Physical-layer Network Coding on the Random-access Channel

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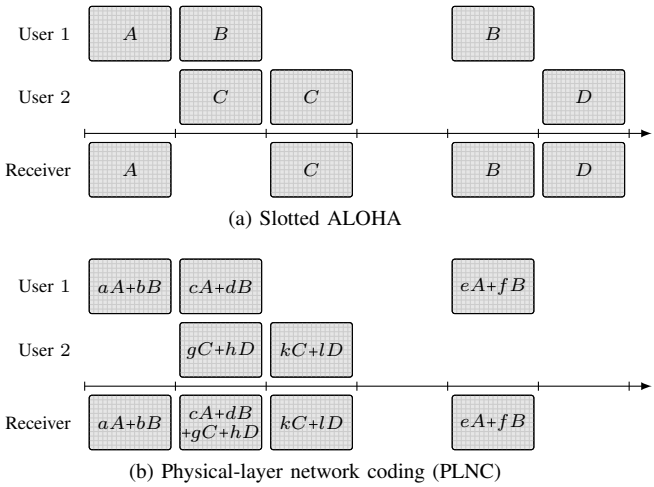
***Abstract*—We consider a physical-layer network coding strategy for the random-access channel, based on compute-and-forward. When packets collide, it is possible to reliably recover a linear combination of the packets at the receiver. Over many rounds of transmission, the receiver can thus obtain many linear combinations and eventually recover all original packets. This is by contrast to slotted ALOHA where packet collisions lead to complete erasures. In previous work we introduced a compute-and-forward strategy for the two-user random-access channel. In the current work we consider an arbitrary number of users. The strategy is shown to be significantly superior to the best known strategies, including multipacket reception.**

I. Introduction

Consider a multiple-access channel with users transmitting according to a random access mechanism. Users are in one of two states, active or inactive, and do not have knowledge of the states of other users. The receiver has complete knowledge of the states of all users. The network is operated in rounds. In each round each user chooses his state at random independently of the state of other rounds and independent of the other users. In [1] we have introduced a new approach to random access, based on *reliable physical-layer network coding*. The basic idea of the approach is that the receiver is decoding in each round a linear combination of messages. Once enough linear combinations are obtained the original messages can be retrieved. The *contribution of the current work* consists of extending the strategy and analysis of [1] from the case of two users to an arbitrary number of users.

The concept of physical-layer network coding is studied in, for instance [2]–[8]. See [9] for a survey of recent results. In [2] and [4] the aim is to obtain these linear combinations reliably. In contrast, in [5]–[8] one is satisfied with a noisy version of these linear combinations, which are then retransmitted. In [10] the approach of [5]–[8] is used in the random access setting. Our strategy is based on *reliable* physical layer network coding, also known as compute-and-forward, which shows that it can be useful to decode linear combinations of messages, even if the receiver is ultimately interested in the messages themselves. We establish the benefits of this approach for the random-access channel.



(a) Slotted ALOHA

(b) Physical-layer network coding (PLNC)

Fig. 1: Illustration of ALOHA compared to the physical-layer network coding approach to random access.

The most thoroughly studied approach to random access is slotted ALOHA, *cf.* [11]. In ALOHA, if more than one user is active, a packet collision occurs and the receiver does not obtain any information about the transmitted packets. This is illustrated in Figure 1a, in which different rounds are represented along the horizontal axis. Packets transmitted by the users are depicted above the axis, the packets below the axis represent the information obtained by the receiver.

It is well known, see for instance [12], that as an alternative to ALOHA, it is possible to carefully choose the communication rates of the users such that the receiver can decode all, or a subset of, the packets of the active users. This is sometimes referred to as multipacket reception and its use for random access has been studied in, for instance, [13]–[15]. Finally, in [16] collisions are interpreted as linear combinations of packets and ZigZag decoding [17] is used to recover the original packets. This approach is different from our work in the sense that linear combinations are not decoded directly.

The *strategy proposed* in [1] is based on another way of

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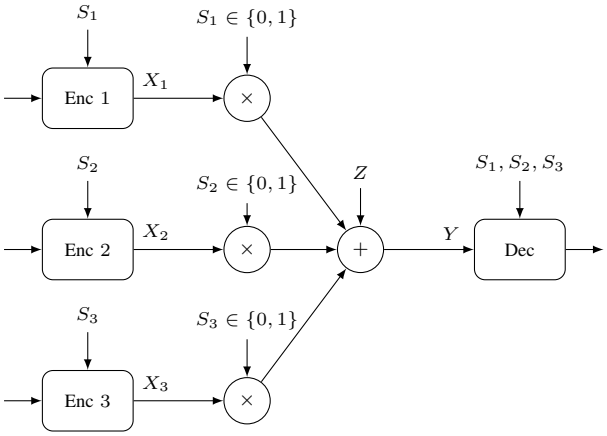

Fig. 2: Model for three users, $K = 3$.

dealing with collisions. Instead of trying to decode any of the packets transmitted by the users, the receiver attempts to decode a linear combination of these packets, using the compute-and-forward approach. In the proposed strategy, the users transmit in each time slot a packet that is itself a linear combination of the messages intended for the receiver. After obtaining a sufficient number of linear combinations the receiver can retrieve the original messages. The physical-layer network coding strategy is illustrated in Figure 1b. For the example in the figure, the information obtained by the receiver can be represented as

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & g & h \\ 0 & 0 & k & l \\ e & f & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \tag{1}$$

where $A, \cdots, D$ denote the messages and $b_1, \ldots, b_4$ the information obtained by the receiver at the channel output. For the case of two users we have demonstrated in [1] that our strategy significantly outperforms existing strategies, including multipacket reception. Similar improvements are demonstrated in the current paper for an arbitrary number of users.

The outline of the remainder of this paper is as follows. In Section II we define the model. The main contributions of the current paper, a description of our approach and an analysis of the resulting throughput are given in Section III. Section IV provides an analysis of the throughput of some other approaches and an evaluation of our result. Finally, in Section V we conclude with a discussion of the results that are presented in the current paper and an outlook on future work.

II. Model and Problem Formulation

We consider a system with K users. Time is slotted. The system is operated in N blocks of B time slots, where B and N are design parameters. User i has a message M_i to transmit, where

$$M_i \in \left\{1, \ldots, 2^{NBR_i}\right\}, \tag{2}$$

i.e., R_i is the rate of user i.

The random access feature of the model is captured by state variables S_i which can be zero or one, depending on whether a user is active (1) or inactive (0). Let $S_i(n)$ denote the state of user i in block n. The state of a user is independently and identically distributed over all blocks and independent of the state of other users. Users are active with probability a, *i.e.*, $\Pr(S_i(n) = 1) = a$ for all $i = 1, \ldots, K$ and $n = 1, \ldots, N$.

Let $X_i[t] \in \mathbb{R}$ and $Y[t] \in \mathbb{R}$ denote the signal transmitted by user i and the signal obtained by the receiver, respectively, in time slot t. We consider an AWGN channel without fading, *i.e.*,

$$Y[t] = \sum_{i=1}^{K} S_i(\lceil t/B \rceil) X_i[t] + Z[t], \tag{3}$$

where $\{Z[t]\}$ is white Gaussian noise with unit variance. In addition let $X_i = (X_i[1], \ldots, X_i[NB])$, with Y and Z defined similarly.

For notational convenience in the description of our proposed strategy we introduce, for $n = 1, \ldots, N$, the notation

$$X_i(n) = (X_i[1 + (n-1)B], \cdots, X_i[nB]), \tag{4}$$
$$Y(n) = (Y[1 + (n-1)B], \cdots, Y[nB]), \tag{5}$$
$$Z(n) = (Z[1 + (n-1)B], \cdots, Z[nB]). \tag{6}$$

The channel model, (3), can alternatively be written as

$$Y(n) = \sum_{i=1}^{K} S_i(n) X_i(n) + Z(n). \tag{7}$$

The model is illustrated for three users, *i.e.*, for $K = 3$, in Figure 2.

Definition 1 (Strategy)**.** *A strategy defines encoders E_i, $i = 1, \ldots, K$, that map the user message M_i and channel states S_i to a signal X_i, i.e.,*

$$E_i : \left\{1, \ldots, 2^{NBR_i}\right\} \times \{0,1\}^N \to \mathbb{R}^{NB}, \tag{8}$$

where we require these mappings to satisfy the following average power constraint:

$$\frac{1}{NB} \sum_{t=1}^{NB} \mathbb{E}\left[X_i^2[t]\right] \leq P_i, \tag{9}$$

for all $i = 1, \ldots, K$, where the expectation is over all messages. We denote the encoder mapping by $X_i = E_i(M_i, S_i)$. Finally, a strategy also defines a decoder D that uses knowledge of user states to map the received signal to an estimate of the user messages, i.e.,

$$D : \mathbb{R}^{NB} \times \{0,1\}^{KN} \to \left\{1, \ldots, 2^{NBR_1}\right\} \times \cdots \times \left\{1, \ldots, 2^{NBR_K}\right\}, \tag{10}$$

We denote the decoder mapping by $\hat{M} \triangleq (\hat{M}_1, \ldots, \hat{M}_K) = D(Y, S_1, \ldots, S_K)$.

For a given strategy, we define the resulting average error probability by

$$P_e = \Pr\left[(\hat{M}_1, \ldots, \hat{M}_K) \neq (M_1, \ldots, M_K)\right]. \tag{11}$$

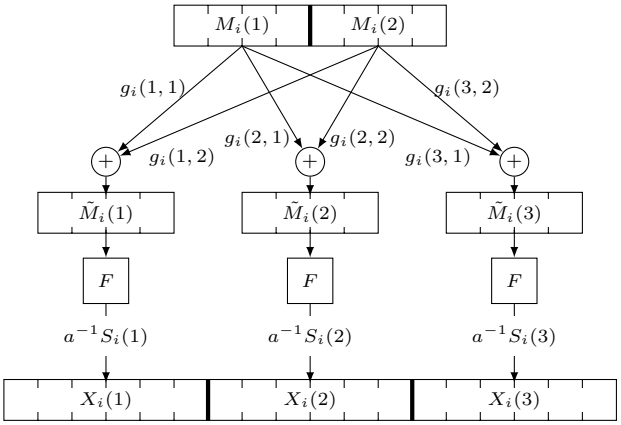

Fig. 3: The encoder for user i, in the case that $N = 3$, $B = 6$, $L_c = 4$, $L_b = 2$.

In the present paper, we restrict attention to the *symmetric* scenario where all rates and all power constraints are equal, i.e.,

$$R = R_1 = \ldots = R_K \quad \text{and} \quad P = P_1 = \ldots = P_K,$$

and we will refer to the *throughput* T of a strategy simply as the sum of all the user rates, namely,

$$T = KR. \tag{12}$$

The goal of the present paper is to characterize *achievable throughput,* which we define in a standard manner:

Definition 2 (Achievable throughput)**.** *Throughput T is achievable if there exists for every $\epsilon_1 > 0$ and $\epsilon_2 > 0$ a strategy with $R = T/K - \epsilon_1$ for which $P_e \leq \epsilon_2$.*

Remark. A few remarks about our modeling assumptions are in order:

1) Users are assumed to know K, the total number of users in the system.
2) There is block synchronization between users.
3) There is no feedback from the receiver to the users.
4) The long-term average power constraint allows to perform power control: Transmit at power $a^{-1}P$ in a block in which a user is active and with zero power otherwise, leading to long-term average power P.

We discuss our assumptions in Section V.

III. Proposed Strategy

The strategy presented in this section forms the main contribution of the current paper.

A. Message structure

Remember from Section II that user i has a message M_i to transmit, where

$$M_i \in \left\{1, \ldots, 2^{NBR}\right\}. \tag{13}$$

The first step of the proposed strategy consists of expressing the message M_i as a string of $NBR/\log_2 q$ symbols from $\mathbb{F}_q$, where q will be suitably chosen. These symbols are grouped in L_b message substrings $M_i(\ell)$, each of length L_c, such that we can express

$$M_i = \left(M_i(1), \ldots M_i(L_b)\right), \tag{14}$$

where $M_i(\ell) \in \mathbb{F}_q^{L_c}$, with

$$L_b L_c = \frac{NBR}{\log_2 q}. \tag{15}$$

The values of L_b and L_c need to be chosen carefully; an analysis is provided in subsection III-D.

B. Encoder

The encoder E_i at user i consists of:

- Matrix $\mathbf{G}_i = [g_i(n,\ell)]$ of size $N \times L_b$ with elements from $\mathbb{F}_q$,
- Linear code $F : \mathbb{F}_q^{L_c} \to \mathbb{R}^B$, *i.e.*, a code of blocklength B that takes L_c message symbols.

Encoder i constructs the signal X_i, by performing the following steps for each block $n = 1, \ldots, N$:

1) The encoder first computes new equivalent message substrings $\tilde{M}_i(n)$ by mixing the original message substrings $M_i(\ell)$, as follows:
$$\tilde{M}_i(n) = \sum_{\ell=1}^{L_b} g_i(n,\ell) M_i(\ell), \tag{16}$$
where all operations on $M_i(\ell)$ are componentwise.
2) Use $\tilde{M}_i(n)$ as the input of code F and take the user state into consideration, *i.e.*,
$$X_i(n) = a^{-1} S_i(n) F(\tilde{M}_i(n)). \tag{17}$$

The encoder strategy is illustrated in Figure 3.

C. Decoder

The receiver decodes as follows:

1) In block n the receiver observes the signal
$$Y(n) = a^{-1} \sum_{\substack{i \in \{1,\ldots,K\}: \\ S_i(n)=1}} F(\tilde{M}_i(n)) + Z(n). \tag{18}$$
2) As demonstrated in the next subsection, if L_c and F are chosen properly, the decoder can perform physical-layer network coding and recover
$$\begin{aligned} b(n) &= \sum_{\substack{i \in \{1,\ldots,K\}: \\ S_i(n)=1}} \tilde{M}_i(n) \\ &= \sum_{\substack{i \in \{1,\ldots,K\}: \\ S_i(n)=1}} \sum_{\ell=1}^{L_b} g_i(n,\ell) M_i(\ell), \end{aligned} \tag{19}$$
for all $n = 1, \ldots, N$.
3) It remains to retrieve the messages by solving the system of linear equations given by (19). It is shown in the next subsection that a carefully chosen L_b results in a full rank system of equations.

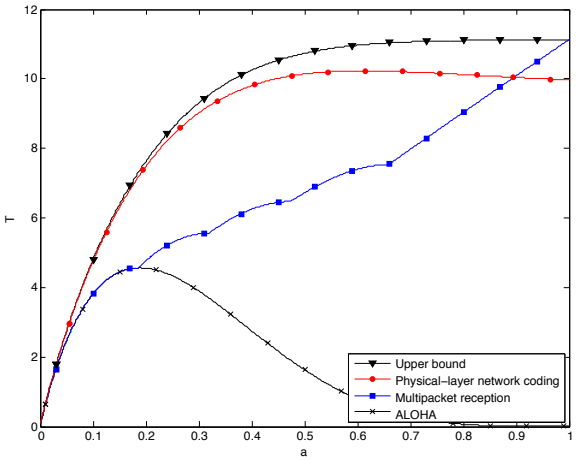

Fig. 4: Five users, $K = 5$.

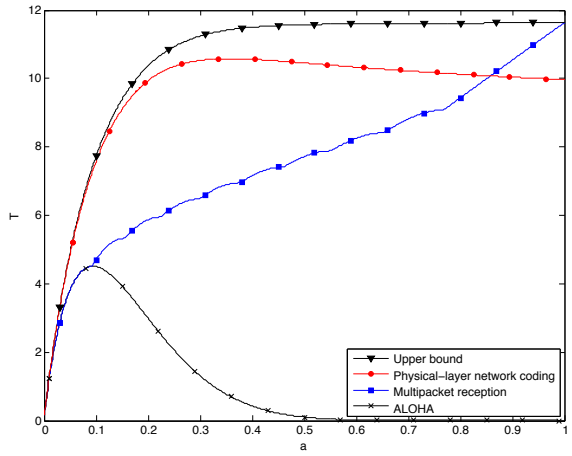

Fig. 5: Ten users, $K = 10$.

D. Achievable throughput

First of all, we consider the use of physical-layer network coding, which in our strategy is done separately in each block of B channel uses, and involves strings of L_c symbols from $\mathbb{F}_q$. From [3] we know that there exists linear codes F that enable physical-layer network coding. More precisely, we have the following result.

Theorem 1 ([3], Thm. 2)**.** *For the standard AWGN multiple-access channel, the following reliable physical-layer network coding rate is achievable:*

$$\frac{L_c \log_2 q}{B} = \frac{1}{2}\log_2\left[\frac{1}{K} + P\right]. \tag{20}$$

In addition to satisfying (20) we have to choose L_b such that the system of linear equations given by (19) is full rank. It can be shown that by choosing

$$\frac{L_b}{N} = \frac{1-(1-a)^K}{K}, \tag{21}$$

we can make this probability arbitrarily close to one if N is chosen sufficiently large.

By (12) and (15), we have that

$$T = KR = \frac{KL_bL_c \log_2 q}{NB}. \tag{22}$$

The above choices of L_b and L_c thus lead to the following theorem, which forms the main contribution of the current paper, the proof of which is omitted due to space constraints.

Theorem 2. *Throughput*

$$T = \left(1-(1-a)^K\right)\frac{1}{2}\log_2\left[\frac{1}{K} + a^{-1}P\right] \tag{23}$$

is achievable using a physical-layer network coding strategy.

IV. Evaluation

In this section we present results on the achievable throughput of various other strategies. A numerical evalutation of the results is given in Figures 4 and 5, where we have plotted the throughput T as a function of the access probability a for various values for the number of users K. Results are presented for avarge power constraint 10^6, *i.e.*, $P = 10^6$.

We start with an upper bound on the throughput, that can be obtained by assuming full CSI at the transmitters, *i.e.*, we assume that all transmitters know which other transmitters are active, and using standard results on the multi-access channels [12]. Details of the proof are omitted.

Theorem 3. *If throughput T is achievable, then*

$$T \le \sum_{i=1}^{K}\binom{K}{i}a^i(1-a)^{K-i}\frac{1}{2}\log_2\left(1+ia^{-1}P\right). \tag{24}$$

Next, we consider slotted ALOHA. Users transmit at the maximum achievable rate, where the maximum is under the condition that there is only a single user in the system, *i.e.*, from [12] we have

$$\frac{L_c \log_2 q}{B} = \frac{1}{2}\log_2(1+a^{-1}P). \tag{25}$$

The receiver is only able to decode if there is a single active user. This directly leads to the well known result, *cf.* [11], that:

Theorem 4. *Slotted ALOHA achieves throughput*

$$T = Ka(1-a)^{K-1}\frac{1}{2}\log_2\left(1+a^{-1}P\right). \tag{26}$$

The figures clearly demonstrate the well-known fact that ALOHA does not perform well for high access probability or many users.

The best strategy that is known in the literature is multi-packet reception together with adaptive rates [14], [15]. This strategy consists of choosing the rate such that the packets of

all active users can be decoded as long as this number is at most $\tilde{K}$, *i.e.*, all users transmit at rate

$$\frac{L_c \log_2 q}{B} = \frac{1}{2\tilde{K}} \log_2 \left[1 + \tilde{K} a^{-1} P\right]. \tag{27}$$

Part of the strategy is to optimize over $\tilde{K}$.

Theorem 5 ([15]). *Multipacket reception achieves throughput*

$$T = \max_{\tilde{K}} \sum_{i=1}^{\tilde{K}} \binom{K}{i} a^i (1-a)^{K-i} \frac{i}{2\tilde{K}} \log_2 \left[1 + \tilde{K} a^{-1} P\right]. \tag{28}$$

V. Discussion

We have presented an approach to random access that is based on physical-layer network coding. The gist of this strategy is that whenever packets collide, the receiver decodes a linear combination of these packets. The throughput that is achieved by this approach is significantly better than that of other approaches.

The strategy as it is presented in the current paper does not employ feedback from the receiver. It can, however, easily be adjusted to incorporate feedback from the receiver. Consider, for instance, the case that the receiver provides an acknowledgement as soon as it decodes the system of linear equations given by (19) and recovers all message substrings from all users. The users can simply continue to transmit linear combinations of their message substrings until they receive the acknowledgement. This results in a rateless strategy that is particularly easy to implement.

Observe that in our model we allow for strategies that perform coding over blocks. This potentially leads to very large delays. In fact, the proof of Theorem 2 is based on using many blocks. We believe that the strategy can be incorporated in a practical implementation, for which the number of blocks that is used is small. In particular, we can, in each block transmit a linear combination of only a small number of message substrings and use a sliding window. This will allow to receiver the decode substrings after a small number of blocks.

Finally, the comparison with the arguments in [15] reveals a further interesting insight. Namely, in [15], an information-theoretic upper bound is presented that applies to any strategy in which *packets* are decoded after every block. It is clear that if we allow to code over many blocks, as proposed in for instance [18] and [19], the throughput can be enlarged beyond this upper bound, but this requires the receiver to store all the physical-layer channel outputs of all the blocks, which is a high price to pay in terms of implementation complexity.

It is interesting to observe that the new strategy introduced in this paper *also* permits to significantly outperform the upper bound presented in [15], *i.e.*, it is beneficial to decode linear combinations of packets, rather than only entire packets. In this sense, the proposed strategy is a simple version of coding over many blocks, where the receiver can discard the physical-layer channel outputs after every block. Moreover, it should be expected that a careful implementation of the strategy, will only require coding over a small number of blocks.

VI. Acknowledgement

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