{\Theta_n,\ell}$ there exists index sets $\mathcal{I}_{\Theta_n,\ell}$ satisfying $\mathcal{C}_{\Theta_n,\ell} = \bigcup_{j \in \mathcal{I}_{\Theta_n,\ell}} \mathcal{C}_{\Psi_n,j}$. The sets $\mathcal{I}_{\Theta_n,\ell}$ partition $\{1,2,\cdots,d_{\Psi_n}\}$. For $1 \leq i \leq d_{\mathcal{S}_n}$, we claim there exists new index sets \mathbb{I}_i and \mathbb{J}_i that satisfy

$$C_{\mathcal{S}_n,i} = \cup_{\ell \in \mathbb{I}_i} C_{\Theta_n,\ell}, \tag{12}$$

$$\mathcal{P}_{\mathcal{S}_n,i} = \cup_{\ell \in \mathbb{J}_i} \mathcal{C}_{\Theta_n,\ell}. \tag{13}$$

If the claim holds, Lemma 1 is easily proved as follows.

By the previously established (11), we can write $A_{\Theta_n,\ell} = \sum_{\beta \in \mathcal{C}_{\Theta_n,\ell}} \rho(\beta)$, where $A_{\Theta_n,\ell}$ is an adjacency matrix of the CC $(\mathcal{S}_n \times \Theta_n, \mathcal{S}_n)$. By (11) again, an adjacency matrix $A_{\mathcal{S}_n,i}$ of the conjugacy CC satisfies $A_{\mathcal{S}_n,i} = \sum_{\beta \in \mathcal{C}_{\mathcal{S}_n,i}} \rho(\beta)$, so (12) implies $A_{\mathcal{S}_n,i} = \sum_{\ell \in \mathbb{I}_i} A_{\Theta_n,\ell}$. Also because $A_{\mathcal{S}_n,i}W = \sum_{\beta \in \mathcal{C}_{\mathcal{S}_n,i}} \rho(\beta w_0)$, by definition of $\mathcal{P}_{\mathcal{S}_n,i}$ then (13) implies $A_{\mathcal{S}_n,i}W = \sum_{\ell \in \mathbb{J}_i} A_{\Theta_n,\ell}$. But because both $A_{\mathcal{S}_n,i}$ and $A_{\mathcal{S}_n,i}W$ are symmetric 0-1 matrices, there must exist sets \mathbb{I}_i and \mathbb{J}_i to express $A_{\mathcal{S}_n,i}$ and $A_{\mathcal{S}_n,i}W$ in terms of symmetrized adjacency matrices $\tilde{A}_{\Theta_n,\ell}$, i.e.

$$\begin{split} A_{\mathcal{S}_n,i} \; &= \; \sum_{\ell \in \tilde{\mathbb{I}}_i} \tilde{A}_{\Theta_n,\ell} = \sum_{\ell=1}^{\tilde{d}_{\Theta_n}} t_{\ell,i} \cdot \tilde{A}_{\Theta_n,\ell}, \\ A_{\mathcal{S}_n,i}W \; &= \; \sum_{\ell \in \tilde{\mathbb{I}}_i} \tilde{A}_{\Theta_n,\ell} = \sum_{\ell=1}^{\tilde{d}_{\Theta_n}} t_{\ell,d_{\mathcal{S}_n}+i} \cdot \tilde{A}_{\Theta_n,\ell}, \end{split}$$

where $t_{\ell,i}$ are coefficients appearing in the lemma statement.

We end by showing the previous claims. The first identity (12) follows easily from the fact $\Theta_n \subseteq \mathcal{S}_n$. The second identity (13) follows by arguing if $\mathcal{C}_{\Theta_n,\ell} \cap \mathcal{P}_{\mathcal{S}_n,i} \neq \emptyset$ then $\mathcal{C}_{\Theta_n,\ell} \subset \mathcal{P}_{\mathcal{S}_n,i}$. Consider some $xw_0 \in \mathcal{C}_{\Theta_n,\ell} \cap \mathcal{P}_{\mathcal{S}_n,i}$, where $x \in \mathcal{C}_{\mathcal{S}_n,i}$. By definition of the conjugacy class $\mathcal{C}_{\Theta_n,\ell} = \{(\alpha,\alpha) \cdot xw_0 : \alpha \in \Theta_n\}$. By definition of the group Θ_n we have $(\alpha,\alpha) \cdot xw_0 = \alpha xw_0\alpha^{-1} = (\alpha x\alpha^{-1})(\alpha w_0\alpha^{-1}) = (\alpha x\alpha^{-1})w_0$, and it follows $(\alpha x\alpha^{-1})w_0$ is also in $\mathcal{P}_{\mathcal{S}_n,i}$ as $\alpha x\alpha^{-1} \in \mathcal{C}_{\mathcal{S}_n,i}$. Hence (13) is shown.

4) Main theorem

Finally, the following verifies that our proposed LP bound (10) indeed provides an upper bound to the optimal value of the dual problem (5).

Proof of Theorem 3: For some $b \in \mathbb{R}^{\tilde{d}_{\Psi_n}}$, $z \in \mathbb{R}^{\tilde{d}_{\Theta_n}}$ and $a \in \mathbb{R}^{2d_{S_n}}$, we have the following chain of equalities

$$\sum_{j=1}^{\tilde{d}_{\Psi_n}} b_j \cdot \tilde{A}_{\Psi_n,j} \stackrel{(a)}{=} \sum_{\ell=1}^{\tilde{d}_{\Theta_n}} z_\ell \cdot \tilde{A}_{\Theta_n,\ell}$$

$$\stackrel{(b)}{=} \sum_{i=1}^d a_i \cdot A_{S_n,i} + \sum_{i=1}^{d_{S_n}} a_{d_{S_n}+i} \cdot A_{S_n,i} W, (14)$$

where (a) follows by setting $b_j = z_\ell$ if $j \in \tilde{\mathcal{I}}_{\Theta_n,\ell}$, and (b) follows by setting $z_\ell = \sum_{i=1}^{2d_{S_n}} t_{\ell,i} \cdot a_i$ for coefficients $t_{\ell,i}$ that satisfy (9). The theorem will be proved by showing for any feasible a in $\mathbb{R}^{2d_{S_n}}$ to (10), there corresponds some some feasible b in $\mathbb{R}^{\tilde{d}_{\Psi_n}}$ to (5) by the above relationship (14).

Firstly the objectives of (5) and (10) are equal because $b_1 = z_1 = \sum_{i=1}^{2d_{S_n}} t_{1,i} \cdot a_i$. Let a satisfy the second constraint of (10) and let b satisfy (14). By $z_\ell = \sum_{i=1}^{2d_{S_n}} t_{\ell,i} \cdot a_i$, if $\gamma_\ell \geq \delta_{\min}$ we have $z_\ell \leq 0$. Because $\gamma_\ell = \max\{\delta_j : j \in \tilde{\mathcal{I}}_{\Theta_n,\ell}\}$, then for any $j \in \tilde{\mathcal{I}}_{\Theta_n,\ell}$ such that $\delta_j \geq \delta_{\min}$, we must have $b_j \leq 0$. Finally $\bigcup_{\ell=2}^{\tilde{d}_{\Theta_n}} \{\delta_j : j \in \tilde{\mathcal{I}}_{\Theta_n,\ell}\} = \{\delta_2,\delta_3,\cdots,\delta_{\tilde{d}_{\Psi_n}}\}$, implying that $b_j \leq 0$ for all $j \geq 2$ whereby $\delta_j \geq \delta_{\min}$, therefore b satisfies the non-positive constraint of (5).

Next consider the matrices $M_{1,j}$ and $M_{2,j}$ given in the theorem statement. Note $M_{1,j}+M_{2,j}=UI_jU^T$ and $M_{1,j}-M_{2,j}=(UI_jU^T)W$. Using $A_{\mathcal{S}_n,i}=\sum_{j=1}^d p_{i,j}\cdot (UI_jU^T)$ in Theorem 2 we express

$$A_{S_{n},i} = \sum_{j=1}^{d_{S_{n}}} p_{i,j} \cdot M_{1,j} + \sum_{j=1}^{d_{S_{n}}} p_{i,j} \cdot M_{2,j},$$

$$A_{S_{n},i}W = \sum_{j=1}^{d_{S_{n}}} p_{i,j} \cdot M_{1,j} - \sum_{j=1}^{d_{S_{n}}} p_{i,j} \cdot M_{2,j},$$

$$J = \sum_{j=1}^{d_{S_{n}}} c_{j} \cdot M_{1,j} + \sum_{j=1}^{d_{S_{n}}} c_{j} \cdot M_{2,j},$$
(15)

where we claim (shown below) that the matrices $M_{1,j}$ and $M_{2,j}$ are i) all symmetric, and ii) have eigenvalues only 0 or 1, and iii) $M_{1,j}M_{2,j}=0$ and iv) $\sum_{j=1}^d (M_{1,j}+M_{2,j})=\sum_{j=1}^d (UI_jU^T)=I$. For any a satisfying the last two constraints of (10), then (15) implies $\sum_{i=1}^d a_i \cdot A_{\mathcal{S}_n,i} + \sum_{i=1}^{d_{\mathcal{S}_n}} a_{d_{\mathcal{S}_n}+i} \cdot A_{\mathcal{S}_n,i}W - J$ is positive semidefinite. Then for b that corresponds by (14) to such an a, we will have $\sum_{j=1}^{\bar{d}_{\Psi_n}} b_j \cdot \tilde{A}_{\Psi_n,j} - J$ satisfying the positive semidefinite constraint in (5).

To finish the proof we address the above claims i) and ii), whereby iii) and iv) will then follow from similar arguments. Claim i) follows because all matrices UI_jU^T commute with all matrices $A_{\mathcal{S}_n,i}$, see Theorem 2. Recall W commutes with all matrices in \mathcal{B} , therefore W commutes with all matrices $A_{\mathcal{S}_n,i}$, which implies W commute with all UI_jU^T . This implies $M_{1,j}$ and $M_{2,j}$ are symmetric, since both W and UI_jU^T are symmetric.

Claim ii) follows because $w_0^{-1} = w_0$, and it can be verified that $W^T = W^{-1} = W$, which implies that the possible eigenvalues of W are -1 and 1. Thus the possible eigenvalues of matrices $M_{1,j}$ and $M_{2,j}$ are 0 or 1.