Spatially-Coupled MacKay-Neal Codes with No Bit Nodes of Degree Two Achieve the Capacity of BEC

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***Abstract*—Obata et al. proved that spatially-coupled (SC) MacKay-Neal (MN) codes achieve the capacity of BEC. However, the SC-MN codes codes have many variable nodes of degree two and have higher error floors. In this paper, we prove that SC-MN codes with no variable nodes of degree two achieve the capacity of BEC.**

I. Introduction

Felström and Zigangirov introduced spatially-coupled (SC) codes defined by sparse parity check matrix. SC codes are based on constitution method for convolutional LDPC codes [1]. Lantmaier et al. confirmed that regular SC LDPC codes achieve MAP threshold of original LDPC block codes by BP decoding in at least certain accuracy [2]. Kudekar et al. proved that SC codes achieve MAP threshold by BP decoding on binary erasure channel (BEC) [3] and binary symmetric channel [4].

Kasai et al. introduced SC MacKay-Neal (MN) codes, and showed that these codes with finite maximum degree achieve capacity of BEC by numerical experiment [5]. Obata et al. proved $(l,2,2)$ SC-MN codes achieve capacity [6]. It has been observed that $(l,2,2)$ SC-MN codes have many bit nodes of degree two. This leads to high error floors.

In this paper, we deal with $(l,3,3)$ SC-MN codes whose bit node degree is greater than two. We prove the codes achieve the capacity of BEC. The codes achieve Shannon limit $\epsilon^{\mathrm{Sha}} = 1-\frac{3}{l}$ for any $l \geq 3$.

II. Background

A. MacKay-Neal Codes

(l,r,g) MN codes are multi-edge type (MET) LDPC codes defined by pair of multi-variables degree distributions (μ,ν) listed below.

$$\nu(\boldsymbol{x};\epsilon) = \frac{r}{l}x_1^l + \epsilon x_2^g,$$
$$\mu(\boldsymbol{x}) = x_1^r x_2^g.$$

In general, the recursion of density evolution of MET-LDPC codes on BEC is given by

$$y_j^{(t)} \quad = \quad 1 - \frac{\mu_j(\mathbf{1}-\boldsymbol{x}^{(t)};1-\epsilon)}{\mu_j(\mathbf{1};1)}, \quad x_j^{(t+1)} = \frac{\nu_j(\boldsymbol{y}^{(t)};\epsilon)}{\nu_j(\mathbf{1};1)},$$

where $x_j^{(t)}$ is probability of erasure message sent along edges of type j at the t-th decoding round. Therefore, density evolution of (l,r,g) MN codes is

$$\begin{aligned} &\boldsymbol{x}^{(t+1)} = \boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{x}^{(t)});\epsilon\big), \qquad (2)\\ &\boldsymbol{f}(\boldsymbol{x};\epsilon) = (x_1^{l-1}, \epsilon x_2^{g-1}),\\ &\boldsymbol{g}(\boldsymbol{x}) = (1-(1-x_1)^{r-1}(1-x_2)^g, 1-(1-x_1)^r(1-x_2)^{g-1}). \end{aligned}$$

B. Spatially-Coupled MacKay-Neal Codes

SC-MN codes of coupling number L and of coupling width w are defined by the Tanner graph constructed by the following process. First, at each section $i \in \mathbb{Z}$, place rM/l bit nodes of type 1 and M bits nodes of type 2. Bit nodes of type 1 and 2 are of degree l and g, respectively. Next, at each section $i \in \mathbb{Z}$, place M check nodes of degree $r+g$. Then, connect edges uniformly at random so that bit nodes of type 1 at section i are connected with check nodes at each section $i \in [i,\dots,i+w-1]$ with rM/w edges, and bit nodes of type 2 at section i are connected with check nodes at each section $i \in [i,\dots,i+w-1]$ with gM/w edges. Bits at section $i \notin [0,L-1]$) are shorten. Bits of type 1 and 2 at section $i \in [0,L-1]$ are punctured and transmitted, respectively. Rate of SC-MN codes R^{MN} is given by

$$R^{\mathrm{MN}} \quad = \quad \frac{r}{l} + \frac{1+w-2\sum_{i=0}^{w}(1-(\frac{i}{w})^{r+g})}{L} = \frac{r}{l} \quad (L \to \infty).$$

C. Vector Admissible System and Potential Function

In this section, we define vector admissible systems and potential functions.

Definition 1. *Define $\mathcal{X} \triangleq [0,1]^d$, and $F : \mathcal{X}\times[0,1] \to \mathbb{R}$ and $G : \mathcal{X} \to \mathbb{R}$ as functionals satisfying $G(\mathbf{0}) = 0$. Let $\boldsymbol{D}$ be a $d \times d$ positive diagonal matrix. Consider a general recursion defined by*

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}^{(t)});\epsilon),$$

where $\boldsymbol{f} : \mathcal{X}\times[0,1] \to \mathcal{X}$ and $\boldsymbol{g} : \mathcal{X} \to \mathcal{X}$ are defined by $F'(\boldsymbol{x};\epsilon) = \boldsymbol{f}(\boldsymbol{x};\epsilon)\boldsymbol{D}$ and $G'(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})\boldsymbol{D}$, where $F'(\boldsymbol{x};\epsilon) \triangleq (\frac{\partial F(\boldsymbol{x})}{\partial x_1},\dots,\frac{\partial F(\boldsymbol{x})}{\partial x_n})$. Then the pair $(\boldsymbol{f},\boldsymbol{g})$ defines a vector admissible system if

1. *$\boldsymbol{f},\boldsymbol{g}$ are twice continuously differentiable,*
2. *$\boldsymbol{f}(\boldsymbol{x};\epsilon)$ and $\boldsymbol{g}(\boldsymbol{x})$ are non-decreasing in $\boldsymbol{x}$ and ϵ with respect to $\preceq$ [1],*

[1]We say $\boldsymbol{x} \preceq \boldsymbol{y}$ if $x_i \leq y_i$ for all $1 \leq i \leq d$

3. $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{0});\epsilon)=\boldsymbol{0}$ *and* $F(\boldsymbol{g}(\boldsymbol{0});\epsilon)=0$.

We say $\boldsymbol{x}$ *is a fixed point if* $\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x});\epsilon)$.

It can be seen that the density evolution $(\boldsymbol{f},\boldsymbol{g})$ of (l,r,g) MN codes given in (2) is a vector admissible system by choosing $F(\boldsymbol{x};\epsilon),G(x)$ and $\boldsymbol{D}$ as below, since this system $(\boldsymbol{f},\boldsymbol{g})$ satisfies the condition in Definition 1.

$$F(\boldsymbol{x};\epsilon)=\frac{r}{l}x_1^l+\epsilon x_2^g,$$
$$G(\boldsymbol{x})=rx_1+gx_2+(1-x_1)^r(1-x_2)^g-1,$$
$$\boldsymbol{D}=\begin{pmatrix} r & 0\\ 0 & g\end{pmatrix}.$$

Definition 2 ([7, Def. 2])**.** *We define the potential function* $U(\boldsymbol{x};\epsilon)$ *of a vector admissible system* $(\boldsymbol{f},\boldsymbol{g})$ *by*

$$U(\boldsymbol{x};\epsilon)\triangleq\boldsymbol{g}(\boldsymbol{x})\boldsymbol{D}\boldsymbol{x}^T-G(\boldsymbol{x})-F(\boldsymbol{g}(\boldsymbol{x});\epsilon).$$

The potential function $U(x_1,x_2,\epsilon)$ of (l,r,g) MN codes is given by

$$\begin{aligned}U(x_1,x_2,\epsilon)=1&-\epsilon\big((1-(1-x_1)^r)(1-x_2)^{g-1}\big)^g\\ &-\frac{r}{l}\big(1-(1-x_1)^{r-1}(1-x_2)^g\big)^l\\ &-(1-x_1)^r(1-x_2)^g\left(1+\frac{rx_1}{1-x_1}+\frac{gx_2}{1-x_2}\right).\end{aligned}$$

Definition 3 ([7, Def. 7])**.** *Let* $\mathcal{F}(\epsilon)\triangleq\{\boldsymbol{x}\in\mathcal{X}\setminus\{\boldsymbol{0}\}\mid\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x});\epsilon)\}$ *be a set of non-zero fixed points for* $\epsilon\in[0,1]$. *The potential threshold* ϵ^* *is defined by*

$$\epsilon^*\triangleq\sup\{\epsilon\in[0,1]\mid\min_{\boldsymbol{x}\in\mathcal{F}(\epsilon)}U(\boldsymbol{x};\epsilon)>0\}.$$

Let ϵ_s^* *be threshold of uncoupled system defined in [7, Def. 6]. For* ϵ *such that* $\epsilon_s^*<\epsilon<\epsilon^*$, *we define energy gap* $\Delta E(\epsilon)$ *as*

$$\Delta E(\epsilon)\triangleq\max_{\epsilon'\in[\epsilon,1]}\inf_{\boldsymbol{x}\in\mathcal{F}(\epsilon')}U(\boldsymbol{x};\epsilon').$$

We define the SC system of a vector admissible system.

Definition 4 ([7, Def. 9])**.** *For a vector admissible system* $(\boldsymbol{f},\boldsymbol{g})$, *we define the SC system of coupling number* L *and coupling width* w *as*

$$\boldsymbol{x}_i^{(t+1)}=\frac{1}{w}\sum_{k=0}^{w-1}\boldsymbol{f}\left(\frac{1}{w}\sum_{j=0}^{w-1}\boldsymbol{g}(\boldsymbol{x}_{i+j-k}^{(t)});\epsilon_{i-k}\right),$$
$$\epsilon_i=\begin{cases}\epsilon, & i\in\{0,\dots,L-1\},\\ 0, & i\notin\{0,\dots,L-1\}.\end{cases}$$

If we define $(\boldsymbol{f},\boldsymbol{g})$ as the density evolution for (l,r,g) MN codes in (2), the SC system gives the density evolution of SC-MN codes with coupling number L and coupling width w.

Next theorem states that if $\epsilon<\epsilon^*$ then fixed points of SC vector system converge towards $\boldsymbol{0}$ for sufficiently large w.

Theorem 1 ([7, Thm. 1])**.** *Consider the constant* $K_{\boldsymbol{f},\boldsymbol{g}}$ *defined in* [7, Lem. 11]. This constant value depends only on $(\boldsymbol{f},\boldsymbol{g})$. If $\epsilon<\epsilon^*$ and $w>(dK_{\boldsymbol{f},\boldsymbol{g}})/(2\Delta E(\epsilon))$, then the SC system of $(\boldsymbol{f},\boldsymbol{g})$ with coupling number L and coupling width w has a unique fixed point $\boldsymbol{0}$.

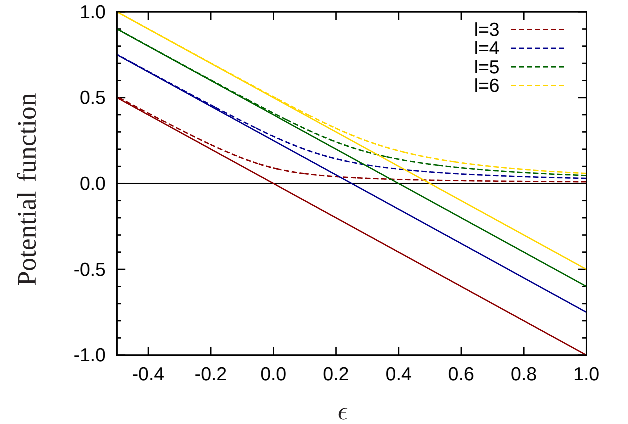

Fig. 1. Potential function $U(\boldsymbol{1};\epsilon)$ and $U(\boldsymbol{x}(x_1);\epsilon(x_1))$ at the trivial fixed points (solid) and non-trivial fixed points (dashed) of $(l,3,3)$ MN codes for $l=3,\dots,6$

We will show that the potential threshold ϵ^* of $(l,r=3,g=3)$ MN codes is $1-R^{\mathrm{MN}}=1-3/l$ for any $l\geq 3$. This is sufficient to show that $(l,3,3)$ SC-MN codes with sufficiently large w and L achieve the capacity of BEC under BP decoding.

III. Proof of Achieving Capacity

In this section, we calculate the potential threshold ϵ^* of $(l,r=3,g=3)$ MN codes. To this end, we first investigate the set of fixed points $\mathcal{F}(\epsilon)$.

The density evolution recursion in (2) can be rewritten as

$$x_1^{(t+1)}=(1-(1-x_1^{(t)})^{r-1}(1-x_2^{(t)})^g)^{l-1},$$
$$x_2^{(t+1)}=\epsilon(1-(1-x_1^{(t)})^r(1-x_2^{(t)})^{g-1})^{g-1}.$$

Fixed points $(x_1,x_2;\epsilon)$ of density evolution with $x_1=0$ and $x_1=1$ are $(0,0;\epsilon)$ and $(1,\epsilon;\epsilon)$, respectively. We define these fixed points as trivial fixed points and all other fixed points as non-trivial fixed points. All non-trivial fixed points $(x_1,x_2(x_1);\epsilon(x_1))$ can be parametrically described as

$$x_2(x_1)=1-\left(\frac{1-x_1^{\frac{1}{l-1}}}{(1-x_1)^{r-1}}\right)^{\frac{1}{g}},$$
$$\epsilon(x_1)=\frac{x_2(x_1)}{\big(1-(1-x_1)^r(1-x_2(x_1))^{g-1}\big)^{g-1}},$$

with $x_1\in(0,1)$.

Next, we shall investigate the value of the potential function at the fixed points. The value of the potential functions at trivial fixed point $(1,\epsilon,\epsilon)$ is respectively given by

$$U(1,\epsilon,\epsilon)\ =\ 1-\frac{r}{l}-\epsilon.$$

Figure 1 draws the potential function of (l,r,g) MN codes at fixed points $\boldsymbol{x}\in\mathcal{F}(\epsilon)$. It appears that the potential function at non-trivial fixed points is always positive. We will prove this.

To be precise, the potential function of (l, r, g) MN codes for non-trivial fixed points satisfies

$$U(x_1, x_2(x_1), \epsilon(x_1)) > 0 \quad \text{for} \quad x_1 \in (0,1). \tag{4}$$

Our strategy of proof is as follows. First change the representation of (4) into a polynomial form by changing variables a few times. Then apply Sturm's theorem for smaller l and bound the polynomial for larger l.

We define $U(z) := U(x_1, x_2(x_1), \epsilon(x_1))|_{x_1=z^{l-1}}$. Obviously, to prove (4), it is sufficient to show $U(z) > 0$ for $z \in (0,1)$.

$$\begin{aligned}U(z) = &-\frac{3z^l}{l} + (1-z)(1-4z^{l-1}) \\ &+ (1-z)^{1/3}(1-z^{l-1})^{-2/3} - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}.\end{aligned}$$

We use next lemma to eliminate fractional power in $U(z)$. The proof is given in Section IV-A.

Lemma 1. *Define $H(u,z)$ as follows.*

$$\begin{aligned}H(u,z) = &\big(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\big)^3 \\ &+ 6(1-z)(1-z^{l-1})\big(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\big) \\ &- (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5.\end{aligned}$$

Then, $H(0,z) < 0$ for $z \in (0,1)$ implies $U(z) > 0$ for $z \in (0,1)$.

Define $I(z) := \dfrac{l^3(1-z^{l-1})^2}{(1-z)z^2} H(0,z)$. Obviously, to prove $H(0,z) < 0$ for $z \in (0,1)$, it is sufficient to prove $I(z) < 0$ for $z \in (0,1)$. We see that $I(z)$ for $l \geq 3$ is a polynomial as follows.

$$\begin{aligned}I(z) = &-l^3 + 27\sum_{i=0}^{l-2}[z^{3l-2+i}(1-z^{l-1})] \\ &- 27l^2 z^{-2+2l}(1-4z^{l-1})(1-z^{l-1})^2 \\ &- 9lz^{-4+l}(1-z^{l-1})^2\big\{(-3+z)z^2 \\ &+ 16(-1+z)z^{2l} - 8(-1+z)z^{1+l}\big\} \\ &- l^3(1-z)z^{-9+l}\big\{8z^{6l} - 56z^{1+5l} + 2z^6(3+7z) \\ &+ 8z^{2+4l}(13+8z) - 8z^{3+3l}(13+22z) \\ &+ 4z^{4+2l}(21+43z) - z^{5+l}(41+73z)\big\}.\end{aligned} \tag{5}$$

We prove $I(z) < 0$ for $3 \leq l < 165$ and $l \geq 165$ in the following lemmas. The proofs are given in Section IV-B and Section IV-C, respectively.

Lemma 2. *For $3 \leq l < 165$, $I(z) < 0$ for $z \in (0,1)$.*

Lemma 3. *For $l \geq 165$, $I(z) < 0$ for $z \in (0,1)$.*

Theorem 2. *For any $l \geq 3$ and $\epsilon < \epsilon^{\text{Sha}} = 1 - \frac{3}{l}$, the unique fixed point of density evolution of $(l,3,3)$ SC-MN codes of coupling number L and coupling width w is $\mathbf{0}$ for sufficiently large w and L.*

Proof: From (4), potential function for non-trivial fixed points is always positive. Therefore, from Definition 3 and potential function for trivial fixed point (3), $\epsilon^* = 1 - \frac{r}{l} = \epsilon^{\text{Sha}}$. From Theorem 1, for $\epsilon < \epsilon^{\text{Sha}}$, the unique fixed point of density evolution for $(l,3,3)$ SC-MN codes is $\mathbf{0}$. □

The case with $l = 3$ implies rate one codes over BEC(0). Some might think this is not interesting. Nevertheless, we included the case with $l = 3$ for comprehensiveness.

IV. Proof of Lemmas

A. Proof of Lemma 1

Partial derivative of $H(u,z)$ with respect to u gives

$$\begin{aligned}\frac{\partial H(u,z)}{\partial u} = &3\big(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\big)^2 \\ &+ 6(1-z)(1-z^{l-1}) \geq 0.\end{aligned} \tag{6}$$

Substituting $u = U(z)$ into $H(u,z)$ gives

$$\begin{aligned}&H(U(z),z) \\ =&\big((1-z)^{1/3}(1-z^{l-1})^{-2/3} - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\big)^3 \\ &+ 6(1-z)(1-z^{l-1})\big\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &- 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\big\} \\ &- (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5 \\ =&(1-z)(1-z^{l-1})^{-2} - 8(1-z)^2(1-z^{l-1})^5 \\ &- 6(1-z)(1-z^{l-1})\big\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &- 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\big\} \\ &+ 6(1-z)(1-z^{l-1})\big\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &- 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\big\} \\ &- (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5 \\ =&0.\end{aligned} \tag{7}$$

From (6), $H(u,z)$ monotonically increasing with respect to u. From (7), $(u,z) = (U(z),z)$ is a root of $H(u,z) = 0$. Therefore $H(0,z) < 0$ for $z \in (0,1)$ implies $U(z) > 0$ for $z \in (0,1)$. □

B. Proof of Lemma 2

From $I(0) = -l^3$ and $I(1) = -l^3$, we see that $z = 0, 1$ are not multiple roots of equation $I(z) = 0$. Let $I_1(z), \ldots, I_m(z)$ be Sturm sequences of $I(x)$. Let $V(z)$ be the number of sign changes in the sequence. Table I lists sign changes of Sturm sequence $I_1(z), \ldots, I_m(z)$ of $I(x)$ in (5) for $l = 3, \ldots, 11$. $V(z)$ is the number of sign changes in the sequence. We see that $V(0) = V(1)$. We observed that $V(0) = V(1)$ for $l < 165$ but not listed all due to the space limit. From Theorem 3, this implies that the number of distinct roots of equation $I(z) = 0$ in $(0,1]$ is $V(0) - V(1) = 0$. Therefore, $I(z) < 0, z \in (0,1)$ for $3, \ldots, 164$. □

TABLE I

SIGN CHANGES OF STURM SEQUENCE $I_1(z), \ldots, I_m(z)$ OF $I(x)$ IN (5) FOR $l = 3, \ldots, 11$. $V(z)$ IS THE NUMBER OF SIGN CHANGES IN THE SEQUENCE.

l	m	$V(z)$	z	$\mathrm{sgn}[I_0(z)], \mathrm{sgn}[I_1(z)], \ldots, \mathrm{sgn}[I_m(z)]$
3	13	5	0	−−+++−−−+−−−++
		5	1	−−+++++−−+−−−+
4	20	10	0	−−+−−+−−−++−+++−−+−−−
		10	1	−−+−−−+−−++−−−−+−−+−−
5	27	12	0	−0+−−−++−−+++−++−−−++−−−+−−−
		12	1	−−++−−−++−++++−−+++−−++−−++−
6	33	16	0	−0+−−−++−+++−−++−−+++−++−−++−−+−−−
		16	1	−−++−−−−+−++−−−−++−−−++−+−−−+−+++−
7	39	18	0	−0+−−−++−−++−++++−+++−−++−−++−−−+++−+++−
		18	1	−−++−−−++−−+−+++++−−−++−−++−−++−+++−−++−
8	45	22	0	−0+−−−+++−+−−−+−−−++−−++−−−++−−+−++++−−+−−−++−
		22	1	−−++−−−+++−+−−+−−−−−++−−++−−−++−+−−−+−−+++−++−
9	51	24	0	−0+−−−++−−++−−+−+++++−+++−−++−−−+−−++−−−+−−+−−−−+++−
		24	1	−−++−−−++−−++−+−−+++++−−+++−−+++−++−+++−++−−+++−+++−
10	57	28	0	−0+−−−++−−−+−+++−+−−−−++−−−++−+++−−−++−+−−−++−++−++++−+++−
		28	1	−−++−−−+++−−+−++−−+−−−−−++−+++−−+++−−−+−+−−−++−++−−−+−−++−
11	63	30	0	−0+−−−++−+++−−++−+−−++++−−+++−−−+−−++++−++−++++−+−+++−++++−++++−
		30	1	−−++−−−−+−−−+−−+−+++−+++++−−+−−−+++−−−−+−−+−+++−+++−++−−−+−++++−

C. Proof of Lemma 3

We first claim that for $z \in (0,1)$,

$$\text{If } \frac{al+b}{al+b+1} \in (0,1), \text{ then}$$

$$q(z) := z^{al+b}(1-z) \le \frac{1}{al+b+1}, \tag{8}$$

$$\text{If } \frac{al+b}{(a+2)l+b-2} \in (0,1), \text{ then}$$

$$r(z) := z^{al+b}(1-z^{l-1})^2 \le \left(\frac{2l-2}{(a+2)l+b-2}\right)^2 \tag{9}$$

Differentiating $q(z)$ gives

$$\frac{\mathrm{d}q(z)}{\mathrm{d}z} = z^{al+b-1}(al+b-(al+b+1)z).$$

Since $\frac{al+b}{al+b+1} \in (0,1)$, we see that $z = \frac{al+b}{al+b+1}$ gives the maximum value of $q(z)$.

$$q(z) \le \left(\frac{al+b}{al+b+1}\right)^{al+b} \frac{1}{al+b+1} < \frac{1}{al+b+1}.$$

Differentiating $r(z)$ gives

$$\frac{\mathrm{d}r(z)}{\mathrm{d}z} = z^{al+b-1}(1-z^{l-1})((al+b)-((a+2)l+b-2)z^{l-1}).$$

Since $\frac{al+b}{(a+2)l+b-2} \in (0,1)$, we see that $z = \left(\frac{al+b}{(a+2)l+b-2}\right)^{\frac{1}{l-1}}$ gives the maximum value of $r(z)$. Thus, next inequality holds.

$$\begin{aligned} r(z) &\le \left(\frac{al+b}{(a+2)l+b-2}\right)^{\frac{al+b}{l-1}} \left(\frac{2l-2}{(a+2)l+b-2}\right)^2 \\ &< \left(\frac{2l-2}{(a+2)l+b-2}\right)^2. \end{aligned}$$

In (a), we eliminate negative terms except for $-l^3$. Next, in (b), we apply (8) and (9) to each term of (5) by using $l \ge 165$.

We obtain an upper bound of $I(z)$ for $z \in (0,1)$ as follows.

$$\begin{aligned} I(z) &\overset{\text{(a)}}{<} -l^3 + 27\sum_{i=0}^{l-2} z^{3l-2+i}(1-z^{l-1}) \\ &\quad + 108l^2 z^{-3+3l}(1-z^{l-1})^2 \\ &\quad + 9lz^{-4+l}(1-z^{l-1})^2(3z^2+16z^{2l}+8z^{2+l}) \\ &\quad + l^3\{(1-z)z^{-9+l}[56z^{1+5l}+8z^{3+3l}(13+22z) \\ &\quad + z^{5+l}(41+73z)]\} \\ &\overset{\text{(b)}}{<} -l^3 + 27\sum_{i=0}^{l-2}[1] + 108l^2\Big(\frac{2l-2}{5l-5}\Big)^2 \\ &\quad + 9l\Big(3\Big(\frac{2l-2}{3l-4}\Big)^2 + 16\Big(\frac{2l-2}{5l-6}\Big)^2 + 8\Big(\frac{2l-2}{4l-4}\Big)^2\Big) \\ &\quad + l^3\Big(\frac{56}{6l-7} + \frac{176}{4l-4} + \frac{104}{4l-5} + \frac{41}{2l-3} + \frac{73}{2l-2}\Big) \\ &\overset{\text{(c)}}{<} -l^3 + 27(l-1) + \frac{432l^2}{25} + 9l\Big(3\frac{5}{9} + 16\frac{1}{5} + 8\frac{1}{4}\Big) \\ &\quad + 5l^3\Big(\frac{59}{29l} + \frac{176}{19l} + \frac{104}{19l} + \frac{73}{9l} + \frac{41}{9l}\Big) \\ &< -l^3 + \frac{6775346}{41325}l^2 + \frac{444}{5}l =: \overline{I}(l). \end{aligned}$$

We used next inequalities valid for $l \ge 165$ in (c).

$$\begin{aligned} &\Big(\frac{2l-2}{3l-4}\Big)^2 \le \frac{5}{9}, &&\Big(\frac{2l-2}{5l-6}\Big)^2 \le \frac{1}{5}, \\ &6l-7 \ge \frac{29l}{5}, &&4l-4 \ge \frac{19l}{5}, \\ &4l-5 \ge \frac{19l}{5}, &&2l-3 \ge \frac{9l}{5}, \\ &2l-2 \ge \frac{9l}{5}. \end{aligned}$$

The roots of $\overline{I}(l) = 0$ are 0 and $\frac{3387673 \pm \sqrt{11627977054429}}{41325} \simeq -0.53984, +164.49$. Thus, we conclude that for $l \ge 165$ and $z \in (0,1)$, $I(z) < \overline{I}(l) < 0$. $\square$

V. Conclusion and Future Work

In this paper, we proved that $(l,3,3)$ SC-MN codes with $l \geq 3$ achieve capacity on the BEC under BP decoding for sufficiently large L and w. This codes do not have bit nodes of degree two and have low error floors. We proved that the potential threshold and Shannon limit of $(l, r=3, g=3)$ MN codes on the BEC are the same.

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Appendix
Sturm's Theorem

Theorem 3 ([8]). *For a polynomial $f(x)$ over $\mathbb{R}$, we define Sturm sequences $f_i(x)$ $(i = 0, \ldots, m)$ as $f(x)$, $f'(x)$ and polynomials obtained by applying Euclid's algorithm to $f(x)$ and $f'(x)$.*

$$\begin{aligned} f_0(x) &= f(x), \\ f_1(x) &= f'(x), \\ f_{n-1}(x) &= q_n(x)f_n(x) - f_{n+1}(x) \quad (n = 1, \ldots, m-1), \\ f_{m-1}(x) &= q_m(x)f_m(x). \end{aligned}$$

For real number c, let $V(c)$ be the number of sign changes in $f_0(c), f_1(c), \ldots, f_m(c)$. If neither $a \in \mathbb{R}$ nor $b \in \mathbb{R}$ is a multiple root of $f(x) = 0$, then the number of distinct roots of $f(x)$ in $(a, b]$ is $V(a) - V(b)$.