Analysis and Practice of Uniquely Decodable One-to-One Code

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Abstract-In this paper, we consider the so-called uniquely decodable one-to-one code (UDOOC) that is formed by inserting a "comma" indicator, termed the unique word (UW), between consecutive one-to-one codewords for separation. Along this research direction, we first investigate several general combinatorial properties of UDOOCs, in particular the enumeration of the number of UDOOC codewords for any (finite) codeword length. Based on the obtained formula on the number of length-n codewords for a given UW, the per-letter average codeword length of UDOOC for the optimal compression of a given source statistics can be computed. Several upper bounds on the average codeword length of such UDOOCs are next established. The analysis on the bounds of average codeword length then leads to two asymptotic bounds for sources having infinitely many alphabets, one of which is achievable and hence tight for a certain source statistics and UW, and the other of which proves the achievability of source entropy rate of UDOOCs when both the block size of source letters for UDOOC compression and UW length go to infinity. Efficient encoding and decoding algorithms for UDOOCs are also given in this paper. Numerical results show that when grouping three English letters as a block, the UDOOCs with UW = 0001, 0000, 000001 and 000000 can respectively reach the compression rates of 3.531, 4.089, 4.115, 4.709 bits per English letter (with the lengths of UWs included), where the source stream to be compressed is the book titled Alice's Adventures in Wonderland. In comparison with the first-order Huffman code, the second-order Huffman code, the third-order Huffman code and the Lempel-Ziv code, which respectively achieve the compression rates of 3.940, 3.585, 3.226 and 6.028 bits per single English letter, the proposed UDOOCs can potentially result in comparable compression rate to the Huffman code under similar decoding complexity and yield a smaller average codeword length than that of the Lempel-Ziv code, thereby confirming the practicability of UDOOCs.

I. INTRODUCTION

The investigation of lossless source coding can be roughly classified into two categories, one for the compression of a sequence of source letters and the other for a single "one shot" source symbol [7]. A well-known representative for the former is the Huffman code, while the latter is usually referred to as the one-to-one code (OOC).

The Huffman code is an optimal entropy code that can achieve the minimum average codeword length for a given statistics of source letters. It obeys the rule of unique decodability and hence the concatenation of Huffman codewords can be uniquely recovered by the decoder. Although optimal in principle, it may encounter several obstacles in implementation. For example, the rare codewords are exceedingly long in length, thereby hampering the efficiency of decoding. Other practical obstacles include

- i) the codebook needs to be pre-stored for encoding and decoding, which might demand a large memory space for sources with moderately large alphabet size,
- ii) the decoding of a sequence of codewords must be done in sequential, not in parallel, and
- iii) erroneous decoding of one codeword could affect the decoding of subsequent codewords, i.e., error propagation.

In contrast to unique decodability, the OOC only requires an assignment of distinct codewords to the source symbols. It has been studied since 1970s [33] and is shown to achieve an average codeword length smaller than the source entropy minus a nontrivial amount of quantity called *anti-redundancy* [29]. Various research works over the years have shown that the anti-redundancy can be as large as the logarithm of the source entropy [1], [5], [6], [13], [21], [24], [26], [27], [29]–[31] In comparison with an entropy coding like Huffman code, the codewords of an OOC can be sequenced alphabetically and hence the practice of an OOC is generally considered to be more computationally convenient.

A question that may arise from the above discussion is whether we could add a "comma" indicator, termed *Unique Word* (UW) in this paper, in-between consecutive OOC codewords, and use the OOC for the lossless compression of a sequence of source letters. A direct merit of such a structure is that the alphabetically sequenced OOC codewords can be manipulated without *a priori* stored codebook at both the encoding and decoding ends. This is however achieved at a price of an additional constraint that the "comma" indicator must not appear as an internal subword¹ in the concatenation of either an OOC codeword with a comma indicator, or a comma indicator with an OOC codeword.

On the one hand, this additional constraint facilitates the fast identification of OOC codewords in a coded bit-stream and makes feasible the subsequent parallel decoding of them. On the other hand, the achievable average codeword length of a UW-forbidden OOC may increase significantly for a bad choice of UWs. Therefore, it is of theoretical importance to investigate the minimum average codeword length of a UW-forbidden OOC, in particular the selection of a proper UW that could minimize this quantity. Since the resultant UW-forbidden OOC coding system satisfies unique decodability

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¹We say $a = a_1 \dots a_m$ is not an *internal subword* of $b = b_1 \dots b_n$ if there does not exist *i* such that $b_i \dots b_{i+m-1} = a$ for all 1 < i < n-m+1. When the same condition holds for all $1 \le i \le n-m+1$ (i.e., with two equalities), we say *a* is not a *subword* of *b*.

(UD), we will refer to it conveniently as the UDOOC in the sequel.

We would like to point out that the conception of inserting UWs between consecutive words might not be new in existing applications. For example, in the IEEE 802.11 standard for wireless local area networks [3], an entity similar to the UW in a bit-stream has been specified as a boundary indicator for a frame, or as a synchronization support, or as a part of error control mechanism. In written English, punctuation marks and spacing are essential to disambiguate the meaning of sentences. However, a complete theoretical study of the UDOOC conception remains undone. This is therefore the main target of this paper. We now give a formal definition of binary UDOOCs.

Definition 1: Given UW $\mathbf{k} = k_1 k_2 \dots k_L \in \mathbb{F} \times \dots \times \mathbb{F} = \mathbb{F}^L$, where $\mathbb{F} = \{0, 1\}$, we say $C_{\mathbf{k}}(n)$ is a UDOOC of length $n \geq 1$ associated with \mathbf{k} if it contains all binary length-n tuples $\mathbf{b} = b_1 \dots b_n$ such that \mathbf{k} is not an internal subword of the concatenated bit-stream $\mathbf{k}\mathbf{b}\mathbf{k}$. As a special case, we set $C_{\mathbf{k}}(0) := \{\text{null}\}^2$ The overall UDOOC associated with \mathbf{k} , denoted by $C_{\mathbf{k}}$, is given by

$$\mathcal{C}_{k} := \bigcup_{n \ge 0} \mathcal{C}_{k}(n)$$

For a better comprehension of Definition 1, we next give an example to illustrate how a UDOOC is generated and how it is used in encoding and decoding. The way to count the number of length-n UDOOC codewords will follow.

Example 1: Suppose the UW k = 00 is chosen. In order to prohibit the concatenation of any UDOOC codeword and UW, regardless of the ordering, from containing 00 as an internal subword, the following constraints must be satisfied.

- Type-I constraints: The UW cannot be a subword of any UDOOC codeword. This means that within any codeword of length n ≥ 1:
 - (C1) "0" can only be followed by "1".
 - (C2) "1" can be followed by either "0" or "1".
- Type-II constraints: Besides the type-I constraints, the UW cannot appear as an internal subword, containing the boundary of any UDOOC codeword and UW, regardless of the ordering. This implies that except for the "null" codeword:

(C3) The first bit of a codeword cannot be "0".

(C4) The last bit of a codeword cannot be "0".

By Constraints (C1)-(C4), we can place the UDOOC codewords on a code tree as shown in Fig. 1, in which each path starting from the root node and ending at a gray-shaded node corresponds to a codeword. Thus, the codewords for UW = 00 include null, 1, 11, 101, 111, 1011, 1101, 1111, etc. It should be noted that we only show the codewords of length up to four, while the code tree actually can grow indefinitely in depth.

At the decoding stage, suppose the received bit-stream is 00100110010100111100, where we add UWs at both the



Fig. 1. UDOOC code tree for UW = 00.

left and the right ends to indicate the margins of the bitstream. This may facilitate, for example, noncoherent bitstream transmission. Then, the decoder first locates UWs and parses the bit-stream into separate codewords as 1, 11, 101 and 1111, after which the four codewords can be decoded separately (possibly in parallel) to their respective source symbols.

With the code tree representation, the number of lengthn codewords in a UDOOC code tree can be straightforwardly calculated. Let the "null"-node be placed at level 0. For $n \ge 1$, denote by a_n and b_n the numbers of "1"-nodes and "0"-nodes at the *n*th level of the code tree, respectively. By the two type-I constraints, the following recursions hold:

$$\begin{cases} a_n = a_{n-1} + b_{n-1} \\ b_n = a_{n-1} \end{cases} \text{ for } n \ge 2$$

With the initial values of $a_1 = 1$ and $a_2 = 1$, it follows that $\{a_n\}_{n=1}^{\infty}$ is the renowned Fibonacci sequence [14], i.e., $a_n = a_{n-1} + a_{n-2}$ for $n \ge 3$. This result, together with the two type-II constraints, implies that the number of length-*n* codewords is $|\mathcal{C}_{00}(n)| = a_n$ for $n \ge 1$, which according to the Fibonacci recursion is given by:

$$a_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ is the Galois conjugate of φ in number field $\mathbb{Q}(\sqrt{5})$. Thus, $|\mathcal{C}_{00}(n)|$ grows exponentially in n with base $\varphi \approx 1.618$.

We can similarly examine the choice of UW = 01 and draw the respective code tree in Fig. 2, where its type-I constraints become:

(C1) "0" can only be followed by "0".

(C2) "1" can be followed by either "0" or "1".

and no type-II constraints are required. We then obtain

$$\begin{cases} a_n = a_{n-1} \\ b_n = a_{n-1} + b_{n-1} \end{cases} \text{ for } n \ge 2,$$

and $|C_{01}(n)| = a_n + b_n = n + 1$. Although from Figs. 1 and 2, taking UW = 01 seems to provide more codewords than taking UW = 00 at small *n*, the linear growth of $|C_{01}(n)|$ with respect to codeword length *n* suggests that such choice is not as good as the choice of UW = 00 when *n* is moderately large.

²In our binary UDOOC, it is allowed to place two UWs side-by-side with nothing in-between in order to produce a *null* codeword.

The above two exemplified UWs point to an important fact that the best UW, which minimizes the average codeword length, depends on the code size required. Thus, the investigation of the efficiency of a UW may need to consider the transient superiority in addition to claiming the asymptotic winner.

In this paper, we provide efficient encoding and decoding algorithms for UDOOCs, and investigate their general combinatorial properties, in particular the enumeration of the number of codewords for any (finite) codeword length. Based on the obtained formula for $|\mathcal{C}_{k}(n)|$, i.e., the number of length-n codewords for a given UW k, the average codeword length of the optimal compression of a given source statistics using UDOOC can be computed. Classifications of UWs are followed, where two types of equivalences are specified, which are (exact) equivalence and asymptotic equivalence. UWs that are equivalent in the former sense are required to yield exactly the same minimum average codeword length for every source statistics, while asymptotic equivalence only dictates the UWs to result in the same asymptotic growth rate as codeword length approaches infinity. Enumeration of the number of asymptotic equivalent UW classes are then studied with the help of methodologies in [17] and [25]. Furthermore, three upper bounds on the average codeword length of UDOOCs are established. The first one is a general upper bound when only the largest probability of source symbols is given. The second upper bound refines the first one under the premise that the source entropy is additionally known. When both the largest and second largest probabilities of source symbols are present apart from the source entropy, the third upper bound can be used. Since these bounds are derived in terms of different techniques, actually none of the three bounds dominates the other two for all statistics. Comparison of these bounds for an English text with statistics from [36] and that with statistics from the book Alice's Adventures in Wonderland will be accordingly provided. The analysis on bounds of the average codeword length gives rise to two asymptotic bounds on ultimate per-letter average codeword length, one of which is tight for a certain choice of source statistics and UW, and the other of which leads to the achievability of the ultimate per-letter average codeword length to the source entropy rate when both the source block length for compression and UW length tend to infinity.

It may be of interest to note that the enumeration of the number of codewords, i.e., $c_{k,n} = |\mathcal{C}_k(n)|$, is actually obtained indirectly via the determination of an auxiliary quantity $s_{k,n}$, which is the number of words satisfying the type-I constraints but not necessarily the type-II constraints. By utilizing the Goulden-Jackson cluster method [16], [20], [22], [23], [32], an explicit formula for $s_{k,n}$ can be established. The desired enumeration formula for the number of length-*n* UDOOC codewords is then obtained by proving that both the so-called linear constant coefficient difference equation (LCCDE) and the asymptotic growth rate of $s_{k,n}$ and $c_{k,n}$ are identical. We next show based on the obtained formula that the all-zero UW has the largest asymptotic growth rate among all UWs of the same length, while the UW with the smallest growth rate is 00...01. Interestingly, the all-zero UW is

often the one that yields the smallest $c_{k,n}$ for small n, in contrast to UW 00...01, whose $c_{k,n}$ tops all other UWs when n is small. We afterwards demonstrate by using these two special UWs that the general encoding and decoding algorithms can be considerably simplified when further taking into consideration the structure of particular UWs. A side result from the enumeration of $c_{k,n}$ is that for all UWs, the codeword growth rate of UDOOCs will tend to $|\mathbb{F}| = 2$ as the length of the UW goes to infinity.

With regard to the compression performance of the proposed UDOOCs, numerical results show that when grouping three English letters as a block and separating the consecutive blocks by UWs, the UDOOCs with UW = 0001, 0000, 000001and 000000 can respectively reach the compression rates of 3.531, 4.089, 4.115, 4.709 bits per English letter (with the length of UWs included), where the source stream to be compressed is the book titled Alice's Adventures in Wonderland. In comparison with the first-order Huffman code, the secondorder Huffman code, the third-order Huffman code³ and the Lempel-Ziv code, which respectively achieve the compression rates of 3.940, 3.585, 3.226 and 6.028 bits per English letter, the proposed UDOOCs can potentially result in comparable compression rate to the Huffman code under similar decoding complexity and yield a smaller average codeword length than that of the Lempel-Ziv code, thereby confirming the practicability of the scheme of separating OOC codewords by UWs.

In the literature, there are a number of publications on enumeration of words in a set that forbids the appearance of a specific pattern [8]–[12]. For example, Doroslova investigated the number of binary length-n words, in which a specific subword like 1010...10 is not allowed [10]. He then extended the result to non-binary alphabet and forbidden subwords of length 3 [9], [12], and forbidden subwords of length 4 [11], as well as the so-called "good" forbidden subwords [8]. The analyses in [8]–[12] however depend on the specific structure of forbidden subwords considered, and no asymptotic examination is performed. On the other hand, algorithmic approaches have been devoted to a problem of similar (but not the same) kind, one of which is called the Goulden-Jackson clustering method [16], [20], [22], [23], [32].

Instead of enumerating the number of words internally without a forbidden pattern, some researchers investigate the inherent characteristic of such patterns. In this literature, Rivals and Rahmann [25] provide an algorithm to account for the number of *overlaps*⁴ for a given set of patterns, for which the definition will be later given in this paper for completeness (cf. Definition 4). Different from the algorithmic approach in [25], Guibas and Odlyzko established upper and lower bounds

 $^{{}^{3}}$ A *k*th-order Huffman code maps a block of *k* source letters onto a variable-length codeword.

⁴In [17] and [25], the authors actually use a different name "autocorrelation" for "overlap" originated from [16]. Specifically, they define the *autocorrelation* $\boldsymbol{v} = v_1 \cdots v_L$ of a binary length-*L* string $\boldsymbol{u} = u_1 \cdots u_L$ as a binary zero-one bit-stream of length *L* such that $v_i = 1$ if *i* is a period of \boldsymbol{u} , where *i* is said to be a period of \boldsymbol{u} when $u_j = u_{i+j}$ for every $1 \le j \le L - i$. Since the term *autocorrelation* is extensively used in other literature ilke digital communications to illustrate similar but different conception, we adopt the name of "overlap" in this paper.



Fig. 2. UDOOC code tree for UW = 01.

for the number of overlaps when the length of the concerned pattern goes to infinity [17].

The rest of the paper is organized as follows. In Section II, construction of general UDOOCs is introduced. In Section III, combinatorial properties of UDOOCs, including the enumeration of the number of codewords, are derived. In Section IV, the encoding and decoding algorithms as well as bounds on average codeword length for general UDOOCs are provided and discussed. In Section V, numerical results on the compression performance of UDOOCs are presented. Conclusion is drawn in Section VI.

II. CONSTRUCTION OF UDOOCS

In the previous section, we have seen that the code tree of a UDOOC with UW = 00 (or UW = 01) is by far a useful tool for devising its properties. Along this line, we will provide a systematic construction of code tree for general UDOOC in this section. Specifically, a digraph [2] whose directional edges meet the type-I and type II constraints⁵ from the UW will be first introduced. By the digraph, the construction of a general UDOOC code tree as well as the determination of the growth rate of UDOOC codewords with respect to the codeword length will follow.

A. Digraphs for UDOOCs

Let $\mathbf{k} = k_1 \dots k_L$ be the chosen UW of length L. Denote by $G_{\mathbf{k}} = (V, E_{\mathbf{k}})$ the digraph for the UDOOC with UW = \mathbf{k} , where $V = \mathbb{F}^{L-1}$ is the set of all binary length-(L-1) tuples, and $E_{\mathbf{k}}$ is the set of directional edges given by

$$E_{k} := \left\{ (i, j) \in V^{2} : i_{2}^{L-1} = j_{1}^{L-2} \text{ and } i_{1} j \neq k \right\}.$$
(1)

Here, we use the conventional shorthand $i_s^t = i_s i_{s+1} \dots i_t$ to denote a binary string from index s to index t, and the elements in V are interchangeably denoted by either $i = i_1 \dots i_{L-1}$ or i_1^{L-1} , depending on whichever is more convenient.



Fig. 3. Digraph G_{010} for UW k = 010.

Define the 2^{L-1} -by- 2^{L-1} adjacency matrix A_k for the digraph G_k by putting its (i + 1, j + 1)th entry as

$$(\mathbf{A}_{\mathbf{k}})_{i+1,j+1} = \begin{cases} 1, & \text{if } (i,j) \in E_{\mathbf{k}}, \\ 0, & \text{otherwise,} \end{cases}$$
(2)

where we abuse the notation by using *i* (resp. *j*) to be the integer corresponding to binary representation of $i = i_1 \dots i_{L-1}$ (resp. $j = j_1 \dots j_{L-1}$) with the leftmost bit being the most significant bit. As an example, for k = 010, we have $V = \mathbb{F}^2 = \{00, 01, 10, 11\},\$

$$E_{010} = \{(00, 00), (00, 01), (01, 11), (10, 00), (10, 01), (11, 10), (11, 11)\},\$$

 $G_{010} = (V, E_{010})$ in Fig. 3, and

$$\mathbf{A}_{010} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We remark that the adjacency matrix A_k will be used for enumerating the number of UDOOC codewords in next section.

B. Code Trees for UDOOCs

Equipped with digraph G_k , constructing the code tree for the UDOOC with UW = k becomes straightforward. Recall that a UDOOC codeword of length n is a binary n-tuple $b = b_1 \dots b_n$, satisfying that k is not an internal subword of the concatenated bit-stream kbk. As such, the traversal of the digraph for constructing a UDOOC code tree should start from the vertex $k_2^L \in V$, which corresponds to the initial "null"node in the code tree. Next, a "0"-node at level 1 is generated if both $(k_2^L, j_1^{L-1}) \in E_k$ and $j_{L-1} = 0$ are satisfied. By the same rule, the "null"-node is followed by a "1"-node at level 1 if $(k_2^L, j_1^{L-1}) \in E_k$ and $j_{L-1} = 1$. We then move the current vertex to j_1^{L-1} and draw a branch from " j_{L-1} "-node at level 1 to a followup "0"-node (resp. "1"-node) at level 2 in the code tree if $(j_1^{L-1}, \ell_1^{L-1}) \in E_k$ and $\ell_{L-1} = 0$ (resp. $\ell_{L-1} = 1$). We move the current vertex again to ℓ_1^{L-1} and re-do the above procedure to generate the nodes in the next level. Repeating

⁵For clarity of its explanation, we introduce the so-called type-I and type-II constraints in Example 1. Listing these constraints for a general UW however may be tedious and less comprehensive. As will be seen from this section, these constraints can actually be absorbed into the construction of the digraph (See specifically Eq. (1)); hence, explicitly listing of constraints becomes of secondary necessity.

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this process will complete the exploration of the nodes in the entire code tree.

Determination of the gray-shaded nodes that end a codeword can be done as follows. Since k cannot be an internal subword of kbk, a node should be gray-shaded if it is immediately followed by a sequence of offspring nodes with their binary marks equal to $k_1 \dots k_{L-1}$. The construction of the UDOOC code tree is accordingly finished.

As an example, we continue from the exemplified UW k = 010 with digraph G_k in Fig. 3 and explore its respective UDOOC code tree in Fig. 4 by following the previously mentioned procedure. By starting from the vertex $k_2^3 = 10$ that corresponds to the "null"-node, two succeeding nodes are generated since both (10,00) and (10,01) are in E_{010} (cf. Fig. 4). Now from vertex 00 that corresponds to the "0"node at level 1, we can reach either vertex 00 or vertex 01 in one transition; hence, both "0"-node and "1"-node are the succeeding nodes to the "0"-node at level 1. However, since vertex 01 can only walk to vertex 11 in one transition, the "1"node at level 1 has only one succeeding node with mark "1." Continuing this process then exhausts all the nodes in the code tree in Fig. 4. Next, all nodes that are followed by $k_1k_2 = 01$ in sequence in the code tree are gray-shaded. The construction of the code tree for the UDOOC with UW k = 010 is then completed.

We end this section by giving the type-I and type-II constraints for the exemplified code tree as follows.

- Type-I constraints:
 - (C1) "0" can be followed by either "0" or "1".

(C2) "1" can be followed by "0" only when the node prior to this "1"-node is not a "0"-node.

• Type-II constraints:

(C3) The first two bits of a UDOOC codeword cannot be "10."

(C4) The last two bits of a UDOOC codeword cannot be "01."

Note that with these constraints (in particular (C4)), one can also perform the node-shading step by first gray-shading all the nodes in the code tree, and then unshade those that end with "01" (in addition to the "1"-node at level 1 for this specific UW). Nevertheless, it may be tedious to perform the nodeunshading for a general UW. For example, when UW $\mathbf{k} =$ 01001, all nodes that end a codeword b_1^n , satisfying either $b_{n-3}b_{n-2}b_{n-1}b_nk_1 = \mathbf{k}$ or $b_{n-2}b_{n-1}b_nk_1k_2 = \mathbf{k}$, should be unshaded. This confirms the superiority of constructing the UDOOC code tree in terms of the digraph over analyzing the explicit listing of constraints from the adopted UW that are perhaps convenient only for some special UWs.

III. COMBINATORIAL PROPERTIES OF UDOOCS

A. The Determination of $|C_{\mathbf{k}}(n)|$

In this subsection, we will see that the conception of digraph G_k , in particular its respective adjacency matrix A_k , can lead to a formula for the number of length-*n* codewords, i.e., $c_{k,n} = |C_k(n)|$.

In accordance with the fact that the traversal of the digraph for constructing a UDOOC code tree should start from vertex



Fig. 4. Code tree for the UDOOC with UW k = 010.

 $k_2^L \in V$, we define a length- 2^{L-1} initial vector as $(\underline{x}_k)_{j+1} = 1$ if integer j has the binary representation k_2^L , and $(\underline{x}_k)_{j+1} = 0$, otherwise, for $0 \le j < 2^{L-1}$. It then follows that the $(\ell+1)$ th entry of row vector $\underline{x}_k^\top \mathbf{A}_k^n$ gives the number of length-n walks that end at vertex ℓ on digraph G_k , where " $^\top$ " denotes the vector/matrix transpose operation, and $\ell = \ell_1 \dots \ell_{L-1}$ is the binary representation of integer index ℓ .

However, not every length-*n* walk produces a codeword. Notably, some nodes on the code tree will be gray-shaded and some will not. Recall that k cannot be an internal subword of kbk if $b = b_1b_2...b_n$ is a codeword. This implies that bis a length-*n* codeword if, and only if, the vertex sequence k_2^L , $k_3^Lb_1$, $k_4^Lb_1^2$, ..., $b_nk_1^{L-2}$, k_1^{L-1} is a valid walk of length n+L-1 on digraph G_k . As a result, the number of length-*n* codewords equals the number of length-(n + L - 1) walks from vertex k_2^L to vertex k_1^{L-1} on digraph G_k . Following the above discussion, we define the length- 2^{L-1} ending vector \underline{y}_k as $(\underline{y}_k)_{j+1} = 1$ if integer j has the binary representation k_1^{L-1} , and $(\underline{y}_k)_{j+1} = 0$, otherwise, for $0 \le j < 2^{L-1}$. Then, the number of length-*n* codewords is given by

$$c_{\boldsymbol{k},n} := |\mathcal{C}_{\boldsymbol{k}}(n)| = \underline{x}_{\boldsymbol{k}}^{\top} \mathbf{A}_{\boldsymbol{k}}^{n+L-1} \underline{y}_{\boldsymbol{k}}.$$
(3)

B. Equivalence among UWs

Two UWs that result in the same minimum average codeword length for every source statistics should be considered *equivalent*. This leads to the following definition.

Definition 2: Two UWs k and k' are said to be *equivalent*, denoted by $k \equiv k'$, if the numbers of their length-n codewords in the corresponding UDOOCs are the same for all n, i.e.,

$$c_{\boldsymbol{k},n} = c_{\boldsymbol{k}',n}$$
 for all $n \ge 0.$ (4)

By this definition, UDOOCs associated with equivalent UWs have the same number of codewords in every code tree level; hence they achieve the same minimum average codeword length in the lossless compression of a sequence of source letters. This equivalence relation allows us to focus only on one UW in every equivalent class. It is however hard to exhaust and identify all equivalent classes of UWs of arbitrary length. Instead, we will introduce a less restrictive notion of *asymptotic equivalence* when the asymptotic compression rate of UDOOCs is concerned, and derive the number of all asymptotically equivalent classes of UWs in Section III-E.

Some properties about the (exact) equivalence of UWs are given below.

Proposition 1 (Equivalence in order reversing): UW $\mathbf{k}' = k_L \dots k_1$ is equivalent to UW $\mathbf{k} = k_1 k_2 \dots k_L$.

Proof: It follows simply from that $\mathbf{b} = b_1 b_2 \dots b_n \in C_{\mathbf{k}}$ if, and only if, $\mathbf{b}' = b_n b_{n-1} \dots b_1 \in C_{\mathbf{k}'}$.

Proposition 2 (Equivalence in binary complement): If \bar{k} is the bit-wise binary complement of k, then \bar{k} and k are equivalent.

Proof: It is a consequence of the fact that the concatenated bit-stream kbk contains k as an internal subword if, and only if, the binary complement \overline{kbk} of kbk contains \overline{k} as an internal subword.

From Propositions 1 and 2, it can be verified that there are at most four equivalent classes for UWs of length L = 4. Representative UWs for these four equivalent classes are 0000, 0001, 0100 and 0101, respectively.

C. Growth Rates of UDOOCs

In this subsection, we investigate the asymptotic growth rate of UDOOCs, of which the definition is given below.

Definition 3: Given UW k, the asymptotic growth rate of the resulting UDOOC is defined as

$$g_{\mathbf{k}} := \lim_{n \to \infty} \frac{c_{\mathbf{k},n+1}}{c_{\mathbf{k},n}}.$$
 (5)

By its definition, the asymptotic growth rate of a UDOOC indicates how fast the number of codewords grows as n increases.

It is obvious that $g_k \leq 2$ for all UWs because the upper bound of 2 is the growth rate for unconstrained binary sequences of length n. In addition, the limit in (5) must exist since it can be inferred from enumerative combinatorics [28], and also from algebraic graph theory [4], that g_k is the largest eigenvalue of adjacency matrix A_k . In the next proposition, we show that the largest eigenvalue of adjacency matrix A_k is unique for all UWs but k = 01.

Proposition 3 (Uniqueness of the largest eigenvalue of A_k): For any UW k of length $L \ge 2$ except k = 01, the largest eigenvalue of adjacency matrix A_k is unique and is real.

Proof: By Perron-Frobenius theorem [15] [19], the largest eigenvalue of adjacency matrix A_k is unique and real with algebraic multiplicity equal to 1 if G_k is a strongly connected diagrph. Thus, we only need to show that G_k is a strongly connected digraph except for k = 01.

We then argue that G_k is a strongly connected diagrph when $L \ge 3$ as follows. According to the definition of E_k in (1),



Fig. 5. Digraph G_k for UW k = 01.

the only situation that a vertex may not be strongly connected to other vertex is when $j = k_2 k_3 \cdots k_L$. This however cannot happen when $L \ge 3$ because vertex $\overline{k_1} k_2 \cdots k_{L-1}$ will connect strongly to $k_2 k_3 \cdots k_L$. The proof is completed after verifying the two cases for L = 2, i.e., G_{00} is strongly connected but G_{01} is not.

The digraph for k = 01 is plotted in Fig. 5. It clearly indicates that there is no directed path from vertex 0 to vertex 1. In fact, the algebraic multiplicity of the largest eigenvalue 1 of A_{01} is two.

By the standard technique of using an indeterminate z in enumerative combinatorics, we can enumerate the numbers $c_{k,n}$ as

$$\sum_{n=0}^{\infty} c_{\boldsymbol{k},n} z^{n} = \sum_{n=0}^{\infty} \underline{x}_{\boldsymbol{k}}^{\top} \mathbf{A}_{\boldsymbol{k}}^{n+L-1} \underline{y}_{\boldsymbol{k}} z^{n}$$

$$= \underline{x}_{\boldsymbol{k}}^{\top} \left(\sum_{n=0}^{\infty} \mathbf{A}_{\boldsymbol{k}}^{n} z^{n} \right) \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}}$$

$$= \underline{x}_{\boldsymbol{k}}^{\top} (\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} z)^{-1} \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}}$$

$$= \frac{\underline{x}_{\boldsymbol{k}}^{\top} \operatorname{adj} (\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} z) \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}}, \quad (6)$$

where the first equality follows from (3) and I denotes the identity matrix of proper size. Equation (6) then implies that det $(I - A_k z)$ can give a linear recursion of $c_{k,n}$ in the form of a linear constant coefficient difference equation (LCCDE).

Now let $\lambda_1, \ldots, \lambda_m$ be distinct nonzero eigenvalues of adjacency matrix \mathbf{A}_k with algebraic multiplicities e_1, \ldots, e_m , respectively, where we assume with no loss of generality that $|\lambda_1| \geq \cdots \geq |\lambda_m|$. In terms of the standard technique of partial fraction for rational functions, we can rewrite (6) as

$$\frac{\underline{x}_{\boldsymbol{k}}^{\top} \operatorname{adj}\left(\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} z\right) \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}}}{\operatorname{det}\left(\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} z\right)} = \sum_{i=1}^{m} \frac{p_i(z)}{(1 - \lambda_i z)^{e_i}}$$
(7)

for some polynomials $p_i(z)$. The next step is expectantly to rewrite the righ-hand-side (RHS) of (7) as a power series of indeterminate z in order to recover the actual values of $c_{k,n}$ for all n. As an example, this can be done by

$$\frac{1}{\left(1-\lambda_{i}z\right)^{e_{i}}} = \sum_{n=0}^{\infty} \binom{n+e_{i}-1}{n} \lambda_{i}^{n} z^{n}$$

(13)

which holds for all $|z| < \min_{1 \le i \le m} \frac{1}{|\lambda_i|}$.

Although the asymptotic growth rate g_k equals exactly the largest eigenvalue of adjacency matrix A_k , it is in general difficult to find a closed-form expression for this value without a proper reshaping of adjacency matrix A_k . Another approach is to consider the following set for $n \ge L$,

$$S_{k}(n) := \{ \boldsymbol{b} \in \mathbb{F}^{n} : \boldsymbol{k} \text{ is not a subword of } \boldsymbol{b} \},$$
 (8)

which, in a way, defines the set of distinct length-*n* walks on digraph $G_{\mathbf{k}}$. Denoting $s_{\mathbf{k},n} := |\mathcal{S}_{\mathbf{k}}(n)|$ and by an argument similar to (6), one can easily show that

$$\sum_{n=0}^{\infty} s_{\boldsymbol{k},n} z^n = \sum_{n=0}^{L-1} 2^n z^n + z^L \frac{\underline{1}^\top \operatorname{adj} (\underline{1} - \underline{A}_{\boldsymbol{k}} z) \underline{1}}{\det (\underline{1} - \underline{A}_{\boldsymbol{k}} z)}, \quad (9)$$

where $\underline{1}$ is the all-one column vector of appropriate length. Equation (9) then implies that the enumeration of $s_{k,n}$ also depends upon the polynomial det $(\mathbb{I} - \mathbb{A}_k z)$ as $c_{k,n}$ does. Based on this observation, we can infer and prove that $s_{k,n}$ has the same asymptotic growth rate as $c_{k,n}$. We summarize this important inference in the proposition below, while the proof will be relegated to the next subsection.

Proposition 4: For any UW k, sequences $\{c_{k,n}\}_{n=0}^{\infty}$ and $\{s_{k,n}\}_{n=0}^{\infty}$ have the same asymptotic growth rate, i.e.,

 $g_{\mathbf{k}} = \mathfrak{g}_{\mathbf{k}},$

where

$$\mathfrak{g}_{k} := \lim_{n \to \infty} \frac{s_{k,n+1}}{s_{k,n}}.$$

Notably, in order to distinguish the asymptotic growth rate of $s_{k,n}$ from that of $c_{k,n}$, a different font \mathfrak{g}_k is used to denote the asymptotic growth rate of $s_{k,n}$.

D. Enumeration of $s_{k,n}$

Enumerating $s_{k,n}$ turns out to be easier than enumerating $c_{k,n}$ due to that there is lesser number of constraints on the sequences in $S_k(n)$. It can be done by an approach similar to the *Goulden-Jackson clustering method* [23]. Before delivering the main theorems, we define the *overlap function* and *overlap vector* of a binary stream k as follows.

Definition 4: For a given k of length L, its overlap function is defined as

$$r_{k}(i) := \begin{cases} 1 & \text{if } k_{i+1}^{L} = k_{1}^{L-i} \text{ and } 0 \le i \le L-1 \\ 0 & \text{otherwise.} \end{cases}$$
(10)

Furthermore, we define its length-L overlap vector as $(\underline{r}_k)_j = r_k(j-1)$ for j = 1...L.

Theorem 1: For a length-L UW \boldsymbol{k} with overlap function $r_{\boldsymbol{k}}(i)$,

$$\sum_{n\geq 0} s_{\mathbf{k},n} z^n = \frac{1 + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^i}{(1 - 2z) \left(1 + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^i\right) + z^L}.$$
 (11)

Moreover, let $h_{k}(z)$ denote the denominator of (11), i.e.,

$$h_{\mathbf{k}}(z) = (1-2z) \left(1 + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^i \right) + z^L.$$
 (12)

Then

$$h_{\boldsymbol{k}}(z) = \det(\mathbf{I} - \mathbf{A}_{\boldsymbol{k}}z),$$

where A_k is the adjacency matrix associated with digraph G_k .

Proof: The result (11) follows from the Goulden-Jackson clustering method [23]. For completeness, a simplified proof to this claim is provided in Appendix A. To establish the second claim, i.e., (13), we combine (9) and (11) to give

$$\frac{1 + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^{i}}{(1 - 2z)(1 + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^{i}) + z^{L}} = \frac{f(z)}{\det(\mathbf{I} - \mathbf{A}_{\mathbf{k}} z)}$$

for some polynomial f(z). Notice that the left-hand-side (LHS) is an irreducible rational function in z. Furthermore, Proposition 10 in Appendix B shows $\deg \det(I - A_k z) = L$. These then imply that

$$\det(\mathbf{I} - \mathbf{A}_{\mathbf{k}}z) = h_{\mathbf{k}}(z)$$

and

$$f(z) = 1 + \sum_{i=1}^{L-1} r_{k}(i) z^{i}.$$

(13) is thus established.

The next example illustrates the usage of the above theorem to the target result of Proposition 4.

Example 2: Consider the case of k = 000. Then from (10), the corresponding overlap function is

$$r_{000}(i) = \begin{cases} 1, & i = 0, 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting the above into (11), we obtain

$$\sum_{n \ge 0} s_{000,n} z^n = \frac{1+z+z^2}{1-z-z^2-z^3}.$$

By regarding the above as an LCCDE, we conclude that the sequence $s_{000,n}$ satisfies the following recursion for all $n \ge 0$:

$$s_{000,n} = s_{000,n-1} + s_{000,n-2} + s_{000,n-3} + \delta_n + \delta_{n-1} + \delta_{n-2},$$

where

 $\delta_n = \begin{cases} 1, & n = 0\\ 0, & \text{otherwise} \end{cases}$

is the Kronecker delta function.

Equipped with Theorem 1, we are now ready to prove Proposition 4.

Proof of Proposition 4: From the proof of Theorem 1, we have seen that the enumeration of $s_{k,n}$ is given by the following irreducible rational function

$$\sum_{n=0}^{\infty} s_{\boldsymbol{k},n} z^n = \frac{1 + \sum_{i=1}^{L-1} r_{\boldsymbol{k}}(i) z^i}{h_{\boldsymbol{k}}(z)}$$

where $h_k(z)$ is the denominator of (11) and is given by (12). Hence, it follows from the standard partial fraction technique and Proposition 3 that

$$\mathfrak{g}_{\boldsymbol{k}} = \max\{|u|^{-1} : h_{\boldsymbol{k}}(u) = 0, u \in \mathbb{C}\},\$$

where \mathbb{C} is the set of complex numbers. Next, noticing that the function $h_{k}(z)$, i.e., $\det(\mathbb{I} - A_{k}z)$, also appears as the denominator of the enumeration function for $c_{k,n}$ (cf. (6)), we get

$$g_{\mathbf{k}} \le \max\{|u|^{-1} : h_{\mathbf{k}}(u) = 0, u \in \mathbb{C}\},\$$

since the rational function in (6) could be reducible. This shows $g_k \leq \mathfrak{g}_k$.

To prove $g_{k} \geq g_{k}$ (which then implies $g_{k} = g_{k}$), it suffices to show that $c_{k,n+2} \geq s_{k,n}$ for $n \geq L$. This can be done by substantiating that for any $b \in S_{k}(n)$, there exist a prefix bit pand a suffix bit q, where $p, q \in \mathbb{F}$, such that $pbq \in C_{k}(n+2)$.

Using the prove-by-contradiction argument, we first assume that k is an internal subword of both kpb and $k\bar{p}b$, where $\bar{p} = 1 - p$. This assumption, together with $b \in S_k(n)$, implies the existence of indices 1 < i < L + 2 and 1 < j < L + 2 such that

$$\underbrace{k_i \cdots k_L p b_1 \cdots b_{i-2}}_{=a} = \underbrace{k_j \cdots k_L \bar{p} b_1 \cdots b_{j-2}}_{=\tilde{a}} = k, \quad (14)$$

where we abuse the notations to let

$$\boldsymbol{a} = \begin{cases} k_i \cdots k_L p, & \text{if } i = 2\\ k_i \cdots k_L p b_1 \cdots b_{i-2}, & \text{if } 2 < i < L+1\\ p b_1 \cdots b_{i-2}, & \text{if } i = L+1 \end{cases}$$
(15)

and similar notational abuse is applied to \tilde{a} and j. Assume without loss of generality that i < j. Then, the sums of the last (j-1) bits of a and \tilde{a} must equal, i.e.,

$$k_{L-(j-i-1)} + \dots + k_L + p + b_1 + \dots + b_{i-2} = \bar{p} + b_1 + \dots + b_{j-2}.$$

Canceling out common terms at both sides gives

$$k_{L-(j-i-1)} + \dots + k_L + p = \bar{p} + b_{i-1} + \dots + b_{j-2}.$$
 (16)

Note again that $\tilde{a} = k$; hence, substituting $b_{(j-2)-\ell}$ by $k_{L-\ell}$ for $\ell = 0, 1, \dots, j-i-1$ in (16) gives $p = \bar{p}$, which contradicts the assumption that $\bar{p} = 1 - p$.

For the suffix bit q, we again assume to the contrary that there exist indices i and j, satisfying n+1-L < i < j < n+2, such that

$$\underbrace{b_{i}\cdots b_{n}qk_{1}\cdots k_{L-n+i-2}}_{=d} = \underbrace{b_{j}\cdots b_{n}\bar{q}k_{1}\cdots k_{L-n+j-2}}_{=\tilde{d}} = k$$
(17)

After canceling out common terms in the respective sums of the first (n+2-i) bits of d and \tilde{d} , we obtain

$$b_i + \dots + b_{j-1} + q = \bar{q} + k_1 + \dots + k_{j-i}.$$

Since d = k, the above implies $q = \bar{q}$, which again leads to a contradiction.

One application of the result $h_k(z) = \det(I - A_k z)$ in Theorem 1 is to obtain a recursion formula for $c_{k,n}$, i.e., an LCCDE for $c_{k,n}$. This is provided in the next corollary. Corollary 1: For a length-L UW k with overlap function $r_{k}(i)$, let $c_{k,n}$ be the number of length-n codewords in the UDOOC C_{k} defined as before. Then, for $n \geq L$,

$$c_{\boldsymbol{k},n} = \left[\sum_{i=1}^{L-1} r_{\boldsymbol{k}}(i) \left(2c_{\boldsymbol{k},n-i-1} - c_{\boldsymbol{k},n-i}\right)\right] + 2c_{\boldsymbol{k},n-1} - c_{\boldsymbol{k},n-L}.$$
(18)

Proof: To prove (18), we first note that the characteristic polynomial for A_k is given by

$$\chi_{\mathbf{A}_{\mathbf{k}}}(z) = \det(z\mathbf{I} - \mathbf{A}_{\mathbf{k}})$$

= $z^{2^{L-1}}h_{\mathbf{k}}(1/z)$
= $z^{2^{L-1}-L}(z^{L}h_{\mathbf{k}}(1/z))$

where $z^{L}h_{k}(1/z)$ is a polynomial with degree

$$\deg h_{\mathbf{k}}(z) = \deg \det(\mathbf{I} - \mathbf{A}_{\mathbf{k}}z) = L$$

Denote

$$m = \min\{p > 0 : \operatorname{Nullity}(\mathbb{A}_{k}^{p}) = 2^{L-1} - L\},$$
 (19)

where Nullity() indicates the dimension of the null space of the square matrix inside parentheses. By Cayley-Hamilton Theorem [18], the following polynomial

$$\mu_{\mathbf{k}}(z) := z^{m}(z^{L}h_{\mathbf{k}}(1/z))$$

$$= z^{m}\left(z^{L} - 2z^{L-1} + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i)z^{L-i-1}(z-2) + 1\right)$$
(20)

is an annihilating polynomial for A_k . We shall remark that $\mu_k(z)$ needs not to be the minimal polynomial for A_k . Plugging (20) into (3) yields that for $n-1 \ge \max\{m, L-1\}$, we have

 $c_{k,n} = r_{\cdot}^{\top}$

$$= \underline{x}_{k}^{\top} \mathbf{A}_{k}^{n+L-1} \underline{y}_{k}$$

$$= \underline{x}_{k}^{\top} \mathbf{A}_{k}^{n-1-m} \mathbf{A}_{k}^{m+L} \underline{y}_{k}$$

$$\begin{bmatrix} & L-1 & \\ & & \end{bmatrix}$$

$$(21)$$

$$= \underline{x}_{\boldsymbol{k}}^{\top} \mathbf{A}_{\boldsymbol{k}}^{n-1} \left[2\mathbf{A}_{\boldsymbol{k}}^{L-1} + \sum_{i=1} r_{\boldsymbol{k}}(i)(2\mathbf{A}_{\boldsymbol{k}}^{L-i-1} - \mathbf{A}_{\boldsymbol{k}}^{L-i}) - \mathbf{I} \right] \underline{y}_{\boldsymbol{k}}$$
$$= \left[\sum_{i=1}^{L-1} r_{\boldsymbol{k}}(i) \left(2c_{\boldsymbol{k},n-i-1} - c_{\boldsymbol{k},n-i} \right) \right] + 2c_{\boldsymbol{k},n-1} - c_{\boldsymbol{k},n-L},$$

where the condition of $n-1 \ge \max\{m, L-1\}$ follows from i) $n-1-m \ge 0$ such that (21) holds, and ii) $n-1 \ge L-1$ such that the last term of the RHS of (22) represents $c_{k,n-L}$. Finally, since rank $(\mathbf{A}_{k}^{p}) \le L$ for p = L - 1 (see Proposition 10), we have $m \le L - 1$, which immediately gives $\max\{m, L-1\} =$ L - 1. The proof is thus completed.

So far, we learn that $c_{k,n}$ and $s_{k,n}$ have the same asymptotic growth rate, and both of their enumerations depend on det($I - A_k z$). Below we will use $s_{k,n}$ to determine the asymptotic growth rates corresponding to two specific UWs, a = 0...00and b = 0...01. We then proceed to show that a has the largest growth rate among all UWs of the same length, while the smallest growth rate is resulted when UW = b. Theorem 2: Among all UWs of the same length, the allzero UW has the largest growth rate, while UW 0...01 achieves the smallest.

Proof: For notational convenience, we set $a = 0 \dots 0$ and $b = 0 \dots 01$. For UW = a, it can be verified from (10) and (11) that

$$h_{a}(z) = 1 - \sum_{i=1}^{L} z^{i},$$
 (23)

and hence the sequence of $\{s_{a,n}\}_{n=1}^{\infty}$ satisfies the following recursion:

$$s_{\boldsymbol{a},n} = \sum_{i=1}^{L} s_{\boldsymbol{a},n-i} \quad \text{for } n \ge L.$$

Similarly, we have $h_{b}(z) = 1 - 2z + z^{L}$, and therefore,

$$s_{\boldsymbol{b},n} = 2s_{\boldsymbol{b},n-1} - s_{\boldsymbol{b},n-L}$$
 for $n \ge L$.

For general UW k of length L, (11) gives the following recursion for $n \ge L$

$$s_{\mathbf{k},n} = \sum_{i=1}^{L-1} \left(2s_{\mathbf{k},n-i-1} - s_{\mathbf{k},n-i} \right) r_{\mathbf{k}}(i) + 2s_{\mathbf{k},n-1} - s_{\mathbf{k},n-L}.$$
(24)

Note that $r_{k}(i) \in \{0, 1\}$ by definition, and $2s_{k,m-1} \ge s_{k,m}$ for all m. From (24), the following bounds hold for any UW k with $n \ge L$:

$$2s_{k,n-1} - s_{k,n-L} \leq s_{k,n} \leq \sum_{i=1}^{L} s_{k,n-i},$$
 (25)

where the lower and upper bounds are respectively obtained by replacing all $r_{k}(i)$ in (24) by 0 and 1. In particular, $s_{k,n}$ equals the upper bound in (25) when $\mathbf{k} = \mathbf{a} = 00 \cdots 0$, and the lower bound is achieved when \mathbf{k} is $\mathbf{b} = 00 \cdots 01$. By dividing all terms in (25) by $s_{k,n-1}$ and taking $n \to \infty$, we obtain

$$2 - g_{\mathbf{k}}^{-(L-1)} \leq g_{\mathbf{k}} \leq 1 + g_{\mathbf{k}}^{-1} + \dots + g_{\mathbf{k}}^{-L+1}.$$
 (26)

To prove our claim that g_a is the largest and g_b is the smallest among all g_k , we first assume to the contrary that there exists \hat{k} with $g_{\hat{k}} > g_a$. Substituting this into (26) leads to the following contradiction

$$g_{\hat{k}} \stackrel{(\mathrm{i})}{<} \sum_{i=1}^{L} g_{\boldsymbol{a}}^{-i+1} \stackrel{(\mathrm{ii})}{=} g_{\boldsymbol{a}},$$

where (i) holds because $g_{\hat{k}}^{-1} < g_{a}^{-1}$ by assumption and (ii) is valid because g_{a}^{-1} is a zero of $h_{a}(z)$ given in (23).

To show g_b achieves the minimum, again assume to the contrary that there exists \hat{k} such that $g_{\hat{k}} < g_b$. Note from (26) that

$$0 \le g_{\hat{k}} - 2 + g_{\hat{k}}^{-(L-1)} = (1 - g_{\hat{k}}) \left(g_{\hat{k}}^{-L+1} + \dots + g_{\hat{k}}^{-1} - 1 \right)$$
(27)

Although $g_{\mathbf{k}} \geq 1$ in general, we claim in this case $g_{\hat{\mathbf{k}}} > 1$. For otherwise, that $h_{\hat{\mathbf{k}}}(z = g_{\hat{\mathbf{k}}}^{-1} = 1) = 0$ according to (12) implies that $r_{\hat{\mathbf{k}}}(i) = 0$ for all *i*; hence, $h_{\hat{\mathbf{k}}}(z) = h_{\mathbf{b}}(z)$ and $g_{\hat{k}} = g_{b}$, a contradiction. Now with $1 < g_{\hat{k}} < g_{b}$, the following series of inequalities lead to the desired contradiction:

$$g_{\boldsymbol{b}} \stackrel{\text{(i)}}{=} g_{\boldsymbol{b}}^{-L+2} + \dots + 1 \stackrel{\text{(ii)}}{<} g_{\hat{\boldsymbol{k}}}^{-L+2} + \dots + 1 \stackrel{\text{(iii)}}{\leq} g_{\hat{\boldsymbol{k}}},$$

where (i) follows from $h_b(z = g_b^{-1}) = 0$ and $g_b > 1$, (ii) holds because $g_b^{-1} < g_{\hat{k}}^{-1}$, and (iii) is due to (27) and $g_{\hat{k}} > 1$.

Using a similar technique in the proof of Theorem 2, we can further devise a general upper bound and a general lower bound for g_k that hold for any k.

Theorem 3: For any UW k of length $L \ge 2$, the asymptotic growth rate g_k satisfies

$$2 - 2^{-(L-2)} \le g_k \le 2 - 2^{-L}.$$
(28)

Proof: It is straightforward to see $s_{k,n-1} \leq s_{k,n} \leq 2s_{k,n-1}$ and hence $1 \leq g_k \leq 2$.

To prove the upper bound, we assume without loss of generality that $g_k > 1$ since the upper bound trivially holds when $g_k = 1$. We then derive

$$g_{\mathbf{k}} - 1 = g_{\mathbf{k}} (1 - g_{\mathbf{k}}^{-1}) \stackrel{(i)}{\leq} 1 - g_{\mathbf{k}}^{-L} \stackrel{(ii)}{\leq} 1 - 2^{-L},$$

where (i) follows from multiplying both sides of the second inequality in (26) by $(1 - g_k^{-1})$ with the fact $g_k > 1$, and (ii) holds since $g_k \leq 2$.

To establish the lower bound, we use the following series of inequalities:

$$g_{\boldsymbol{k}}(1 - g_{\boldsymbol{k}}^{-1}) = g_{\boldsymbol{k}} - 1$$

$$\stackrel{(i)}{\geq} 1 - g_{\boldsymbol{k}}^{-(L-1)}$$

$$= (1 - g_{\boldsymbol{k}}^{-1}) \left(1 + g_{\boldsymbol{k}}^{-1} + g_{\boldsymbol{k}}^{-2} + \dots + g_{\boldsymbol{k}}^{-(L-2)}\right)$$

$$\stackrel{(ii)}{\geq} (1 - g_{\boldsymbol{k}}^{-1}) \left(1 + 2^{-1} + 2^{-2} + \dots + 2^{-(L-2)}\right)$$

$$= (1 - g_{\boldsymbol{k}}^{-1}) \left(2 - 2^{-(L-2)}\right), \qquad (29)$$

where (i) is from the first inequality in (26), and (ii) holds because $g_k \leq 2$. Equipped with (29), we next distinguish two cases to complete the proof.

- 1) When L = 2, the lower bound is trivially valid and is actually achieved by taking UW = 01 as $g_{01} = 1$ is the multiplicative inverse of the smallest zero of polynomial $h_{01}(z) = 1 - 2z + z^2 = (1 - z)^2$.
- For L > 2, it suffices to show g_k > 1. Assume to the contrary that there exists k of length L > 2 such that g_k = 1. By h_k(z = g_k⁻¹ = 1) = 0 and (12), we have r_k(i) = 0 for all i and hence h_k(z) = 1-2z+z^L. Since g_k is the multiplicative inverse of the smallest zero of h_k(z), the absolute values of all the remaining zeros of h_k(z), say λ₁,..., λ_{L-1}, must be strictly larger than 1. It then follows from the splitting of h_k(z), i.e.,

$$h_{k}(z) = (z-1) \prod_{i=1}^{L-1} (z-\lambda_{i}),$$

the constant term of $h_k(z)$ must have absolute value $\prod_{i=1}^{L-1} |\lambda_i| > 1$, contradicting to the fact that the constant term in polynomial $h_k(z) = 1 - 2z + z^L$ is 1.

Theorem 3 provides concrete explicit expressions for both upper and lower bounds on g_k . Although the bounds are asymptotically tight and well approximate the true g_k for moderately large L, they are not sharp in general. We can actually refine them using Theorem 2 and obtain that $g_b \leq g_k \leq g_a$, where from the proof of Theorem 2, we have

$$g_{a} = \max\left\{ |t| : h_{a}(z = t^{-1}) = 0, t \in \mathbb{C} \right\}$$

and

$$g_{\mathbf{b}} = \max\left\{ |t| : h_{\mathbf{b}}(z = t^{-1}) = 0, t \in \mathbb{C} \right\}$$

The determination of g_a and g_b can be done via finding the largest |s| and |t|, $0 \neq s, t \in \mathbb{C}$, such that $h_a(s^{-1}) = 0$ and $h_b(t^{-1}) = 0$, respectively. By noting that

$$(z-1) [z^{L}h_{a} (z^{-1})] = (z-1)(z^{L} - z^{L-1} - \dots - 1)$$

= $z^{L+1} - 2z^{L} + 1$

and

$$z^{L}h_{b}(z^{-1}) = z^{L} - 2z^{L-1} + 1$$

we conclude the following corollary.

Corollary 2: Let $a = 0 \dots 0$ and $b = 0 \dots 01$ be binary streams of length L. Then for any k of the same length to a and b,

$$g_{\boldsymbol{b}} \leq g_{\boldsymbol{k}} \leq g_{\boldsymbol{a}}$$

In addition, $g_a = \alpha_{L+1}$ and $g_b = \alpha_L$, where

$$\alpha_L := \max\{|t| : t^L - 2t^{L-1} + 1 = 0, t \in \mathbb{C}\}.$$

In particular, we have $\alpha_L \approx 2 - 2^{-L+1}$ for large L.

Based on Theorem 3, the following corollary is immediate by taking L to infinity.

Corollary 3: For any UW k of length L, the asymptotic growth rate of the corresponding UDOOC approaches 2 as $L \rightarrow \infty$, i.e.,

$$\lim_{L \to \infty} g_{k} = 2$$

In Table I, we illustrate the asymptotic growth rates of UDOOCs for UWs a and b with lengths up to 8. Also shown are the bounds in Theorem 3. It is seen that for moderately large L, all UDOOCs have roughly the same asymptotic growth rate, and hence are about the same good in terms of compressing sources of large size. Furthermore, having $g_k \rightarrow 2$ as $L \rightarrow \infty$ means that for very large L, UDOOCs can have asymptotic growth rates comparable to the unconstrained OOC, whose asymptotic growth rate equals 2.

TABLE I
The asymptotic growth rates for UWs $m{a}$ and $m{b}$ and the bounds
IN THEOREM 3 WITH VARIOUS L

L	2	3	4	5	6	7	8
$2 - 2^{-L}$	1.75	1.875	1.938	1.969	1.984	1.992	1.996
g_{a}	1.618	1.839	1.928	1.966	1.984	1.992	1.996
$g_{\mathbf{b}}$	1	1.618	1.839	1.928	1.966	1.984	1.992
$2 - 2^{-(L-2)}$	1	1.5	1.75	1.875	1.938	1.969	1.984

E. Asymptotic Equivalence

After presenting the results on asymptotic growth rates, we proceed to define the asymptotic equivalence for UWs and show that the number of asymptotic equivalent UW classes is upper bounded by the number of different overlap vectors in Definition 4.

Definition 5: Two UWs k and k' are said to be asymptotically equivalent, denoted by $k \stackrel{\text{a.e.}}{\equiv} k'$, if they have the same growth rate, i.e., $g_k = g_{k'}$.

Following the definition, we have the next proposition.

Proposition 5: Fix the length L of UWs, and denote by N_L the number of all possible overlap vectors of length L, i.e., $N_L = |\{\underline{r}_k : k \in \mathbb{F}^L\}|$. Then, the number of asymptotically equivalent UW classes is upper-bounded by N_L .

Proof: Since the growth rate of $s_{k,n}$ is given by $\max \{|t| : h_k (z = t^{-1}) = 0, t \in \mathbb{C}\}$, in which the polynomial $h_k(z)$, defined in (12), is completely determined by the respective overlap vector \underline{r}_k . As two different polynomials $h_k(z)$ and $h_{k'}(z)$, resulting respectively from two different overlap vectors \underline{r}_k and $\underline{r}_{k'}$, could yield the same growth rate, the number of distinct asymptotic growth rates of $s_{k,n}$ for various k must be upper-bounded by N_L . The proof is then completed after invoking the result from Proposition 4 that $s_{k,n}$ and $c_{k,n}$ have the same growth rate.

One may find the number of asymptotically equivalent UW classes by a brutal force algorithm when L is small. With the help of Proposition 5, an efficient algorithm for its upper bound N_L is available in [25], in which \underline{r}_k is regarded as *(auto)correlations* of a string. Values of N_L for various L are accordingly listed in Table II. This table shows the trend, as being pointed out in [17], that $\ln N_L$ grows at the speed of $(\ln L)^2$, or specifically,

$$\frac{1}{2\ln 2} \le \liminf_{L \to \infty} \frac{\ln N_L}{(\ln L)^2} \le \limsup_{L \to \infty} \frac{\ln N_L}{(\ln L)^2} \le \frac{1}{2\ln \frac{3}{2}}.$$
 (30)

TABLE II

 N_L values for various L. It is stated in [25] that the lower asymptotic bound $1/(2\ln(2))\approx 0.72$ in (30) only holds for very large L; hence, this lower bound is not valid for $L\leq 13$ in this table .

L	1	2	3	4	5	6	7	8	9	10	11	12	13
N_L	1	2	3	4	6	8	10	13	17	21	27	30	37
$\frac{\ln N_L}{(\ln L)^2}$	-	1.4	4.91	.72	.69	.65	.61	.59	59	.57	.57	.55	.55

IV. ENCODING AND DECODING ALGORITHMS OF UDOOCS

In this section, the encoding and decoding algorithms of UDOOCs are presented. Also provided are upper bounds for the averaged codeword length of the resulting UDOOC.

Denote by $\mathcal{U} = \{u_1, u_2, \dots, u_M\}$ the source alphabet of size M to be encoded. Assume without loss of generality that $p_1 \ge p_2 \ge \dots \ge p_M$, where p_i is the probability of occurrence for source symbol u_i .

Then, an optimal lossless source coding scheme for UDOOCs associated with UW k should assign codewords of shorter lengths to messages with higher probabilities and reserve longer codewords for less likely messages. By following this principle, the encoding mapping ϕ_k from \mathcal{U} to \mathcal{C}_k should satisfy $\ell(\phi_k(u_i)) \leq \ell(\phi_k(u_j))$ whenever $i \leq j$, where $\ell(\phi_k(u_i))$ denotes the length of bit stream $\phi_k(u_i)$. The coding system thus requires an ordering of the words in \mathcal{C}_k according to their lengths. This can be achieved in terms of the recurrence equation for $c_{k,n}$ (for example, (22)). As such, $\phi_k(u_1)$ must be the null word, and the mapping ϕ_k must always form a bijection mapping between $\{u_i : F_{k,n-1} < i \leq F_{k,n}\}$ and $\mathcal{C}_k(n)$ for every integer $n \geq 1$, where

$$F_{\boldsymbol{k},n} := \begin{cases} \sum_{i=0}^{n} c_{\boldsymbol{k},i}, & \text{if } n \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$
(31)

This optimal assignment results in average codeword length:

$$L_{\boldsymbol{k}} = \ell(\boldsymbol{k}) + \sum_{i=1}^{M} p_i \cdot \ell(\phi_{\boldsymbol{k}}(u_i)), \qquad (32)$$

where the first term $\ell(k)$ accounts for the insertion of UW k to separate adjacent codewords.

A. Upper Bounds on Average Codeword Length of UDOOCs

The average codeword length L_k is clearly a function of the source distributions and does not in general exhibit a closed-form formula. In order to understand the general compression performance of UDOOCs, three upper bounds on L_k are established in this subsection. The first upper bound is applicable to the situation when the largest probability p_1 of source symbols is given. Other than p_1 , the second upper bound additionally requires the knowledge of the source entropy. When both the largest and second largest probabilities (i.e., p_1 and p_2) of source symbols are present apart from the source entropy, the third upper bound can be used. Note that the third upper bound holds for all UWs and requires no knowledge about k; therefore, one might predict that the third upper bound could be looser than the other two. Experiments using English text from Alice's Adventures in Wonderland however indicate that such an intuitive prediction is not always valid. Nevertheless, the second upper bound is better than the first one in most cases we have examined. Details are given below.

Proposition 6 (The first upper bound on L_k): For UW k of length L, the average codeword length L_k is upper-bounded as follows:

$$L_{\boldsymbol{k}} \le L + (1 - p_1)N_{\boldsymbol{k}} \tag{33}$$

where N_{k} is the smallest integer such that $F_{k,N_{k}} \ge M$.

Proof: It can be derived from (32) and $\ell(\phi_k(u_1)) = 0$ that

$$L_{\boldsymbol{k}} = L + \sum_{i=2}^{M} p_i \ell(\phi_{\boldsymbol{k}}(u_i))$$

$$\leq L + \sum_{i=2}^{M} p_i \ell(\phi_{\boldsymbol{k}}(u_M))$$

$$= L + (1 - p_1) N_{\boldsymbol{k}}.$$

Proposition 7 (The second upper bound on L_k): Suppose $g_k > 1$. Then

$$L_{k} \leq L + \frac{\mathrm{H}(\mathcal{U}) + p_{1} \log_{2}(p_{1})}{\log_{2}(g_{k})} + (1 - p_{1})(1 - \log_{g_{k}}(K_{k})) \quad (34)$$

where $H(\mathcal{U}) = \sum_{i=1}^{M} p_i \log_2(1/p_i)$ is the source entropy with units in bits, K_k is a constant given by

$$K_{\mathbf{k}} = \min\left\{g_{\mathbf{k}}^{1-n_{i}}F_{\mathbf{k},n_{i}-1} : i = 2, \cdots, M\right\},\tag{35}$$

and n_i is the smallest integer satisfying $F_{k,n_i} \ge i$. *Proof:* From the definitions of n_i and K_k we have

$$K_{k} g_{k}^{n_{i}-1} \le F_{k,n_{i}-1} < i \le \frac{1}{p_{i}}$$

where the last inequality follows from that $p_i \leq \frac{1}{i}$ for $1 \leq i \leq M$ as $p_1 \geq p_2 \geq \cdots \geq p_M$. By $g_k > 1$ the above implies

$$h_i \leq 1 - \log_{g_k} (K_k p_i) = 1 - \log_{g_k} (p_i) - \log_{g_k} (K_k).$$

Note that $\ell(\phi_k(u_i)) \leq n_i$ by the property of optimal lossless compression function ϕ_k . Consequently, we have

$$L_{k} = L + \sum_{i=2}^{M} p_{i} \ell(\phi_{k}(u_{i}))$$

$$\leq L + \sum_{i=2}^{M} p_{i} n_{i}$$

$$\leq L + \sum_{i=2}^{M} p_{i} \left(1 - \log_{g_{k}}(p_{i}) - \log_{g_{k}}(K_{k})\right)$$

$$= L - \sum_{i=2}^{M} p_{i} \log_{g_{k}}(p_{i}) + \sum_{i=2}^{M} p_{i} \left(1 - \log_{g_{k}}(K_{k})\right)$$

$$= L + \frac{H(\mathcal{U}) + p_{1} \log_{g_{k}}(p_{1})}{\log_{2}(g_{k})} + (1 - p_{1})(1 - \log_{g_{k}} K_{k}).$$

The previous two upper bounds require the computations of either N_k , or g_k and K_k ; hence, they are functions of UW k. Next we provide a simple third upper bound that holds universally for all UWs.

Proposition 8 (The third upper bound on L_k): For UW k of length L > 2,

$$L_{k} \le L + \frac{\mathrm{H}(\mathcal{U}) + p_1 \log_2(p_1) + p_2 \log_2(p_2)}{\log_2(2 - 2^{2-L})}$$

Proof: First, we claim that

$$c_{k,n} \ge 2^{n-2}$$
 for $2 \le n \le L+1$. (37)

This claim can be established by showing that for any binary sequence $b = b_1 \dots b_m \in \mathbb{F}^m$, where $0 \le m = n - 2 \le L - 1$, there exist a prefix bit p and a suffix bit q, where $p, q \in \mathbb{F}$, such that k is not an internal subword of kpbqk. This can be done in two steps: i) there exists $q \in \mathbb{F}$ such that k is not a subword of $u := bqk_1^{L-1}$, and ii) there exists $p \in \mathbb{F}$ such that k is not an internal subword of kpu.

Because the first step trivially holds when m = 0, we only need to focus on the case of m > 0. Utilizing the prove-bycontradiction argument, we suppose that k is a subword of both bqk_1^{L-1} and $b\bar{q}k_1^{L-1}$, where $\bar{q} = 1 - q$. This implies the existence of indices $1 \le i < j \le m + 1$ such that

$$\underbrace{b_i \cdots b_m q k_1 \cdots k_{L-m+i-2}}_{= \mathbf{d}} = \underbrace{b_j \cdots b_m \bar{q} k_1 \cdots k_{L-m+j-2}}_{= \bar{\mathbf{d}}} = \mathbf{k}$$

where we abuse the notations to let

$$\boldsymbol{d} = \begin{cases} \boldsymbol{b}qk_1 \cdots k_{L-m-1}, & \text{if } i = 1\\ b_i \cdots b_m qk_1 \cdots k_{L-m+i-2}, & \text{if } 1 < i < m+1\\ qk_1 \cdots k_{L-1}, & \text{if } i = m+1 \end{cases}$$

and similar notational abuse is applied to \tilde{d} and j. After canceling out common terms in the respective sums of the first (m+2-i) bits of d and \tilde{d} , we obtain

$$b_i + \dots + b_{j-1} + q = \bar{q} + k_1 + \dots + k_{j-i}.$$

Since d = k, the above then implies $q = \bar{q}$, which leads to a contradiction. The validity of the first step is verified.

After verifying $u = bqk_1^{L-1} \in S_k(m+L)$, we can follow the proof of Proposition 4 to confirm the second step (See the paragraph regarding (14) and (15)). The claim in (37) is thus validated. Note that the equality in (37) holds when k is all-zero or all-one.

Next, we note also from the proof of Proposition 4 that $c_{k,n+2} \geq s_{k,n}$ for $n \geq L$. Since $s_{k,L} = 2^L - 1$, we immediately have $c_{k,L+2} \geq 2^L - 1$. On the other hand, we

This concludes:

$$c_{\boldsymbol{k},n} \ge \begin{cases} 1, & \text{if } 0 \le n \le 1, \\ 2^{n-2}, & \text{if } 2 \le n \le L+1, \\ (2-2^{2-L})^{n-L-2}(2^L-1), & \text{if } n \ge L+2, \end{cases}$$
(39)

 $\frac{s_{k,n}}{s_{k,n-1}} \ge 2 - 2^{2-L}$ for $n \ge L$.

where $c_{k,0} = 1$ because $C_k(0)$ contains only the null codeword, and $c_{k,1} \ge 1$ can be verified again by that k cannot be the internal subword of both kpk and $k\bar{p}k$.⁷ The lower bound (39) then indicates that if $2^L - 1 \ge (2 - 2^{2-L})^L$ for L > 2, we can immediately have the following exponential lower bound for $c_{k,n}$, i.e.,

$$c_{\boldsymbol{k},n} \ge \begin{cases} 1, & \text{if } 0 \le n \le 1, \\ (2 - 2^{2-L})^{n-2}, & \text{if } n \ge 2. \end{cases}$$
(40)

A stronger claim of $2^L - 1 \ge (2 - 2^{2-L})^L$ for L > 0 simply follows from

$$2^{L} - 1 - (2 - 2^{2-L})^{L} > 2^{L} - 1 - 2^{L-1}(2 - 2^{2-L}) = 1.$$

Hence, codeword lengths of the optimal UDOOC code must satisfy: ⁸

$$\ell(\phi_{k}(u_{i})) \leq \log_{2-2^{2-L}}(i) + 2 \text{ for } i \geq 3.$$

Consequently,

$$L_{\boldsymbol{k}} = L + \sum_{i=2}^{M} p_i \ell(\phi_{\boldsymbol{k}}(u_i))$$

⁶We can prove (38) by induction. Extending the definition of $S_{\mathbf{k}}(n)$ in (8), we obtain that $s_{\mathbf{k},n} = 2^n$ for $0 \le n < L$. This implies

$$\frac{s_{\boldsymbol{k},L}}{s_{\boldsymbol{k},L-1}} = \frac{2^L - 1}{2^{L-1}} = 2 - 2^{1-L} \ge 2 - 2^{2-L}$$

and

$$\frac{s_{\boldsymbol{k},m}}{s_{\boldsymbol{k},m-1}} = \frac{2^m}{2^{m-1}} = 2 \ge 2 - 2^{2-L} \text{ for all } 1 \le m < L.$$

Now we suppose that for some $n \ge L$ fixed, (38) is true for all $1 \le m \le n$, i.e.,

$$\frac{s_{\boldsymbol{k},m}}{s_{\boldsymbol{k},m-1}} \ge 2 - 2^{2-L} \text{ for all } 1 \le m \le n.$$

Then, we derive by (25) that

$$\begin{array}{ll} \frac{\mathbf{k}, n+1}{s_{\mathbf{k},n}} & \geq & 2 - \frac{s_{\mathbf{k},n-L+1}}{s_{\mathbf{k},n}} \geq 2 - \frac{s_{\mathbf{k},n-L+1}}{s_{\mathbf{k},n-L+1}(2-2^{2-L})^{L-1}} \\ & = & 2 - (2-2^{2-L})^{1-L} \geq 2 - 2^{2-L}. \end{array}$$

This completes the proof of (38).

⁷If it were not true, then there exist indices *i* and *j*, $2 \le i < j \le L+1$, such that $\mathbf{k} = k_i \cdots k_L p k_1 \cdots k_{i-2} = k_j \cdots k_L \bar{p} k_1 \cdots k_{j-2}$; hence, $p - k_i = \bar{p} - k_j$ with $k_i = k_j = k_1$. The desired contradiction is obtained. ⁸Pu; (21) and (40), we have that for $i \ge 2$ and $r_i = \ell(f_i - f_i)$.

⁸By (31) and (40), we have that for $i \ge 3$ and $n_i = \ell(\phi_k(u_i))$,

$$i > F_{\mathbf{k},n_i-1} = \sum_{t=0}^{n_i-1} c_{\mathbf{k},t} \ge 2 + \frac{(2-2^{2-L})^{n_i-2}-1}{1-2^{2-L}}$$

which implies $\log_{2-2^{2-L}}[(i-2)(1-2^{2-L})+1]+2 > n_i = \ell(\phi_k(u_i))$. Since $(i-2)(1-2^{2-L})+1 \le i$ for $i \ge 2-2^{L-2}$, we obtain

$$\ell(\phi_{\mathbf{k}}(u_i)) < \log_{2-2^{2-L}}\left[(i-2)(1-2^{2-L})+1\right] + 2 \le \log_{2-2^{2-L}}(i) + 2.$$

(38)

$$= L + p_{2} + \sum_{i=3}^{M} p_{i}\ell(\phi_{k}(u_{i}))$$

$$\leq L + p_{2} + \sum_{i=3}^{M} p_{i} \left(\log_{2-2^{2-L}}(i) + 2\right)$$

$$= L + 2 - 2p_{1} - p_{2} + \sum_{i=3}^{M} p_{i} \log_{2-2^{2-L}}(i)$$

$$\leq L + 2 - 2p_{1} - p_{2} + \sum_{i=3}^{M} p_{i} \log_{2-2^{2-L}}\left(\frac{1}{p_{i}}\right)$$
(41)

$$= L + 2 - 2p_1 - p_2 + \frac{\mathbf{H}(\mathcal{U}) + p_1 \log_2(p_1) + p_2 \log_2(p_2)}{\log_2(2 - 2^{2-L})}$$

where (41) follows from that $p_1 \ge p_2 \ge \cdots \ge p_i$ implies $p_i \le \frac{1}{i}$ for $1 \le i \le M$.

We next study the asymptotic compression performance of UDOOCs, i.e., the situation when the source has infinitely many alphabets. Note first that with complete knowledge of the source statistics $\{p_i : i = 1, ..., M\}$, the upper bound (34) in Proposition 7 can be reformulated using similar arguments as

$$L_{k} \leq L + \frac{\mathrm{H}(\mathcal{U}) + p_{1} \log_{2}(p_{1})}{\log_{2}(g_{k})} + (1 - p_{1})(1 - \log_{g_{k}}(T_{k})) \quad (42)$$

where T_k is given by

$$T_{k} = \min \left\{ g_{k}^{1-n_{i}} F_{k,n_{i}-1} : i = 2, \cdots, M \right\}, \quad (43)$$

and n_i is the smallest integer satisfying $F_{k,n_i} \geq \frac{1}{p_i}$. Secondly, we can further extend the above upper bound (42) to the case of grouping t source symbols (with repetition) to form a new "grouped" source for UDOOC compression. The alphabet set of the new source is therefore \mathcal{U}^t of size M^t . Let $L_{k,t}$ be the per-letter average codeword length of UDOOCs for the t-grouped source. Then, applying (42) to the t-grouped source yields the following upper bound on $L_{k,t}$

$$L_{k,t} \leq \frac{1}{t} \left(L + \frac{\mathrm{H}(\mathcal{U}^{t}) + q_1 \log_2(q_1)}{\log_2(g_k)} + (1 - q_1)(1 - \log_{g_k}(T_{k,t})) \right), \quad (44)$$

where

$$T_{k,t} = \min\left\{g_{k}^{1-n_{i,t}}F_{k,n_{i,t}-1} : i = 2, \cdots, M^{t}\right\}, \quad (45)$$

 $n_{i,t}$ is the smallest integer satisfying $F_{\mathbf{k},n_{i,t}} \geq \frac{1}{q_i}$, and q_i is the *i*th largest probability of the grouped source. For independent and identically distributed (i.i.d.) source, we have $H(\mathcal{U}^t) = t H(\mathcal{U})$. Moreover, assuming M > 1 and $p_1 < 1$ for the nontrivial i.i.d. sources, we have $q_1 = p_1^t \to 0$ as $t \to \infty$, and $T_{\mathbf{k},t}$ can be shown to converge to some finite positive constant

$$T_{k,\infty}$$
 := $\lim_{t\to\infty} T_{k,t}$

$$-\frac{\underline{x}_{\boldsymbol{k}}^{\mathsf{T}}\operatorname{adj}\left(\mathbb{I}-\mathbf{A}_{\boldsymbol{k}}z\right)\mathbf{A}_{\boldsymbol{k}}^{L-1}\underline{y}_{\boldsymbol{k}}}{\det\left(\mathbb{I}-\mathbf{A}_{\boldsymbol{k}}z\right)}\left(1-g_{\boldsymbol{k}}z\right)\right|_{z=g_{\boldsymbol{k}}^{-1}},$$

where the last step follows from the conventional expansion theory for power series and also from the fact of g_k being the unique maximal eigenvalue of the adjacency matrix A_k under L > 2 (cf. Proposition 3). To elaborate, from the power series expansion, we have that $c_{k,n} = \sum_i a_{i,n} \lambda_i^n + c$, where c is some constant, $\{\lambda_i\}$ is the set of nonzero distinct eigenvalues of A_k , and $a_{i,n}$ is the coefficient associated with λ_i (which could a polynomial function of n if λ_i has algebraic multiplicity larger than one). In particular, assuming λ_1 is the largest eigenvalue, we can establish that $T_{k,\infty} = a_{1,n} = a_1$, where the second equality emphasizes that $a_{1,n}$ is a constant independent of n since $\lambda_1^{-1} = g_k^{-1}$ is a simple zero for $h_k(z)$ when the digraph G_k is strongly connected.

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By taking limits (letting $t \to \infty$) on both sides of (44) and by noting that $\lim_{t\to\infty} q_1 = \lim_{t\to\infty} p_1^t = 0$ and $T_{k,\infty}$ is some finite positive constant, we summarize the asymptotic compression performance of UDOOCs in the next proposition.

Proposition 9: Given $g_k > 1$ and a nontrivial i.i.d. source, we have

$$\lim_{t \to \infty} L_{\boldsymbol{k},t} \le \frac{\mathrm{H}(\mathcal{U})}{\log_2(g_{\boldsymbol{k}})} \le \frac{\mathrm{H}(\mathcal{U})}{\log_2(2-2^{2-L})}.$$
 (46)

Two remarks are made based on Proposition 9. First, the larger asymptotic bound $H(\mathcal{U})/\log_2(2-2^{2-L})$ in (46) immediately gives

$$\lim_{L \to \infty} \lim_{t \to \infty} L_{\boldsymbol{k},t} = \mathbf{H}(\mathcal{U}).$$

Hence, if both t and L are sufficiently large, the per-letter average codeword length of UDOOCs can achieve the entropy rate $H(\mathcal{U})$ of the i.i.d. source. Secondly, the bound of $H(\mathcal{U})/\log_2(g_k)$ in (46) is actually achievable by taking the all-zero UW with the source being uniformly distributed. In other words,

$$\lim_{t \to \infty} L_{\boldsymbol{a},t} = \frac{\mathrm{H}(\mathcal{U})}{\log_2(g_{\boldsymbol{a}})} = \frac{\log_2(M)}{\log_2(g_{\boldsymbol{a}})},\tag{47}$$

where $a = 0 \dots 0$. For better readability, we relegate the proof of (47) to Appendix D.

Tables III and IV evaluate the bounds for the English text source with letter probabilities from [36] and a true text source from Alice's Adventure in Wonderland with empirical frequencies directly obtained from the book, respectively. The source alphabet of the English text and that from Alice's Adventure in Wonderland is of size 27, where letters of upper and lower cases are regarded the same and all symbols other than the 26 English letters are treated as one. It can be observed from Table III that bound (33) is always the best among all three bounds but still has a visible gap to the resultant average codeword length L_k . Table IV however shows that the three bounds may take turn to be on top of the other two. For example, under k = a, (33), (34) and (36) are the lowest when (L, t) = (3, 1), (L,t) = (5,2) and (L,t) = (6,3), respectively. Table IV also indicates that enlarging the value of t may help improving the per-letter average codeword length as well as the bounds of UDOOCs. Comparison of the per-letter average codeword length of UDOOCs with the source entropy will be provided later in the simulation section.

TABLE III UPPER BOUNDS (33), (34) AND (36) ON THE AVERAGE CODEWORD LENGTH $L_{\mathbf{k}}$ OF UDOOCS FOR ENGLISH TEXT SOURCE WITH LETTER PROBABILITIES FROM [36]. HERE, $\mathbf{a} = 0 \cdots 0$ and $\mathbf{b} = 0 \cdots 01$.

\boldsymbol{k}		L = 3	L = 4	L = 5	L = 6
	L_{a}	6.432	7.411	8.411	9.411
\boldsymbol{a}	(33)	8.330	9.330	10.330	11.330
	(34)	9.606	10.496	11.488	12.484
	L _b	5.215	6.185	7.185	8.185
b	(33)	6.553	7.553	8.553	9.553
	(34)	10.385	10.206	10.889	11.769
-	(36)	10.831	10.140	10.652	11.456

TABLE IVUPPER BOUNDS (33), (34) AND (36) ON THE PER-LETTER AVERAGECODEWORD LENGTH $L_{k,t}$ of UDOOCS FOR ENGLISH TEXT SOURCEFROM Alice's Adventure in Wonderland. HERE, $a = 0 \cdots 0$ and

	0 = 0 01.											
\mathbf{k}			L = 3	L = 4	L = 5	L = 6						
		t = 1	5.773	6.757	7.757	7.757						
	$L_{a,t}$	t=2	4.498	4.920	5.397	5.891						
		t = 3	3.862	4.089	4.388	4.709						
		t = 1	7.459	8.459	9.459	10.459						
\boldsymbol{a}	(33)	t=2	6.569	7.069	7.569	7.608						
		t = 3	5.770	5.786	6.119	6.134						
		t = 1	8.700	9.596	10.585	11.580						
	(34)	t=2	6.548	6.886	7.333	7.813						
		t = 3	5.771	5.586	6.120	6.135						
		t = 1	4.792	5.774	6.774	7.774						
	$L_{\boldsymbol{b},t}$	t=2	3.791	4.134	4.598	5.090						
		t = 3	3.455	3.532	3.802	4.115						
		t = 1	6.716	6.973	7.973	8.973						
b	(33)	t=2	6.108	6.147	6.647	7.147						
		t = 3	5.452	5.150	5.483	5.816						
		t = 1	9.399	9.366	10.089	10.984						
	(34)	t=2	7.356	6.819	7.040	7.435						
		t = 3	5.453	5.150	5.483	5.817						
		t = 1	9.676	9.221	9.801	10.632						
-	(36)	t=2	8.106	7.035	7.399	7.816						
		t - 3	6 9/17	5 815	5 726	5 880						

B. General Encoding and Decoding Mappings for UDOOCs

In this subsection, the encoding and decoding mappings for a UDOOC with general UW are introduced.

The practice of UDOOC requires the encoding function $\phi_{\mathbf{k}}$ to be a bijective mapping between the subset of source letters $\mathcal{U}_{\mathbf{k}}(n) := \{u_m : F_{\mathbf{k},n-1} < m \leq F_{\mathbf{k},n}\}$ and the set of lengthn codewords $\mathcal{C}_{\mathbf{k}}(n)$ for all n. Since the resulting average codeword length will be the same for any such bijective mapping from $\mathcal{U}_{\mathbf{k}}(n)$ to $\mathcal{C}_{\mathbf{k}}(n)$, we are free to devise one that facilities efficient encoding and decoding of message u_m . The bijective encoding mapping $\phi_{\mathbf{k}}$ that we propose is described in the following.

We define for any binary stream d of length $\leq n$,

$$\mathcal{C}_{\boldsymbol{k}}(\boldsymbol{d},n) := \{ \boldsymbol{c} \in \mathcal{C}_{\boldsymbol{k}}(n) : \boldsymbol{d} \text{ is a prefix of } \boldsymbol{c}, \text{ or } \boldsymbol{c} = \boldsymbol{d} \}.$$
(48)

Obviously, $C_k(d, n) \cap C_k(\tilde{d}, n) = \emptyset$ for every pair of distinct dand \tilde{d} of the same length, and for any fixed i with $1 \le i \le n$,

$$\mathcal{C}_{\boldsymbol{k}}(n) = \bigcup_{\boldsymbol{d} \in \mathbb{P}^i} \mathcal{C}_{\boldsymbol{k}}(\boldsymbol{d}, n).$$
(49)

Then, given message $u_m \in \mathcal{U}_{\mathbf{k}}(n)$, i.e., the number n is chosen such that $F_{\mathbf{k},n-1} < m \leq F_{\mathbf{k},n}$, the proposed encoding mapping $\phi_{\mathbf{k}}$ produces the codeword $\phi_{\mathbf{k}}(u_m) = c_1 c_2 \cdots c_n$ for source letter u_m recursively according to the rule that for $i = 1, 2, \ldots, n$,

$$c_{i} = \begin{cases} 0, & \text{if } \rho_{i-1} \leq |\mathcal{C}_{k}(c_{1}\cdots c_{i-1}0, n)| \\ 1, & \text{if } \rho_{i-1} > |\mathcal{C}_{k}(c_{1}\cdots c_{i-1}0, n)| \end{cases}$$
(50)

where the progressive metric ρ_i is also maintained recursively as:

$$\begin{aligned}
\rho_i &:= \rho_{i-1} - c_i \left| \mathcal{C}_{k}(c_1 \cdots c_{i-1} 0, n) \right| \\
&= \begin{cases} \rho_{i-1}, & \text{if } c_i = 0\\ \rho_{i-1} - \left| \mathcal{C}_{k}(c_1 \cdots c_{i-1} 0, n) \right|, & \text{if } c_i = 1 \end{aligned}
\end{aligned}$$
(51)

with an initial value $\rho_0 = m - F_{k,n-1}$. This encoding mapping actually assigns codewords according to their lexicographical ordering.

Example 3: Taking $\mathbf{k} = 010$ as an example, we can see from Fig. 4 that the seven codewords of length 4, i.e., 0000, 0011, 0110, 0111, 1100, 1110 and 1111, will be respectively assigned to source letters u_9 , u_{10} , u_{11} , u_{12} , u_{13} , u_{14} and u_{15} . The progressive metrics ρ_0 , ρ_1 , ρ_2 , ρ_3 for source letter u_{11} are 3, 3, 1, 1, respectively, with $|\mathcal{C}_k(0,4)| = 4$, $|\mathcal{C}_k(00,4)| = 2$, $|\mathcal{C}_k(010,4)| = 0$ and $|\mathcal{C}_k(0110,4)| = 1$.

Note again that given m (equivalently, u_m), n can be determined via $F_{k,n-1} < m \leq F_{k,n}$. At the end of the *n*th recursion, we must have

$$m = F_{\boldsymbol{k},n-1} + \sum_{i=1}^{n} c_i \left| \mathcal{C}_{\boldsymbol{k}}(c_1^{i-1}0,n) \right| + 1.$$
 (52)

We emphasize that (52) actually gives the corresponding computation-based decoding function $\psi_{\mathbf{k}} : C_{\mathbf{k}}(n) \to \mathcal{U}_{\mathbf{k}}(n)$ for codewords c of length n.

One straightforward way to implement ϕ_k and ψ_k is to pre-store the value of $|\mathcal{C}_k(d0, n)|$ for every d and n. By considering the huge number of all possible prefixes d for each n, this straightforward approach does not seem to be an attractive one.

Alternatively, we find that $|C_k(d0, n)|$ can be obtained through adjacency matrix A_k introduced in Section II. The advantage of this alternative approach is that there is no need to pre-store or pre-construct any part of the codebook C_k , and the value of $|C_k(d0, n)|$ is computed only when it is required during the encoding or decoding processes. Moreoever, for specific UWs such as 00...0, 00...01, and their binary complements, we can further reduce the required computations.

In the following subsections, we will first introduce the encoding and decoding algorithms for specific UWs as they can be straightforwardly understood. Algorithms for general UWs require an additional computation of $|C_k(d0, n)|$ and will be presented in subsequent subsections.

C. Encoding and Decoding Algorithms for $UW = 11 \dots 1$

It has been inferred from Proposition 2 that the encoding and decoding of the UDOOC with UW $\mathbf{k} = 00...0$ can be equivalently done through the encoding and decoding of the UDOOC with UW k = 11...1 as one can be obtained from the other by binary complementing. Thus, we only focus on the case of k = 11...1 in this subsection.

For this specific UW, we observe that a codeword $c = d0b \in C_k(d0, n)$ if, and only if, $0b \in C_k(n - \ell(d))$ is a codeword of length $n - \ell(d)$, where $\ell(d)$ is the length of prefix bitstream d. We thus obtain

$$|\mathcal{C}_{\boldsymbol{k}}(\boldsymbol{d}0,n)| = c_{\boldsymbol{k},n-\ell(\boldsymbol{d})}.$$
(53)

It can be shown that the LCCDE for $c_{k,n}$ with $k = 11 \dots 1$ is

$$c_{k,n} = \sum_{i=1}^{L} c_{k,n-i}, \text{ for all } n > L,$$
 (54)

where the initial values are

$$c_{\mathbf{k},n} = \begin{cases} 1, & n = 0, 1, 2, \\ 2^{n-2}, & n = 3, \dots, L. \end{cases}$$
(55)

Based on (53), (54) and (55), the algorithmic encoding and decoding procedures can be described below.

Algorithm 1 Encoding of UDOOC with $k = 11 \dots 1$

Input: Index m for message u_m

Output: Codeword $\phi_{k}(u_{m}) = c_{1} \dots c_{n}$

1: Compute $c_{k,0}, c_{k,1}, c_{k,2}, \ldots$ using (54) and (55) to determine the smallest n such that $F_{k,n} \ge m$. If n = 0, then $\phi_k(u_m) =$ null and stop the algorithm.

2: Initialize $\rho_0 \leftarrow m - F_{k,n-1}$

- 3: for i = 1 to n do
- 4: **if** $\rho_{i-1} \leq c_{k,n-i+1}$ **then** 5: $c_i \leftarrow 0$ and $\rho_i \leftarrow \rho_{i-1}$ 6: **else**
- 7: $c_i \leftarrow 1 \text{ and } \rho_i \leftarrow \rho_{i-1} c_{k,n-i+1}$ 8: **end if**
- 9: end for

Algorithm 2 Decoding of UDOOC with k = 11...1Input: Codeword $c = c_1 ... c_n$ Output: Index m for message $u_m = \psi_k(c)$ 1: Compute $c_{k,0}, c_{k,1}, ..., c_{k,n}$ using (54) and (55) 2: Initialize $m \leftarrow F_{k,n-1} + 1$ 3: for i = 1 to n do 4: if $c_i = 1$ then 5: $m \leftarrow m + c_{k,n-i+1}$

- 6: end if
- 7: end for

D. Encoding and Decoding Algorithms for $11 \cdots 10$

Again, Proposition 2 infers that the encoding and decoding of the UDOOC with UW $\mathbf{k} = 00...01$ can be equivalently done through the encoding and decoding of the UDOOC with UW $\mathbf{k} = 11...10$. We simply take $\mathbf{k} = 11...10$ for illustration. It can be derived from (12) that for $k = 11 \cdots 10$,

$$c_{k,n} = 2c_{k,n-1} - c_{k,n-L}$$
 (56)

with initial condition

$$c_{\boldsymbol{k},n} = \begin{cases} 1, & n = 0\\ 2^n, & n = 1, \dots, L - 1\\ 2^L - 1, & n = L. \end{cases}$$
(57)

It remains to determine $|C_{k}(d0, n)|$. Observe that $d0b \in C_{k}(d0, n)$ if, and only if, $b \in C_{k}(n - \ell(d) - 1)$; hence, $|C_{k}(d0, n)| = c_{k,n-\ell(d)-1}$. We summarize the encoding and decoding algorithms of UDOOCs with k = 11...10 in Algorithms 3 and 4, respectively.

Algorithm 3 Encoding of UDOOC with $k = 11 \cdots 10$

	6
In	put: Index m for message u_m
Oı	itput: Codeword $\phi_{k}(u_{m}) = c_{1} \dots c_{n}$
1	: Compute $c_{k,0}, c_{k,1}, c_{k,2}, \dots$ using (56) and (57) to deter-
	mine the smallest n such that $F_{k,n} \ge m$. If $n = 0$, then
	$\phi_{\mathbf{k}}(u_m) = \text{null and stop the algorithm.}$
2	: Initialize $\rho_0 \leftarrow m - F_{k,n-1}$
3	: for $i = 1$ to n do
4	: if $\rho_{i-1} \leq c_{k,n-i}$ then
5	$c_i \leftarrow 0 \text{ and } \rho_i \leftarrow \rho_{i-1}$

- 6: **else**
- 7: $c_i \leftarrow 1 \text{ and } \rho_i \leftarrow \rho_{i-1} c_{\boldsymbol{k},n-i}$

8: **end if**

9: end for

Algorithm 4 Decoding	of UDOOC with	$\mathbf{k} = 11 \cdots 10$
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Input: Codeword $c = c_1 \dots c_n$ Output: Index m for message $u_m = \psi_k(c)$ 1: Compute $c_{k,0}, c_{k,1}, \dots, c_{k,n}$ using (56) and (57) 2: Initialize $m \leftarrow F_{k,n-1} + 1$ 3: for i = 1 to n do 4: if $c_i = 1$ then 5: $m \leftarrow m + c_{k,n-i}$ 6: end if 7: end for

E. Encoding and Decoding Algorithms for General UW k

It is clear from the discussions in the previous two subsections as well as from (50) that to determine c_i in the encoding algorithm, we only need to keep track of the most recent ρ_{i-1} , instead of retaining sequentially all of $\rho_0, \ldots, \rho_{i-2}$. We address the recursion for the update of ρ_i in (51) only to facilitate our interpretation on the operation of the progressive metric. The same approach will be followed in the presentation of the general encoding algorithm below, where a progressive matrix D_i is used in addition to the progressive metric ρ_i .

The encoding algorithm for general UWs consists of two phases. Given the index m, we first identify the smallest n such that $F_{k,n} \ge m$. Note that the computation of $F_{k,n}$ requires the knowledge of $c_{k,n}$, which can be recursively obtained using

the LCCDE in (18). In the second phase, as seen from the two previous subsections, we need to determine the cardinality of $C_k(d, n)$ for any prefix d with $\ell(d) \leq n$. Thus, our target in this subsection is to provide an expression for $|C_k(d, n)|$ that holds for general k and d.

Define $E_{k,0}$ and $E_{k,1}$ for digraph $G_k = (V, E_k)$ as

$$E_{k,0} := \{ (i, j) \in E_k : j_{L-1} = 0 \}$$
(58)

$$E_{k,1} := \{ (i, j) \in E_k : j_{L-1} = 1 \}.$$
(59)

Literally speaking, $E_{k,0}$ (resp. $E_{k,1}$) is the set of edges in E_k , whose ending vertex has its last bit j_{L-1} equal to 0 (resp. 1). Let $A_{k,0}$ and $A_{k,1}$ be the adjacency matrices respectively for digraphs $G_{k,0} = (V, E_{k,0})$ and $G_{k,1} = (V, E_{k,1})$. Obviously, $A_k = A_{k,0} + A_{k,1}$. Based on the two adjacency matrices, we derive

$$|\mathcal{C}_{k}(\boldsymbol{d},n)| = \underline{x}_{\boldsymbol{k}}^{\top} \mathsf{D}_{i} \mathsf{A}_{\boldsymbol{k}}^{(n+L-1)-i} \underline{y}_{\boldsymbol{k}}$$
(60)

where for a prefix stream $d = d_1 \dots d_i$,

$$\mathsf{D}_i := \prod_{t=1}^i \mathsf{A}_{\boldsymbol{k}, d_t},\tag{61}$$

and \underline{x}_k and \underline{y}_k are the initial and ending vectors for digraph G_k defined in Section III-A. With (60) and (61), the general encoding and decoding algorithms are given in Algorithms 5 and 6, respectively. Verification of the two algorithms is relegated to Appendix C for better readability.

Algorithm 5 Encoding of UDOOC with General k

Input: Index m for message u_m

Output: Codeword $\phi_k(u_m) = c_1 \dots c_n$

- Compute c_{k,0}, c_{k,1}, c_{k,2},... using (18) and the method in Section IV-F to determine the smallest n such that F_{k,n} ≥ m. If n = 0, then φ_k(u_m) = null and stop the algorithm.
 Initialize ρ₀ ← m − F_{k,n-1} and D₀ ← I
- 3: for i = 1 to n do

4: Compute
$$dummy \leftarrow \underline{x}_{k}^{\top} D_{i-1} \mathbf{A}_{k,0} \mathbf{A}_{k}^{(n+L-1)-i} \underline{y}_{k}$$

5: **if** $\rho_{i-1} \leq dummy$ **then**

6:
$$c_i \leftarrow 0, \ \rho_i \leftarrow \rho_{i-1} \text{ and } \mathsf{D}_i \leftarrow \mathsf{D}_{i-1}\mathsf{A}_{k,0}$$

- 7: **else**
- 8: $c_i \leftarrow 1, \ \rho_i \leftarrow \rho_{i-1} dummy \text{ and } \mathsf{D}_i \leftarrow \mathsf{D}_{i-1}\mathsf{A}_{k,1}$
- 9: **end if**
- 10: **end for**

Algorithm 6 Decoding of UDOOC with General k

Input: Codeword $c = c_1 \dots c_n$

Output: Index m for message $u_m = \psi_k(c)$

- 1: Compute $c_{k,0}, c_{k,1}, \ldots, c_{k,n}$ using (18) and the method in Section IV-F.
- 2: Initialize $m \leftarrow F_{k,n-1} + 1$ and $D_0 \leftarrow I$
- 3: for i = 1 to n do
- 4: **if** $c_i = 1$ **then**

5:
$$m \leftarrow m + \underline{x}_{k}^{\dagger} \mathsf{D}_{i-1} \mathsf{A}_{k,0} \mathsf{A}_{k}^{(n+2-1)} \underbrace{y}_{k}$$

- 6: **end if**
- 7: $D_i \leftarrow D_{i-1}A_{k,c_i}$
- 8: end for

F. Exemplified Realization of the Encoding and Decoding Algorithms for General UW k

The matrix expressions in (60) and (61) facilitate the presentation of Algorithms 5 and 6 for general UW; however, their implementation involves extensive computation of matrix multiplications. Since the entries in each row or column of A_k are all 0's except for at most two 1's, the complexity of computing

$$c_{\boldsymbol{k},n} = \underline{x}_{\boldsymbol{k}}^{\top} \mathbb{A}_{\boldsymbol{k}}^{n+L-1} \underline{y}_{\boldsymbol{k}}$$
(62)

and

$$|\mathcal{C}_{\boldsymbol{k}}(\boldsymbol{d}0,n)| = \underline{x}_{\boldsymbol{k}}^{\top} \mathsf{D}_{\ell(\boldsymbol{d})} \mathsf{A}_{\boldsymbol{k},0} \mathsf{A}_{\boldsymbol{k}}^{n+L-\ell(\boldsymbol{d})-2} \underline{y}_{\boldsymbol{k}}$$
(63)

is in fact relatively small. Furthermore, it is much easier to compute $c_{k,n}$ than $|\mathcal{C}_k(d0, n)|$. To see this, note from (6) that we have the following enumeration for $c_{k,n}$

$$\sum_{n=0}^{\infty} c_{\boldsymbol{k},n} \boldsymbol{z}^n = \frac{\underline{\boldsymbol{x}}_{\boldsymbol{k}}^{\top} \operatorname{adj} \left(\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} \boldsymbol{z} \right) \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{\boldsymbol{y}}_{\boldsymbol{k}}}{\det \left(\mathbf{I} - \mathbf{A}_{\boldsymbol{k}} \boldsymbol{z} \right)}$$

Thus, simply evaluating the RHS of the above equation gives the values of $c_{k,n}$ for n = 1, 2, ..., L - 1. The remaining values of $c_{k,n}$ for $n \ge L$ can be easily determined through the recursion formula (18).

Another way to compute the values of $c_{k,n}$ can be easily obtained by modifying the algorithm for computing the values of $|C_k(d0, n)|$, which we now discuss. The first step to compute $|C_k(d0, n)|$ is to break up formula (63) into:

$$|\mathcal{C}_{k}(\boldsymbol{d}0,n)| = \underbrace{(\underline{x}_{\boldsymbol{k}}^{\top} \mathsf{D}_{\ell(\boldsymbol{d})})}_{\underline{u}^{\top}} \mathsf{A}_{\boldsymbol{k},0} \mathsf{A}_{\boldsymbol{k}}^{n-\ell(\boldsymbol{d})-1} \underbrace{(\mathsf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}})}_{\underline{w}_{\boldsymbol{k}}}.$$
 (64)

We note that from the choice of d in the encoding algorithm 5, we must have $|C_k(d, n)| \geq 1.^9$ Hence, $\underline{u} = \mathsf{D}_{\ell(d)}^\top \underline{x}_k$ is actually a zero-one indication vector of length 2^{L-1} for the rightmost (L-1) bits of $k_2^L d$, i.e., all components of vector $\underline{u} = [u_1 \ u_2 \ \cdots \ u_{2^{L-1}}]^\top$ are 0's except the (j+1)th component (being 1's), where j is the integer corresponding to the binary representation of the rightmost (L-1) bits of $k_2^L d$. Hence, \underline{u} can be directly determined without any computation. In addition, we can pre-compute \underline{w}_k since it is the same for all n and d. Our task is therefore reduced to computing the value of

$$|\mathcal{C}_{k}(\boldsymbol{d}\boldsymbol{0},n)| = \underline{\boldsymbol{u}}^{\top} \mathbf{A}_{k,0} \mathbf{A}_{k}^{n-\ell(\boldsymbol{d})-1} \underline{\boldsymbol{w}}_{k}.$$

Below, we demonstrate how to utilize a finite state machine based on the digraph G_k to evaluate $|C_k(d0, n)|$ without resorting to matrix operations.

Notations that are used to describe the finite state machine are addressed first. Let $\mathfrak{S} = \{s_{00\dots 0}, s_{00\dots 01}, \dots, s_{11\dots 1}\}$ be the set of states indexed by all binary bit-streams of length L-1. We say $s_i = s_{i_1\dots i_{L-1}}$ is a *counting state* if the (i+1)th component of \underline{w}_k is 1, where *i* is the integer corresponding to

⁹Given any choice of prefix d, it is possible that $|\mathcal{C}_{k}(d,n)| = 0$ if $d \notin \mathcal{C}_{k}$, and in this case we have $\underline{u} = \underline{0}$ in (64). However, the prefix d considered in our encoding algorithm, Algorithm 5, is always a prefix of some codeword; hence we have $|\mathcal{C}_{k}(d,n)| > 0$.

binary representation of $i = i_1 \dots i_{L-1}$.¹⁰ Denote by \mathfrak{C}_k the set of all counting states corresponding to k. Also, for each state $s_k \in \mathfrak{S}$ we define

$$\begin{aligned} \mathcal{I}(s_{i}) &= \{s_{j} : (j, i) \in E_{k}\}, \\ \mathcal{O}(s_{i}) &= \{s_{j} : (i, j) \in E_{k}\}, \\ \mathcal{I}_{b}(s_{i}) &= \{s_{j} : (j, i) \in E_{k, b}\}, \\ \mathcal{O}_{b}(s_{i}) &= \{s_{j} : (i, j) \in E_{k, b}\}, \end{aligned}$$

for b = 0, 1, where the edge-sets $E_{k,0}$ and $E_{k,1}$ are defined in (58) and (59), respectively. Literally speaking, from digraph G_k , $\mathcal{I}(s_i)$ is the set of states that link directionally to s_i , $\mathcal{O}(s_i)$ is the set of states that are linked directionally by s_i , and $\mathcal{I}_0(s_i)$ is the set of states that link to s_i via a so-called 0-edge in $E_{k,0}$. The sets $\mathcal{I}_1(s_i)$, $\mathcal{O}_0(s_i)$ and $\mathcal{O}_1(s_i)$ all have in a similar meaning.

In our state machine, we associate each state s_i with an integer. Without ambiguity, we use s_i to also denote the integer associated with it. Define an operator $\Xi_k : \mathfrak{S} \to \mathfrak{S}$, which updates the value associated with each state according to:

$$\Xi_{k} : s_{i} \leftarrow \sum_{s_{j} \in \mathcal{I}(s_{i})} s_{j} \text{ for all } s_{i} \in \mathfrak{S}.$$

It should be noted that the operator Ξ_k updates all states in \mathfrak{S} in a parallel fashion. Also, if $\mathcal{I}(s_i)$ is an empty set, operator Ξ_k would set $s_i \leftarrow 0$. We similarly define operators $\Xi_{k,0}$ and $\Xi_{k,1}$ respectively as

$$\Xi_{\boldsymbol{k},0} \ : \ s_{\boldsymbol{i}} \leftarrow \sum_{s_{\boldsymbol{j}} \in \mathcal{I}_0(s_{\boldsymbol{i}})} s_{\boldsymbol{j}} \text{ and } \ \Xi_{\boldsymbol{k},1} \ : \ s_{\boldsymbol{i}} \leftarrow \sum_{s_{\boldsymbol{j}} \in \mathcal{I}_1(s_{\boldsymbol{i}})} s_{\boldsymbol{j}}.$$

An example is provided below to help clarify these notations.

Example 4: For UW $\mathbf{k} = 000$ of length L = 3, there are four possible states in $\mathfrak{S} = \{s_{00}, s_{01}, s_{10}, s_{11}\}$. Because $\mathbb{A}_{\mathbf{k}}^{L-1} \underline{y}_{\mathbf{k}} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^{\top}$, we have $\mathfrak{C}_{\mathbf{k}} = \{s_{01}, s_{11}\}$. Create the 0-edges and 1-edges of the digraph in Fig. 6. Table V then shows $\mathcal{I}(s_i)$, $\mathcal{O}(s_i)$, $\mathcal{I}_0(s_i)$, $\mathcal{I}_1(s_i)$, $\mathcal{O}_0(s_i)$ and $\mathcal{O}_1(s_i)$ for each s_i .

TABLE V Various state sets for UW $\mathbf{k} = 000$

s_i	s_{00}	s_{01}	s_{10}	s_{11}
$\mathcal{I}(s_i)$	$\{s_{10}\}$	$\{s_{00}, s_{10}\}$	$\{s_{01}, s_{11}\}$	$\{s_{01}, s_{11}\}$
$\mathcal{O}(s_i)$	$\{s_{01}\}$	$\{s_{10}, s_{11}\}$	$\{s_{00}, s_{01}\}$	$\{s_{10}, s_{11}\}$
$\mathcal{I}_0(s_i)$	$\{s_{10}\}$	{}	$\{s_{01}, s_{11}\}$	{}
$\mathcal{I}_1(s_i)$	{}	$\{s_{00}, s_{10}\}$	{}	$\{s_{01}, s_{11}\}$
$\mathcal{O}_0(s_i)$	{}	$\{s_{10}\}$	$\{s_{00}\}$	$\{s_{10}\}$
$\mathcal{O}_1(s_i)$	$\{s_{01}\}$	$\{s_{11}\}$	$\{s_{01}\}$	$\{s_{11}\}$

According to the first row in Table V, the operator Ξ_k simultaneously updates all states in \mathfrak{S} according to

$$\Xi_{\boldsymbol{k}} : \begin{array}{c} s_{00} \leftarrow s_{10} \\ s_{01} \leftarrow s_{00} + s_{10} \\ s_{10} \leftarrow s_{01} + s_{11} \\ s_{11} \leftarrow s_{01} + s_{11} \end{array}$$



Fig. 6. Digraph G_{000} for UW $\mathbf{k} = 000$

Likewise, the operators $\Xi_{k,0}$ and $\Xi_{k,1}$ simultaneously update all states in \mathfrak{S} according to

$$\Xi_{k,0} : \begin{array}{c} s_{00} \leftarrow s_{10} \\ s_{01} \leftarrow 0 \\ s_{10} \leftarrow s_{01} + s_{11} \\ s_{11} \leftarrow 0 \end{array} \text{ and } \begin{array}{c} s_{00} \leftarrow 0 \\ s_{01} \leftarrow s_{00} + s_{10} \\ s_{10} \leftarrow 0 \\ s_{11} \leftarrow s_{01} + s_{11}. \end{array}$$

With the above, we now demonstrate how to compute $c_{k,i}$ and $|\mathcal{C}_k(d0, n)|$ using the finite state machine. Note $c_{k,n} = \underline{x}_k^\top \mathbf{A}_k^n \underline{w}_k$; hence to compute $c_{k,n}$, the states are initialized such that $s_{k_2...k_L} = 1$ and $s_i = 0$ for all remaining $i \neq k_2^L$. Note that these initial values correspond exactly to the component values of vector \underline{x}_k . Next we apply n times the operator Ξ_k to update the states correspond exactly to the contents of the row vector $\underline{x}_k^\top \mathbf{A}_k^n$. Thus, the value of $c_{k,n}$ can be obtained by summing the values of the counting states. Again, we remark that we only need the finite state machine for computing the values of $c_{k,n}$ for $n = 1, 2, \ldots, L - 1$, as the values of $c_{k,n}$ for $n \ge L$ can be easily determined by the recursion formula (18).

On the other hand, to compute

$$|\mathcal{C}_{k}(\boldsymbol{d}\boldsymbol{0},n)| = \underline{\boldsymbol{u}}^{\top} \mathbf{A}_{k,0} \mathbf{A}_{k}^{n-\ell(\boldsymbol{d})-1} \underline{\boldsymbol{w}}_{k}$$

for a given prefix d, we initialize the values associated with all states to be zero except $s_{u_{m-L+2}\cdots u_m} = 1$, where $u_{m-L+2}\cdots u_m$ is the rightmost (L-1) elements in u = $u_1 \dots u_m = k_2^L d$. Apply the operator $\Xi_{k,0}$ to all states in \mathfrak{S} once, followed by updating all the states $(n - \ell(d) - 1)$ times via operator Ξ_k . Then, the sum of the values of all counting states equals $|\mathcal{C}_k(d0, n)|$.

V. PRACTICE AND PERFORMANCE OF UDOOCS

In Fig. 7, we compare the numbers of length-n codewords for all UWs of lengths L = 2, 3, 4 and 5. These numbers are plotted in logarithmic scale and are normalized against the number of length-n codewords for the all-zero

¹⁰Here we implicitly use a fact that \underline{w}_{k} is a binary zero-one vector. Note that the (i+1)th component of $\underline{w}_{k} = [w_{1} \ w_{2} \ \cdots \ w_{2^{L-1}}]^{\top} = \mathbf{A}_{k}^{L-1} \underline{y}_{k}$ is equal to the number of distinct walks from vertex $i_{1} \cdots i_{L-1}$ to vertex $k_{1} \cdots k_{L-1}$ on digraph G_{k} . This fact follows since there is at most one walk of length (L-1) between the above two vertexes.

UW k = 0...00 to facilitate their comparison. By the equivalence relation defined in Definition 2, only one UW in each equivalence class needs to be illustrated. We have the following observations.

- The logarithmic ratio log₂(c_{k,n}/c_{a,n}), where a = 0...00, exhibits some transient fluctuation for n ≤ L but becomes a steady straight line of negative slope after n > L. This hints that c_{k,n} has a steady exponential growth when n is beyond L.
- 2) The number $c_{b,n}$, where b = 00...01, is always the largest among all $c_{k,n}$ when n is small. However, this number has an apparent trend to be overtaken by those of other UWs as n grows and will be eventually smaller than the number of length-n codewords for the all-zero UW. This result matches the statement of Theorem 2.
- As a contrary, the number c_{a,n} for the all-zero UW a = 0...00 is the smallest among all c_{k,n} for UWs of the same length when n is small. Although Theorem 2 indicates that this number will eventually be the largest, Fig. 7 shows that such would happen only when n is very large.
- 4) As a result of the two previous observations, UW b = 00...01 perhaps remains a better choice in the compression of sources with practical number of source letters even though it is asymptotically the worst. We will confirm this inference by the later practice of UDOOCs on a real text source from the book *Alice's Adventure in Wonderland*.

We next investigate the compression rates of UDOOCs and compare them with those of the Huffman and Lempel-Ziv (specifically, LZ77 and LZ78) codes. In this experiment, the standard Huffman code in the communication toolbox of Matlab is used instead of the adaptive Huffman code. The LZ77 executable is obtained from the basic compression library in [34], while the LZ78 is self-implemented using C++ programing language. As a convention, the data is binary ASCII encoded before it is fed into the two Lempel-Ziv compression algorithms. The sliding window for the LZ77 is set as 10,000 bits, and the tree-structured LZ78 is implemented without any windowing.

Three different English text sources are used, in which the uppercase and lowercase of each English letter are treated as the same symbol. The first English text source is distributed uniformly over the 26 symbols. The second English text source is assumed independent and identically distributed (i.i.d.) with marginal statistics from [36]. The third one is a realistic English text source from *Alice's Adventure in Wonderland*, in which any symbols other than the 26 English alphabets are regarded as a "space." In addition, the effect of grouping t symbols as a grouped source for compression is studied, which will be termed t-grouper in remarks below. The results are summarized in Tables VI and VII, in which the average codeword length of UDOOCs has already taken into account the length of UWs. We remark on the experimental results as follows.

1) First of all, it can be observed from Table VI that the length-2 UW k = 01 gives a good per-letter average

codeword length only when t = 1. When the size of source alphabet increases by grouping t = 2 or t = 3letters as one symbol for UDOOC compression, the per-letter average codeword length dramatically grows. Note that k = 01 is the only UW, whose number of length-*n* codewords has a linear growth with respect to *n*, i.e., we have $c_{01,n} = n + 1$. Since the size of source alphabets increases exponentially in *t* when *t*grouper is employed, the resulting per-letter average codeword length also increases exponentially as *t* grows. Therefore, when UW = 01, *t*-grouper will result in an extremely poor performance for moderately large *t*.

- 2) By independently generating 10^6 letters according to the statistics in [36] for compression, we record the per-letter average codeword in the second row of Table VI. As expected, the Huffman coding scheme gives the smallest per-letter average codeword length of 4.253 bits per letter, when 3-grouper is used. The gap of perletter average codeword lengths between the 3-grouper Huffman and the 3-grouper UDOOC however can be made as small as 4.795 - 4.253 = 0.542 bits per source letter if UW = 0001. This is in contrast to the gap of 1.007 bits when uniform independent English text source is the one to be compressed (cf. the first row in Table VI). We would like to point out that the error propagation of UDOOCs is limited firmly by at most two codewords, while that of the Huffman code may be statistically beyond this range. In comparison with the LZ77 and LZ78, the UDOOC clearly performs better in compression rate for usual independent English text source.
- 3) When the compression of a source with memory such as the book titled *Alice's Adventures in Wonderland* [35] is concerned, the third row in Table VI shows that the gap of per-letter average codeword lengths between the optimal 3-grouper Huffman and the 3-grouper UDOOC with UW = 0001 is narrowed down to 0.305 bits per letter. The 3-grouper UDOOC with the all-zero UW also performs well for this source. Note that part of the per-letter average codeword length of UDOOCs is contributed by the UW, i.e., L/t; hence, in a sense, a larger t and a smaller L are favored (except for L = 2). As can be seen from Table VI, the best compression performance is given by t = 3, L = 4, and UW = 0001.
- 4) For the third English text source, the LZ77 performs better than all of the 1-grouper UDOOC compression schemes but one. We then compare the running time of both algorithms. We reduce the window size of LZ77 so that it has a similar running time to the 1-grouper UDOOC scheme. The compression performance of LZ77 degrades down to 5.234 bits per letter, which is larger than that of the 1-grouper UDOOC. Note that we only compare their running time in encoding in Table VII as the decoding efficiency of UDOOCs is seemingly better than that of the LZ77. Considering also the low memory consumption of UDOOCs when a specific UW is pre-given in addition to its simplicity in implementation, the UDOOC can be regarded as a cost-



Fig. 7. Normalized numbers of length-n codewords for UWs of lengths L = 2, 3, 4, 5

effective compression scheme for practical applications.

TABLE VII AVERAGE CODEWORD LENGTHS IN BITS PER SOURCE SYMBOL AND RUNNING TIME IN SECONDS FOR THE UDOOC ENCODING AND THE LZ77 ENCODING ON *Alice's Adventures in Wonderland*. THE PROGRAMS ARE IMPLEMENTED USING C++, AND ARE EXECUTED IN A MICROSOFT WINDOWS-BASED DESKTOP WITH INTEL-CORE7 2.4G CUP AND 8G MEMORY.

	Туре	Average Codewrod length	Running Time
UDOOC	UW $\boldsymbol{k} = 00$	4.887	0.0162 sec
	UW $k = 01$	4.068	0.0158 sec F
LZ77 -	Window Size = 10^4 bits	4.661	0.0328 sec
	Window Size $= 3000$ bits	5.234	0.01607 sec

VI. CONCLUSION

In this paper, we have provided a general construction of UDOOCs with arbitrary UW. Combinatorial properties of UDOOCs are subsequently investigated. Based on our studies, the appropriate UW for the UDOOC compression of a given source can be chosen. Various encoding and decoding algorithms for general UDOOCs, as well as their efficient counterparts for specific UWs like k = 00...0, 00...01, are also provided. Performances of UDOOCs are then compared with the Huffman and Lempel-Ziv codes. Our experimental results show that the UDOOC can be a good practical candidate for lossless data compression when a costefficient solution is desired.

APPENDIX A Proof of Theorem 1

In this section, we will prove (11), the enumeration of $s_{k,n}$ in Theorem 1. Our proof technique is similar to that in [23].

Let $\mathbb{F}^{\infty} := \bigcup_{n \geq 0} \mathbb{F}^n$ be the set of all binary sequences. For a word $w = w_1 \dots w_n \in \mathbb{F}^{\infty}$ of length n, let $\mathcal{F}_k(w)$ be the set of index pairs indicating the places that w contains k as a subword, i.e.,

$$\mathcal{F}_{\boldsymbol{k}}(\boldsymbol{w}) = \left\{ (i,j) : \boldsymbol{k} = w_i^j \right\}.$$

Further denote by $\ell(w)$ the length of word w. Then

$$f(z) = \sum_{n \ge 0} s_{\mathbf{k},n} z^n$$

$$\stackrel{(i)}{=} \sum_{\mathbf{w} \in \mathbb{F}^\infty} z^{\ell(\mathbf{w})} 0^{|\mathcal{F}_{\mathbf{k}}(\mathbf{w})|}$$

$$= \sum_{\mathbf{w} \in \mathbb{F}^\infty} z^{\ell(\mathbf{w})} \prod_{a \in \mathcal{F}_{\mathbf{k}}(\mathbf{w})} (1 + (-1))$$

$$\stackrel{(ii)}{=} \sum_{\mathbf{w} \in \mathbb{F}^\infty} z^{\ell(\mathbf{w})} \sum_{A \subseteq \mathcal{F}_{\mathbf{k}}(\mathbf{w})} (-1)^{|A|}$$
(65)

where in (i) we have adopted the convention of $0^0 = 1$, and (ii) follows from the inclusion-exclusion principle. In light of (65), we will regard the pair (w, A) with $A \subseteq \mathcal{F}_k(w)$ as a *marked word*. The set of all marked words is thus defined as

$$\mathcal{M}_{\boldsymbol{k}} \ := \ \left\{ (\boldsymbol{w}, A) \ : \ \boldsymbol{w} \in \mathbb{F}^{\infty} \ \text{and} \ A \subseteq \mathcal{F}_{\boldsymbol{k}}(\boldsymbol{w}) \right\}.$$

Define the following weight function for elements in \mathcal{M}_{k}

$$\pi(\boldsymbol{w}, A) := z^{\ell(\boldsymbol{w})}(-1)^{|A|}; \tag{66}$$

TABLE VI

AVERAGE CODEWORD LENGTHS IN BITS PER SOURCE SYMBOL FOR THE COMPRESSION OF THREE DIFFERENT SOURCES. THE BEST ONE AMONG 1-GROUPER, 2-GROUPER AND 3-GROUPER OF THE SAME COMPRESSION SCHEME IS BOLDFACED.

Туре		Entropy		LZ77	LZ78		Huffman			U	$W = 00 \cdot \cdot$	$\cdot 0$	UV	$\mathbf{V} = 00 \cdot \cdot$	· 01
	t = 1	t = 2	t = 3			t = 1	t = 2	t = 3		t = 1	t = 2	t = 3	t = 1	t = 2	t = 3
Independent									L = 2	6.961	6.820	6.790	5.846	12.76	41.99
English Letter with	4.700	4.700	4.700	7.992	7.178	4.768	4.738	4.702	L = 4	8.576	6.831	6.213	7.000	5.899	5.709
Uniform distribution									L = 6	10.58	7.748	6.746	9.000	6.768	6.104
	t = 1	t = 2	t = 3			t = 1	t = 2	t = 3		t = 1	t = 2	t = 3	t = 1	t = 2	t = 3
Independent									L = 2	5.591	5.550	5.637	4.557	7.771	20.907
English Letter with	4.246	4.246	4.246	7.925	6.626	4.274	4.261	4.253	L = 4	7.411	5.872	5.351	6.185	4.970	4.795
Usual distribution									L = 6	9.411	6.818	5.924	8.185	5.882	5.274
	t = 1	t = 2	t = 3			t = 1	t = 2	t = 3		t = 1	t = 2	t = 3	t = 1	t = 2	t = 3
Alice's									L = 2	4.887	4.340	3.958	4.068	4.975	7.573
Adventures	3.914	3.570	3.215	4.661	6.028	3.940	3.585	3.226	L = 4	6.757	4.920	4.089	5.774	4.133	3.531
in Wonderland									L = 6	8.757	5.890	4.709	7.774	5.089	4.115

then (65) can be rewritten as

$$f(z) = \sum_{(\boldsymbol{w},A)\in\mathcal{M}_{k}} \pi(\boldsymbol{w},A).$$
(67)

To determine f(z), below we introduce the concept of a *cluster*.

Definition 6 (Cluster): We say the marked word (w, A) is a cluster if, and only if,

$$\bigcup_{(i_t,j_t)\in A} [i_t,j_t] = [1,\ell(\boldsymbol{w})]$$

where by [a, b] we mean the closed interval $\{x \in \mathbb{R} : a \le x \le b\}$ on the real line. The set of all clusters is thus

$$\mathcal{T}_{k} = \{(\boldsymbol{w}, A) \in \mathcal{M}_{k} : (\boldsymbol{w}, A) \text{ is a cluster} \}.$$

Definition 7 (Concatenation of sets of marked words): For any two sets of marked words \mathcal{A}_k and \mathcal{B}_k , we define the concatenation of \mathcal{A}_k and \mathcal{B}_k as

$$\mathcal{A}_{\boldsymbol{k}} \lor \mathcal{B}_{\boldsymbol{k}} := \{ (\boldsymbol{a}\boldsymbol{b}, A \cup \mathfrak{J}(B, \ell(\boldsymbol{a})) : (\boldsymbol{a}, A) \in \mathcal{A}_{\boldsymbol{k}}, (\boldsymbol{b}, B) \in \mathcal{B}_{\boldsymbol{k}} \}$$

where by ab we meant the usual concatenation of strings a and b, and the function $\mathfrak{J}(B, \ell(a))$ is

$$\mathfrak{J}(B, \ell(a)) := \{ (i_t + \ell(a), j_t + \ell(a)) : (i_t, j_t) \in B \}.$$

Having defined the concatenation operation \lor for sets of marked words, we next claim the following decomposition for the set \mathcal{M}_{k}

$$\mathcal{M}_{\boldsymbol{k}} = \{ (\operatorname{null}, \emptyset) \} \cup (\mathcal{M}_{\boldsymbol{k}} \vee \mathcal{F}) \cup (\mathcal{M}_{\boldsymbol{k}} \vee \mathcal{T}_{\boldsymbol{k}}), \quad (68)$$

where $\mathcal{F} := \{(b, \emptyset) : b \in \mathbb{F}\}.$

To show (68), for any $(w, A) \in \mathcal{M}_k$ we distinguish the following three disjoint cases:

- 1) If $\ell(w) = 0$, it is obvious that w is a null word and $A = \emptyset$ from the definition of $\mathcal{F}_{k}(w)$.
- 2) For $\ell(w) \ge 1$, appending an arbitrary binary word to w results in another marked word (wb, A), which cannot be a cluster since

$$\bigcup_{(i_t,j_t)\in A} [i_t,j_t] \subset [1,\ell(\boldsymbol{w})+1].$$

Conversely, take any marked word (w, A) from \mathcal{M}_k with $\ell(w) = n$. If $j_t < \ell(w) = n$ for all $(i_t, j_t) \in A$, then we can delete the rightmost bit from w, and the resulting pair (w_1^{n-1}, A) is still a marked word. Summarizing the above gives the following equalities between two sets of marked words

$$\{(\boldsymbol{w}, A) \in \mathcal{M}_{\boldsymbol{k}} : j_t < \ell(\boldsymbol{w}) \text{ for all } (i_t, j_t) \in A)\}$$

= $\{(\boldsymbol{w}b, A) : (\boldsymbol{w}, A) \in \mathcal{M}_{\boldsymbol{k}}, b \in \mathbb{F}\}$
= $\mathcal{M}_{\boldsymbol{k}} \lor \mathcal{F},$ (69)

where the last equality follows from the definition of concatenation operation \lor .

3) The last case concerns the situation when (*w*, *A*) satisfies ℓ(*w*) = n ≥ 1, A = {(i₁, j₁), ..., (i_m, j_m)} and i₁ < ··· < i_m < j_m = n. In other words, this is the case when max{j_t : (i_t, j_t) ∈ A} = ℓ(*w*), which is disjoint from the second case. For this, let u be the smallest index such that [i_{u+t}, j_{u+t}] ∩ [i_{u+t+1}, j_{u+t+1}] ≠ ∅ for all t = 0, 1, ..., m - u + 1. Then obviously we have the following de-concatenation of (*w*, A)

$$(\boldsymbol{w}, A) = (w_1^{i_u-1}, \{(i_t, j_t) : t = 1, \dots, u-1\}) \\ \vee (w_{i_u}^n, \{(i_t - i_u + 1, j_t - i_u + 1) : t = u, \dots, m\}).$$

Clearly, the first marked word $(w_1^{i_u-1}, \{(i_t, j_t) : t = 1, \ldots, u - 1\}) \in \mathcal{M}_k$. The second marked word $(w_{i_u}^n, \{(i_t - i_u + 1, j_t - i_u + 1) : t = u, \ldots, m\})$ is a cluster since

$$\bigcup_{t=u}^{m} [i_t - i_u + 1, j_t - i_u + 1] = [1, n - i_u + 1]$$

by the choice of u. Hence we arrive at the following equality between two sets of marked words

$$\{(\boldsymbol{w}, A) \in \mathcal{M}_{\boldsymbol{k}} : \max\{j_t : (i_t, j_t) \in A\} = \ell(\boldsymbol{w})\} = \mathcal{M}_{\boldsymbol{k}} \lor \mathcal{T}_{\boldsymbol{k}}.$$
(70)

Combining the case of null word and equations (69) and (70) proves the desired claim of (68).

Using the decomposition in (68), we can rewrite (67) in terms of the three sets, i.e., the set for null word, $\mathcal{M}_{k} \vee \mathcal{F}$,

$$\sum_{(\boldsymbol{w},W)\in\mathcal{M}_{\boldsymbol{k}}\vee\mathcal{T}_{\boldsymbol{k}}} \pi(\boldsymbol{w},W)$$

$$= \sum_{(\boldsymbol{a},A)\in\mathcal{M}_{\boldsymbol{k}}} \sum_{(\boldsymbol{b},B)\in\mathcal{T}_{\boldsymbol{k}}} z^{\ell(\boldsymbol{a}\boldsymbol{b})}(-1)^{|A\cup\mathfrak{J}(B,\ell(\boldsymbol{a}))|}$$

$$= \sum_{(\boldsymbol{a},A)\in\mathcal{M}_{\boldsymbol{k}}} \sum_{(\boldsymbol{b},B)\in\mathcal{T}_{\boldsymbol{k}}} z^{\ell(\boldsymbol{a})+\ell(\boldsymbol{b})}(-1)^{|A|+|B|}$$

$$= \left(\sum_{(\boldsymbol{a},A)\in\mathcal{M}_{\boldsymbol{k}}} \pi(\boldsymbol{a},A)\right) \left(\sum_{(\boldsymbol{b},B)\in\mathcal{T}_{\boldsymbol{k}}} \pi(\boldsymbol{b},B)\right). \quad (71)$$
where one can show that

Similarly, one can show that

$$\sum_{(\boldsymbol{w},A)\in\mathcal{M}_{\boldsymbol{k}}\vee\mathcal{F}_{2}}\pi(\boldsymbol{w},A) = 2z\sum_{(\boldsymbol{w},A)\in\mathcal{M}_{\boldsymbol{k}}}\pi(\boldsymbol{w},A).$$
(72)

Substituting (71) and (72) into (67) gives

$$f(z) = \sum_{(\boldsymbol{w}, A) \in \mathcal{M}_{\boldsymbol{k}}} \pi(\boldsymbol{w}, A) = 1 + 2zf(z) + f(z)T(z),$$

or equivalently,

$$f(z) = \frac{1}{1 - 2z - T(z)},$$
(73)

where T(z) is the weight enumerator of elements in \mathcal{T}_{k} given by

$$T(z) := \sum_{(\boldsymbol{b},B)\in\mathcal{T}_{\boldsymbol{k}}} \pi(\boldsymbol{b},B).$$
(74)

Determining T(z) is now relatively easy. Recall that the overlap function $r_{\mathbf{k}}(i) = \mathbf{1}(k_1^{L-i} = k_{i+1}^L)$, where $\mathbf{1}(\cdot)$ is the usual indicator function, shows exactly whether the length-(L-i) prefix of \mathbf{k} is also a suffix of \mathbf{k} . Let $\mathcal{R}_{\mathbf{k}} = \{i: 1 \leq i \leq L-1, r_{\mathbf{k}}(i) = 1\}$. For any cluster $(\mathbf{b}, B) \in \mathcal{T}_{\mathbf{k}}$ with $\mathbf{b} = b_1 \dots b_n$, we must have $b_{n-L+1}^n = \mathbf{k}$ by Definition 6. So for any $i \in \mathcal{R}_{\mathbf{k}}$, i.e., $r_{\mathbf{k}}(i) = 1$, we have $b_{n-L+i+1}^n = k_{i+1}^{L-i} = k_1^{L-i}$. Hence the pair

$$(bk_{L-i+1}...k_L, B \cup \{(n+i-L+1, n+i)\})$$

is a cluster in \mathcal{T}_k . It implies that for $i \in \mathcal{R}_k$, the set

$$\mathcal{T}_{\boldsymbol{k},i} := \left\{ \begin{array}{c} \left(\boldsymbol{b} \boldsymbol{k}_{L-i+1}^{L}, B \cup \{ (n+i-L+1, n+i) \} \right) : \\ (\boldsymbol{b}, B) \in \mathcal{T}_{\boldsymbol{k}}, n = \ell(\boldsymbol{b}) \end{array} \right\}$$
(75)

is a subset of \mathcal{T}_k .

On the other hand, take any $(\boldsymbol{b}, B) \in \mathcal{T}_{\boldsymbol{k}}$ with $\ell(\boldsymbol{b}) = n$ and $B = \{(i_t, j_t) : t = 1, \ldots, m\}$, where $1 = i_1 < i_2 < \cdots < i_m < j_m = n$ and $i_m = n - L + 1$. If m = 1, then $\boldsymbol{b} = \boldsymbol{k}$ and $B = \{(1, L)\}$. Hence we consider the case when m > 1. As (\boldsymbol{b}, B) is a cluster, $[i_{m-1}, j_{m-1}] \cap [i_m, j_m] \neq \emptyset$ and $b_{i_m-1}^{j_{m-1}} = b_{i_m}^{j_m} = \boldsymbol{k}$. Therefore, we must have $b_{i_m}^{j_{m-1}} = k_1^v = k_{L-v+1}^L$, where $v = j_{m-1} - i_m + 1$. Thus, $r_{\boldsymbol{k}}(L - v) = 1$ and $(\boldsymbol{b}, B) \in \mathcal{T}_{\boldsymbol{k}, L-v}$. The above discussion then gives the following decomposition for $\mathcal{T}_{\boldsymbol{k}}$

$$\mathcal{T}_{\boldsymbol{k}} = \{ (\boldsymbol{k}, \{(1,L)\}) \} \cup \left(\bigcup_{i \in \mathcal{R}_{\boldsymbol{k}}} \mathcal{T}_{\boldsymbol{k},i} \right).$$
(76)

For enumerating the weights of elements in \mathcal{T}_{k} , we further claim that $\mathcal{T}_{k,i} \cap \mathcal{T}_{k,j} = \emptyset$ for all $i \neq j$. This simply follows

from the definition of $\mathcal{T}_{k,i}$ in (75) that for any $(\boldsymbol{b}, B) \in \mathcal{T}_{k,i}$ and $(\boldsymbol{b}', B') \in \mathcal{T}_{k,j}$, say $B = \{(i_t, j_t) : t = 1, \ldots, m\}$ and $B' = \{(i_t, j_t) : t = 1, \ldots, m'\}$, where the pairs (i_t, j_t) are arranged in ascending order, we have that $j_m - j_{m-1} = i$ for B and $j_{m'} - j_{m'-1} = j$ for B'. This proves our claim. Finally, using (76) and the fact that the sets $\{\mathcal{T}_{k,i}\}$ are disjoint, we obtain

$$T(z) = \pi(\mathbf{k}, \{(1, L)\}) + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) \sum_{(\mathbf{b}, B) \in \mathcal{T}_{\mathbf{k}}, i} \pi(\mathbf{b}, B)$$
$$= z^{\ell(\mathbf{k})}(-1) + \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) \sum_{(\mathbf{b}, B) \in \mathcal{T}_{\mathbf{k}}} z^{\ell(\mathbf{b})+i} (-1)^{|B|+1}$$
$$= -z^{L} - \sum_{i=1}^{L-1} r_{\mathbf{k}}(i) z^{i} T(z).$$

Hence

$$T(z) = -\frac{z^L}{1 + \sum_{i=1}^{L-1} r_k(i) z^i}.$$

Substituting the above into (73) proves (11) of Theorem 1.

APPENDIX B DEGREE OF det $(I - A_k z)$

In this section, we will determine the degree of polynomial det $(I - A_k z)$ that is required in the proof of Theorem 1.

Proposition 10: Let A_k be the adjacency matrix for the digraph G_k associated with UW k defined in Section III. Then

$$\deg \det \left(\mathbf{I} - \mathbf{A}_{k} z \right) = L. \tag{77}$$

Proof: First, from (9) and (11), the two equivalent formulas for the enumeration of $s_{k,n}$, we see det $(\mathbf{I} - \mathbf{A}_k z)$ is divisible by $h_k(z) = (1 - 2z)(1 + \sum_{i=1}^{L-1} r_k(i)z^i) + z^L$. It follows that

$$\deg \det \left(\mathbf{I} - \mathbf{A}_{k} z \right) \geq L.$$

To establish the converse of the above inequality, i.e., deg det $(\mathbf{I} - \mathbf{A}_{k}z) \leq L$, it suffices to show that $\operatorname{rank}(\mathbf{A}_{k}^{L-1}) \leq L$, which in turns implies $\operatorname{rank}(\mathbf{A}_{k}^{L}) \leq L$. As a result, the algebraic multiplicity of eigenvalue 0 for \mathbf{A}_{k} is at least $2^{L-1} - L$. Hence, the degree of det $(\mathbf{I} - \mathbf{A}_{k}z)$ is at most L.

To prove the claim, given the UW $\mathbf{k} = k_1 \dots k_L$ of length L and the corresponding adjacency matrix A_k for digraph G_k , let

$$\mathbf{H} = \mathbf{A}_{\mathbf{k}} + \underline{e}_{\mathbf{k}_1} \underline{e}_{\mathbf{k}_2}^{\top}$$

where $\mathbf{k}_1 = k_1^{L-1}$ and $\mathbf{k}_2 = k_2^L$, and where by $\underline{e}_{\mathbf{d}} \in \mathbb{F}^{2^{L-1}}$ with $\mathbf{d} = d_1 \dots d_{L-1} \in \mathbb{F}^{L-1}$ we mean $(\underline{e}_{\mathbf{d}})_{j+1} = 1$ if j has the binary representation \mathbf{d} , and $(\underline{e}_{\mathbf{d}})_{j+1} = 0$, otherwise.

Apparently, H is the adjacency matrix for the digraph without UW forbidden constraint and is therefore independent of the choice of k. As an example, if L = 3, then

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Furthermore, it can be easily verified that $\mathbb{H}^{L-1} = \underline{\mathbf{1}} \underline{\mathbf{1}}^{\top}$ is the all-one matrix. Armed with the above, we now have

$$\begin{aligned} \mathbf{A}_{\mathbf{k}}^{L-1} &= \left(\mathbf{H} - \underline{e}_{\mathbf{k}_{1}} \underline{e}_{\mathbf{k}_{2}}^{\top}\right)^{L-1} \\ &= \mathbf{H}^{L-1} - \sum_{i=0}^{L-2} \mathbf{H}^{L-2-i} \left(\underline{e}_{\mathbf{k}_{1}} \underline{e}_{\mathbf{k}_{2}}^{\top}\right) \mathbf{A}_{\mathbf{k}}^{i}, \quad (78) \end{aligned}$$

where the last equality is due to the following identity for square matrices A and B:

$$(A-B)^{L-1} = A^{L-1} - \sum_{i=0}^{L-2} A^{L-2-i} B(A-B)^i.$$

Applying the standard rank inequality of $rank(A + B) \leq rank(A) + rank(B)$ [19] to (78) yields

$$\begin{aligned} \operatorname{rank}\left(\mathbf{A}_{\boldsymbol{k}}^{L-1}\right) \\ &\leq \operatorname{rank}\left(\mathbf{H}^{L-1}\right) + \sum_{i=0}^{L-2} \operatorname{rank}\left(\mathbf{H}^{L-2-i}\left(\underline{e}_{\boldsymbol{k}_{1}}\underline{e}_{\boldsymbol{k}_{2}}^{\mathsf{T}}\right)\mathbf{A}_{\boldsymbol{k}}^{i}\right) \\ &= 1 + \sum_{i=0}^{L-2} 1 = L, \end{aligned}$$

and the proof is completed.

APPENDIX C Verification of Algorithms 5 and 6

For completeness, we verify Algorithms 5 and 6 in this section.

For message u_1 , i.e., the most likely message, we have from line 1 in Algorithm 5 that m = 1 and n = 0 since $F_{k,0} = c_{k,0} = 1$. This results in the encoding output of the null codeword. In parallel, when receiving the null codeword, we have n = 0. Algorithm 6 then sets m = 1 at line 2 as $F_{k,-1} = 0$. This verifies the correctness of Algorithms 5 and 6 for message u_1 .

For $m \ge 2$, we shall show that for each $n \ge 1$, the encoding function $\phi_{\mathbf{k}}$ is a bijection between $\mathcal{U}_{\mathbf{k}}(n) = \{u_m : F_{\mathbf{k},n-1} < m \le F_{\mathbf{k},n}\}$ and $\mathcal{C}_{\mathbf{k}}(n)$, and the decoding function $\psi_{\mathbf{k}}$ is the functional inverse of $\phi_{\mathbf{k}}$. Equivalently, it suffices to show that

- 1) ψ_{k} is a bijection between $C_{k}(n)$ and $U_{k}(n)$ for each $n \geq 1$, and
- 2) $\phi_{\mathbf{k}}$ is the functional inverse of $\psi_{\mathbf{k}}$

We will proceed with this approach.

Prior to establishing the claims, we first introduce below a well-ordering of binary sequences. This is in fact a key concept embedded in Algorithms 5 and 6.

Definition 8 (Lexicographical ordering): For any two binary sequences $\mathbf{a} = a_1 \dots a_i$ and $\mathbf{b} = b_1 \dots b_j$, we say $\mathbf{a} \succ \mathbf{b}$ if i > j, or if i = j and there exists a smallest integer s, $1 \le s \le i$, such that $a_u = b_u$ for $u = 1, \dots, s - 1$, $a_s = 1$, and $b_s = 0$.

Obviously, such ordering is a total-ordering of binary sequences. How the lexicographical ordering of binary sequences plays a key role in the encoding and decoding of UDOOCs is due to the following lemma. Lemma 1: For any two length-*n* codewords $a, b \in C_k(n)$, we have $a \succ b$ if, and only if,

$$\sum_{i=1}^{n} a_i \left| \mathcal{C}_{\mathbf{k}}(a_1^{i-1}0, n) \right| > \sum_{i=1}^{n} b_i \left| \mathcal{C}_{\mathbf{k}}(b_1^{i-1}0, n) \right|.$$
(79)

Proof: As $\ell(a) = \ell(b)$ and $a \succ b$, there exists a smallest integer $s, 1 \le s \le n$, such that $a_u = b_u$ for $u = 1, \ldots, s - 1$, $a_s = 1$, and $b_s = 0$. Thus,

$$\begin{split} &\sum_{i=1}^{n} a_{i} \left| \mathcal{C}_{\mathbf{k}}(a_{1}^{i-1}0,n) \right| \\ &\geq \sum_{i=1}^{s-1} a_{i} \left| \mathcal{C}_{\mathbf{k}}(a_{1}^{i-1}0,n) \right| + \left| \mathcal{C}_{\mathbf{k}}(a_{1}^{s-1}0,n) \right| \\ &> \sum_{i=1}^{s-1} a_{i} \left| \mathcal{C}_{\mathbf{k}}(a_{1}^{i-1}0,n) \right| + \sum_{i=s+1}^{n} b_{i} \left| \mathcal{C}_{\mathbf{k}}(a_{1}^{s-1}0b_{s+1}^{i-1}0,n) \right| \\ &= \sum_{i=1}^{n} b_{i} \left| \mathcal{C}_{\mathbf{k}}(b_{1}^{i-1}0,n) \right|, \end{split}$$

where the second inequality follows from the fact that the sets $C_k(a_1^{s-1}0b_{s+1}^{i-1}0, n)$, where $i = s + 1, \ldots, n$ and $b_i = 1$, are disjoint proper subsets of $C_k(a_1^{s-1}0, n)$.

With the above lemma, given a codeword $c = c_1 \dots c_n$, Algorithm 6 outputs $\psi_k(c) = m$ with

$$m = \sum_{i=1}^{n} c_i \underline{x}_{\mathbf{k}}^{\top} \left(\prod_{j=1}^{i-1} \mathbf{A}_{\mathbf{k},c_j} \right) \mathbf{A}_{\mathbf{k},0} \mathbf{A}_{\mathbf{k}}^{(n+L-1)-i} \underline{y}_{\mathbf{k}} + F_{\mathbf{k},n-1} + 1$$
$$= \sum_{i=1}^{n} c_i \left| \mathcal{C}_{\mathbf{k}}(c_1^{i-1}0,n) \right| + F_{\mathbf{k},n-1} + 1.$$
(80)

We remark that the first term in the above, i.e., $\sum_{i=1}^{n} c_i |\mathcal{C}_{\mathbf{k}}(c_1^{i-1}0, n)|$, is the only term dependent on c, and it also appears in (79). It means that the encoding and decoding algorithms of UDOOC given in Algorithms 5 and 6 are indeed based on the lexicographical ordering of length-n codewords in $\mathcal{C}_{\mathbf{k}}(n)$. Using Lemma 1 we can establish the range of $\psi_{\mathbf{k}}$ when restricted to $\mathcal{C}_{\mathbf{k}}(n)$.

Corollary 4: The range of $\psi_{\mathbf{k}}$ when restricted to $C_{\mathbf{k}}(n)$ is the set $\mathcal{U}_{\mathbf{k}}(n) = \{u_m : F_{\mathbf{k},n-1} < m \leq F_{\mathbf{k},n}\}$. Therefore, $\psi_{\mathbf{k}}$ is a bijection between $C_{\mathbf{k}}(n)$ and $\mathcal{U}_{\mathbf{k}}(n)$ for all $n \geq 1$.

Proof: Given $C_{k}(n)$, let **b** be the smallest member and **d** be the largest member according to the lexicographical ordering, i.e. $\mathbf{b} \leq \mathbf{c} \leq \mathbf{d}$ for all $\mathbf{c} \in C_{k}(n)$. It then follows from Lemma 1 that

$$\min_{\boldsymbol{c}\in\mathcal{C}_{\boldsymbol{k}}(n)}\psi_{\boldsymbol{k}}(\boldsymbol{c}) = \psi_{\boldsymbol{k}}(\boldsymbol{b}) \quad \text{and} \quad \max_{\boldsymbol{c}\in\mathcal{C}_{\boldsymbol{k}}(n)}\psi_{\boldsymbol{k}}(\boldsymbol{c}) = \psi_{\boldsymbol{k}}(\boldsymbol{d}).$$

For the minimum, from (80) we have

$$\psi_{\mathbf{k}}(\mathbf{b}) = \sum_{i=1}^{n} b_i \left| \mathcal{C}_{\mathbf{k}}(b_1^{i-1}0, n) \right| + F_{\mathbf{k}, n-1} + 1.$$

Since **b** is the smallest member, it follows that for all i, i = 1, ..., n, $|C_k(b_1^{i-1}0, n)| = 0$ if $b_i = 1$. Hence

$$\min_{\boldsymbol{c}\in\mathcal{C}_{\boldsymbol{k}}(n)}\psi_{\boldsymbol{k}}(\boldsymbol{c}) = \psi_{\boldsymbol{k}}(\boldsymbol{b}) = F_{\boldsymbol{k},n-1}+1.$$

To see the maximum, again from (80)

$$\psi_{\mathbf{k}}(\mathbf{d}) = \sum_{i=1}^{n} d_i \left| \mathcal{C}_{\mathbf{k}}(d_1^{i-1}0, n) \right| + F_{\mathbf{k}, n-1} + 1.$$

Since d is the largest member in $C_k(n)$, the sets $C_k(d_1^{i-1}0, n)$, where i = 1, ..., n and $d_i = 1$, are disjoint and proper subsets of $C_k(n)$. Moreover, for any $c \in C_k(n)$ and $c \prec d$, there exists a smallest integer $s, 1 \leq s \leq n$, such that $d_u = c_u$ for u = 1, ..., s - 1, $d_s = 1$, and $c_s = 0$. This in turn implies $c \in C_k(d_1^{s-1}0, n)$. Therefore,

$$\bigcup_{\substack{i=1\\d_i=1}}^n \mathcal{C}_{\boldsymbol{k}}(d_1^{i-1}0,n) = \mathcal{C}_{\boldsymbol{k}}(n) \setminus \{\boldsymbol{d}\}$$

and

$$\psi_{\mathbf{k}}(\mathbf{d}) = c_{\mathbf{k},n} - 1 + F_{\mathbf{k},n-1} + 1 = F_{\mathbf{k},n}.$$

Finally, noting that $|C_k(n)| = |U_k(n)|$ and that ψ_k is injective by Lemma 1, we conclude that ψ_k is bijective.

So far we have established the first claim that $\psi_{\mathbf{k}}$ is a bijection between $C_{\mathbf{k}}(n)$ and $\mathcal{U}_{\mathbf{k}}(n)$. To prove the second claim that $\phi_{\mathbf{k}}$ is the functional inverse of $\psi_{\mathbf{k}}$, given a codeword $\mathbf{c} = c_1 \dots c_n$, Algorithm 6 outputs

$$m = \psi_{\mathbf{k}}(\mathbf{c}) = \sum_{i=1}^{n} c_i \left| \mathcal{C}_{\mathbf{k}}(c_1^{i-1}0, n) \right| + F_{\mathbf{k}, n-1} + 1$$

and $F_{k,n-1} < m \leq F_{k,n}$. Line 2 of Algorithm 5 would produce the correct n for m. Then, from line 3 of Algorithm 5, we get

$$\rho_0 = \sum_{i=1}^n c_i \left| \mathcal{C}_{k}(c_1^{i-1}0, n) \right| + 1.$$

For the loop of lines 3-10 of Algorithm 5, when i = 1, dummy has value

$$dummy = \underline{x}_{\boldsymbol{k}}^{\top} \mathbf{A}_{\boldsymbol{k},0} \mathbf{A}_{\boldsymbol{k}}^{n+L-2} \underline{y}_{\boldsymbol{k}} = |\mathcal{C}_{\boldsymbol{k}}(0,n)|.$$

We distinguish two cases:

1) if $c_1 = 0$, then we must have

$$\rho_0 = \sum_{i=2}^n c_i \left| \mathcal{C}_k(0c_2^{i-1}0, n) \right| + 1 \le dummy$$

since $\sum_{i=2}^{n} c_i |\mathcal{C}_k(0c_2^{i-1}0, n)|$ is the sum of the cardinalities of certain disjoint subsets (with different prefixes) of $\mathcal{C}_k(0, n)$. Hence lines 5-9 of Algorithm 5 output $c_1 = 0$ as desired.

2) if $c_1 = 1$, then

$$\rho_0 = |\mathcal{C}_k(0,n)| + \sum_{i=2}^n c_i \left| \mathcal{C}_k(1c_2^{i-1}0,n) \right| + 1 > dummy$$

and lines 5-9 of Algorithm 5 gives the correct $c_1 = 1$. Furthermore, it can be seen that at the end of line 9, we have

$$\rho_1 = \sum_{i=2}^{n} c_i \left| \mathcal{C}_{k}(c_1^{i-1}0, n) \right| + 1$$

for the next iteration. Now suppose we are at the *t*th iteration of Algorithm 5 for some integer *t* with 1 < t < n. We have already determined $c_1, c_2, \ldots, c_{t-1}$, and have

$$\rho_{t-1} = \sum_{i=t}^{n} c_i \left| \mathcal{C}_{k}(c_1^{i-1}0, n) \right| + 1.$$

Line 4 of Algorithm 5 then gives

$$dummy = \underline{x}_{k}^{\top} \left(\prod_{i=1}^{t-1} \mathbf{A}_{k,c_{i}} \right) \mathbf{A}_{k,0} \mathbf{A}_{k}^{(n+L-1)-t} \underline{y}_{k}$$
$$= \left| \mathcal{C}_{k}(c_{1}^{t-1}0,n) \right|.$$

Using the same reasoning as the above it can be easily shown that lines 5-9 of Algorithm 5 always produce the correct value for c_t . Finally at the *n*th iteration we have

$$\rho_{n-1} = c_n \left| \mathcal{C}_k(c_1^{n-1}0, n) \right| + 1$$

and

$$dummy = \underline{x}_{\boldsymbol{k}}^{\top} \left(\prod_{i=1}^{n-1} \mathbf{A}_{\boldsymbol{k},c_i} \right) \mathbf{A}_{\boldsymbol{k},0} \mathbf{A}_{\boldsymbol{k}}^{L-1} \underline{y}_{\boldsymbol{k}} = \left| \mathcal{C}_{\boldsymbol{k}}(c_1^{n-1}0,n) \right|.$$

It should be noted that $c_1^{n-1}0$ is a length-*n* word, hence dummy = 0 or 1. We distinguish the following cases:

1) If dummy = 0, then $c_1^{n-1}0$ cannot be a valid codeword for UDOOC. Lines 5-9 of Algorithm 5 achieve exactly the above, since we have

$$\rho_{n-1} = c_n \cdot dummy + 1 = 1 > dummy = 0$$

and the algorithm always outputs $c_n = 1$.

2) If dummy = 1, then $\rho_{n-1} = c_n + 1$. The same reasoning as the above shows that lines 5-9 of Algorithm 5 always produce the correct value for c_n .

We therefore complete the proof that ϕ_k is the functional inverse of ψ_k .

Appendix D $\lim_{t\to\infty} L_{k,t}$ for all-zero UW and uniform I.I.D. Source

Let a = 00...0 be the all-zero UW of length L. From (23), (54), and (55), it can be easily verified that

$$\sum_{n=0}^{\infty} c_{\boldsymbol{a},n} z^n = 1 + \frac{z}{1 - \sum_{i=1}^{L} z^i}.$$
(81)

Furthermore, from (23) we have $g(z) = (1-z)h_a(z) = 1 - 2z + z^{L+1}$. It is straightforward to show that the two polynomials g(z) and $\frac{d}{dz}g(z)$ are co-prime to each other; hence there are no repeating zeros in $h_a(z)$. It then implies that all the nonzero eigenvalues of A_a are simple.

Denote by $\lambda_1 \cdots \lambda_L$ the nonzero eigenvalues of A_a , and assume without loss of generality that $|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_L|$. Then

$$c_{\boldsymbol{a}.n} = \delta_n + \sum_{i=1}^{L} a_i (\lambda_i)^n$$

where $a_1 \cdots a_L$ are constants such that (81) holds. We can also obtain the closed-form expression for $F_{a,n}$ as

$$F_{a,n} = 1 + \sum_{i=1}^{L} a_i \frac{\lambda_i^{n+1} - 1}{\lambda_i - 1}, \text{ for all } n \ge 0$$

Consider a uniform i.i.d. source \mathcal{U} of alphabet size M with M > 1. Let \mathcal{U}^t be the grouped source obtained by grouping any t source symbols (with repetition) in \mathcal{U} . It is clear that \mathcal{U}^t is also a uniform i.i.d. source. The per-letter average codeword length is given by

$$L_{\boldsymbol{a},t} = \frac{1}{t} \left(L + \sum_{i=2}^{M^t} p_i \ell(\phi(u_i)) \right)$$
$$= \frac{1}{t} \left(L + \frac{1}{M^t} \sum_{i=2}^{M^t} \ell(\phi(u_i)) \right).$$

Let N be the smallest integer such that $F_{a,N} \ge M^t > F_{a,N-1}$. Then

$$\begin{split} L_{\boldsymbol{a},t} &\geq \frac{1}{t} \left(L + \frac{1}{F_{\boldsymbol{a},N}} \sum_{i=2}^{M^{t}} \ell(\phi(u_{i})) \right) \\ &\geq \frac{1}{t} \left(L + \frac{1}{F_{\boldsymbol{a},N}} \sum_{i=2}^{N-1} i \cdot c_{\boldsymbol{a},i} \right) \\ &\geq \frac{1}{\log_{M}(F_{\boldsymbol{a},N})} \left(L + \frac{1}{F_{\boldsymbol{a},N}} \sum_{i=2}^{N-1} i \cdot c_{\boldsymbol{a},i} \right). \end{split}$$

Consequently,

$$\begin{split} \lim_{t \to \infty} L_{a,t} \\ &\geq \lim_{N \to \infty} \frac{1}{\log_M(F_{a,N})} \left(L + \frac{1}{F_{a,N}} \sum_{i=2}^{N-1} i \cdot c_{a,i} \right) \\ &= \lim_{N \to \infty} \frac{\sum_{i=2}^{N-1} i \cdot c_{a,i}}{F_{a,N} \log_M(F_{a,n})} \\ &= \lim_{N \to \infty} \frac{\sum_{i=1}^{L} a_i \frac{\lambda_i [(N-1)\lambda_i^N - N\lambda_i^{N-1} + 1]}{(\lambda_i - 1)^2}}{\left(1 + \sum_{i=1}^{L} a_i \frac{\lambda_i^{N+1} - 1}{\lambda_i - 1} \right) \log_M \left(1 + \sum_{i=1}^{L} a_i \frac{\lambda_i^{N+1} - 1}{\lambda_i - 1} \right)} \\ &= \frac{1}{\log_M(\lambda_1)} = \frac{1}{\log_M(g_a)}. \end{split}$$

This implies

$$\lim_{t \to \infty} L_{\boldsymbol{a},t} \ge \frac{\log_2(M)}{\log_2(g_{\boldsymbol{a}})} = \frac{\mathrm{H}(\mathcal{U})}{\log_2(g_{\boldsymbol{a}})}.$$

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