

Moderate-Deviations of Lossy Source Coding for Discrete and Gaussian Sources

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Abstract—We study the moderate-deviations (MD) setting for lossy source coding of stationary memoryless sources. More specifically, we derive fundamental compression limits of source codes whose rates are $R(D) \pm \epsilon_n$, where $R(D)$ is the rate-distortion function and ϵ_n is a sequence that dominates $\sqrt{1/n}$. This MD setting is complementary to the large-deviations and central limit settings and was studied by Altug and Wagner for the channel coding setting. We show, for finite alphabet and Gaussian sources, that as in the central limit-type results, the so-called dispersion for lossy source coding plays a fundamental role in the MD setting for the lossy source coding problem.

Index Terms—Moderate-deviations, rate-distortion, dispersion.

I. INTRODUCTION

Rate-distortion theory [1] consists in finding the optimal compression rate for a source $X \sim P$ subject to the condition that there exists a code which can reproduce the source to within a distortion level D . The optimal compression rate for the distortion level D is known as the *rate-distortion function* $R(P, D)$. This function can be expressed as the minimization of mutual information over test channels [1].

It is also of interest to study the excess distortion probability for codes at rate $R > R(P, D)$. This is the probability that the average distortion between X^n and its reconstruction \hat{X}^n exceeds D . The exact exponential rate of decay of this probability was derived by Marton [2] for discrete memoryless sources (DMSs). This was extended to Gaussian [3] and general sources [4]. These results belong to the theory of *large-deviations* (LD) and are reviewed in Section II.

With the revival of interest in second-order coding rates and dispersion analysis [5]–[7], various researchers have also studied the fundamental limit of lossy compression subject to the condition that the probability of excess distortion is no larger than $\epsilon > 0$. In particular, it was shown in [8] and independently in [9], [10] that

$$R(n, D, \epsilon) \approx R(P, D) + \sqrt{\frac{V(P, D)}{n}} Q^{-1}(\epsilon), \quad (1)$$

where $R(n, D, \epsilon)$ is the optimal rate of compression of a memoryless source at blocklength n and $V(P, D)$ is known as the *dispersion* of the source. Eq. (1) holds true for both discrete and Gaussian sources and belongs to the realm of *central limit theorem* (CLT)-style results.

In this paper, we operate in a moderate-deviations (MD) regime [11, Section 3.7] that “interpolates between” the LD

and CLT regimes. In particular, we study the performance of source codes of rates $R_n = R(P, D) \pm \epsilon_n$ where ϵ_n is a sequence that is asymptotically larger than $\sqrt{1/n}$ (cf. (1)). Our results apply to both finite alphabet and Gaussian sources but do not reduce to the LD or CLT settings. Moreover, neither the LD nor CLT results specialize to our setting. We show that the dispersion $V(P, D)$ also plays a fundamental role in this MD setting. Besides studying the excess distortion probability, we also study the complementary probability (also termed the probability of correct decoding) for codes whose rates are *below* the rate-distortion function. Similarly, the fundamental nature of the dispersion is revealed.

This work is inspired by the work on MD in the context of channel coding [12], [13]. It was shown in [12] that for *positive* discrete memoryless channels (i.e., $W(y|x) > 0$ for all x, y), the dispersion also governs the “MD exponent”

$$\lim_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, W) = -\frac{1}{2V(W)}. \quad (2)$$

The direct part was proved by considering the Taylor expansion of Gallager’s random coding exponent. We also use this proof strategy. In [13], several assumptions in [12] were relaxed and the relations between the MD and CLT were clarified. Concurrent to this work, Sason [14] studied MD for binary hypothesis testing. Finally, we mention that He et al. [15] studied the redundancy of the Slepian-Wolf problem which is also related to [8]–[10] and to the current problem.

II. SYSTEM MODEL AND BASIC DEFINITIONS

Let $\mathcal{P}(\mathcal{X})$ be the set of probability mass functions supported on the finite alphabet \mathcal{X} . Let $\mathcal{P}_n(\mathcal{X}) \subset \mathcal{P}(\mathcal{X})$ be the set of n -types. For a type $Q \in \mathcal{P}_n(\mathcal{X})$, let \mathcal{T}_Q^n be the set of sequences x^n of type Q , i.e., the type class. The reproduction alphabet is denoted as $\hat{\mathcal{X}}$. In addition, let $d : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow \mathbb{R}^+$ be a distortion measure such that for every $x \in \mathcal{X}$, there exists an $\hat{x}_0 \in \hat{\mathcal{X}}$ for which $d(x, \hat{x}_0) = 0$. The average distortion is $d(x^n, \hat{x}^n) := \frac{1}{n} \sum_{i=1}^n d(x_i, \hat{x}_i)$. For a function $f : \mathcal{A} \rightarrow \mathcal{B}$, the notation $\|f\| := |f(\mathcal{A})|$ denotes the cardinality of its range.

A DMS $X^n \sim \prod_{i=1}^n P(x_i)$ is described at rate R by an encoder. The decoder receives the description index over a noiseless link and generates a reconstruction sequence $\hat{X}^n \in \hat{\mathcal{X}}^n$. We now remind the reader of the rate-distortion problem.

Definition 1. A rate-distortion code consists of (i) an encoder $f_n : \mathcal{X}^n \rightarrow \mathcal{M}_n$ and (ii) a decoder $\varphi_n : \mathcal{M}_n \rightarrow \hat{\mathcal{X}}^n$. The rate

of the code is $R_n := \frac{1}{n} \log |\mathcal{M}_n|$.

The rate-distortion function $R(P, D)$ is defined as the infimum of all numbers R for which there exists codes $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ for which the probability of excess distortion

$$e(f_n, \varphi_n, P, D) := \mathbb{P}(d(X^n, \varphi_n(f_n(X^n))) > D) \quad (3)$$

is arbitrarily small for sufficiently large blocklengths n . The rate-distortion function [1] can be expressed as

$$R(P, D) = \min_{W: \mathbb{E}[d(X, \hat{X})] \leq D} I(P, W), \quad (4)$$

where $\mathbb{E}[d(X, \hat{X})] := \sum_{x, \hat{x}} P(x) W(\hat{x}|x) d(x, \hat{x})$. Another fundamental quantity introduced by Ingber and Kochman [8] is the *dispersion for lossy source coding*

$$V(P, D) := \text{Var}_X[R'(X; P, D)], \quad (5)$$

where $R'(x; P, D) = \frac{\partial}{\partial P(x)} R(P, D)$ for $x \in \mathcal{X}$ is the partial derivative of the rate-distortion function w.r.t. $P(x)$ (assuming it exists). In (5), the variance is taken w.r.t. the distribution P and $R'(X; P, D)$ is a function of the random variable X . In fact, the term *dispersion* is usually an operational one but since it was shown in [8] that the operational definition coincides with the one in (5), we will abuse terminology and use the generic term dispersion for both quantities.

We analyze $e(f_n, \varphi_n, P, D)$ in the so-called MD regime where the rate of the code $R_n := \frac{1}{n} \log \|f_n\| = R(P, D) + \epsilon_n$ for some sequence ϵ_n . Clearly, if $\epsilon_n \rightarrow 0$, then $R_n \rightarrow R(P, D)$. When the rate of the code R is a constant strictly above $R(P, D)$, Marton [2] showed that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log e(f_n, \varphi_n, P, D) = -F(P, R, D), \quad (6)$$

where *Marton's exponent* is defined as

$$F(P, R, D) := \min_{Q \in \mathcal{P}(\mathcal{X}): R(Q, D) \geq R} D(Q \| P). \quad (7)$$

The exponent is positive for $R > R(P, D)$. One can also consider the *probability of correct decoding* $1 - e(f_n, \varphi_n, P, D)$. In [16, pp. 156], it was shown that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log (1 - e(f_n, \varphi_n, P, D)) = -G(P, R, D), \quad (8)$$

where the *exponent for correct decoding* is

$$G(P, R, D) := \min_{Q \in \mathcal{P}(\mathcal{X}): R(Q, D) \leq R} D(Q \| P). \quad (9)$$

The exponent is positive for $R < R(P, D)$. These limits and exponents are Sanov-like LD results [11]. We present MD versions of Marton's and Iriyama's results where the normalizations in (6) and (8) need not be $\frac{1}{n}$.

III. DISCRETE MEMORYLESS SOURCES (DMS)

Our main result for a DMS with bounded distortion measure (i.e. $d: \mathcal{X} \times \hat{\mathcal{X}} \rightarrow [0, d_{\max}]$) is stated as follows:

Theorem 1. *Let ϵ_n be any positive sequence satisfying*

$$\lim_{n \rightarrow \infty} \epsilon_n = 0, \quad \lim_{n \rightarrow \infty} \frac{n\epsilon_n^2}{\log n} = \infty. \quad (10)$$

That is, $\epsilon_n = \omega((\frac{\log n}{n})^{1/2}) \cap o(1)$. Assume that $R(Q, D)$ is twice differentiable w.r.t. Q in a neighborhood of P and $V(P, D) > 0$. There exists a rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \leq R(P, D) + \epsilon_n$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, P, D) \leq -\frac{1}{2V(P, D)}. \quad (11)$$

Furthermore, every rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \leq R(P, D) + \epsilon_n$ must satisfy

$$\liminf_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, P, D) \geq -\frac{1}{2V(P, D)}. \quad (12)$$

Though somewhat ungainly, the log factor in (10) appears to be essential because the proof hinges on the method of types. So our analysis does not completely close the gap between the CLT and LD regimes. This log factor is unnecessary in the Gaussian case as will be seen in Theorems 5 and 6. Theorem 1 means that if the dispersion $V(P, D)$ is small, the “MD exponent” $(2V(P, D))^{-1}$ is large, corresponding to a faster decay in the excess distortion probability. This has the same interpretation as in the CLT regime (1). As an example, for the Bernoulli source with Hamming distortion, the dispersion can be computed as

$$V(\text{Bern}(\alpha), D) = \alpha(1 - \alpha) \log^2 \left(\frac{1 - \alpha}{\alpha} \right). \quad (13)$$

The parameter that maximizes (resp. minimizes) $V(P, D)$ is $\alpha \approx 0.0832$ (resp. $\alpha = 0, 0.5$). Thus, the “MD exponent” is maximized when the source is deterministic or has maximum entropy. The proof uses the following lemma, whose proof is essentially identical to that of [17, Theorem 8], where the divergence and the constraint set in (7) are approximated by a quadratic and an affine subspace respectively.

Lemma 2. *If the limit exists, Marton's exponent satisfies*

$$\lim_{\delta \rightarrow 0} \frac{F(P, R(P, D) + \delta, D)}{\delta^2} = \frac{1}{2V(P, D)}. \quad (14)$$

In the sequel, we assume that the limit in (14) exists. Otherwise, the results are modified accordingly by considering the upper and lower limits in (14) and replacing the dispersion by its upper and lower limit versions. We first prove the direct part of Theorem 1 in (11) followed by the converse in (12).

Proof: The code construction proceeds along the lines of that in [8]. Fix a sequence ϵ_n satisfying (10). From the refined type covering lemma by Berger (stated in [18]), for every type $Q \in \mathcal{P}_n(\mathcal{X})$ there exists a set \mathcal{C}_Q that completely D -covers \mathcal{T}_Q^n (i.e., for every $x^n \in \mathcal{T}_Q^n$ there exists an $\hat{x}^n \in \mathcal{C}_Q$ such that $d(x^n, \hat{x}^n) \leq D$) and \mathcal{C}_Q has rate

$$\frac{1}{n} \log |\mathcal{C}_Q| \leq R(Q, D) + J(|\mathcal{X}|, |\hat{\mathcal{X}}|) \frac{\log n}{n}. \quad (15)$$

where J is some function of the size of the alphabets. Consider the set \mathcal{C} that is the union of all sets that D -cover the types $Q \in \mathcal{U}_n(D, \epsilon_n)$, defined as

$$\mathcal{U}_n(D, \epsilon_n) := \{Q \in \mathcal{P}_n(\mathcal{X}) : R(Q, D) < R(P, D) + \epsilon'_n, \quad \|Q - P\|_1 \leq \epsilon_n / \sqrt{V(P, D)}\}. \quad (16)$$

where $\epsilon'_n := \epsilon_n - J(|\mathcal{X}|, |\hat{\mathcal{X}}|) \frac{\log n}{n} - |\mathcal{X}| \frac{\log(n+1)}{n}$. The second constraint on the ℓ_1 distance of the type Q to the true distribution P is to ensure that $R(\cdot, D)$ is differentiable. This is also done in [15, Theorem 4]. Note that if ϵ_n satisfies (10) so does ϵ'_n . Now, consider the size of \mathcal{C} :

$$\begin{aligned} |\mathcal{C}| &= \sum_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) < R(P, D) + \epsilon'_n} |\mathcal{C}_Q| \\ &\leq (n+1)^{|\mathcal{X}|} \exp \left[n \left(R(Q^*, D) + J(|\mathcal{X}|, |\hat{\mathcal{X}}|) \frac{\log n}{n} \right) \right] \\ &\leq \exp [n(R(P, D) + \epsilon_n)] \end{aligned} \quad (17)$$

The first inequality applies (15) and the type counting lemma. Furthermore, Q^* is the dominating type. The second inequality applies the definitions of \mathcal{U}_n and ϵ'_n . Take f_n to be the function that maps a sequence $x^n \in \mathcal{X}^n$ with type P_{x^n} to a predefined index in $\mathcal{C} = \cup_{Q \in \mathcal{U}_n} \mathcal{C}_Q$ and take φ_n to be the function that maps the index to the reproduction sequence in $\mathcal{C}_{P_{x^n}}$ that D -covers x^n . Now, we evaluate the error probability, which is the P^n -probability of the types not in $\mathcal{U}_n(D, \epsilon_n)$. Consider,

$$\begin{aligned} &\mathbb{P}(R(P_{X^n}, D) \geq R(P, D) + \epsilon'_n) \\ &\leq \sum_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) \geq R(P, D) + \epsilon'_n} P^n(\mathcal{T}_Q^n) \\ &\leq \sum_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) \geq R(P, D) + \epsilon'_n} \exp(-nD(Q \| P)) \\ &\leq (n+1)^{|\mathcal{X}|} \exp[-nF(P, R(P, D) + \epsilon'_n, D)], \end{aligned} \quad (18)$$

where we applied the type counting lemma and the definition of Marton's exponent in the last line. Next, from [19],

$$\mathbb{P}(\|P_{X^n} - P\|_1 > \epsilon_n / \sqrt{V}) \leq 2^{|\mathcal{X}|} \exp[-n\epsilon_n^2 / (2V)]. \quad (19)$$

Combining (18) and (19) with the union bound,

$$\begin{aligned} &e(f_n, \varphi_n, P, D) \\ &\leq 2 \exp \left[-n \left(\frac{\epsilon_n'^2}{2V(P, D)} - o(\epsilon_n'^2) - \frac{|\mathcal{X}| \log(n+1)}{n} \right) \right], \end{aligned}$$

where we invoked Lemma 2 with $\epsilon'_n = o(1)$ in the role of δ . Now, we take the logarithm and normalize by $n\epsilon_n'^2$ to assert the achievability part of the theorem in (11). Note that we used the fact that $\frac{\log n}{n\epsilon_n'^2} \rightarrow 0$.

Now for the converse, we fix a code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ of rate $R_n = \frac{1}{n} \log \|f_n\| \leq R(P, D) + \epsilon_n$ and observe that

$$e(f_n, \varphi_n, P, D) \geq \mathbb{P}(d(X^n, \hat{X}^n) > D | \mathcal{E}_{\Psi_n}) \mathbb{P}(\mathcal{E}_{\Psi_n}). \quad (20)$$

where the event $\mathcal{E}_{\Psi_n} := \{R(P_{X^n}, D) \geq R_n + \Psi_n\}$ and P_{X^n} is the type of X^n . From the converse of the type covering lemma [20, Lemma 3], for any type $Q \in \mathcal{P}_n(\mathcal{X})$ such that $R(Q, D) > R$, the fraction of \mathcal{T}_Q^n that is covered by any set is no greater than $\exp[-n(R(Q, D) - R + K(|\mathcal{X}|, |\hat{\mathcal{X}}|) \frac{\log n}{n})]$. Hence, the first term above can be bounded as

$$\begin{aligned} &\mathbb{P}(d(X^n, \hat{X}^n) > D | \mathcal{E}_{\Psi_n}) \\ &\geq 1 - \exp \left[-n \left(\Psi_n + K(|\mathcal{X}|, |\hat{\mathcal{X}}|) \frac{\log n}{n} \right) \right] \end{aligned} \quad (21)$$

Put $\Psi_n := (K(|\mathcal{X}|, |\hat{\mathcal{X}}|) + 1) \frac{\log n}{n}$. Then, (21) yields

$$\mathbb{P}(d(X^n, \hat{X}^n) > D | \mathcal{E}_{\Psi_n}) \geq 1 - \frac{1}{n} \geq \frac{1}{2}. \quad (22)$$

Hence, it remains to bound the second term in (20). Let $\epsilon'_n := \epsilon_n + \Psi_n$ and consider,

$$\begin{aligned} P^n(\mathcal{E}_{\Psi_n}) &= \mathbb{P}(R(P_{X^n}, D) - R(P, D) \geq R - R(P, D) + \Psi_n) \\ &\geq \mathbb{P}(R(P_{X^n}, D) - R(P, D) \geq \epsilon_n + \Psi_n) \\ &= \sum_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) \geq R(P, D) + \epsilon'_n} P^n(\mathcal{T}_Q^n) \\ &\geq \sum_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) \geq R(P, D) + \epsilon'_n} \frac{\exp(-nD(Q \| P))}{(n+1)^{|\mathcal{X}|}} \\ &\geq (n+1)^{-|\mathcal{X}|} \exp[-nD(Q^{(n)} \| P)] \end{aligned} \quad (23)$$

where the first inequality is from the definition of $R_n \leq R(P, D) + \epsilon_n$ and in the last inequality we defined the type $Q^{(n)} := \arg \min_{Q \in \mathcal{P}_n(\mathcal{X}): R(Q, D) \geq R(P, D) + \epsilon'_n} D(Q \| P)$. In the appendix, we prove the following key continuity statement.

Lemma 3. *If ϵ'_n satisfies (10), the types $Q^{(n)}$ satisfy*

$$\lim_{n \rightarrow \infty} \frac{D(Q^{(n)} \| P)}{F(P, R(P, D) + \epsilon'_n, D)} = 1. \quad (24)$$

Let $\eta > 0$. For n large enough, the ratio in (24) is smaller than $1 + \eta$. Uniting (20) – (24) yields

$$\begin{aligned} &e(f_n, \varphi_n, P, D) \\ &\geq \frac{1}{2} (n+1)^{-|\mathcal{X}|} \exp[-n(1 + \eta)F(P, R(P, D) + \epsilon'_n, D)] \\ &\geq \frac{1}{2} (n+1)^{-|\mathcal{X}|} \exp \left[-n(1 + \eta) \left(\frac{\epsilon_n'^2}{2V(P, D)} + o(\epsilon_n'^2) \right) \right]. \end{aligned}$$

The last inequality is an application of Lemma 2 with $\epsilon'_n = o(1)$ in the role of δ . Now, we take the logarithm and normalize by $n\epsilon_n'^2$ to establish the converse noting that η is arbitrary, $\Psi_n = O(\frac{\log n}{n})$ and $\frac{\log n}{n\epsilon_n'^2} \rightarrow 0$. The latter allows us to assert that $\epsilon'_n / \epsilon_n \rightarrow 1$. ■

Note that the *multiplicative* nature of (24) is necessary to establish Theorem 1. The analysis for the probability of correct decoding $1 - e(f_n, \varphi_n, P, D)$ in the MD regime is analogous and is stated in the following:

Theorem 4. *Let ϵ_n be any positive sequence satisfying (10). Assume that $R(Q, D)$ is twice differentiable w.r.t. Q in a neighborhood of P and $V(P, D) > 0$. There exists a rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \geq R(P, D) - \epsilon_n$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log(1 - e(f_n, \varphi_n, P, D)) \geq -\frac{1}{2V(P, D)}. \quad (25)$$

Furthermore, every rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \geq R(P, D) - \epsilon_n$ must satisfy

$$\limsup_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log(1 - e(f_n, \varphi_n, P, D)) \leq -\frac{1}{2V(P, D)}. \quad (26)$$

Proof: Similar to Theorem 1. ■

IV. QUADRATIC GAUSSIAN SOURCE CODING

We now turn our attention to the quadratic Gaussian setting where X^n is a length- n vector whose entries are identically distributed as zero-mean Gaussians with variance σ^2 . The distortion measure is $d(x, \hat{x}) := (x - \hat{x})^2$. It is known [1] that in this case, the rate-distortion function takes the form

$$R(\sigma^2, D) = \frac{1}{2} \log \max \left\{ 1, \frac{\sigma^2}{D} \right\}. \quad (27)$$

Furthermore, Ihara and Kubo [3] showed that the analogue of Marton's exponent in (7) also holds in the Gaussian setting. Indeed, it is shown that the excess distortion exponent is

$$F(\sigma^2, R, D) = \frac{1}{2} \left[\frac{D}{\sigma^2} e^{2R} - 1 - \log \left(\frac{D}{\sigma^2} e^{2R} \right) \right]. \quad (28)$$

whenever $R > R(\sigma^2, D)$ and zero otherwise. The exponent for correct decoding $G(\sigma^2, R, D)$ takes the same form as in (28) when $R < R(\sigma^2, D)$ and zero otherwise. In this case, it is easy to show by direct differentiation of $F(\sigma^2, R, D)$ (or $G(\sigma^2, R, D)$) that the dispersion for lossy source coding is

$$V(\sigma^2, D) = \frac{1}{2}, \quad (29)$$

for all σ^2 and all D . In analogy to Theorem 1, we have the following in the quadratic Gaussian setting:

Theorem 5. *Let ϵ_n be any positive sequence satisfying*

$$\lim_{n \rightarrow \infty} \epsilon_n = 0, \quad \lim_{n \rightarrow \infty} n\epsilon_n^2 = \infty. \quad (30)$$

There exists a rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \leq R(\sigma^2, D) + \epsilon_n$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, \sigma^2, D) \leq -1. \quad (31)$$

Furthermore, every rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \leq R(P, D) + \epsilon_n$ must satisfy

$$\liminf_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, \sigma^2, D) \geq -1. \quad (32)$$

In contrast to the DMS case, the dispersion for the quadratic Gaussian case (29) is constant. Hence, the exponents in (31) and (32) are also constant. Also note from (30) that the requirement on ϵ_n is less stringent than in the DMS case (10). In particular, the log factor is no longer required. This is because the method of types is not used in the proof.

Proof: Fix the sequence ϵ_n . For the direct part, let us consider the set of “empirical variances”

$$\mathcal{U}_n(D, \epsilon_n) := \{\hat{\sigma}^2 : |R(\hat{\sigma}^2, D) - R(\sigma^2, D)| < \epsilon'_n\}, \quad (33)$$

where $\epsilon'_n := \epsilon_n - \frac{5 \log n}{2n} - \frac{\log 6}{n}$. By using the definition of $R(\sigma^2, D)$ in (27), it is easy to see that $\hat{\sigma}^2 \in \mathcal{U}_n$ if and only if $e^{-2\epsilon'_n} < \hat{\sigma}^2/\sigma^2 < e^{2\epsilon'_n}$. We now use a result by Verger-Gaugry [21, Theorem 1.2], which in our context, says that $6n^{5/2}(\sigma^2 e^{2\epsilon'_n}/D)^{n/2}$ reconstruction points suffice to D -cover length- n vectors x^n whose empirical variance $\frac{1}{n} \sum_i x_i^2 \in \mathcal{U}_n$. Hence, the size of the code is bounded as

$$|\mathcal{C}| \leq 6n^{5/2}(\sigma^2 e^{2\epsilon'_n}/D)^{n/2} \leq \exp(n(R(\sigma^2, D) + \epsilon_n)), \quad (34)$$

where we used the definition of ϵ'_n . Hence, the rate $R_n \leq R(\sigma^2, D) + \epsilon_n$ as required. For the probability of excess distortion, we have

$$\begin{aligned} e(f_n, \varphi_n, \sigma^2, D) &= \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n X_i^2 \notin \mathcal{U}_n \right) \\ &\leq \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n X_i^2 > \sigma^2 e^{2\epsilon'_n} \right) + \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n X_i^2 < \sigma^2 e^{-2\epsilon'_n} \right) \\ &\leq 4 \exp \left[-\frac{n}{2} (e^{2\epsilon'_n} - 1 - 2\epsilon'_n) \right]. \end{aligned} \quad (35)$$

The first inequality is by the definition of \mathcal{U}_n and the union bound. The second is an application of the upper bound of Cramér's theorem [11] applied to the χ_1^2 -random variables X_i^2/σ^2 . Now note from Taylor's theorem that $e^{2\epsilon'_n} - 1 - 2\epsilon'_n = 2\epsilon_n'^2 + o(\epsilon_n'^2)$. Taking the logarithm, normalizing by $n\epsilon_n'^2$ and taking the upper limit of (35) yields the desired result in (31).

We now turn our attention to the converse. The gist of the proof follows from the converse in [3] but, as we shall see, the error probability analysis is more intricate. Fix codes of rates $R_n = \frac{1}{n} \log \|f_n\| \leq R(\sigma^2, D) + \epsilon_n$. Let the reproduction sequences be denoted as $\hat{x}^n(m), m \in \mathcal{M}_n$. Also, let $\mathcal{A}_n := \cup_{m \in \mathcal{M}_n} \mathcal{B}_n(\hat{x}^n(m), \sqrt{D})$ where $\mathcal{B}_n(c^n, r)$ is the n -dimensional ball centered at c^n with radius r . Now, let $\gamma_n > 0$ be such that $\text{Vol}(\mathcal{B}_n(0, \gamma_n)) = \text{Vol}(\mathcal{A}_n)$. Clearly, $\text{Vol}(\mathcal{A}_n) \leq |\mathcal{M}_n| \text{Vol}(\mathcal{B}_n(0, \sqrt{D}))$. Since $R_n = \frac{1}{n} \log |\mathcal{M}_n|$,

$$e^{nR_n} \geq \frac{\text{Vol}(\mathcal{A}_n)}{\text{Vol}(\mathcal{B}_n(0, \sqrt{D}))} = \frac{\text{Vol}(\mathcal{B}_n(0, \gamma_n))}{\text{Vol}(\mathcal{B}_n(0, \sqrt{D}))} = \left(\frac{\gamma_n}{\sqrt{D}} \right)^n.$$

Hence, we have $R(\sigma^2, D) + \epsilon_n \geq R_n \geq \frac{1}{2} \log \frac{\gamma_n^2}{D}$, i.e.,

$$\gamma_n \leq \sigma^2 e^{2\epsilon_n}. \quad (36)$$

The probability of excess distortion can be lower bounded as:

$$e(f_n, \varphi_n, \sigma^2, D) = \mathbb{P}(X^n \notin \mathcal{A}_n) \geq \mathbb{P}(X^n \notin \mathcal{B}_n(0, \gamma_n)).$$

Now define the random variables $Y_i := X_i^2/\sigma^2$ and note that the Y_i 's are χ_1^2 -distributed. With this notation, and using (36),

$$e(f_n, \varphi_n, \sigma^2, D) \geq \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n Y_i > \frac{\gamma_n}{\sigma^2} \right) \geq \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n Y_i > e^{2\epsilon_n} \right).$$

Recall that for the χ_1^2 -distribution, the cumulant generating function is $\Lambda(\theta) = -\frac{1}{2} \log(1 - 2\theta)$ and the rate function is $I(y) = \max_{\theta} \{\theta y - \Lambda(\theta)\} = \frac{1}{2}(y - 1) - \frac{1}{2} \log y$. Furthermore, $\theta^*(y) := \frac{1}{2}(1 - \frac{1}{y})$ is the maximizer. Using the standard change of measure technique for the lower bound in Cramér's theorem (see proof of [11, Theorem 2.2.3]),

$$e(f_n, \varphi_n, \sigma^2, D) \geq \beta_n \exp \left[-nI(e^{2\epsilon_n}) - \frac{n}{2}(1 - e^{-2\epsilon_n})\tau_n \right],$$

where $\beta_n := \mathbb{P}(\frac{1}{n} \sum_{i=1}^n \tilde{Y}_i \in (e^{2\epsilon_n}, e^{2\epsilon_n} + \tau_n))$ and τ_n is a sequence to be chosen. The random variables \tilde{Y}_i have (tilted) distribution $q(\tilde{y}) := \exp[\theta^*(e^{2\epsilon_n})\tilde{y} - \Lambda(\theta^*(e^{2\epsilon_n}))]p(\tilde{y})$ where

$p(\cdot)$ is the χ_1^2 distribution of the Y_i 's. By the choice of $q(\cdot)$, $\mathbb{E}_q[Y_i] = e^{2\epsilon_n}$. Put $\tau_n := \zeta\epsilon_n$ for some $\zeta > 0$. Then,

$$1 - \beta_n \leq \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n \tilde{Y}_i \leq e^{2\epsilon_n}\right) + \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n \tilde{Y}_i \geq e^{2\epsilon_n} + \tau_n\right) \\ \leq \frac{1}{2} + \frac{32}{\sqrt{n}} + \frac{2}{n\tau_n^2} = \frac{1}{2} + \frac{32}{\sqrt{n}} + \frac{2}{n\zeta^2\epsilon_n^2},$$

where in the second inequality, we applied the Berry-Esséen theorem to the first term (the third moment of \tilde{Y}_i is $15e^{-7\epsilon_n}$) and Chebyshev's inequality to the second. By (30), $\beta_n \rightarrow \frac{1}{2}$ from below. With this choice of τ_n , for n sufficiently large,

$$e(f_n, \varphi_n, \sigma^2, D) \geq \frac{1}{4} \exp[-n\epsilon_n^2(1 + \zeta + o(1))], \quad (37)$$

where applied the facts $I(e^{2\epsilon_n}) = \epsilon_n^2 + o(\epsilon_n^2)$ and $1 - e^{-2\epsilon_n} = 2\epsilon_n + o(\epsilon_n)$. The converse in (32) follows by taking the logarithm, normalizing by $n\epsilon_n^2$, taking $n \rightarrow \infty$, and finally taking $\zeta \rightarrow 0$. ■

The MD setting for the probability of correct decoding of Gaussian sources can also analyzed analogously:

Theorem 6. *Let ϵ_n be any positive sequence satisfying (30). There exists a rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \geq R(\sigma^2, D) - \epsilon_n$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log(1 - e(f_n, \varphi_n, \sigma^2, D)) \geq -1. \quad (38)$$

Furthermore, every rate-distortion code $\{(f_n, \varphi_n)\}_{n \in \mathbb{N}}$ with rates $\frac{1}{n} \log \|f_n\| \geq R(\sigma^2, D) - \epsilon_n$ must satisfy

$$\limsup_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log(1 - e(f_n, \varphi_n, \sigma^2, D)) \leq -1. \quad (39)$$

Proof: Similar to Theorem 5 and uses ideas in [3]. ■

V. CONCLUSION

In this paper, we analyzed the MD regime for lossy source coding. In analogy to (2), we showed for discrete sources that

$$\lim_{n \rightarrow \infty} \frac{1}{n\epsilon_n^2} \log e(f_n, \varphi_n, P, D) = -\frac{1}{2V(P, D)} \quad (40)$$

and for Gaussian sources the RHS of (40) is equal to -1 independent of the variance σ^2 and the distortion level D . As in [8]–[10], this reveals that the fundamental nature of the dispersion in the lossy source coding context. There are at least three avenues for future research: (i) Can the results be applied to, for instance, general sources as in [4]? (ii) Can similar analysis of the MD setting be applied to lossy source coding problems with *side information*, e.g., the Wyner-Ziv problem? (iii) What is the exact relationship between the MD and CLT regimes cf. [13]?

APPENDIX: PROOF OF LEMMA 3

Proof: The rate-distortion function is uniformly continuous. Specifically, $R(Q, D) - R(P, D) = O(\|Q - P\|_1 \log \|Q - P\|_1)$ [22]. Also, $\min_{Q \in \mathcal{P}_n(\mathcal{X})} \|Q - P\|_1 \leq |\mathcal{X}|/n$ for any $P \in \mathcal{P}(\mathcal{X})$ [11, Lemma 2.1.2] so $\min_{Q \in \mathcal{P}_n(\mathcal{X})} R(Q, D) - R(P, D) = O(\frac{\log n}{n})$ which is asymptotically dominated by

$\epsilon'_n = \omega((\frac{\log n}{n})^{1/2})$. Thus, there exist n -types in the regular-closed set $\{Q \in \mathcal{P}(\mathcal{X}) : R(Q, D) - R(P, D) \geq \epsilon'_n\}$ for n large. Let Marton's exponent be $D(Q_M^{(n)} \| P) = F(P, R(P, D) + \epsilon'_n, D)$. Then, notice that

$$\frac{D(Q^{(n)} \| P)}{D(Q_M^{(n)} \| P)} = \frac{D(Q^{(n)} \| P) - D(Q_M^{(n)} \| P)}{D(Q_M^{(n)} \| P)} + 1. \quad (41)$$

The numerator of the first term on the RHS in (41) is $O(\frac{1}{n})$ because $|D(Q^{(n)} \| P) - D(Q_M^{(n)} \| P)| = O(\|Q^{(n)} - Q_M^{(n)}\|_1)$ and $\|Q^{(n)} - Q_M^{(n)}\|_1 = O(\frac{1}{n})$. From Lemma 2, the denominator (Marton's exponent) scales as $\epsilon_n'^2/(2V(P, D)) = \omega(\frac{\log n}{n})$. Thus, the first term in (41) tends to zero and the ratio of the divergence in (23) and Marton's exponent tends to one. ■

REFERENCES

- [1] C. E. Shannon, "Coding theorems for a discrete source with a fidelity criterion," *IRE Int. Conv. Rec.*, vol. 7, pp. 142–163, 1959.
- [2] K. Marton, "Error exponent for source coding with a fidelity criterion," *IEEE Trans. on Inf. Th.*, vol. 20, no. 2, pp. 197–199, Mar 1974.
- [3] S. Ihara and M. Kubo, "Error exponent of coding for memoryless gaussian sources with a fidelity criterion," *IEICE Transactions*, vol. 83-A, no. 10, pp. 1891–1897, 2000.
- [4] K. Iriyama, "Probability of error for the fixed-length lossy source coding of general sources," *IEEE Trans. on Inf. Th.*, vol. 51, no. 4, pp. 1498–1507, Apr 2005.
- [5] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding in the finite blocklength regime," *IEEE Trans. on Inf. Th.*, pp. 2307–59, May 2010.
- [6] M. Hayashi, "Information spectrum approach to second-order coding rate in channel coding," *IEEE Trans. on Inf. Th.*, pp. 4947–66, Nov 2009.
- [7] V. Y. F. Tan and O. Kosut, "On the dispersions of three network information theory problems," *arXiv:1201.3901*, Feb 2012, [Online].
- [8] A. Ingber and Y. Kochman, "The dispersion of lossy source coding," in *Data Compression Conference (DCC)*, 2011.
- [9] V. Kostina and S. Verdú, "Fixed-length lossy compression in the finite blocklength regime: Discrete memoryless sources," in *Int. Symp. Inf. Th.*, 2011.
- [10] —, "Fixed-length lossy compression in the finite blocklength regime: Gaussian source," in *Information Theory Workshop*, 2011.
- [11] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd ed. Springer, 1998.
- [12] Y. Altug and A. B. Wagner, "Moderate deviation analysis of channel coding: Discrete memoryless case," in *Int. Symp. Inf. Th.*, 2010.
- [13] Y. Polyanskiy and S. Verdú, "Channel dispersion and moderate deviations limits for memoryless channels," in *Allerton Conference*, 2010.
- [14] I. Sason, "On Refined Versions of the Azuma-Hoeffding Inequality with Applications in Information Theory," *arXiv:1111.1977*, Nov 2011.
- [15] D.-K. He, L. A. Lastras-Montaña, E.-H. Yang, A. Jagmohan, and J. Chen, "On the redundancy of Slepian-Wolf coding," *IEEE Trans. on Inf. Th.*, vol. 55, no. 12, pp. 5607–27, Dec 2009.
- [16] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Akademiai Kiado, 1997.
- [17] V. Y. F. Tan, A. Anandkumar, L. Tong, and A. S. Willsky, "A large-deviation analysis for the maximum likelihood learning of Markov tree structures," *IEEE Trans. on Inf. Th.*, vol. 57, no. 3, pp. 1714–35, Mar 2011.
- [18] B. Yu and T. P. Speed, "A rate of convergence result for a universal d-semifaithful code," *IEEE Trans. on Inf. Th.*, vol. 39, no. 3, pp. 813–820, Mar 1997.
- [19] T. Weissman, E. Ordentlich, G. Seroussi, S. Verdú, and M. L. Weinberger, "Inequalities for the l_1 deviation of the empirical distribution," Hewlett-Packard Labs, Tech. Rep., 2003.
- [20] Z. Zhang, E.-H. Yang, and V. K. Wei, "The redundancy of source coding with a fidelity criterion: Known statistics," *IEEE Trans. on Inf. Th.*, vol. 43, no. 1, pp. 71–91, Jan 1997.
- [21] J. L. Verger-Gaugry, "Covering a ball with smaller equal balls in \mathbb{R}^n ," *Disc. and Comp. Geom.*, vol. 33, no. 1, pp. 143–155, 2005.
- [22] H. Palaiyanur and A. Sahai, "On the uniform continuity of the rate-distortion function," in *Int. Symp. Inf. Th.*, 2008.