# SUPER: Sparse signals with Unknown Phases Efficiently Recovered 

Sheng Cai, Mayank Bakshi, Sidharth Jaggi, Minghua Chen<br>\{cs010, mayank, jaggi, minghua\}@ie.cuhk. edu. hk<br>The Chinese University of Hong Kong, Hong Kong SAR China


#### Abstract

Suppose $\mathbf{x}$ is any exactly $k$-sparse vector in $\mathbb{C}^{n}$. We present a class of phase measurement matrix $A$ in $\mathbb{C}^{m \times n}$, and a corresponding algorithm, called SUPER, that can resolve $\mathbf{x}$ up to a global phase from intensity measurements $|A \mathbf{x}|$ with high probability over $A$. Here $|A \mathbf{x}|$ is a vector of component-wise magnitudes of $A \mathbf{x}$. The SUPER algorithm is the first to simultaneously have the following properties: (a) it requires only $\mathcal{O}(k)$ (order-optimal) measurements, (b) the computational complexity of decoding is $\mathcal{O}(k \log k)$ (near order-optimal) arithmetic operations.


## I. Introduction

Phase Retrieval: In many applications, it's difficult to measure the phase information of the underlying signal. Instead, we recover the signal by its intensity measurements. For instance, in X-ray crystallography, optics [1] and image reconstruction for astronomy [2], signal/image is reconstructed from the intensity measurements of its Fourier transform.

Let $A \in \mathbb{C}^{m \times n}$ be used to denote the phase measurement matrix, and $\mathbf{x} \in \mathbb{C}^{n}$ be used to denote the unknown underlying signal. Instead of linear measurements of the form $y=A \mathbf{x}$ as in the compressive sensing literature (see, for instance, [3]) in the phase retrieval problem we have $m$ non-linear intensity measurements of the form $b_{i}=\left|<A_{i}, \mathbf{x}>\right|$. Here the index $i$ is an integer in $\{1, \ldots, m\}$ (or $[\mathrm{m}]$ for short), $A_{i}$ is the $i$-th row of phase measurement matrix $\left.A,<\cdot\right\rangle$ is the inner product and $|\cdot|$ is the absolute value.

Problems of this kind have been studied over the last decades. A good survey of some of the algorithms via non-convex process can be found in [4], [5]. Recently, two convex optimization methods, PhaseLift [6] and PhaseCut [7], have been proposed by Candès et al. and Waldspurger et al.. PhaseLift is inspired by finding the low-rank matrix (specifically for the phase retrieval problem, rank-one matrices) by minimizing the trace norm (SDP) [8]. PhaseLift is able to reconstruct $\mathbf{x}$ with $\mathcal{O}(n \log n)$ intensity measurements by solving semidefinite programming with high probability. The $A_{i}$ 's are independently sampled on the unit sphere of $\mathbb{C}^{n}$. Later, it's shown that the number of intensity measurements can be improved to $\mathcal{O}(n)$ where $A_{i}$ 's are independently and identically distributed with the uniform distribution on the sphere of radius $\sqrt{n}$, or the complex normal distribution [9]. PhaseCut is inspired by solving max-cut problem via SDP. The decoding complexity for both PhaseLift and PhaseCut is $\mathcal{O}\left(n^{3}\right)$, which is still computationally costly when $n$ is large.

Besides SDP-based approach, more computationally efficient algorithms are proposed such as [10], [11]. For instance, in [11], the number of intensity measurements required is $\mathcal{O}\left(n \log ^{3} n\right)$. However, the decoding complexity is $\mathcal{O}\left(n^{2} \log ^{3} n\right)$ which is less than that of SDP-based approach.
Compressive Phase Retrieval: Suppose $\mathbf{x}$ is "sparse", i.e., the number of non-zero components of $\mathbf{x}$ is at most $k$, which is much less than the length $n$ of $\mathbf{x}$. This assumption is not uncommon in many applications like X-ray crystallography. Then, given $A$ and $b$, the goal of compressive phrase retrieval is to reconstruct $\mathbf{x}$ as $\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ equals $\mathbf{x}$ up to a global phase. That is, $\hat{\mathbf{x}}=\mathbf{x} e^{\iota \Theta}$ for some arbitrary fixed $\Theta \in[0,2 \pi)$. Here $\iota$ denotes the positive square root of -1 . The reason we allow this degeneracy in $\hat{\mathbf{x}}$, up to a global phase factor, is that all such $\hat{\mathbf{x}}$ 's result in the same measurement vector under intensity measurements. If $\hat{\mathbf{x}}$ does indeed equal $\mathbf{x}$ up to a global phase, then we denote this "equality" as $\hat{\mathbf{x}} \hat{=} \mathbf{x}$.

It is shown that $4 k-1$ intensity measurements suffice to uniquely reconstruct $\mathbf{x}$ in [12] (for $\mathbf{x} \in \mathbb{R}^{n}$ ) and [13] (for $\mathbf{x} \in \mathbb{C}^{n}$ ). However, no efficient algorithms is given. The $\ell_{1}$-regularized PhaseLift method is introduced in the compressive phase retrieval problem in [14]. In [15], it is shown that if the number of Gaussian intensity measurements is $\mathcal{O}\left(k^{2} \log n\right)$, $\mathbf{x}$ can be correctly reconstructed via $\ell_{1}$-regularized PhaseLift.

The works in [16] and the works by Jaganathan et al. [17], [18], [19] study the case when the phase measurement matrix is a Fourier transform matrix. In [20], it is explained that SDP-based methods can reconstruct $\mathbf{x}$ with sparsity up to $o(\sqrt{n})$. In [18], the algorithm based on reweighted $\ell_{1}$-minimization with $\mathcal{O}\left(k^{2} \log n\right)$ phaseless Fourier measurements is proposed to go beyond this bottleneck. When the phase measurement matrix is allowed to be designed, a matrix ensemble and a corresponding combinatorial algorithm is proposed in [18] such that $\mathbf{x}$ is correctly reconstructed with $\mathcal{O}(k \log n)$ intensity measurements in $\mathcal{O}(k n \log n)$ time.

To our best knowledge, in the literature, there is no construction of a measurement matrix $A$ and a corresponding reconstruction algorithm that correctly reconstructs $\mathbf{x}$ with an order-optimal number of measurements and with near-optimal decoding complexity simultaneously.

| Notation | Definition |
| :---: | :--- |
| $\mathbf{x}$ | Length- $n$ signal over $\mathbb{C}$ with sparsity $k$ |
| $A$ | Dimension- $n \times m$ phase measurement matrix over $\mathbb{C}$. |
| $b$ | Length- $m$ Intensity measurement vector over $\mathbb{R}^{+}$. |
| $A_{i}$ | The $i$-th row of phase measurement matrix $A$ for all $\forall i \in[m]$. |
| $b_{i}$ | $b_{i}=\left\|\left\langle A_{i}, \mathbf{x}\right\rangle\right\|$, the $i$-th intensity measurement $\forall i \in[m]$. |
| $k$ | $k=\\|\mathbf{x}\\|_{0}$, the number of non-zero components (sparsity) of $\mathbf{x}$. |

Table I
TABLE OF NOTATION FOR THE MODEL

## A. Our Contribution

In this work, we describe a randomized design of the phase measurement matrix $A$ and a corresponding decoding algorithm achieving the following guarantees:

Theorem 1. (Main theorem) There exists a measurement ensemble $\{A\}$ and a corresponding decoding algorithm for compressive phase retrieval with the following performance:

1) For every $\mathbf{x} \in \mathbb{C}^{n}$, with probability $1-o(1)$ over the randomized design of $A$, the algorithm exactly reconstructs $\mathbf{x}$ up to a global phase;
2) The number of measurements $m=\mathcal{O}(k)$;
3) The decoding complexity is $\mathcal{O}(k \log k)$.

The rest of this paper is organized as follows. We first present the high-level overview of our algorithm in Section II In Section II-C, we introduce the graphs used for measurement structure. Section IV and Section VI contain actual measurement design. Section $V$ and Section VII discuss the reconstruction algorithm and the performance of it. Section VIII concludes this paper.

## II. Overview/High-Level Intuition

Our SUPER algorithm is non-adaptive. There are three phase 1 in our decoding algorithm. In the first phase (called seeding phase), we are able to recover the magnitudes and relative phases of constant fraction of non-zero components of $\mathbf{x}$. In the second phase (called geometric-decay phase), there are $\mathcal{O}(\log (\log k))$ stages. In each stage, we recovery the magnitudes and relative phases of constant fraction of unresolved non-zero components of $\mathbf{x}$. In the third phase (called cleaning-up phase), the remaining $\mathcal{O}(k / \log k)$ unresolved non-zero components are decoded.

## A. Pieces of the puzzle

We first define some useful terminology.

## Singletons:

If a measurement $b_{i}$ involves only a single non-zero component of $\mathbf{x}$, then we say that such a measurement is a singleton 2 Singletons are important since they can be used to pin down the magnitude (though not the phase) of components of $\mathbf{x}$. There are several challenges, however. One lies in even identifying whether a measurement is a singleton or not. The second lies in identifying which of the $\mathbf{x}$ components being measured in $b_{i}$ corresponds to the singleton. The third is to be able to do all this blindingly fast, in fact in constant time (independent of $n$ and $k!$ ). Each of these challenges can be handled by using ideas from the our prior work on compressive sensing [22]. For details, see Sections [IV] and $\nabla$ below.

## Doubletons:

Similarly, if a measurement $b_{i}$ involves exactly two non-zero components of $\mathbf{x}$, then we say that such a measurement is a doubleton. Doubletons, especially doubletons measuring two non-zero components of $\mathbf{x}$ which have already been measured by singletons (we call such doubletons resolvable doubletons), are useful since they can be used to deduce the relative phases of the two non-zero components of $\mathbf{x}$. For example, if one is given the magnitudes $\left|x_{i}\right|,\left|x_{j}\right|$, and $\left|x_{i}+x_{j}\right|$, then one can determine the angle $\theta$ between the phases of the complex numbers $x_{i}$ and $x_{j}$ (up to degeneracy of sign of $\theta$ ). In fact, even this degeneracy can be resolved by an additional judiciously chosen measurement. Similar challenges to those mentioned above vis-a-vis singletons (identifying whether or not a measurement is a doubleton/resolvable doubleton, identifying which components of $\mathbf{x}$ it corresponds to, and doing so in constant time) also hold for doubletons. See Sections IV and $\square$ for details. Mutual resolvability:

We say our decoding algorithm has thus far mutually resolved two non-zero components $x_{i}$ and $x_{j}$ of $\mathbf{x}$ if the magnitudes of both $x_{i}$ and $x_{j}$ have been deduced, and also the relative phase between $x_{i}$ and $x_{j}$ has been deduced (for instance via resolvable doubleton measurements roughly described above). Note that mutual resolvability is an equivalence relation - it is

[^0]reflexive, symmetric and transitive. Note therefore that if $x_{i}$ and $x_{i^{\prime}}$ have been mutually resolved, it is not necessary that they even are involved in the same measurement; it is sufficient that $x_{i}$ and $x_{i^{\prime}}$ are part of a chain of non-zero components of $\mathbf{x}$ that are pairwise mutually resolved. Finally, we note that as our decoding algorithm progresses, if it is successful, in fact all the non-zero components of $\mathbf{x}$ are eventually mutually resolved. Hence this property of mutual resolvability is perhaps most interesting in the intermediate stages of our decoding algorithm.

## Giant component:

We say that a subset of the non-zero components of $\mathbf{x}$ form a giant component if it is the largest subset satisfying the two properties:

- The subset is of size linear in $k$.
- Any pair of components in the subset have been mutually resolved (thus far) by the decoding algorithm.

Non-zero components of $\mathbf{x}$ that have not (yet) been mutually resolved with respect to an element of the giant component by the decoding algorithm are said to be unresolved.

Essentially, our algorithm proceeds by iteratively enlarging the giant component until it engorges all the non-zero components of x .
Resolvable multiton:
We say that a measurement $b_{i}$ is a resolvable multiton if it is the case that exactly one (say $x_{i}$ ) of the non-zero components of $\mathbf{x}$ involved in the measurement $b_{i}$ is outside the giant component, and at least one of non-zero components of $\mathbf{x}$ is inside the giant component. Such measurements are useful since, in the latter parts of our algorithm, there are not enough resolvable doubletons. By carefully choosing the parameters of the algorithm, one can guarantee that a constant fraction of measurements are resolvable mutitons.

Judiciously designed measurements (see Section (IV) enable one to mutually resolve the component $x_{i}$ that is outside the giant component, with the components of x inside the giant component, by solving a quadratic equation. Care is indeed required in choosing the measurements since the amplitude measurement process is inherently non-linear, and there may not be a "clean" manner to mutually resolve $x_{i}$ via arbitrary measurements - indeed the design of such a measurement process is also one of the intellectual contributions we wish to highlight in this work. We call this process "cancelling out" the already resolved components of $\mathbf{x}$.

## B. Putting the pieces together

## Seeding phase:

In the first phase, called the seeding phase, there are $\mathcal{O}(k)$ "sparse" measurements (each measurement involves, in expectation, $\mathcal{O}(n / k)$ components of $\mathbf{x})$. We demonstrate that by first examining the measurements corresponding to this phase, the decoding algorithm is already able to decode a constant fraction (say $1 / 2 \sqrt{3}$ of the components of $\mathbf{x}$ up to a global phase. The algorithm is able to do this since we are able to show that a "significant" fraction of measurements are singletons and resolvable doubletons. Standard results in percolation theory [23] then lead one to conclude that the number of non-zero nodes that are mutually resolvable is linear in $k$, i.e., that there is a giant component. Hence this phase is called the "seeding" phase, since the giant component forms the nucleus on which the remainder of the algorithm builds upon.

Prior work ( $[18]$ ) closest to our work here comprises essentially only of the seeding phase, but with $\mathcal{O}(k \log (k))^{4}$ measurements. The reason that prior work needs this many measurements is essentially due to what happens at the tail end of a "coupon collection" process [24] (wherein one has to collect at least one copy of each of $k$ coupons by sampling with replacement) when most of the coupons have already been collected/the giant component is of size close to $k$, then the growth rate slows down. Specifically, this is because the fraction of resolvable doubletons decays slowly to zero, and an additional multiplicative factor of $\log (k)$ measurements is required so as to ensure the giant component subsumes all non-zero components of $\mathbf{x}$.

The key technique used in our work, then, is to segue to a different sampling process outlined below, and using resolvable multitons rather than doubletons. The challenge is to make the numbers work - unlike [18], not only do we require only $\mathcal{O}(k)$ measurements, but we also require our decoding complexity to be $\mathcal{O}(k \log (k))$.
Geometric-decay phase:
This phase itself comprises of $\mathcal{O}(\log (\log (k)))$ separate stages. Each stage has half the number of measurements compared to the previous stage, but measurements in each stage are twice as "dense" as the measurements in the previous stage. So, for instance, if in the first stage of the geometric-decay phase, there are say $c k$ measurements, with each measurement involving $n / k$ components of $\mathbf{x}$, then in the second stage of the geometric-decay phase, there are $c k / 2$ measurements, but each measurement involves $2 n / k$ components of $\mathbf{x}$.

There are two reasons for this choice of parameters. Firstly, with such a geometric decay in the number of measurements in each stage, the overall number of measurements in the geometric-decay phase is still $\mathcal{O}(k)$. Secondly, we show that with

[^1]the geometric increase in the density of measurements, a significant fraction of measurements in each stage lead to resolvable multitons, and use this to show that the number of unresolved components decays geometrically.

The reason we run the geometric-decay phase for only $\mathcal{O}(\log (\log (k)))$ stages is also two-fold. Firstly, after that many stages, with the number of unresolved components halving at every stage, the number of unresolved components of $\mathbf{x}$ is, in expectation, $\mathcal{O}(k / \log (k))$. Hence the concentration inequalities (which depend on the number of unresolved components) we use to control the probabilities of error get progressively weaker (though they still result in good concentration at the last stage of the geometric-decay phase). Secondly, and more importantly, the number of non-zero components in each resolvable multiton increases geometrically as the number of stages increases. This has implications for the time-complexity of the decoding algorithm, since the time-complexity depends directly on the number of non-zero components in each measurement that need to be "cancelled out". By terminating the geometric-decay phase after $\mathcal{O}(\log (\log (k)))$ stages ensures that, in expectation, the number of such "cancellations" is at most $\mathcal{O}(\log (k))$, and hence the overall time-complexity of the algorithm scales as $\mathcal{O}(k \cdot \log (k))$.
Cleaning-up phase:
Finally, we segue to what we call the "cleaning-up" phase. As noted above, after the geometric-decay phase the number of unresolved components of $\mathbf{x}$ is, in expectation, $k^{\prime} \triangleq \mathcal{O}(k / \log (k))$. To fit our budget of $\mathcal{O}(k)$ measurements, and $\mathcal{O}(k \log (k))$ decoding time, we now segue to using "coupon collection" as a primitive. This may be viewed as restarting the seeding (first) phase, but with different parameters. In particular, the problem dimension has now been significantly reduced (since there are now only $k^{\prime}$ unresolved components of $\mathbf{x}$ ). Therefore we can now afford to pay the coupon collection penalty that we avoided in the seeding phase by moving to the geometric-decay phase.

Specifically, in this cleaning-up phase we take $\mathcal{O}\left(k^{\prime} \log \left(k^{\prime}\right)\right)$ measurements so as to resolve the remaining $k^{\prime}$ unresolved components of $\mathbf{x}$. Note that $\mathcal{O}\left(k^{\prime} \log \left(k^{\prime}\right)\right)$ scales as $\mathcal{O}(k)$. Each measurement we take has the same density as the measurements in the last stage of the geometric decay phase, and hence the time-complexity of resolving measurements also scales in the same manner. However, since there are many more measurements than in the last stage of the geometric-decay phase, by standard arguments corresponding to the coupon collection problem we are able to argue that for each unresolved component of $\mathbf{x}$ there is at least one resolvable multiton that helps resolve it.

## C. Summary of the overview

As the above discussion outlines, to make the numbers work (i.e., to ensure $\mathcal{O}(k)$ number of measurements and $\mathcal{O}(k \log (k))$ time-complexity), one has to delicately choose the parameters of the measurement ensemble. Our analysis indicates that having a phase in which the sparsity actually geometrically increases, at least for a while, significantly improves performance. To take advantage of this, however, we have to carefully design the measurements, so that one can resolve unresolved components of $\mathbf{x}$ via judiciously designed non-linear measurements. In this work we have not attempted to optimize the constant factors we expect further constant-factor improvements are possible via further careful tuning.

## III. GRAPH PROPERTIES

We construct a series of bipartite graphs with some desirable properties outlined in this section. We then use the structure of the bipartite graphs to generate our measurement matrix $A$ in Section $I V$ and design the corresponding reconstruction algorithm in Section $\mathbf{V}$ Each left nodes of a bipartite graph represents a component of $\mathbf{x}$ and each right node represents a set of intensity measurements.

## A. Seeding Phase

The properties of the bipartite graph, $\mathcal{G}_{I}$, in the first phase are as follows:

1) There are $n$ left nodes and $c k$ right nodes, where $c$ is a constant.
2) Each edge in $\mathcal{G}_{I}$ appears with probability $1 / k$. For each right node, the degree, in expectation, is $n / k$.
3) For each edge in $\mathcal{G}_{I}$, it is assigned different weights which are discussed in the measurement design (See Section IV).
4) Many singleton nodes: Singleton nodes are right nodes which involves exactly one non-zero component of x. Singleton nodes help to recover the magnitude of non-zero component. See Section VII for details.
5) Many resolvable doubleton nodes: Doubleton nodes are right nodes which involve exactly two non-zero components of $\mathbf{x}$. Resolvable doubletons are the doubletons which involve exactly two non-zero components whose magnitudes are recovered by singleton nodes. See Section (III for details.
Another graph $\mathcal{H}$ is implied by $\mathcal{G}_{I}$. Each vertex in $\mathcal{H}$ represents a non-zero component of $\mathbf{x}$ and there is an edge in $\mathcal{H}$ if and only if two left nodes involved are mutually resolved by a resolvable doubleton node. The property of $\mathcal{H}$ is as follows:
6) $\mathcal{H}$ has a giant connected component: The connected component, $\mathcal{H}^{\prime}$ contains a constant fraction of nodes in $\mathcal{H}$. This property is formally stated in Section (VII)

## B. Geometric-decay phase

There are $\mathcal{O}(\log \log k)$ separate bipartite graphs/stages in this phase.
The properties of the $l$-th bipartite graph, $\mathcal{G}_{I I, l}(l=1,2, \ldots, L=\mathcal{O}(\log \log k))$, are as follows:

1) There are $n$ left nodes and $c f_{I I, l-1} k$ right nodes, where $f_{I I, l-1}$ is the expected fraction of unresolved non-zero components of $\mathbf{x}$ after the $(l-1)$-th stage of decoding process in the second phase. $f_{I I, 0}=f_{I}$ is the expected fraction of unresolved non-zero components after seeding phase. The 0-th stage of geometric-decay phase is seeding-phase. The value of $f_{I I, l}$ is discussed in Section VI.
2) Each edges in $\mathcal{G}_{I I, l}$ appears with probability $1 /\left(f_{I I, l-1} k\right)$.
3) For each edge in $\mathcal{G}_{I I, l}$, it is assigned different weights which are discussed in the measurement design.
4) Many resolvable multiton nodes: The resolvable multiton nodes are right nodes which involve exactly one unresolved non-zero component of $\mathbf{x}$ and at least one of the resolved non-zero components. Each resolvable multiton node helps to recover both the magnitude and the relative phase of the corresponding unresolved non-zero component via "Cancelling out" process (See Section $\nabla$ ).
For a newly resolved non-zero component, the corresponding node in $\mathcal{H}$ is appended to the giant connected component, $\mathcal{H}^{\prime}$. In expectation, there are $\left(f_{I I, l-1}-f_{I I, l}\right) k$ non-zero components decoded in the $l$-th stage of decoding. We show in Section VII that we are able to reconstruct a constant fraction of undecoded non-zero components with high probability at each stage. After $\mathcal{O}(\log \log k)$ stages, there are $\mathcal{O}(k / \log k)$ unresolved non-zero components of $\mathbf{x}$ left.

## C. Cleaning-up phase

The properties of the bipartite graph, $\mathcal{G}_{I I I}$, in the last phase are as follows:

1) There are $n$ left nodes and $c(k / \log k) \log (k / \log k)=\mathcal{O}(k)$ right nodes.
2) Each edges in $\mathcal{G}_{\text {III }}$ appears with probability $\log k / k$.
3) For each edge in $\mathcal{G}_{I I I}$, it is assigned different weights which are discussed in the measurement design.
4) Many resolvable multiton nodes.

In this stage, all the resolved non-zero components of $\operatorname{size} \mathcal{O}(k / \log k)$ are finally recovered using resolvable multiton nodes by "Cancelling out" process and a Coupon Collection argument.

| Notation | Definition |
| :---: | :--- |
| $\mathcal{G}_{I}$ | The bipartite graph used in the seeding phase with $n$ left nodes and $c k$ right nodes. <br> Each edge appears with probability $1 / k$. |
| $\mathcal{H}$ | Implied graph by $\mathcal{G}_{I}$. |
| $\mathcal{H}^{\prime}$ | Connected component of $\mathcal{H}$. |
| $\mathcal{G}_{I I, l}$ | The $l$-th bipartite graph used in the $l$-th stage in geometric-decay phase with $n$ left nodes <br> and $c f_{I I, l-1} k$ right nodes for $l \in[L]$. Each edge appears with probability $1 / f_{I I, l-1} k$. |
| $f_{I}$ | The expected fraction of unresolved non-zero components of $\mathbf{x}$ after the seeding phase. |
| $f_{I I, l}$ | The expected fraction of unresolved non-zero components of $\mathbf{x}$ after the $l$-th stage <br> of the geometric-decay phase. Let $f_{I I, 0}=f_{I}$. |
| $\mathcal{G}_{I I I}$ | The bipartite graph used in the cleaning-up phase with $n$ left nodes and <br> $c(k / \log k) \log (k / \log k)$ right nodes. Each edge appears with probability $\log k / k$. |

TABLE OF NOTATION USED IN THE DESIGN OF BIPARTITE GRAPHS

## IV. Measurement Design

For a bipartite graph $\mathcal{G}\left(\mathcal{G}\right.$ is one of the $\mathcal{G}_{I}, \mathcal{G}_{I I, l}$ 's and $\left.\mathcal{G}_{I I I}\right)$, there are $n$ nodes on the left and $m_{\mathcal{G}}^{\prime}$ nodes on the right . $A(\mathcal{G})^{\prime}$ is the dimension- $m_{\mathcal{G}}^{\prime} \times n$ adjacent matrix of $\mathcal{G}$ where the entry at $i$-th row and $j$-th column equals to 1 if and only if $i$-th right node connects to the $j$-th left node for $j \in[n]$ and $i \in\left[m_{\mathcal{G}}^{\prime}\right]$. The dimension- $m_{\mathcal{G}} \times n$ phase measurement matrix $A(\mathcal{G})$ is designed based on $A(\mathcal{G})^{\prime}$ where $m_{\mathcal{G}}=5 m_{\mathcal{G}}^{\prime}$. By appending all the matrix $A(\mathcal{G})$ sequentially, we get the actual $m \times n$ measurement matrix $A$ where $m=\Sigma_{\mathcal{G}} m_{\mathcal{G}}$. For $i$-th row $A(\mathcal{G})_{i}^{\prime}$ of $A(\mathcal{G})^{\prime}$, a set of rows (of size 5) of $A(\mathcal{G})$ are designed for $i \in\left[m_{\mathcal{G}}^{\prime}\right]$. If the $j$-th entry of $A(\mathcal{G})_{i}^{\prime}$ is zero, then corresponding set of entries of $A(\mathcal{G})$ are all zero for all $j \in[n]$. In the following measurement matrix design, we design the entries corresponding to non-zero entries in $A(\mathcal{G})^{\prime}$. See Section $\nabla$ for how these measurements are used for decoding.

1) Trigonometric entries: The $j$-th entries of the $(5 i-4)$-th and $(5 i-3)$-th rows of $A(\mathcal{G})$ are denoted by $a_{i, j}^{(\mathcal{G}, 1)}$ and $a_{i, j}^{(\mathcal{G}, 2)}$ . The values are set as follows:

$$
\begin{aligned}
a_{i, j}^{(\mathcal{G}, 1)} & =\cos \left(\frac{j \pi}{2 n}\right) \\
a_{i, j}^{(\mathcal{G}, 2)} & =\iota \sin \left(\frac{j \pi}{2 n}\right)
\end{aligned}
$$

where $\iota$ denotes the positive square root of -1 and $\pi / 2 n$ can be treated as the unit phase of the entry design. In particular, the phase $j \pi / 2 n$ will be critical for our algorithm. The first two entries are used in singleton node identification and "cancelling out" process of resolvable multiton node.
2) Structured unit complex entries: The $j$-th entry of the $(5 i-2)$-th row of $A(\mathcal{G})$ is denoted by $a_{i, j}^{(\mathcal{G}, 3)}$. The value is set as follows:

$$
a_{i, j}^{(\mathcal{G}, 3)}=\exp \left(\iota \frac{j \pi}{2 n}\right)
$$

This type of measurement will be used only in "cancelling out" process of resolvable multiton node.
3) Unit entries: The $j$-th entry of the $(5 i-1)$-th row of $A$ is denoted by $a_{i, j}^{(\mathcal{G}, 4)}$. The value is set to be 1 . This measurement is used in resolvable doubleton identification and "cancelling out" process of resolvable multiton node.
4) Random unit complex entries: The $j$-th entry of the $5 i$-th row of $A$ is denoted by $a_{i, j}^{(\mathcal{G}, 5)}$ used as verification. The value is set as follows:

$$
a_{i,(j)}^{(\mathcal{G}, 5)}=\exp \left(\iota \phi_{i, j}\right)
$$

where $\iota$ denotes the positive square root of -1 and $\phi_{i, j}$ is chosen uniformly at random from $[0, \pi / 2]$. This measurement is used in resolve the degeneracy when resolvable multiton and resolvable doubleton are used for decoding. Also, it helps to verify our identification and estimation of magnitude and relative phase.

| Notation | Definition |
| :---: | :--- |
| $m_{\mathcal{G}}^{\prime}$ | The number of right nodes for the bipartite graph $\mathcal{G} . \mathcal{G}$ is one of $\mathcal{G}_{I}, \mathcal{G}_{I I, l}$ for $l \in[L]$, and $\mathcal{G}_{I I I .}$ |
| $A(\mathcal{G})^{\prime}$ | The dimension- $m_{\mathcal{G}}^{\prime} \times n$ adjacent matrix of $\mathcal{G}$. |
| $A(\mathcal{G})_{i}^{\prime}$ | The $i$-th row of matrix $A(\mathcal{G})^{\prime}$ for $i \in\left[m_{\mathcal{G}}^{\prime}\right]$. |
| $A(\mathcal{G})$ | The dimension- $m_{\mathcal{G}} \times n$ measurement matrix generated by $A(\mathcal{G})^{\prime}$. Here $m_{\mathcal{G}}=5 m_{\mathcal{G}}^{\prime}$. |
| $A$ | The dimension- $m \times n$ phase measurement matrix generated by all $A(\mathcal{G})$ 's. Here, $m=\Sigma_{\mathcal{G}} m_{\mathcal{G}}$. |
| $a_{i, j}^{(\mathcal{G}, q)}$ | The $j$-th entry of the $[5(i-1)+q]$-th the rows of $A(\mathcal{G})$. Here, $i \in\left[m_{\mathcal{G}}\right], j \in[n]$, and $q \in[5]$. |
| Table III |  |
| TABLE OF NOTATION FOR MEASUREMENTS DESIGN |  |

## V. Reconstruction Algorithm

Let $b_{i}^{(\mathcal{G}, q)}$ denote the $[5(i-1)+q]$-th measurement generated by $A(\mathcal{G})$. Here, $\mathcal{G}$ is one of the $\mathcal{G}_{I}, \mathcal{G}_{I I, l}$ 's and $\mathcal{G}_{I I I}, i \in\left[m_{\mathcal{G}}\right]$, and $q \in[5]$.

## A. Seeding phase

1) Overview:
2) Preprocessing: Each right node is attached to a list to record its neighbors (left nodes) in the decoding process.
3) Magnitude Recovery and Doubleton Identification: Check every right node to see whether it's singleton node or not. If yes, we locate the corresponding non-zero component and measure the magnitude of it. After decoding the non-zero component (only the magnitude), each list of its neighbors' (right nodes') is inserted the location of the decoded non-zero component if the length of the list is no larger than one. For the list whose length is 3 after insertion, it will be discarded and won't be considered in the following iteration since it definitely is not a doubleton. So far we get the potential resolvable doubletons. Later, we use the verification measurement to find the actual resolvable doubletons. The reason why we need the verification step is that the potential resolvable doubletons may involve other non-zero components which have not been resolved yet.
In this step, we decode the magnitudes of constant fraction of all the non-zero components and locate these non-zero components. We also identify the potential resolvable doubletons by checking whether its list is of length 2 and the actual resolvable doubletons by verification measurement.
4) Relative Phase Recovery: For each resolvable doubleton, it's used to resolve the phase between the two non-zero components whose locations lie in the neighbor list.
Breadth first search (BFS) or Depth first search (DFS) [25] algorithm would guide us to explore the connected components in graph $\mathcal{H}$ efficiently. We only care about the largest connected component, $\mathcal{H}^{\prime}$. After this step, any pair of nodes in $\mathcal{H}^{\prime}$ are mutually resolved.
5) The formal description of reconstruction algorithm:
6) Initialization: We initialize by setting the signal estimate vector $\hat{\mathbf{x}}$ to all-zeros vector $0^{n}$. Each right node $i \in\left[m_{\mathcal{G}_{I}}^{\prime}\right]$ attaches an empty neighbor list $\mathcal{N}(i)$. let $\mathcal{D}$ denote a list of the resolvable doubletons. Initially, $\mathcal{D}$ is empty. Set $i=1$.
7) Singleton Identification, Magnitude Recovery and Doubleton Identification:
a) Compute the ratio of Trigonometric measurements:

$$
s_{i}=\frac{\arctan \left(\frac{b_{i}^{\left(\mathcal{G}_{I}, 2\right)}}{\iota b_{i}^{\left(\mathcal{G}_{I}, 1\right)}}\right)}{\frac{\pi}{2 n}}
$$

i) Check if $s_{i}$ is an integer. If so, we tentatively identifies that $i$ is a singleton, $s_{i}$-th entry of $\mathbf{x}$ is non-zero and

$$
\left|\hat{x}_{s_{i}}\right|= \begin{cases}\frac{b_{i}^{\left(\mathcal{G}_{I}, 1\right)}}{a_{i, s_{i}}^{\left(\mathcal{G}_{I}, 1\right)}} & \text { if } a_{i, s_{i}}^{\left(\mathcal{G}_{I}, 1\right)} \neq 0 \\ \iota \frac{b_{i}^{\left(\mathcal{G}_{I}, 2\right)}}{a_{i, s_{i}}^{\left(\mathcal{G}_{I}, 2\right)}} & \text { if } a_{i, s_{i}}^{\left(\mathcal{G}_{I}, 2\right)} \neq 0\end{cases}
$$

b) We verify our estimate from the previous step. If $\left|\hat{x}_{s_{i}}\right| \neq\left|b_{i}^{\left(\mathcal{G}_{I}, 5\right)}\right|$, the verification fails. We increment $i$ by 1 and go back to step a) to start a new iteration. If verification passes, we do the following steps:
i) $s_{i}$ is appended to the neighbor lists of all its neighbors. For $i \in\left[m_{\mathcal{G}_{I}}^{\prime}\right]$, it is no longer considered in the later process if $|\mathcal{N}(i)| \geq 3$ since in the next step we only care about doubleton whose neighbor list size equals 2 .
ii) Increment $i$ by 1 and go back to step a) to start a new iteration.
c) For each $i$ whose neighbor list is of size 2 , it is appended to the resolvable doubleton list $\mathcal{D}$ where $\mathcal{N}(i)[1]$ and $\mathcal{N}(i)[2]$ are the two indices of non-zero components whose magnitudes have been recovered.
3) Relative Phase Recovery:
a) Compute connected component of $\mathcal{H}$ : Breadth first search or depth first search for adjacent list representation of $\mathcal{H}$ is applied in this step. For each $i \in \mathcal{D}$, the elements in $\mathcal{N}(i)$ tell which two vertices in $\mathcal{H}$ are connected. BFS or DFS outputs connected components of graph $\mathcal{H}$. We run the BFS or DFS, for each edge in $\mathcal{H}$, with additional steps b), c), and d) stated below:
b) Law of Cosine: Suppose $i$ 's two neighbors are denoted by $\mathcal{N}(i)[1]$ and $\mathcal{N}(i)[2]$. The fourth measurement is used to derive the phase between $\mathcal{N}(i)[1]$-th and $\mathcal{N}(i)[2]$-th components of $\mathbf{x}, \theta=\left|\theta_{\mathcal{N}(i)[1]}-\theta_{\mathcal{N}(i)[2]}\right|$, by Law of Cosind .
c) The verification measurement helps to resolve the degeneracy of sign of $\theta$ (i.e., whether $\theta$ or $-\theta$ is the actual phase difference we are interested in.) by checking whether

$$
\left|\left|\hat{\mathbf{x}}_{\mathcal{N}(i)[1]}\right| \exp \left(\iota \phi_{i, \mathcal{N}(i)[1]}\right)+\left|\hat{\mathbf{x}}_{\mathcal{N}(i)[2]}\right| \exp \left(\iota \phi_{i, \mathcal{N}(i)[2]}+\iota \theta\right)\right|=\left|b_{i}^{\left(\mathcal{G}_{I}, 5\right)}\right|
$$

or

$$
\left|\left|\hat{\mathbf{x}}_{\mathcal{N}(i)[1]}\right| \exp \left(\iota \phi_{i, \mathcal{N}(i)[1]}\right)+\left|\hat{\mathbf{x}}_{\mathcal{N}(i)[2]}\right| \exp \left(\iota \phi_{i, \mathcal{N}(i)[2]}-\iota \theta\right)\right|=\left|b_{i}^{\left(\mathcal{G}_{I}, 5\right)}\right| .
$$

If neither of the above equations holds, then $i$ is not a resolvable doubleton.
d) For the first node in a connected component, its phase is set to be zero.
e) When the BFS or DFS terminates, we can find the largest connected component of $\mathcal{H}, \mathcal{H}^{\prime}$. For all the node pairs in $\mathcal{H}^{\prime}$, they are mutually resolved.

## B. Geometric-decay and Cleaning-up phases

Claim 2. ("Cancelling out" Process) For a bipartite graph $\mathcal{G}$ in geometric-decay phase or cleaning-up phase, if a right node $i$ is a resolvable multiton node, it involves exactly one (unknown) undecoded non-zero component, $\mathbf{x}_{j}$, and at least one (known) resolved non-zero components. Then, we are able to find the location of $\mathbf{x}_{j}, j$, and resolve $\mathbf{x}_{j}$ (both magnitude and relative phase).

Proof: We will use four measurements in the "cancelling out" process,

[^2]\[

$$
\begin{aligned}
b_{i}^{(\mathcal{G}, 1)} & =\left|A+\mathbf{x}_{j} \cos \left(\frac{j \pi}{2 n}\right)\right| \\
b_{i}^{(\mathcal{G}, 2)} & =\left|B+\mathbf{x}_{j} \iota \sin \left(\frac{j \pi}{2 n}\right)\right| \\
b_{i}^{(\mathcal{G}, 3)} & =\left|C+\mathbf{x}_{j} \exp \left(\iota \frac{j \pi}{2 n}\right)\right| \\
b_{i}^{(\mathcal{G}, 5)} & =\left|D+\mathbf{x}_{j} \exp \left(\iota \phi_{i, j}\right)\right|
\end{aligned}
$$
\]

where $A, B, C$, and $D$ are calculated from the decoded non-zero components which connect to right node $i$ in $\mathcal{G}$. We find that by the measurements design

$$
A+B=C
$$

and

$$
\mathbf{x}_{j} \cos \left(\frac{j \pi}{2 n}\right)+\mathbf{x}_{j \iota \sin }\left(\frac{j \pi}{2 n}\right)=\mathbf{x}_{j} \exp \left(\iota \frac{j \pi}{2 n}\right)
$$

Let

$$
\begin{aligned}
\frac{j \pi}{2 n} & =\alpha \\
A+\mathbf{x}_{j} \cos \left(\frac{j \pi}{2 n}\right) & =U \\
B+\mathbf{x}_{j} \iota \sin \left(\frac{j \pi}{2 n}\right) & =V \\
C+\mathbf{x}_{j} \exp \left(\iota \frac{j \pi}{2 n}\right) & =W
\end{aligned}
$$

we have

$$
\begin{aligned}
b_{i}^{(\mathcal{G}, 1)} & =|U| \\
b_{i}^{(\mathcal{G}, 2)} & =|V| \\
b_{i}^{(\mathcal{G}, 3)} & =|W| \\
& =|U+V|
\end{aligned}
$$

Finding the relation between $U$ and $V$ :
We know that

$$
U=V \times \frac{b_{i}^{(\mathcal{G}, 1)}}{b_{i}^{(\mathcal{G}, 2)}} \exp (\iota \psi)
$$

or

$$
U=V \times \frac{b_{i}^{(\mathcal{G}, 1)}}{b_{i}^{(\mathcal{G}, 2)}} \exp (-\iota \psi)
$$

where $\psi$ is the phase between $U$ and $V$ and $\cos \psi=\frac{|U|^{2}+|V|^{2}-|U+V|^{2}}{2|U||V|}$.
Finding the relation between $\mathbf{x}$ and $\alpha$ :
For simplicity, we only consider the case that

$$
\begin{aligned}
U & =V \times \frac{b_{i}^{(\mathcal{G}, 1)}}{b_{i}^{(\mathcal{G}, 2)}} \exp (\iota \psi) \\
& \triangleq V \times M
\end{aligned}
$$

So,

$$
A+\mathbf{x}_{j} \cos \alpha=\left[B+\mathbf{x}_{j} \iota \sin \alpha\right] M
$$

We have

$$
\mathbf{x}_{j}=\frac{B M-A}{\cos \alpha-\iota M \sin \alpha}
$$

$\underline{\text { Solving } \cos ^{2} \alpha \text { by quadratic equation: }}$
Replacing $\mathbf{x}_{j}$ in

$$
b_{i}^{(\mathcal{G}, 1)}=|U|
$$

we know that

$$
\begin{aligned}
b_{i}^{(\mathcal{G}, 1)} & =\left|A+\frac{B M-A}{\cos \alpha-\iota M \sin \alpha} \cos \alpha\right| \\
& =\left|\frac{B M \cos \alpha-\iota A M \sin \alpha}{\cos \alpha-\iota M \sin \alpha}\right| \\
& =\left|\frac{B \cos \alpha-\iota A \sin \alpha}{\cos \alpha-\iota M \sin \alpha}\right||M| \\
& =\left|\frac{B \cos \alpha-\iota A \sin \alpha}{\cos \alpha-\iota M \sin \alpha}\right| \frac{b_{i}^{(\mathcal{G}, 1)}}{b_{i}^{(\mathcal{G}, 2)}} .
\end{aligned}
$$

So,

$$
b_{i}^{(\mathcal{G}, 2)}|\cos \alpha-\iota M \sin \alpha|=|B \cos \alpha-\iota A \sin \alpha|
$$

Let

$$
\begin{aligned}
A & =A_{1}+\iota A_{2} \\
B & =B_{1}+\iota B_{2} \\
M & =M_{1}+\iota M_{2}
\end{aligned}
$$

where $A_{1}, A_{2}, B_{1}, B_{2}, M_{1}$, and $M_{2}$ are real numbers. We have

$$
\begin{array}{r}
b_{i}^{(\mathcal{G}, 2)}\left|\cos \alpha-\iota\left(M_{1}+\iota M_{2}\right) \sin \alpha\right| \\
=\left|\left(B_{1}+\iota B_{2}\right) \cos \alpha-\iota\left(A_{1}+\iota A_{2}\right) \sin \alpha\right| .
\end{array}
$$

Squaring both sides, we get

$$
\begin{aligned}
& {\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2}\left[\left(\cos \alpha+M_{2} \sin \alpha\right)^{2}+\left(M_{1} \sin \alpha\right)^{2}\right] } \\
= & \left(B_{1} \cos \alpha+A_{2} \sin \alpha\right)^{2}+\left(B_{2} \cos \alpha-A_{1} \sin \alpha\right)^{2}
\end{aligned}
$$

After reorganizing the above equation, we have

$$
\begin{aligned}
& \left(\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2}-|B|^{2}\right) \cos ^{2} \alpha+\left(\left[b_{i}^{(\mathcal{G}, 1)}\right]^{2}-|A|^{2}\right) \sin ^{2} \alpha \\
& \quad=2 \cos \alpha \sin \alpha\left(A_{2} B_{1}-A_{1} B_{2}-2\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2} M_{2}\right)
\end{aligned}
$$

Let

$$
\begin{aligned}
P & =\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2}-|B|^{2} \\
Q & =\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2}-|A|^{2} \\
R & =A_{2} B_{1}-A_{1} B_{2}-2\left[b_{i}^{(\mathcal{G}, 2)}\right]^{2} M_{2} \\
S & =\cos ^{2} \alpha
\end{aligned}
$$

and square both sides, we have

$$
[P S+Q(1-S)]^{2}=4 R^{2} S(1-S)
$$

After reorganizing the above equation, we get

$$
\begin{aligned}
& \left(P^{2}+Q^{2}-2 P Q+4 R^{2}\right) S^{2} \\
+ & \left(2 P Q-2 Q^{2}-4 R^{2}\right) S+Q^{2}=0
\end{aligned}
$$

We are able to solve $S$ (quadratic equation) in constant time and similarly for the case that $U=V \times \frac{b_{i}^{(\mathcal{G}, 1)}}{b_{i}^{(\mathcal{G}, 2)}} \exp (-\iota \psi)$. Resolving the degeneracy via random unit complex measurements:

After deriving the value of $S=\cos ^{2} \alpha$, we can get the constant (4) possible value of $j$ and $\mathbf{x}_{j}$ (both magnitude and the relative phase in $\mathcal{H}^{\prime}$ ) pairs.

Last, we check which pairs of solution that satisfies the following equation to resolve the degeneracy

$$
b_{i}^{(\mathcal{G}, 5)}=\left|D+\mathbf{x}_{j} \exp \left(\iota \phi_{i, j}\right)\right| .
$$

Note that if "cancelling out" fails (i.e., none of the pairs of $j$ and $\mathbf{x}_{j}$ satisfies the last equation in the proof), then $i$ is not a resolvable multiton. In each stage at geometric-decay phase and cleaning-up phase, we go through all the right nodes, find resolvable multitons and use them to recover unresolved non-zero components by the "cancelling out" process. For a newly resolved component of $\mathbf{x}$, the corresponding node in $\mathcal{H}$ is appended to $\mathcal{H}^{\prime}$. In the end, the size of the node set of $\mathcal{H}^{\prime}$ should be $k$.

| Notation | Definition |
| :---: | :--- |
| $b_{i}^{(\mathcal{G}, q)}$ | The $[5(i-1)+q]$-th intensity measurement generated by measurement matrix $A(\mathcal{G})$. Here, $i \in\left[m_{\mathcal{G}}\right]$, and $q \in[5]$. |
| $\mathcal{D}$ | Resolvable doubleton list used in the seeding phase. |
| $\mathcal{S}$ | Singleton List. used in the seeding phase. |
| $\mathcal{N}(i)$ | The neighbor list for $i$-th node in $\mathcal{G}_{I}$ for $i \in\left[m_{\mathcal{G}}[\right.$. |
| Table IV |  |

TABLE OF NOTATION FOR MEASUREMENTS DESIGN

## VI. Parameters Design

All the parameters designed in this section are calculated based on expectation. The actual performance of our algorithm will be discussed in Section VII

## A. Seeding phase

1) Magnitude Recovery by singletons:

- The probability of a right node being a singleton node:

$$
\begin{aligned}
P_{S} & =\binom{k}{1} \frac{1}{k}\left(1-\frac{1}{k}\right)^{k-1} \\
& =\left(1-\frac{1}{k}\right)^{k-1} \\
& \doteq e^{-1}
\end{aligned}
$$

- The expected number of singletons is $=c k \times P_{S} \doteq e^{-1} c k$.
- The expected number of different non-zero components whose magnitudes are recovered:

Lemma 3. (Generalized coupon collection) Given $V$ different coupons and $V \log \frac{V}{V-U}$ picks with repetition $(U<V)$, the expected number of different coupons picked is $U$ for $V \rightarrow+\infty$. With probability at least $1-2 \exp \left(-\frac{2 \epsilon^{2}(V-U) U}{V}\right)$, the number of different coupons picked is between $(1-\epsilon) U$ and $(1+\epsilon) U$ for any $\epsilon>0$.

By Lemma 3 (let $V=k$ and $V \log \frac{V}{V-U}=c k \times P_{S}$ ), we know that the expected number of non-zero components of $x$ whose magnitudes are recovered is $k\left(1-e^{-c P_{S}}\right)$.
2) Relative Phase Recovery by resolvable doubletons:

- The probability of a right node being doubleton:

$$
\begin{aligned}
P_{D} & =\binom{k}{2}\left(\frac{1}{k}\right)^{2}\left(1-\frac{1}{k}\right)^{k-2} \\
& =\frac{1}{2}\left(1-\frac{1}{k}\right)^{k-1} \\
& \doteq \frac{e^{-1}}{2}
\end{aligned}
$$

- The expected number of doubletons is $=c k \times P_{D} \doteq e^{-1} c k / 2$.
- The expected number of resolvable doubletons:

Note that only the doubleton which involves two non-zero components whose magnitudes have been recovered is useful to recover the relative phase.

$$
\begin{aligned}
\text { \# resolvable doubletons } & =\frac{\binom{k\left(1-e^{-c P_{S}}\right)}{2}}{\binom{k}{2}} \times \operatorname{ck} P_{D} \\
& \doteq \frac{\left(1-e^{-c P_{S}}\right)^{2} c k e^{-1}}{2}
\end{aligned}
$$

- The expected number of different pairs of components whose relative phase is recovered by resolvable doubletons:

By Lemma 3, given $k\left(1-e^{-c P_{S}}\right)$ nodes and $\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}$ edges with repetition in $\mathcal{H}$, there are

$$
(1+\mathcal{O}(1 / k))\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

distinct edges.
3) The giant connected components:

Theorem 4. [23] For a random graph $\mathcal{G}_{N, M}$ with $N$ nodes and $M$ edges chosen at random among the $\binom{N}{2}$ possible edges. Let $\mathcal{Z}_{N, M}$ denote the size of the greatest component of $\mathcal{G}_{N, M}$. If $r=2 M / N>1$, we have for any $\epsilon>0$

$$
\operatorname{Pr}\left(\left|\frac{\mathcal{Z}_{N, M}}{N}-\beta\right|<\epsilon\right)=1-\mathcal{O}\left(\frac{1}{\epsilon^{2} N}\right)
$$

where $\beta$ is the unique solution to $\beta+\exp (-\beta r)=1$.
We need to find the size of giant connected component of a random graph with $k\left(1-e^{-c P_{S}}\right)$ nodes and $\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}$ edges (with repetition) and therefore $(1+\mathcal{O}(1 / k))\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}$ distinct edges (implied by Lemma 3). Let's say the size is $\left(1-f_{I}\right) k$ where $f_{I}$ is the function of $c$.

By Theorem 4, when $2\left(1-e^{-c P_{S}}\right) c P_{D}>1$, the giant connected component exists (this inequality holds when constant $c$ is large enough) and the size of the giant component is $\left(1-f_{I}\right) k=\beta_{c} k\left(1-e^{-c P_{S}}\right)$ where $\beta_{c}$ is the unique solution to $\beta+\exp \left[-\beta \cdot 2\left(1-e^{-c P_{S}}\right) c P_{D}\right]=1$.

## B. Geometric-decay phase:

Let $f_{I I, l}$ denote the expected fraction of unresolved non-zero components after the $l$-th stages in this phase. Let $f_{I}=f_{I I, 0}$.

1) Stage $l+1(0 \leq l \leq L-1)$ :

- The probability that a right node being a resolvable multiton:

$$
\begin{aligned}
P_{M}^{(I I, l+1)} & =\binom{f_{I I, l} k}{1} \frac{1}{f_{I I, l} k}\left(1-\frac{1}{f_{I I, l} k}\right)^{f_{I I, l} k-1}\left[1-\left(1-\frac{1}{f_{I I, l} k}\right)^{\left(1-f_{I I, l}\right) k}\right] \\
& =\left(1-\frac{1}{f_{I I, l} k}\right)^{f_{I I, l} k-1}-\left(1-\frac{1}{f_{I I, l} k}\right)^{k-1} \\
& \doteq e^{-1}-e^{-\frac{1}{f_{I I, l}}} .
\end{aligned}
$$

- The expected number of resolvable multitons is $c f_{I I, l} k P_{M}^{(I I, l+1)}$.
- The expected number of non-zero components which are resolved (both magnitude and phase):

By Lemma 3, let

$$
f_{I I, l} k \log \frac{f_{I I, l} k}{f_{I I} k-\left(f_{I I, l}-f_{I I, l+1}\right) k}=c f_{I I, l} k P_{M}^{(I I, l+1)}
$$

we know that $f_{I I, l}-f_{I I, l+1}=f_{I I, l}\left(1-e^{-c P_{M}^{(I I, l+1)}}\right)$.
Therefore, $f_{I I, l+1}=e^{-c P_{M}^{(I I, l+1)}} f_{I I, l}$. We can compute the value of $f_{I I, l}$ recursively.
Note that $P_{M}^{(I I, l)}$ increases as $l$ increases. So, $P_{M}^{(I I, l)}$ is bounded by $e^{-1}-e^{-\frac{1}{f_{I}}}(l=0)$ and $e^{-1}(l=+\infty)$.
2) End of this phase: There are $\mathcal{O}(\log \log k)$ stages in the geometric-decay phase. We already show that in each step we expect to recover constant fraction of remaining unresolved non-zero components. In the end of this phase, the number of unresolved non-zero components is $\mathcal{O}(k / \log k)$.

## C. Cleaning-up phase

Recall that, in this phase, each edges appears with probability $\log k / k$ and there are $c(k / \log k) \log (k / \log k)=\mathcal{O}(k)$ right nodes in $\mathcal{G}_{\text {III }}$.

## VII. Performance of algorithm (Proof of Main Theorem)

## Number of measurements:

In Section VI, we already know that $f_{I I, l+1}=e^{-c P_{M}^{(I I, l+1)}} f_{I I, l}$. And $P_{M}^{(I I, l)}$ increases as $l$ increases.

$$
\begin{aligned}
f_{I I, l+1} & =\exp \left[-c P_{M}^{(I I, l+1)}\right] f_{I I, l} \\
& =\exp \left[-c \Sigma_{t=1}^{l+1} P_{M}^{(I I, t)}\right] f_{I I, 0} \\
& =\exp \left[-c \Sigma_{t=1}^{l+1} P_{M}^{(I I, t)}\right] f_{I} \\
& \leq \exp \left[-c(l+1) P_{M}^{(I I, 1)}\right] f_{I}
\end{aligned}
$$

Then, the total number of measurements in three phases is

$$
\begin{aligned}
\underbrace{c k}_{\text {Seeding }}+\underbrace{\Sigma_{l=1}^{L} c f_{I I, l-1} k}_{\text {geometric-decay }}+\underbrace{c(k / \log k) \log (k / \log k)}_{\text {cleaning-up }} & =\mathcal{O}(k)+\left(\Sigma_{l=1}^{L} c f_{I I, l-1}\right) k+\mathcal{O}(k) \\
& \leq \mathcal{O}(k)+\left(\Sigma_{l=1}^{L} \exp \left[-c l P_{M}^{(I I, 1)}\right]\right) f_{I} c k \\
& =\mathcal{O}(k)
\end{aligned}
$$

## Decoding complexity:

Almost all the operations take constant time except for DFS in the seeding phase and "Cancelling out" process in the geometric-decay and cleaning-up phases.

For DFS, the time complexity is linear in the size of node set and edge set. Since there are $k$ nodes and $\mathcal{O}(k)$ edges involved in the seeding phase, the time complexity is $\mathcal{O}(k)$.

For "Cancelling out" process, the time complexity is dominated by calculating the value of $A, B, C$ and $D$ (See Section V). And the complexity depends on the number of resolved non-zero components which corresponds to the resolvable multiton.

In the later stage/phase, more non-zero components are associated with a measurement. Since the number of measurements is $\mathcal{O}(k)$, it suffices to show that each measurement involves at most $\mathcal{O}(\log k)$ non-zero components (even if they are unresolved) in the cleaning-up phase with probability at least $1-o(1 / k)$.

Let $N Z$ be the number of non-zero components involved in a measurement in cleaning-up phase. By Chernoff bound, for any $\epsilon_{N Z} \geq 0$, we have

$$
\operatorname{Pr}\left[N Z \geq\left(1+\epsilon_{N Z}\right) k \cdot \log k / k\right] \leq \exp \left(-\frac{\epsilon_{N Z}^{2}}{2+\epsilon_{N Z}^{2}} k \cdot \log k / k\right)
$$

Therefore,

$$
\operatorname{Pr}[N Z=\mathcal{O}(\log k)] \geq 1-\mathcal{O}(1 / \operatorname{poly}(k))
$$

Thus, we know that the decoding complexity is at most $\mathcal{O}(k \cdot N Z)=\mathcal{O}(k \log k)$ with probability at least $1-k$. $\mathcal{O}(1 / \operatorname{poly}(k))=1-o(1)$ by Union bound.

## Correctness:

The actual performance of our algorithm is slightly different from expectation. Here, "slightly" means that the actual number of resolved non-zero components in each phase/stage deviates from the expected value but it can be concentrated around expectation with high probability.

Let $g_{I}$ denote the actual fraction of unresolved non-zero components after seeding phase. Let $g_{I I, l}$ denote the actual fraction of unresolved non-zero components after the $l$-th stage in geometric-decay phase. Let $g_{I I, 0}=g_{I}$.

Just recall the properties of the bipartite graphs for measurement design. In the seeding phase, each edge appears with probability $1 / k$ and there are $c k$ right nodes. In the geometric-decay phase, each edge appears with probability $1 / f_{I I, l} k$ and there are $c f_{I I, l} k$ right nodes in $(l+1)$-th step for $l \geq 0$. In the cleaning-up phase, each edge appears with probability $\log k / k$ and there are $c(k / \log k) \log (k / \log k)$ right nodes.

## A. Seeding Phase

1) Magnitude recovery:

- By Chernoff bound, the probability that the number of singletons is larger than $\left(1+\epsilon_{S}\right) c k \times P_{S}$ or smaller than $\left(1-\epsilon_{S}\right) c k \times P_{S}$ is less than $2 e^{-\left(\epsilon_{S}\right)^{2} c k P_{S} / 2}=\mathcal{O}\left(\exp \left(-\epsilon_{S}^{2} k\right)\right)$ for any $\epsilon_{S}>0$.
- By Lemma 3 and Union bound, we know that, for any $\epsilon_{D S}>0$, the number of different non-zero components whose magnitudes are recovered is between $\left(1-\epsilon_{D S}\right)\left[1-e^{-\left(1-\epsilon_{S}\right) c P_{S}}\right] k$ and $\left(1+\epsilon_{D S}\right)\left[1-e^{-\left(1+\epsilon_{S}\right) c P_{S}}\right] k$ with probability

$$
1-\mathcal{O}\left[\exp \left(-\epsilon_{D S}^{2} k\right)+\exp \left(-\epsilon_{S}^{2} k\right)\right]
$$

Note that

$$
\left(1+\beta_{S}\right)\left[1-e^{-\left(1+\epsilon_{S}\right) c P_{S}}\right] k
$$

and

$$
\left(1-\beta_{S}\right)\left[1-e^{-\left(1-\epsilon_{S}\right) c P_{S}}\right] k
$$

scale as

$$
\left(1-e^{-c P_{S}}\right)\left[1+\left(\epsilon_{D S}+\epsilon_{S}\right)+o\left(\epsilon_{D S}+\epsilon_{S}\right)\right] k
$$

and

$$
\left(1-e^{-c P_{S}}\right)\left[1-\left(\epsilon_{D S}+\epsilon_{S}\right)-o\left(\epsilon_{D S}+\epsilon_{S}\right)\right] k
$$

respectively.
2) Relative phase recovery:

- By Chernoff bound, the probability that the number of doubletons is larger than $\left(1+\epsilon_{D}\right) c k \times P_{D}$ or smaller than $\left(1-\epsilon_{D}\right) c k \times P_{D}$ is less than $2 e^{-\left(\epsilon_{D}\right)^{2} c k P_{D} / 2}=\mathcal{O}\left(\exp \left(-\epsilon_{D}^{2} k\right)\right)$ for any $\epsilon_{D}>0$.
- The resolvable doubletons

$$
\begin{aligned}
\text { \# resolvable doubletons } \leq & \frac{\binom{\left(1+\epsilon_{D S}\right)\left[1-e^{-\left(1+\epsilon_{S}\right) c P_{S}}\right] k}{2}}{\binom{k}{2}} \\
& \times\left(1+\epsilon_{D}\right)\left(1+\epsilon_{R D}\right) c k P_{D}
\end{aligned}
$$

and

$$
\begin{aligned}
\text { \# resolvable doubletons } \geq & \frac{\binom{\left(1-\epsilon_{D S}\right)\left[1-e^{\left.-\left(1-\epsilon_{S}\right) c P_{S}\right] k}\right.}{2}}{\binom{k}{2}} \\
& \times\left(1-\epsilon_{D}\right)\left(1-\epsilon_{R D}\right) c k P_{D}
\end{aligned}
$$

with probability

$$
1-\mathcal{O}\left[\exp \left(-\epsilon_{D S}^{2} k\right)+\exp \left(-\epsilon_{S}^{2} k\right)+\exp \left(-\epsilon_{D}^{2} k\right)+\exp \left(-\epsilon_{R D}^{2} k\right)\right]
$$

for any $\epsilon_{R D}>0$, by Chernoff bound and Union bound. Again, the upper bound and the lower bound on the number of resolvable doubletons scale as

$$
\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

and

$$
\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

respectively.

- Number of distinct edges in giant component: By Lemma 3 and Union bound, for any $\epsilon_{D R D}>0$, with probability

$$
1-\mathcal{O}\left[\exp \left(-\epsilon_{D S}^{2} k\right)+\exp \left(-\epsilon_{S}^{2} k\right)+\exp \left(-\epsilon_{D}^{2} k\right)+\exp \left(-\epsilon_{R D}^{2} k\right)+\exp \left(-\epsilon_{D R D}^{2} k\right)\right]
$$

the number of pairs of relative phase resolved by all the resolvable doubletons will be bounded by

$$
\left(1+\epsilon_{D R D}\right)\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right) / k\right]\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

and

$$
\left(1-\epsilon_{D R D}\right)\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right) / k\right]\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

which scale as

$$
\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

and

$$
\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D}
$$

3) The giant connected component: Let $N^{+}$and $N^{-}$be the upper bound and lower bound on the number of nodes in giant component.

Let $M^{+}$and $M^{-}$be the upper bound and lower bound on the number of edges in giant component. Then, $r^{+}=2 M^{+} / N^{-}$ is the upper bound on twice the size of edges over size of nodes and the $r^{-}=2 M^{-} / N^{+}$is the lower bound. $\beta_{c}^{+}$and $\beta_{c}^{-}$are the solution to the equation $\beta+\exp \left(-\beta r^{+}\right)=1$ and $\beta+\exp \left(-\beta r^{-}\right)=1$.

We know that

$$
\begin{aligned}
r^{+}= & 2 M^{+} / N^{-} \\
= & 2\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D} \\
& /\left(1-e^{-c P_{S}}\right)\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}\right)\right] k \\
= & 2\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right) c P_{D}
\end{aligned}
$$

and

$$
\begin{aligned}
r^{-}= & 2 M^{-} / N^{+} \\
= & 2\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right)^{2} c k P_{D} \\
& /\left(1-e^{-c P_{S}}\right)\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}\right)\right] k \\
= & 2\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(1-e^{-c P_{S}}\right) c P_{D}
\end{aligned}
$$

Since $r=-\log (1-\beta) / \beta, d r / d \beta=(-\log (1-\beta) / \beta)_{\beta=\beta_{c}}^{\prime}$ is a constant.
By Theorem 4

$$
\operatorname{Pr}\left(\left|\frac{\mathcal{Z}_{N, M}}{N}-\beta_{c}\right| \leq \epsilon_{G C}\right)=\mathcal{O}\left(\frac{1}{\epsilon_{G C}^{2} k}\right)
$$

for any $\epsilon_{G C}>0$.
Therefore,

$$
\beta_{c}^{+}=\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(\beta_{c}+\epsilon_{G C}\right)
$$

and

$$
\beta_{c}^{-}=\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(\beta_{c}-\epsilon_{G C}\right) .
$$

The upper bound on the size of giant component is

$$
\begin{aligned}
N^{+} \beta_{c}^{+} & =\left(1-e^{-c P_{S}}\right)\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}\right)\right] k\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}\right)\right]\left(\beta_{c}+\epsilon_{G C}\right) \\
& =\left[1+\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}+\epsilon_{G C}\right)\right] \beta_{c}\left(1-e^{-c P_{S}}\right) k
\end{aligned}
$$

and the lower bound on the size of giant component is

$$
N^{-} \beta_{c}^{-}=\left[1-\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}+\epsilon_{G C}\right)\right] \beta_{c}\left(1-e^{-c P_{S}}\right) k
$$

with probability

$$
1-\mathcal{O}\left[\exp \left(-\epsilon_{D S}^{2} k\right)+\exp \left(-\epsilon_{S}^{2} k\right)+\exp \left(-\epsilon_{D}^{2} k\right)+\exp \left(-\epsilon_{R D}^{2} k\right)+\exp \left(-\epsilon_{D R D}^{2} k\right)+\mathcal{O}\left(1 / \epsilon_{G C}^{2} k\right)\right]
$$

Recall that $\left(1-f_{I}\right) k=\beta_{c}\left(1-e^{-c P_{S}}\right) k$, we conclude that, with probability

$$
1-\mathcal{O}\left[\exp \left(-\epsilon_{D S}^{2} k\right)+\exp \left(-\epsilon_{S}^{2} k\right)+\exp \left(-\epsilon_{D}^{2} k\right)+\exp \left(-\epsilon_{R D}^{2} k\right)+\exp \left(-\epsilon_{D R D}^{2} k\right)+\mathcal{O}\left(1 / \epsilon_{G C}^{2} k\right)\right]
$$

there exists $\epsilon_{I}$ such that

$$
\left(1-\epsilon_{I}\right) f_{I} \leq g_{I} \leq\left(1+\epsilon_{I}\right) f_{I}
$$

Here, $\epsilon_{I}$ scales as $\mathcal{O}\left(\epsilon_{D S}+\epsilon_{S}+\epsilon_{D}+\epsilon_{R D}+\epsilon_{D R D}+\epsilon_{G C}\right)$. Choose all the $\epsilon^{\prime}$ s to be $k^{-1 / 3}$. Then, $\epsilon_{I}$ scales as $\mathcal{O}\left(k^{-1 / 3}\right)$ with probability $1-\mathcal{O}\left(k^{-1 / 3}\right)$.

## B. Geometric-decay Phase

1) Stage $l+1(0 \leq l \leq L-1)$ :

- The probability that a right node being a resolvable multiton:

$$
\begin{aligned}
Q_{M}^{(I I, l+1)} & =\binom{g_{I I, l} k}{1} \frac{1}{f_{I I, l} k}\left(1-\frac{1}{f_{I I, l} k}\right)^{g_{I I, l} k-1}\left[1-\left(1-\frac{1}{f_{I I, l} k}\right)^{\left(1-g_{I I, l}\right) k}\right] \\
& =\frac{g_{I I, l}}{f_{I I, l}}\left[\left(1-\frac{1}{f_{I I, l} k}\right)^{g_{I I, l} k-1}-\left(1-\frac{1}{f_{I I, l} k}\right)^{k-1}\right] \\
& \doteq \frac{g_{I I, l}}{f_{I I, l}}\left(e^{\left.-\frac{g_{I I, l}}{f_{I I, l}}-e^{-\frac{1}{f_{I I, l}}}\right) .} \$ .\right.
\end{aligned}
$$

- The number of resolvable multitons is bounded by

$$
\left(1+\epsilon_{M}^{(I I, l+1)}\right) c f_{I I, l} k Q_{M}^{(I I, l+1)}
$$

and

$$
\left(1-\epsilon_{M}^{(I I, l+1)}\right) c f_{I I, l} k Q_{M}^{(I I, l+1)}
$$

with probability $1-\mathcal{O}\left(\exp \left(-\left[\epsilon_{M}^{(I I, l+1)}\right]^{2} f_{I I, l} k\right)\right)$ for any $\epsilon_{M}^{(I I, l+1)}>0$.

- The number of non-zero components which are recovered (both magnitude and phase)

Let $\epsilon_{I I, 0}=\epsilon_{I}$. By Lemma 3, we know that

$$
\begin{aligned}
g_{I I, l+1}-g_{I I, l} & \leq\left(1+\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{-\left(1+\epsilon_{M}^{(I I, l+1)}\right) c f_{I I, l} Q_{M}^{(I I, l+1)} / g_{I I, l}}\right] g_{I I, l} \\
& \doteq\left(1+\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{-\left(1+\epsilon_{M}^{(I I, l+1)}\right) c\left(e^{\left.-\frac{g_{I I, l}}{f_{I I, l}}-e^{-\frac{1}{f_{I I, l}}}\right)}\right] g_{I I, l}}\right. \\
& \leq\left(1+\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{\left.-\left(1+\epsilon_{M}^{(I I, l+1)}\right) c\left(e^{-\left(1-\epsilon_{I I, l}\right)}-e^{-\frac{1}{f_{I I, l}}}\right)\right] g_{I I, l}}\right.
\end{aligned}
$$

and

$$
\begin{aligned}
g_{I I, l+1}-g_{I I, l} & \geq\left(1-\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{-\left(1-\epsilon_{M}^{(I I, l+1)}\right) c f_{I I, l} Q_{M}^{(I I, l+1)} / g_{I I, l}}\right] g_{I I, l} \\
& \doteq\left(1-\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{-\left(1-\epsilon_{M}^{(I I, l+1)}\right) c\left(e^{\left.-\frac{g_{I I, l}}{f_{I I, l}}-e^{-\frac{1}{f_{I I, l}}}\right)}\right] g_{I I, l}}\right. \\
& \geq\left(1-\epsilon_{D M}^{(I I, l+1)}\right)\left[1-e^{-\left(1-\epsilon_{M}^{(I I, l+1)}\right) c\left(e^{-\left(1-\epsilon_{I I, l}\right)}-e^{\left.-\frac{1}{f_{I I, l}}\right)}\right] g_{I I, l}}\right.
\end{aligned}
$$

with probability $1-\mathcal{O}\left(\exp \left(-\left[\epsilon_{M}^{(I I, l+1)}\right]^{2} f_{I I, l} k\right)+\exp \left(-\left[\epsilon_{D M}^{(I I, l+1)}\right]^{2} g_{I I, l} k\right)+k^{-1 / 3}\right)$ by Lemma 3 and Union bound for any $\epsilon_{D M}^{(I I, l+1)}>0$.

We conclude there exists $\epsilon_{I I, l+1}^{\prime}$ such that

$$
\begin{aligned}
e^{-c P_{M}^{(I I, l+1)}}\left(1-\epsilon_{I I, l+1}^{\prime}\right) g_{I I, l} & \leq g_{I I, l+1} \\
& \leq e^{-c P_{M}^{(I I, l+1)}}\left(1+\epsilon_{I I, l+1}^{\prime}\right) g_{I I, l}
\end{aligned}
$$

Here $\epsilon_{I I, l+1}^{\prime}$ scales as $\mathcal{O}\left[\epsilon_{D M}^{(I I, l+1)}+\epsilon_{M}^{(I I, l+1)}+\epsilon_{I I, l}\right]$.
Since $\left(1-\epsilon_{I I, l}\right) f_{I I, l} \leq g_{I I, l}=\left(1+\epsilon_{I I, l}\right) f_{I I, l}$, we have

$$
\begin{aligned}
e^{-c P_{M}^{(I I, l+1)}}\left(1-\epsilon_{I I, l+1}^{\prime}\right)\left(1-\epsilon_{I I, l}\right) f_{I I, l} & \leq g_{I I, l+1} \\
& \leq e^{-c P_{M}^{(I I, l+1)}}\left(1+\epsilon_{I I, l+1}^{\prime}\right)\left(1+\epsilon_{I I, i}\right) f_{I I, l} .
\end{aligned}
$$

Since $f_{I I, l+1}=e^{-c P_{M}^{(I I, l+1)}} f_{I I, l}$, we have

$$
\begin{aligned}
\left(1-\epsilon_{I I, l+1}^{\prime}\right)\left(1-\epsilon_{I I, l}\right) f_{I I, l+1} & \leq g_{I I, l+1} \\
& \leq\left(1+\epsilon_{I I, l+1}^{\prime}\right)\left(1+\epsilon_{I I, l}\right) f_{I I, l+1}
\end{aligned}
$$

Thus, we get that

$$
\begin{aligned}
\left(1-\epsilon_{I I, l+1}\right) f_{I I, l+1} & \leq g_{I I, l+1} \\
& \leq\left(1+\epsilon_{I I, l+1}\right) f_{I I, l+1}
\end{aligned}
$$

Here, $\epsilon_{I I, l+1}$ scales as $\mathcal{O}\left[\epsilon_{D M}^{(I I, l+1)}+\epsilon_{M}^{(I I, l+1)}+2 \epsilon_{I I, l}\right]$.
Choose $\epsilon_{D M}^{(I I, l+1)}$ and $\epsilon_{M}^{(I I, l+1)}$ to be $k^{-1 / 3}$ for all $l$. The error probability in each stage is $\mathcal{O}\left(k^{-1 / 3}\right)$ (which is the dominant term).

After $L$ stages $(L=\mathcal{O}(\log \log k))$,

$$
\begin{equation*}
\left(1-\epsilon_{I I, L}\right) f_{I I, L} \leq g_{I I, L} \leq\left(1+\epsilon_{I I, L}\right) f_{I I, L} \tag{1}
\end{equation*}
$$

holds with probability $1-\mathcal{O}\left(\log \log k \cdot k^{-1 / 3}\right)$ by Union bound. Here $\epsilon_{I I, L}$ scales as $\mathcal{O}\left(\log k \cdot k^{-1 / 3}\right)$ (since each stage $\epsilon_{I I, l}$ doubles).
C. Cleaning-up phase

Theorem 5. [Folklore](Coupon Collection) Let the random variable $X$ denote the minimum number of trials for collecting each of the $V$ types of coupons. Then, we have $\operatorname{Pr}[X>\eta V \log V] \leq V^{-\eta+1}$ for any $\eta>0$.

- The probability that a right node being a resolvable multiton:

$$
Q_{M}^{I I I} \doteq \frac{g_{I I, L}}{f_{I I, L}}\left(e^{-\frac{g_{I I, L}}{f_{I I, L}}}-e^{-\frac{1}{f_{I I, L}}}\right)
$$

- $f_{I I I}=1 / \log k$. The number of resolvable multitons is lower bounded by

$$
\left(1-\epsilon_{M}^{I I I}\right) c f_{I I I} k \log \left(f_{I I I} k\right) Q_{M}^{I I I}
$$

with probability $1-\mathcal{O}\left(\exp \left(-\left[\epsilon_{M}^{I I I}\right]^{2} k / \log k\right)\right)$ given Equation (11) holds for any $\epsilon_{M}^{(I I I)}>0$.

- The number of unresolved components in this phase is $\mathcal{O}(k / \log k)$. By Theorem 5, all the components are resolved with probability $1-\mathcal{O}(\log k / k)$ for large enough $c$ given Equation (1) holds.
By Union bound, the overall error probability is $o(1)$.


## VIII. Conclusion

In this paper, we present the first algorithm for compressive phase retrieval problem whose number of measurements is order-optimal and computational complexity is nearly order-optimal.

## IX. Appendix

## A. Proof for Lemma 3

Proof: Let $Y_{i}$ be the indicator random variable which represents whether $i$-th coupon is picked in $M$ trials. We know that $Y_{i}^{\prime} s$ are dependent and

$$
Y_{i}= \begin{cases}1 & \text { with probability } 1-\left(1-\frac{1}{V}\right)^{M} \\ 0 & \text { with probability }\left(1-\frac{1}{V}\right)^{M}\end{cases}
$$

Then, $Y=Y_{1}+\cdots+Y_{V}$ is the total number of different types of coupons picked in $M$ trials.
By the linearity of expectation, we have

$$
\begin{aligned}
E[Y] & =\sum_{i=1}^{V} E\left[Y_{i}\right] \\
& =V\left[1-\left(1-\frac{1}{V}\right)^{M}\right] \\
& \doteq V\left(1-e^{-\frac{M}{V}}\right) \\
& =V\left(1-\frac{V-U}{V}\right) \\
& =U
\end{aligned}
$$

Let $Z_{1}, \ldots, Z_{M}$ be independent random variables all taking values in [ $V$ ] uniformly at random representing each pick for $V$ types of coupon.

Let $f\left(Z_{1}, \ldots, Z_{M}\right)$ be the number of different types of coupons picked. Then, $\mathbf{E}[f]=\mathbf{E}[Y] \doteq U$.
Also, $\forall i \in[M]$,

$$
\left|f\left(Z_{1}, \ldots, Z_{i}, \ldots, Z_{M}\right)-f\left(Z_{1}, \ldots, Z_{i}^{\prime}, \ldots, Z_{M}\right)\right| \leq 1
$$

For all $\beta>0$, by McDiarmid's Inequality (Theorem 7), we have

$$
\operatorname{Pr}(f-\mathbf{E}[f] \leq-\beta) \leq \exp \left(-\frac{2 \beta^{2}}{M}\right)
$$

and

$$
\operatorname{Pr}(f-\mathbf{E}[f] \geq \beta) \leq \exp \left(-\frac{2 \beta^{2}}{M}\right)
$$

Thus,

$$
\operatorname{Pr}\left(f \leq V\left(1-e^{-M / V}\right)-\beta\right) \leq \exp \left(-\frac{2 \beta^{2}}{M}\right)
$$

Let $M=V \log \frac{V}{V-U}$ and $\beta=\epsilon U$, we know that

$$
\begin{aligned}
\operatorname{Pr}(f \leq(1-\epsilon) U) & \leq \exp \left(-\frac{2(\epsilon U)^{2}}{V \log \frac{V}{V-U}}\right) \\
& \leq \exp -\frac{2(\epsilon U)^{2}}{V \frac{U}{V-U}} \\
& =\exp \left(-\frac{2 \epsilon^{2} U(V-U)}{V}\right)
\end{aligned}
$$

and

$$
\operatorname{Pr}(f \geq(1+\epsilon) U) \leq \exp \left(-\frac{2 \epsilon^{2} U(V-U)}{V}\right)
$$

Therefore,

$$
\operatorname{Pr}((1-\epsilon) U \leq f \leq(1+\epsilon) U) \geq 1-2 \exp \left(-\frac{2 \epsilon^{2} U(V-U)}{V}\right)
$$

## B. Proof of Theorem 5

Proof: Let $Z_{i}$ be the event that $i$-th coupon has not yet picked in $M$ trials. We know that

$$
\begin{aligned}
\operatorname{Pr}\left(Z_{i}\right) & =\left(1-\frac{1}{V}\right)^{M} \\
& \leq \exp (-M / V)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\operatorname{Pr}(X>M) & =\operatorname{Pr}\left(\cup_{i=1}^{v} Z_{i}\right) \\
& \leq \sum_{i=1}^{v} \operatorname{Pr}\left(Z_{i}\right) \\
& \leq V \exp (-M / V) .
\end{aligned}
$$

Let $M=\eta V \log V$, we get

$$
\operatorname{Pr}(X>\eta V \log V) \leq V^{-\eta+1}
$$

## C. Chernoff Bound and McDiamid's Inequality

Theorem 6. (Chernoff Bound) Let $X_{1}, \ldots, X_{n}$ be independent random variables. Assume that $0 \leq X_{i} \leq 1$ for each $i \in[n]$. Let $X=X_{1}+\cdots+X_{n} . \mu=\mathbf{E}[X]=\mathbf{E}\left[X_{1}\right]+\cdots+\mathbf{E}\left[X_{n}\right]$. Then for any $\epsilon \geq 0$,

$$
\operatorname{Pr}[X \leq(1-\epsilon) \mu] \leq \exp \left(-\frac{\epsilon^{2}}{2} \mu\right)
$$

and

$$
\operatorname{Pr}[X \geq(1+\epsilon) \mu] \leq \exp \left(-\frac{\epsilon^{2}}{2+\epsilon} \mu\right)
$$

Theorem 7. $\left[26 /(M c D i a r m i d ' s ~ I n e q u a l i t y) ~ L e t ~ X_{1}, \ldots, X_{m}\right.$ be independent random variables all taking values in the set $\mathcal{X}$. Further, let $f: \mathcal{X}^{m} \rightarrow \mathbb{R}$ be a function of $X_{1}, \ldots, X_{m}$ that satisfies $\forall i, \forall x_{1}, \ldots, x_{m}, x_{i}^{\prime} \in \mathcal{X}$,

$$
\left|f\left(x_{1}, \ldots, x_{i}, \ldots, x_{m}\right)-f\left(x_{1}, \ldots, x_{i}^{\prime}, \ldots, x_{m}\right)\right| \leq c_{i} .
$$

Then for all $\epsilon>0$,

$$
\operatorname{Pr}(f-\mathbf{E}[f] \geq \epsilon) \leq \exp \left(\frac{-2 \epsilon^{2}}{\sum_{i=1}^{m} c_{i}^{2}}\right)
$$

and

$$
\operatorname{Pr}(f-\mathbf{E}[f] \leq-\epsilon) \leq \exp \left(\frac{-2 \epsilon^{2}}{\sum_{i=1}^{m} c_{i}^{2}}\right)
$$

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[^0]:    ${ }^{1}$ All the measurements are designed before the decoding process, so it is still non-adaptive.
    ${ }^{2}$ We borrow this terminology (of singletons, doubletons, multitons, etc) from the compressive sensing work of Pawar et al [21].

[^1]:    ${ }^{3}$ Here, $1 / 2$ is arbitrarily chosen to simplify the presentation of intuition. The actual fraction of resolved non-zero components in the seeding phase is different from $1 / 2$. See Section VII for details. Here, the parameter $1 / 2$ for the geometric-decay phase in this section is due to the same reason.
    ${ }^{4}$ The combinatorial algorithm in [18] can be modified to have $\mathcal{O}(k \log (k))$ measurement with error probability $\mathcal{O}(1 / p o l y(k))$ instead of $1 / n$ in the paper. Also, based on our reconstruction algorithm, the decoding complexity can be reduced to $\mathcal{O}(k \log k)$.

[^2]:    ${ }^{5}$ Given the lengths of two complex number $A$ and $B$, we can deduce the phase between $A$ and $B, \Delta$, by Law of Cosine if we also know the length of $A+B$. To be more explicit, $-\cos \Delta=\frac{|A|^{2}+|B|^{2}-|A+B|^{2}}{2|A||B|}$.

