Bounds on Locally Recoverable Codes with Multiple Recovering Sets

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Abstract—A locally recoverable code (LRC code) is a code over a finite alphabet such that every symbol in the encoding is a function of a small number of other symbols that form a recovering set. Bounds on the rate and distance of such codes have been extensively studied in the literature. In this paper we derive upper bounds on the rate and distance of codes in which every symbol has $t \geq 1$ disjoint recovering sets.

I. INTRODUCTION

Locally recoverable (LRC) codes currently form one of the rapidly developing topics in coding theory because of their applications in distributed and cloud storage systems. Recently LRC codes have been the subject of a large number of publications, among them [2], [3], [5], [7], [8], [12], [4]. We say that a code $\mathcal{C} \subset \mathbb{F}_q^n$ has locality r if every symbol of the codeword $x \in \mathcal{C}$ can be recovered from a subset of r other symbols of x (i.e., is a function of some other r symbols $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$). In other words, this means that, given $x \in \mathcal{C}, i \in [n]$, there exists a subset of coordinates $R_i \subset [n] \backslash i, |R_i| \leq r$ such that the restriction of \mathcal{C} to the coordinates in R_i enables one to find the value of x_i . The subset R_i is called a *recovering set* for the symbol x_i .

Now assume that every symbol of the code $\mathcal C$ can be recovered from t disjoint subsets of symbols of size r_1,\ldots,r_t respectively, called recovering sets of the symbol. Below we shall restrict ourselves to the case $r_1=\cdots=r_t=r$ which makes the expressions of the bounds more compact. At the same time, we note that the technique presented below enables us to treat the general case as well. Given a code $\mathcal C$ with t disjoint recovering sets of size r, we use the notation (n,k,r,t) to refer to its parameters. If the values of n,k,r are understood, we simply call $\mathcal C$ a t-LRC code.

More formally, denote by \mathcal{C}_I the restriction of the code \mathcal{C} to a subset of coordinates $I \subset [n]$. Given $a \in \mathbb{F}_q$ define the set of codewords $\mathcal{C}(i,a) = \{x \in \mathcal{C} : x_i = a\}, \ i \in [n]$.

DEFINITION: A code $\mathcal C$ is said to have t disjoint recovering sets if for every $i \in [n]$ there are t pairwise disjoint subsets $R_{i,1}, \ldots, R_{i,t} \subset [n]$ such that for all $j = 1, \ldots, t$

$$C_{R_{i,j}}(i,a) \cap C_{R_{i,j}}(i,a') = \emptyset, \quad a \neq a'.$$

Having more then one recovering set is beneficial in practice because it enables more users to access a given portion of data, thus enhancing data availability in the system. One of the main questions studied for LRC codes is related to estimates of the largest possible minimum distance of codes with locality r.

Theorem 1.1: Let $\mathcal C$ be an (n,k,r,t=1) LRC code, then: The rate of $\mathcal C$ satisfies

$$\frac{k}{n} \le \frac{r}{r+1}.\tag{1}$$

The minimum distance of C satisfies

$$d \le n - k - \left\lceil \frac{k}{r} \right\rceil + 2. \tag{2}$$

These upper bounds on the distance and rate of LRC codes were proved in [3], [5]. Recently codes that generalize Reed-Solomon codes and achieve the bound (2) for any n were constructed in [12]. Other bounds on the distance of LRC codes appear in [1], [6].

A graph-theoretic proof of Theorem 1.1 was recently presented in [12]. Developing the ideas of this paper, here we prove the following results.

Theorem 1.2: Let C be an (n, k, r, t) LRC code with t disjoint recovering sets of size r. Then the rate of C satisfies

$$\frac{k}{n} \le \frac{1}{\prod_{j=1}^{t} (1 + \frac{1}{jr})}.$$
(3)

The minimum distance of C is bounded above as follows:

$$d \le n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor. \tag{4}$$

REMARKS:

- 1. For t=1 the bound on the rate (3) reduces to (1). For general t the expression on the right-hand side of (3) approximately equals $t^{-1/r}$ (more precise results are established below in the paper; see Lemma 2.2).
- 2. ON TIGHTNESS OF THE BOUND ON THE RATE (3). For a code with a single recovering set for every symbol, inequality (3) provides a tight bound on the rate (1). For two recovering sets the bound (3) takes the form

$$\frac{k}{n} \le \frac{2r^2}{(r+1)(2r+1)}. (5)$$

Addressing the question of the tightness of the bound consider a binary code which is the product of two single-parity-check codes with r message symbols each. The resulting rate equals $r^2/(r+1)^2$ which is only slightly less than the right-hand side of (5).

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Generalizing, we can construct a t-fold power of the binary (r+1,r) single-parity-check code and obtain a code with t disjoint recovering sets that has the rate $(r/(r+1))^t$. We believe that the rate $(r/(r+1))^t$ is the largest possible for a code with t disjoint recovering sets as long as t is not too large (e.g., $O(\log n)$).

3. On TIGHTNESS OF THE BOUND ON THE DISTANCE (4). For t=1 the bound (4) reduces to (2), and there exist large families of codes that meet this bound with equality [12], [8], [10]. The next interesting case, in particular for applications, is t=2. From (4) we obtain the bound

$$d \le n - \left(k - 1 + \left\lfloor \frac{k - 1}{r} \right\rfloor + \left\lfloor \frac{k - 1}{r^2} \right\rfloor\right). \tag{6}$$

Interestingly, this bound is also tight. Indeed, consider the shortened binary Hamming code of length 6 with the parity-check matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

It is easily seen that this is a $(6, 3, \{2, 2\})$ LRC code, and its distance d = 3 meets the bound (6) with equality.

II. AN UPPER BOUND ON THE RATE OF LRC CODES

A. The recovering graph

Assume that coordinate i has t disjoint recovering sets $R_1^i,...R_t^i$, each of size r, where $R_j^i \subset [n] \backslash i$. Define a directed graph G as follows. The set of vertices V = [n] corresponds to the set of n coordinates of the LRC code. The ordered pair of vertices (i,j) forms a directed edge $i \to j$ if $j \in R_l^i$ for some $l \in [t]$. We color the edges of the graph with t distinct colors in order to differentiate between the recovering sets of each coordinate. More precisely, let $F_e : E(G) \to [t]$ be a coloring function of the edges, given by F((i,j)) = l if $j \in R_l^i$. Thus, the out-degree of each vertex $i \in V = V(G)$ is $\sum_l |R_l^i| = tr$, and the edges leaving i are colored in t colors. We call G the recovering graph of the code \mathcal{C} .

The following lemma will be used to prove the main theorem of this section.

Lemma 2.1: There exists a subset of vertices $U \subseteq V$ of size at least

$$|U| \ge n \left(1 - \frac{1}{\prod_{i=1}^{t} (1 + \frac{1}{ir})}\right)$$
 (7)

such that for any $U' \subseteq U$, the induced subgraph $G_{U'}$ on the vertices U' has at least one vertex $v \in U'$ such that its set of outgoing edges $\{(v,j), j \in U'\}$ is missing at least one color.

Proof: For a given permutation τ of the set of vertices V=[n], we define the coloring of some of the vertices as follows: The color $j\in [t]$ is assigned to the vertex v if

$$\tau(v) > \tau(m)$$
 for all $m \in R_i^v$. (8)

If this condition is satisfied for several recovering sets R_j^v , the vertex v is assigned any of the colors j corresponding to these

sets. Finally, if this condition is not satisfied at all, then the vertex \boldsymbol{v} is not colored.

Let U be the set of colored vertices, and consider one of its subsets $U'\subseteq U$. Let $G_{U'}$ be the induced subgraph on U'. We claim that there exists $v\in U'$ such that its set of outgoing edges is missing at least one color in $G_{U'}$. Assume toward a contradiction that every vertex of $G_{U'}$ has outgoing edges of all t colors. Choose a vertex $v\in U'$ and construct a walk through the vertices of $G_{U'}$ according to the following rule. If the path constructed so far ends at some a vertex with color j, choose one of its outgoing edges also colored in j and leave the vertex moving along this edge. By assumption, every vertex has outgoing edges of all t colors, so this process, and hence this path can be extended indefinitely. Since the graph $G_{U'}$ is finite, there will be a vertex, call it v_1 , that is encountered twice. The segment of the path that begins at v_1 and returns to it has the form

$$v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_l$$
,

where $v_1 = v_l$. For any i = 1, ..., l-1 the vertex v_i and the edge (v_i, v_{i+1}) are colored with the same color. Hence by the definition of the set U we conclude that $\tau(v_i) > \tau(v_{i+1})$ for all i = 1, ..., l-1, a contradiction.

In order to show that there exists such a set U of large cardinality, we choose the permutation τ randomly and uniformly among all the n! possibilities and compute the expected cardinality of the set U

Let $A_{v,j}$ be the event that (8) holds for the vertex v and the color j. Since $Pr(A_{v,j})$ does not depend on v, we suppress the subscript v, and write

$$\Pr(v \in U) = \Pr(\bigcup_{i=j}^{t} A_j).$$

Let us compute the probability of the event $\bigcup_{j=1}^t A_j$. Note that for any set $S \subseteq [t]$ the probability of the event that all the $A_j, j \in S$ occur simultaneously, equals

$$P(\cap_{j\in S} A_j) = \frac{1}{|S|r+1},$$

Hence by the inclusion exclusion formula we get

$$\Pr(\cup_{j=1}^{t} A_{j}) = \sum_{j=1}^{t} (-1)^{j-1} {t \choose j} P(A_{1} \cap ... \cap A_{j})$$

$$= \sum_{j=1}^{t} (-1)^{j-1} {t \choose j} \frac{1}{jr+1}$$

$$= \frac{-1}{r} \left(\sum_{j=0}^{t} (-1)^{j} {t \choose j} \frac{1}{j+\frac{1}{r}} - r \right)$$

$$= 1 - \frac{1}{r} \sum_{j=0}^{t} (-1)^{j} {t \choose j} \frac{1}{j+\frac{1}{r}}$$

$$= 1 - \frac{1}{r} \frac{t!}{\frac{1}{r}(1+\frac{1}{r})...(t+\frac{1}{r})}$$

$$= 1 - \frac{1}{\prod_{j=1}^{t} (1+\frac{1}{jr})},$$
(9)

where (9) follows from [13, p. 188]. Now let X_v be the indicator random variable for the event that $v \in U$, then

$$\mathsf{E}(|U|) = \sum_{v \in V} \mathsf{E}(X_v)$$
$$= \sum_{v \in V} \Pr(v \in U)$$

$$= n \Pr(\bigcup_{j=1}^{t} A_j) = n \left(1 - \frac{1}{\prod_{j=1}^{t} \left(1 + \frac{1}{jr}\right)}\right).$$

The proof is completed by observing that there exists at least one choice of τ for which $|U| \ge \mathsf{E}(|U|)$.

B. Proof of the bound on the rate (3)

Let $U \subseteq [n]$ be the set of vertices of cardinality as in (7) constructed in Lemma 2.1 and let $\overline{U} = [n] \setminus U$ be its complement in [n]. We claim that the value of every coordinate $i \in U$ can be recovered by accessing the coordinates in \overline{U} . To show this, we construct the following iterative procedure, which in each step is applied to the subset $U' \subseteq U$ formed of the coordinates whose values are still unknown. In the first step U' = U. By Lemma 2.1 the induced subgraph $G_{U'}$ contains a vertex $v \in U'$ that is missing one color, call it i. This means that the *i*-th recovering set of v is entirely contained in $\overline{U'}$. Hence one can recover the value of the coordinate v of the codeword by knowing the values of the coordinates in $\overline{U'}$. In the next step use the same argument for the set of coordinates $U'\setminus\{v\}$. In this way all the coordinates in U are recovered step by step relying only on the values of the coordinates in \overline{U} . Therefore,

$$k \le |\overline{U}| \le \frac{n}{\prod_{i=1}^{t} (1 + \frac{1}{ir})}$$

and the proof of (3) is complete.

To get a clearer impression of the bound on the rate derived, observe that

$$\log \prod_{j=1}^t \left(1 + \frac{1}{jr}\right) = \sum_{j=1}^t \log \left(1 + \frac{1}{jr}\right) \approx \sum_{j=1}^t \frac{1}{jr} \approx \frac{1}{r} \log t.$$

Therefore, the value of the product in (3) is about $\sqrt[r]{t}$. More precisely, we have

Lemma 2.2:

$$\sqrt[r]{t+1} \le \prod_{i=1}^{t} \left(1 + \frac{1}{jr}\right) \le \sqrt[r]{t+1} \left(1 + \frac{1}{r}\right).$$

Therefore the rate of a t-LRC code (3) satisfies

$$\frac{k}{n} \le \frac{1}{\sqrt[r]{t+1}}$$

Proof: For i = 0, ..., r - 1 define the quantity

$$f_i = \prod_{j=1}^t \left(1 + \frac{1}{i+jr}\right).$$

It can be easily seen that for any i,

$$f_{i} \leq f_{0} \leq f_{i} \left(1 + \frac{1}{r} \right) \left(1 + \frac{1}{(t+1)r} \right)^{-1}$$

$$= f_{i} \left(1 + \frac{t}{(t+1)r+1} \right).$$
(10)

Hence

$$\prod_{i=0}^{r-1} f_i = \prod_{i=0}^{r-1} \prod_{j=1}^{t} \left(1 + \frac{1}{i+jr}\right)$$

$$= \prod_{j=r}^{(t+1)r-1} \left(1 + \frac{1}{j}\right)$$

$$= t+1.$$
(11)

Using the inequalities (10) in (11), we obtain

$$\sqrt[r]{t+1} = \sqrt[r]{\prod_{i=0}^{r-1} f_i} \le \sqrt[r]{\prod_{i=0}^{r-1} f_0}$$

$$= \prod_{j=1}^{t} \left(1 + \frac{1}{jr}\right)$$

$$\le \sqrt[r]{\prod_{i=0}^{r-1} f_i (1 + \frac{t}{(t+1)r+1})}$$

$$= \sqrt[r]{t+1} (1 + \frac{t}{(t+1)r+1})$$

$$\le \sqrt[r]{t+1} (1 + \frac{1}{r}).$$

III. AN UPPER BOUND ON THE MINIMUM DISTANCE OF LRC CODES: PROOF OF (4)

Consider the recovering graph G of an (n, k, r, t) LRC code \mathcal{C} with t recovering sets, defined in Sect. II-A. Define the following coloring procedure of the vertices. Start with an arbitrary subset of vertices $S \subseteq V$ and color it in some fixed color, call it red. Now let us color some of the remaining uncolored vertices according to the following rule. A vertex is colored red if at least one of its recovering sets is completely colored in red. This process continues until no more vertices can be colored (recall that G is finite). Call the set of redcolored vertices obtained at this point the *closure* of the set S and call the quantity $|\operatorname{Cl}(S)|/|S|$ expansion ratio of the set S. Since the expansion ratio equals to the quotient of the number of coordinates whose value is determined by the set S and the size of the set S itself, it is clear that large expansion ratio means that the set S contains a large amount of information about the other coordinates of the code. In other words, a large number of values of coordinates outside S is determined by the values of the coordinates in S.

Recall the definition of the distance of the code C of length n and cardinality q^k over an alphabet of size q:

$$d = n - \max_{I \subseteq [n]} \{ |I| : |\mathcal{C}_I| < q^k \},$$

where C_I is the restriction of the code to coordinates in I. Using the recovering graph and the expansion ratio concept, we will show that there exists a large set $I \subseteq [n]$ of coordinates such that $|C_I| < q^k$.

We need the following two lemmas whose proofs are deferred to the end of the section.

Lemma 3.1: Let G be the recovering graph of a (n, k, r, t) LRC code \mathcal{C} . For any vertex $v \in G$ there exists a set S of size at most r^t such that $v \in \mathrm{Cl}(S)$, and the expansion ratio of S is at least

$$e_t = \frac{r^{t+1} - 1}{r^{t+1} - r^t}. (12)$$

Lemma 3.2: Let m be an integer whose base-r representation is

$$m = \sum_{i} \alpha_i r^i,$$

then for an integer t,

$$\left\lfloor \frac{m}{r^t} \right\rfloor r^t e_t + \sum_{i=0}^{t-1} \alpha_i r^i e_i = \sum_{i=0}^t \left\lfloor \frac{m}{r^i} \right\rfloor,$$

where e_t is defined in (12)

Proof of the upper bound on the distance (4): We need to prove that the distance of an (n, k) code with t disjoint recovering sets of size r satisfies the inequality

$$d \le n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor.$$

Let G be the recovering graph of the code. We will use Lemma 3.1 several times for the graph G. Assume that we are allowed to color k-1 vertices and would like to color them in the way that guarantees a large expansion ratio with respect to their closure. We begin by using Lemma 3.1 for the graph $G_1 = G$. According to it, G_1 contains a subset S_1 of vertices of size at most r^t whose expansion ratio is at least e_t . Color the vertices in S_1 and $Cl(S_1)$. Then call G_2 the subgraph induced on the subset of vertices $V \setminus Cl(S_1)$ and apply Lemma 3.1 to G_2 , etc. Continuing this process, suppose that in the *i*-th round there are b_i vertices still to be colored, and let G_i be the induced subgraph of G on the set of vertices that have not been colored in the previous i-1 rounds. Each vertex in G_i has outgoing edges of all t colors because if not, then one of its recovering sets has been already removed, but then this vertex itself cannot be present because of the definition of the closure. Let $m \leq t$ be the largest integer such that $r^m \leq b_i$. Now apply Lemma 3.1 for the graph G_i to find a set S_i of vertices of size at most

$$|S_i| < r^m \tag{13}$$

and expansion ratio at least e_m . Now color the set S_i . Continue this process until we have used all the k-1 vertices and call the obtained set of k-1 vertices S. In each step the cardinality of S_i is at most r^m according to (13), and hence

$$|\operatorname{Cl}(S)| \ge \lfloor \frac{k-1}{r^t} \rfloor r^t e_t + \sum_{i=0}^{t-1} \alpha_i r^i e_i, \tag{14}$$

where

$$k - 1 = \sum_{i} \alpha_i r^i,$$

is the r-ary representation of k-1. Using Lemma 3.2, (14) becomes

$$|\operatorname{Cl}(S)| \ge \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor.$$

Since the value of the coordinates in $\mathrm{Cl}(S)$ is determined by the value of the coordinates in S which is of size k-1, the size of the restriction of the code $\mathcal C$ to coordinates $I=\mathrm{Cl}(S)$ is at most $|\mathcal C_I|\leq q^{k-1}< q^k$, hence

$$d \le n - |I| = n - |\operatorname{Cl}(S)| \le n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor.$$

Proof of Lemma 3.1: We apply induction on t. For t=0 there are no edges in the graph. Define $S=\{v\}$ and note that $\mathrm{Cl}(S)=S=\{v\}$, and the expansion ratio is 1 as needed. Now assume that the claim is correct for t recovering sets. Let us prove it for t+1 recovering sets. Remove from G the vertex v. For each other vertex $u\neq v$ we remove the edges that correspond to one of its recovering sets. Specifically, if u has a recovering set that contains v, we remove all of its edges that correspond to this recovering set; otherwise, remove the edges that correspond to any one of its recovering sets. Denote the resulting graph by G_1 , and observe that each vertex of G_1 has exactly t recovering sets.

Let $v_1, ..., v_l$ be the vertices of one of the recovering sets of v, where $l \leq r$. Our plan is to apply the induction hypothesis successively l times for some induced subgraphs of G_1 which we denote below by $G_i, i = 1, ..., l$. We also use the notation $\operatorname{Cl}_i(S), i = 1, ..., l$ to refer to the closure operation of the set S in the graph G_i , and use the notation $\operatorname{Cl}(S)$ to refer to the closure operation in the original graph G. Upon performing the ith step we will have the vertex v_i colored.

In the first step, we use the induction hypothesis to find a set S_1 of size at most r^t in the graph G_1 whose expansion ratio is at least e_t , and such that $v_1 \in \operatorname{Cl}_1(S_1)$. Suppose that S_1, \ldots, S_{i-1} sets of vertices have been constructed in the first i-1 steps, $2 \le i \le l$. Denote by G_i the graph G_1 obtained upon removing the set of vertices $\operatorname{Cl}_1(S_1 \cup \ldots \cup S_{i-1})$.

Let us describe the construction of the set S_i . If $v_i \in \operatorname{Cl}_1(S_1 \cup \ldots \cup S_{i-1})$, put $S_i = \emptyset$. Otherwise $v_i \in V(G_i)$. Note that each vertex u in G_i has outgoing edges of all t colors because otherwise, if u is missing one color, then it has a recovering set that is contained in $\operatorname{Cl}_1(S_1 \cup \ldots \cup S_{i-1})$, and then also $u \in \operatorname{Cl}_1(S_1 \cup \ldots \cup S_{i-1})$. Apply the induction hypothesis for G_i to find a set S_i of size at most r^t and expansion ratio at least e_t such that $v_i \in \operatorname{Cl}_i(S_i)$. Notice that since $\operatorname{Cl}_i(S_i)$ is a subset of the vertices of the graph G_i , it is disjoint from the set $\operatorname{Cl}_1(S_1 \cup \ldots \cup S_{i-1})$. We claim that

$$S = \bigcup_{i=1}^{l} S_i$$

is the desired set. Observe that

$$Cl_1(S) = Cl_1(S_1 \cup ... \cup S_l)$$

= $\bigcup_{i=1}^l Cl_1(S_1 \cup ... \cup S_i) \setminus Cl_1(S_1 \cup ... \cup S_{i-1}).$

Since

$$\operatorname{Cl}_1(S_1,\ldots,S_i) = \operatorname{Cl}_1(S_1,\ldots,S_{i-1}) \cup \operatorname{Cl}_i(S_i)$$

(disjoint union), we obtain

$$\operatorname{Cl}_1(S) = \bigcup_{i=1}^l \operatorname{Cl}_i(S_i), \tag{15}$$

where the union is also disjoint.

We claim that for any i=1,...,l the vertex v_i belongs to $\operatorname{Cl}_1(S)$. Indeed, by construction, if S_i is the empty set, then $v_i \in \operatorname{Cl}_1(S_1 \cup ... \cup S_{i-1})$, otherwise $v_i \in \operatorname{Cl}_i(S_i)$. We conclude that $\operatorname{Cl}_1(S)$ contains a complete recovering set $v_1,...,v_l$ of the vertex v, and therefore,

$$Cl(S) = Cl_1(S) \cup \{v\}. \tag{16}$$

The size of S satisfies

$$|S| = |\cup_{i=1}^{l} S_i| = \sum_{i=1}^{l} |S_i| \le r \cdot r^t = r^{t+1},$$

and all is left to show is the expansion ratio. By (15) and (16)

$$|\operatorname{Cl}(S)| = |\cup_{i=1}^{l} \operatorname{Cl}_{i}(S_{i}) \cup \{v\}| = 1 + \sum_{i=1}^{l} |\operatorname{Cl}_{i}(S_{i})|.$$

Hence the expansion ratio of the set S satisfies

$$\frac{|\operatorname{Cl}(S)|}{|S|} = \frac{1 + \sum_{i=1}^{l} |\operatorname{Cl}_{i}(S_{i})|}{|S|}$$

$$\geq \frac{1}{r^{t+1}} + \frac{\sum_{i=1}^{l} |\operatorname{Cl}_{i}(S_{i})|}{|S|}$$

$$= \frac{1}{r^{t+1}} + \sum_{i=1}^{l} \frac{|S_{i}|}{|S|} \frac{|\operatorname{Cl}_{i}(S_{i})|}{|S_{i}|}$$

$$\geq \frac{1}{r^{t+1}} + \sum_{i=1}^{l} \frac{|S_{i}|}{|S|} e_{t}$$

$$= \frac{1}{r^{t+1}} + e_{t}$$

$$= e_{t+1}, \tag{18}$$

where (17) follows since the set S_i has expansion ratio of at least e_t in G_i .

Proof of Lemma 3.2: We apply induction on t. For t=0 the equality can be easily checked. We assume correctness for t and prove it for t+1.

$$\left[\frac{m}{r^{t+1}}\right]r^{t+1}e_{t+1} + \sum_{i=0}^{t} \alpha_i r^i e_i$$

$$= \sum_{i=0}^{t+1} \left[\frac{m}{r^{t+1}}\right]r^i + \sum_{i=0}^{t} \alpha_i r^i e_i$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=1}^{t+1} \left\lfloor \frac{m}{r^{t+1}} \right\rfloor r^{i} + \sum_{i=0}^{t} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=1}^{t+1} \left\lfloor \frac{m}{r^{t+1}} \right\rfloor r^{i} + \alpha_{t} r^{t} e_{t} + \sum_{i=0}^{t-1} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=1}^{t+1} \left\lfloor \frac{m}{r^{t+1}} \right\rfloor r^{i} + \alpha_{t} \left(\sum_{i=0}^{t} r^{i} \right) + \sum_{i=0}^{t-1} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=0}^{t} r^{i} \left(\left\lfloor \frac{m}{r^{t+1}} \right\rfloor r + \alpha_{t} \right) + \sum_{i=0}^{t-1} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=0}^{t} r^{i} \left\lfloor \frac{m}{r^{t}} \right\rfloor + \sum_{i=0}^{t-1} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \left\lfloor \frac{m}{r^{t}} \right\rfloor r^{t} e_{t} + \sum_{i=0}^{t-1} \alpha_{i} r^{i} e_{i}$$

$$= \left\lfloor \frac{m}{r^{t+1}} \right\rfloor + \sum_{i=0}^{t} \left\lfloor \frac{m}{r^{i}} \right\rfloor, \tag{19}$$

where (19) follows from the induction hypothesis, and the result follows.

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