On the Finite Length Scaling of Ternary Polar Codes

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Abstract—The polarization process of polar codes over a ternary alphabet is studied. Recently it has been shown that the scaling of the blocklength of polar codes with prime alphabet size scales polynomially with respect to the inverse of the gap between code rate and channel capacity. However, except for the binary case, the degree of the polynomial in the bound is extremely large. In this work, it is shown that a much lower degree polynomial can be computed numerically for the ternary case. Similar results are conjectured for the general case of prime alphabet size.

Keywords—polar codes, scaling, non-binary channels

I. INTRODUCTION

Polar codes for transmission over binary discrete memoryless channels (DMCs) were introduced by Arikan [1], and were further analyzed in [2]. These results were extended to q-ary polarization for an arbitrary prime q in [3]–[5].

For the binary case it was shown that the blocklength required to transmit reliably scales polynomially with respect to the inverse of the gap between code rate and channel capacity [6]–[8]. This result was recently extended to q-ary channels for an arbitrary prime q [9] but in the new bound, the degree of this polynomial is extremely large.

In this paper we obtain numerically a much better bound for q = 3. For that purpose we obtain numerically a lower bound on the size of a basic polarization step which is higher than the one for the binary case. We conjecture similar results for any prime value of the alphabet size, q.

II. PRELIMINARIES

A. General definitions and results

We follow the notations of [5, Lemma 5]. For the q-ary channel $W(y \mid x)$, we define $W(y) \triangleq (1/q) \sum_{x \equiv 0}^{q-1} W(y \mid x)$ and the vector $\mathbf{v}(y) \triangleq [v_0(y), v_1(y), \dots, v_{q-1}(y)]^T$ where

$$\forall x \in \{0, 1, \dots, q-1\} : v_x(y) \triangleq \frac{W(y \mid x)}{qW(y)}.$$
 (1)

Note that $\sum_{x=0}^{q-1} v_x(y) = 1$ and the symmetric capacity is

$$I(W) = \sum_{y} W(y) \{ 1 - H[\mathbf{v}(y)] \}$$
(2)

where

$$H\left[\mathbf{v}(y)\right] \triangleq -\sum_{x=0}^{q-1} v_x(y) \log_q v_x(y) .$$
(3)

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We can rewrite (2) as
$$I(W) = \sum_{G} \hat{W}(G) G$$
, where

$$\hat{W}(G) \triangleq \sum_{y:H[\mathbf{v}(y)]=1-G} W(y) \tag{4}$$

A basic polarization transformation of a channel W forms two channels, $W^- = W \circledast W$ and $W^+ = W \circledast W$. Recall that given two channels, W_a and W_b , $W_{a \boxtimes b} \stackrel{\triangle}{=} W_a \boxtimes W_b$ is defined by

$$W_{a \boxtimes b}(y_1, y_2 \mid u) \triangleq \frac{1}{q} \sum_{u'=0}^{q-1} W_b(y_2 \mid u') W_a(y_1 \mid u+u')$$

Hence $W_{a \otimes b}(y_1, y_2) = W_a(y_1) W_b(y_2)$ and [5, Proof of Lemma 6]

$$v_{a \not\equiv b, u} (y_1, y_2) = \sum_{u'=0}^{q-1} v_{b, u'} (y_2) v_{a, u+u'} (y_1)$$

which can be rewritten as

$$\mathbf{v}_{a \otimes b} \left(y_1, y_2 \right) = \mathbf{v}_b \left(y_2 \right) \star \mathbf{v}_a \left(y_1 \right) \tag{5}$$

where \star denotes circular cross-correlation with period *q*. Defining

$$g\left(G_{1},G_{2}\right) \triangleq 1 - \min_{\substack{H\left[\mathbf{v}_{a}\left(y_{1}\right)\right]=1-G_{1}\\H\left[\mathbf{v}_{b}\left(y_{2}\right)\right]=1-G_{2}}} H\left[\mathbf{v}_{b}\left(y_{2}\right) \star \mathbf{v}_{a}\left(y_{1}\right)\right] \quad (6)$$

we obtain

$$I(W_{a \boxtimes b}) = \sum_{y_1, y_2} W_{a \boxtimes b} (y_1, y_2) \{1 - H[\mathbf{v}_{a \boxtimes b} (y_1, y_2)]\}$$

$$\leq \sum_{\substack{G_1, G_2 \ y_1: H[\mathbf{v}_a(y_1)] = 1 - G_1 \\ y_2: H[\mathbf{v}_b(y_2)] = 1 - G_2}} W_a(y_1) W_b(y_2) g(G_1, G_2)$$

$$= \sum_{\substack{G_1, G_2 \ W_a}} \hat{W}_a(G_1) \hat{W}_b(G_2) g(G_1, G_2)$$

where the first equality is an application of (2), the inequality follows from (5), (6) and $W_{a \boxtimes b}(y_1, y_2) = W_a(y_1) W_b(y_2)$, and (4) yields the last equality. If $g(G_1, G_2)$ is concave in G_1 and separately, not necessarily jointly, in G_2

$$I(W_{a \otimes b}) \leq g\left[\sum_{G_{1}} \hat{W}_{a}(G_{1}) G_{1}, \sum_{G_{2}} \hat{W}_{a}(G_{2}) G_{2}\right] = g\left[I(W_{a}), I(W_{b})\right] \quad (7)$$

and since $W^- = W \otimes W$, $I(W^-) \leq g[I(W), I(W)]$. If $g(G_1, G_2)$ is not concave in G_1 and in G_2 , we can replace it with a concave upper-bound, and (7) will remain true.

Note that by (5), $v_{a \boxtimes b, u}(y_1, y_2) = v_{b \boxtimes a, -u}(y_2, y_1)$, where the subtraction is modulo q. Combining this with (6) yields $g(G_1, G_2) = g(G_2, G_1)$.

B. Proved results about the QSC channel

A q-ary symmetric channel (QSC) $W(y \mid x)$ with error probability p is defined by

$$W(y \mid x) = \begin{cases} 1-p & y=x\\ p/(q-1) & y \neq x \end{cases}$$

Although the QSC channel does not maximize (6) for some pair (G_1, G_2) , we observed that for q = 3 it provides an excellent approximation to the maximum, and we conjecture that this holds true for any prime q.

Lemma 1. If W_a and W_b are QSC channels, then $W_{a \boxtimes b}$ is a QSC channel as well. Furthermore, $I(W_{a \boxtimes b}) = g_{QSC}[I(W_a), I(W_b)]$ for

$$g_{QSC}(G_1, G_2) \triangleq 1 - h_q \left[h_q^{-1} \left(1 - G_1 \right) + h_q^{-1} \left(1 - G_2 \right) - \frac{q}{q - 1} h_q^{-1} \left(1 - G_1 \right) h_q^{-1} \left(1 - G_2 \right) \right]$$
(8)

with $h_q(p) \triangleq -(1-p)\log_q(1-p) - p\log_q\left(\frac{p}{q-1}\right)$ and h_q^{-1} is the inverse of h_q , that yields values in $\left[0, \frac{q-1}{q}\right]$.

The proof of this Lemma is a straightforward application of (1) and (5).

Lemma 2. Using QSC channels W_a and W_b yields an extreme point in the Lagrangian related to (6) for $G_1, G_2 > 0$.

The proof of this Lemma is also straightforward.

III. ANALYSIS AND NUMERICAL RESULTS

Observe the similar to (6) problem

$$\tilde{g}\left(G_{1},G_{2}\right) \triangleq 1 - \min_{\substack{H\left[\mathbf{v}_{a}\left(y_{1}\right)\right] \geq 1-G_{1}\\H\left[\mathbf{v}_{b}\left(y_{2}\right)\right] \geq 1-G_{2}}} H\left[\mathbf{v}_{b}\left(y_{2}\right) \star \mathbf{v}_{a}\left(y_{1}\right)\right]$$

First, we prove the following.

Lemma 3. Define $f(\mathbf{u}) \triangleq \min_{H(\mathbf{v}) \geq 1-G} H(\mathbf{u} \star \mathbf{v})$. Then, $f(\mathbf{u})$ is concave.

Proof: By definition, $f(\mathbf{u}_0) \triangleq \min_{H(\mathbf{v}) \geq 1-G} H(\mathbf{u}_0 \star \mathbf{v})$ and $f(\mathbf{u}_1) \triangleq \min_{H(\mathbf{v}) \geq 1-G} H(\mathbf{u}_1 \star \mathbf{v})$. Then

$$f (\alpha \mathbf{u}_{0} + (1 - \alpha)\mathbf{u}_{1})$$

$$= \min_{H(\mathbf{v}) \ge 1 - G} H (\alpha \mathbf{u}_{0} \star \mathbf{v} + (1 - \alpha)\mathbf{u}_{1} \star \mathbf{v})$$

$$\geq \min_{H(\mathbf{v}) \ge 1 - G} [\alpha H (\mathbf{u}_{0} \star \mathbf{v}) + (1 - \alpha)H (\mathbf{u}_{1} \star \mathbf{v})]$$

$$\geq \alpha \min_{H(\mathbf{v}) \ge 1 - G} H (\mathbf{u}_{0} \star \mathbf{v}) + (1 - \alpha) \min_{H(\mathbf{v}) \ge 1 - G} H (\mathbf{u}_{1} \star \mathbf{v})$$

$$= \alpha f (\mathbf{u}_{0}) + (1 - \alpha)f (\mathbf{u}_{1})$$

where the first inequality follows from concavity of H, and the added degree of freedom to the minimization yields the second inequality.

Since the constraints in this problem form a convex region, and by Lemma 3 we minimize a concave function, $f(\mathbf{u})$, the result is obtained on the boundary of the convex region, and $\tilde{g} = g$. Note that Lemma 3 enables us to compute gefficiently using known algorithms for concave minimization over a convex region [10]. This algorithm generates linear programs whose solutions minimize the convex envelope of the original function over successively tighter polytopes enclosing the feasible region. As the polytopes become more complex and more tight, the generated solution becomes more precise.

We can now prove the following.

Lemma 4. $g(G_1, G_2)$ has the following properties:

1) $g(x_1, y_1) \le g(x_2, y_2)$ for $x_1 \le x_2$ and $y_1 \le y_2$.

2)
$$g(1,G_2) = G_2$$

3)
$$g(G_1, G_2) \leq \min(G_1, G_2)$$
.

4) $\lim_{x \to 1} \frac{\tilde{\partial}g(x,G_2)}{\partial x} = 0$

Proof: Since $x_1 \leq x_2$ and $y_1 \leq y_2$, the constraints for $\tilde{g}(x_1, y_1)$ are tighter than the constraints for $\tilde{g}(x_2, y_2)$. Since it is a maximization problem $(1 - \min)$, the maximum for (x_1, y_1) would be smaller than the maximum for (x_2, y_2) , i.e. $\tilde{g}(x_1, y_1) \leq \tilde{g}(x_2, y_2)$. Since $\tilde{g} = g$, statement 1 follows. Statement 2 follows since for $G_1 = 1$, $\mathbf{v}_a(y_1)$ is a circular permutation of $[1, 0, \ldots, 0]^T$, so by (3) and (5), $H[\mathbf{v}_{a \boxplus b}(y_1, y_2)] = H[\mathbf{v}_b(y_2)]$ Now, $g(G_1, G_2) \leq g(G_1, 1) = G_1$ and $g(G_1, G_2) \leq g(1, G_2) = G_2$, which yields statement 3. Since (6) is a maximization problem, Lemma 2 yields that $g(x, G_2) \geq g_{QSC}(x, G_2)$, where g_{QSC} is defined in (8). By parts 1) and 2), $g(x, G_2) \leq g(1, G_2) = G_2 = g_{QSC}(1, G_2)$. Also, straightforward calculations show that $\lim_{x\to 1} \frac{\partial g_{QSC}(x, G_2)}{\partial x} = 0$. Combining the above yields statement 4.

Next, we calculate $g(G_1, G_2)$ for $G_1, G_2 \approx 0$ and for $G_1, G_2 \approx 1$. To simplify the notation, we will denote $\mathbf{v}_a(y_1) = \mathbf{v}_a = [v_{a,0}, v_{a,1}, \dots, v_{a,q-1}]^T$, $\mathbf{v}_b(y_2) =$ $\mathbf{v}_b = [v_{b,0}, v_{b,1}, \dots, v_{b,q-1}]^T$ and $\mathbf{v}_{a \boxplus b}(y_1, y_2) = \mathbf{v}_t = [v_{t,0}, v_{t,1}, \dots, v_{t,q-1}]^T$.

Lemma 5. For sufficiently small values of G_1 and G_2 and q = 3, $g(G_1, G_2) = \ln 3 \cdot G_1 G_2$.

Proof: Consider (6). For G_2 sufficiently small, $v_{b,i} = 1/q + \epsilon_i$ where ϵ_i are sufficiently small and $\sum_{i=0}^{q-1} \epsilon_i = 0$. Using Taylor's approximation, and $\gamma \triangleq q/(2 \ln q)$, $H[\mathbf{v}_b] = 1 - \gamma \sum_{i=0}^{q-1} \epsilon_i^2$. We shall first solve the minimization problem in (6) for a fixed \mathbf{v}_a and $G_2 \approx 0$, so $v_{t,i} = 1/q + \sum_{k=0}^{q-1} \epsilon_k v_{a,i+k}$ and $H[\mathbf{v}_t] = 1 - \gamma \sum_{i=0}^{q-1} \left(\sum_{k=0}^{q-1} \epsilon_k v_{a,i+k} \right)^2$. Hence, $g(G_1, G_2) = \gamma \max \sum_{i=0}^{q-1} \left(\sum_{k=0}^{q-1} \epsilon_k v_{a,i+k} \right)^2 = \gamma \max \epsilon^T A \epsilon$ s.t. $\epsilon^T \epsilon = G_2/\gamma$ and $\sum_{i=0}^{q-1} \epsilon_i = 0$. Here $\epsilon = [\epsilon_0, \ldots, \epsilon_{q-1}]^T$ and $A = \sum_{i=0}^{q-1} \mathbf{v}_{a,i} \mathbf{v}_{a,i}^T$ where $\mathbf{v}_{a,i}$ is a cyclic shift by i of \mathbf{v}_a . Hence,

$$g(G_1, G_2) = G_2 \max \boldsymbol{\epsilon}^T A \boldsymbol{\epsilon} \text{ s.t. } \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1, \sum_{i=0}^{q-1} \boldsymbol{\epsilon}_i = 0.$$
 (9)

Note that A is a circulant matrix, and for q = 3

$$a_{i,j} = \begin{cases} \sum_{k=0}^{2} v_{a,k}^2 & i = j\\ \sum_{k=0}^{2} v_{a,k} v_{a,k+1} & i \neq j \end{cases}$$

so A has only two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = \sum_{k=0}^2 v_{a,k}^2 - \sum_{l=0}^2 v_{a,k} v_{a,k+1} < \lambda_1$. The eigenvector associated with λ_1 is $\mathbf{u}_1 = c[1,1,1]^T$ so the linear constraint can be expressed as $\boldsymbol{\epsilon}^T \mathbf{u}_1 = 0$. Following [11, page 411, Th. 7], the solution to (9) is $G_2 \lambda_2$. The eigenvector associated with λ_2 is $\boldsymbol{\epsilon} = c[1, -0.5, -0.5]^T$, making W_b a QSC channel. Substituting it into (9) yields

$$g(G_1, G_2) = G_2\left(\sum_{i=0}^2 v_{a,i}^2 - \sum_{i=0}^2 v_{a,i}v_{a,i+1}\right)$$
(10)

For $G_1 \approx 0$, $v_{a,i} = 1/3 + \delta_i$, $\sum_{i=0}^2 \delta_i = 0$ and $\sum_{i=0}^2 v_{a,i}^2 - \sum_{i=0}^2 v_{a,i} v_{a,i+1} = \sum_{i=0}^2 \delta_i^2 - \sum_{i=0}^2 \delta_i \delta_{i+1}$.

Since $\sum_{i=0}^{2} \delta_i = 0$, $\sum_{i=0}^{2} \delta_i^2 = 2 \left(\delta_1^2 + \delta_2^2 + \delta_1 \delta_2 \right)$ and $\sum_{i=0}^{2} \delta_i \delta_{i+1} = - \left(\delta_1^2 + \delta_2^2 + \delta_1 \delta_2 \right)$. Therefore, $\sum_{i=0}^{2} \delta_i^2 - \sum_{i=0}^{2} \delta_i \delta_{i+1} = 1.5 \sum_{i=0}^{2} \delta_i^2 = \frac{3G_1}{2\gamma}$. Combining this with (10) yields the stated result.

Lemma 6. For G_1 and G_2 sufficiently close to 1, and q = 3, $g(G_1, G_2) = G_1 + G_2 - 1$

Proof: Consider (6). For G_1 sufficiently close to 1, we can assume without loss of generality that $v_{a,i} = \delta_i$, $i = 1, \ldots, q-1$, where δ_i are small, and $v_{a,0} = 1 - \sum_{i=1}^{q-1} \delta_i$. Similarly, for G_2 sufficiently close to 1, we can assume without loss of generality that $v_{b,i} = \epsilon_i$, $i = 1, \ldots, q-1$, where ϵ_i are small, and $v_{b,0} = 1 - \sum_{i=1}^{q-1} \epsilon_i$. Now, $1 - G_1 = H[\mathbf{v}_a] = -\sum_{i=1}^{q-1} \delta_i \log_q \delta_i$ and $1 - G_2 = H[\mathbf{v}_b] = -\sum_{i=1}^{q-1} \epsilon_i \log_q \epsilon_i$. For G_1 and G_2 sufficiently close to 1, $\mathbf{v}_t \approx [1 - \delta_1 - \delta_2 - \epsilon_1 - \epsilon_2, \delta_1 + \epsilon_2 + \epsilon_1 \delta_2, \delta_2 + \epsilon_1 + \epsilon_2 \delta_1]^T$. Hence, $H[\mathbf{v}_t] = -(\delta_1 + \epsilon_2 + \epsilon_1 \delta_2) \log_3(\delta_1 + \epsilon_2 + \epsilon_1 \delta_2) - (\delta_2 + \epsilon_1 + \epsilon_2 \delta_1) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_2) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_1) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_1) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_1) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_2) \log_3(\delta_2 + \epsilon_1 + \epsilon_2 \delta_2)$

Note that the same proof applies for a general q.

We calculated the actual value of g numerically. We calculated g(0.01n, 0.01m) for q = 3, $n = 1, 2, \ldots, 99$ and $m = 1, 2, \ldots, 99$. In Figure 1 we plot the contour of this function. This figure shows that $g(G_1, G_2) = g(G_2, G_1)$ as noted above, and, as proved in Lemma 4, $g(1, G_2) = G_2$.

Plotting the numeric $\frac{\partial g(G_1,G_2)}{\partial G_1}$ in Figure 2 shows that $g(G_1,G_2)$ is increasing in G_1 (and by symmetry, in G_2), as proved in Lemma 4. Next, using the calculated points, we estimate $\frac{\partial^2 g(x,G_2)}{\partial x^2}$. This estimated second derivative is shown in Figure 3, suggesting the following conjecture (since the bottom line represents $\frac{\partial^2 g(G_1,G_2)}{\partial G_1^2} = 0$, so below it $\frac{\partial^2 g(G_1,G_2)}{\partial G_1^2} > 0$ and $\frac{\partial^2 g(G_1,G_2)}{\partial G_1^2} < 0$ above that line):

Property 1. $g(G_1, G_2)$ is concave in G_1 (and by symmetry, in G_2), except for small values of G_1 and G_2 . In other words,

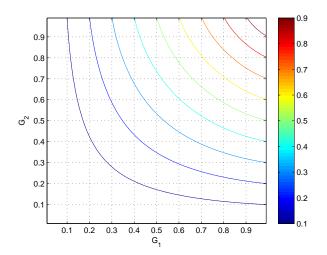


Fig. 1. Numerically calculated $g(G_1, G_2)$ for q = 3

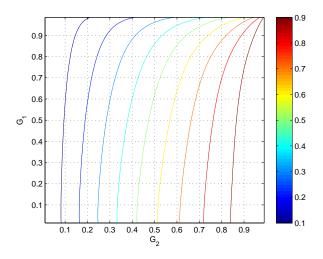


Fig. 2. Numerically calculated $\frac{\partial g(G_1,G_2)}{\partial G_1}$ for q=3

for each $G_2 \in (0,1)$ there exists x^* s.t. $\frac{\partial^2 g(x,G_2)}{\partial x^2}$ is positive for $x < x^*$ and negative for $x > x^*$.

Therefore, the convex hull of $g(G_1, G_2)$ for a given G_2 is

$$\max_{x \in [G_1, 1]} \frac{G_1}{x} g\left(x, G_2\right) = \begin{cases} \frac{G_1}{G_1^*} g\left(G_1^*, G_2\right) & G_1 \le G_1^* \\ g\left(G_1, G_2\right) & G_1 \ge G_1^* \end{cases}$$

where $G_1^* = \operatorname{argmax}_{x \in [0,1]} \frac{g(x,G_2)}{x}$. Finding G_1^* is equivalent to solving $\frac{\partial g(x,G_2)}{\partial x} = \frac{g(x,G_2)}{x}$ s.t. $\frac{\partial^2 g(x,G_2)}{\partial x^2} < 0$, i.e. finding a tangent to $g(x,G_2)$ at x s.t. $\frac{\partial^2 g(x,G_2)}{\partial x^2} < 0$, that passes through (0,0).

Lemma 7. If Property 1 holds, the problem $x \cdot \frac{\partial g(x,G_2)}{\partial x} = g(x,G_2)$ s.t. $\frac{\partial^2 g(x,G_2)}{\partial x^2} < 0$ has a single solution.

The proof of this Lemma follows from analysis of $x \cdot \frac{\partial g(x,G_2)}{\partial x} - g(x,G_2)$.

However, we want an upper bound on $g(G_1, G_2)$ that would be concave in G_1 and G_2 . Similarly to the case of

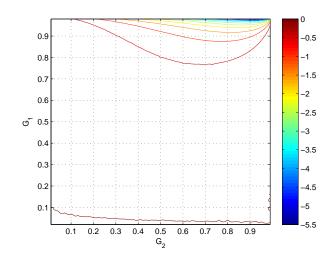


Fig. 3. Numerically calculated $\frac{\partial^2 g(G_1, G_2)}{\partial G_1^2}$ for q = 3

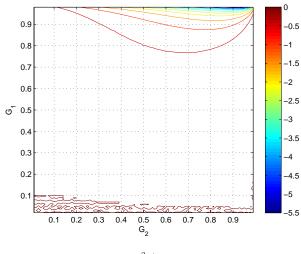


Fig. 4. Numerically calculated $\frac{\partial^2 g^*(G_1,G_2)}{\partial G_1^2}$ for q=3

fixed G_2 ,

$$g^{*}(G_{1}, G_{2}) = \max_{\substack{x_{1} \in [G_{1}, 1] \\ x_{2} \in [G_{2}, 1]}} \frac{G_{1}G_{2}}{x_{1}x_{2}} g(x_{1}, x_{2})$$
(11)

Clearly, $g^*(G_1, G_2) \ge g(G_1, G_2)$ and Figure 4 shows that $g^*(G_1, G_2)$ is concave in G_1 and in G_2 (the lines at the bottom of the figure stand for the area where $\frac{\partial^2 g(G_1, G_2)}{\partial G_1^2} = 0$.

Proposition 1. There exists $\epsilon_l^*(x)$ s.t. $I(W^-) + \epsilon_l^*[I(W)] \le I(W) \le I(W^+) - \epsilon_l^*[I(W)].$

Proof: Set $\epsilon_l^*(x) = x - g^*(x, x)$, where $g^*(x, x)$ was defined in (11). Recalling that $I(W^-) \leq g^*[I(W), I(W)]$ and $I(W^-) + I(W^+) = 2I(W)$ yields the stated result.

The minimal polarization step size is $\epsilon_l^*(x)$ rather than $\epsilon_l(x) = x - g(x, x)$. However, $\epsilon_l(x) - \epsilon_l^*(x)$ is very small, as seen in Figure 5, so we can use $\epsilon_l(x)$, which is easier to calculate. In Figure 6 we plot $\epsilon_l(x)$ for different values of q, and see that for q = 3, $\epsilon_l(x)$ is close, but not equal to

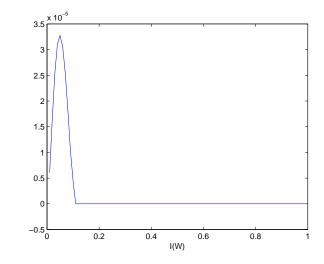


Fig. 5. Numerically calculated $\epsilon_l(I(W)) - \epsilon_l^*(I(W))$ for q = 3

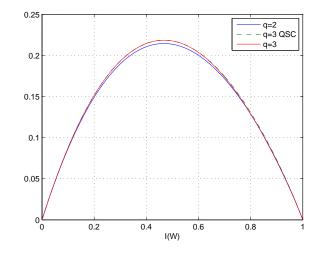


Fig. 6. The lower bound on $I(W^+) - I(W)$, which is also a lower bound on $I(W) - I(W^-)$, for different values of q, and for the QSC channel

$$\begin{split} \epsilon_{l,QSC}(x) &= x + h_q \left\{ h_q^{-1} \left(1 - x \right) \left[2 - \frac{q}{q-1} h_q^{-1} \left(1 - x \right) \right] \right\} - 1 \\ \text{which is marked as "}q &= 3 \text{ QSC". From Lemma 5, } \epsilon_l(x) \approx \\ x - \ln 3 \cdot x^2 \text{ for } x \to 0 \text{, so } \lim_{x \to 0} \frac{\partial \epsilon_l(x)}{\partial x} = 1 \text{, as seen in Figure 6. Lemma 6 yields } \epsilon_l(x) \approx 1 - x \text{ for } x \to 1 \text{, as can be seen in Figure 6. Note that for } q = 2 \text{, we would obtain the same } \\ \epsilon_l(x) &= \epsilon_l^*(x) = \epsilon_{l,QSC}(x) \text{ as in [7].} \end{split}$$

Given some function $f_0(x)$, defined over [0, 1] s.t. $f_0(x) > 0$ for $x \in (0, 1)$, and $f_0(0) = f_0(1) = 0$, we define $f_k(x)$ for $k = 1, 2, \ldots$ recursively as follows,

$$f_k(x) \triangleq \sup_{\epsilon_l(x) \le \epsilon \le \epsilon_h(x)} \frac{f_{k-1}(x+\epsilon) + f_{k-1}(x-\epsilon)}{2}$$

where $\epsilon_l(x) = x - g(x, x)$ and $\epsilon_h(x) = \min(x, 1 - x)$.

Define $L_k(x) = \frac{f_k(x)}{f_0(x)}$ and $L_k = \sup_{z \in (0,1)} L_k(z)$. With the definition of $f_k(x)$, $\sqrt[k]{L_k} \leq L_1$ still holds as in [8]. Simi-

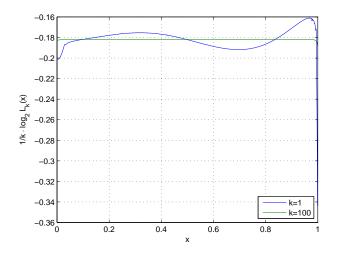


Fig. 7. A plot of $\frac{1}{k} \log L_k(x)$ for k = 1 and k = 100, q = 3 and $f_0(x) = (0.26x^2 + 1) x^{0.8} (1-x)^{0.6}$. The functions $L_k(x)$ were calculated numerically.

larly to [8, Equation (11)] we have, for an integer 0 < k < n,

$$\operatorname{E}\left[f_{0}\left(I_{n}\right)\right] \leq \left(\frac{L_{1}}{\sqrt[k]{L_{k}}}\right)^{k-1} \cdot \left(\sqrt[k]{L_{k}}\right)^{n} \cdot f_{0}\left[I(W)\right] .$$
(12)

Similarly to [8] we define $J_n \triangleq \min(I_n, 1 - I_n)$. Using $f_0(z) = (0.26x^2 + 1) x^{0.8}(1 - x)^{0.6}$ similarly to [8, Lemma 3], we obtain $P(J_n > \delta) \le \frac{\alpha_1}{2\delta} \cdot 2^{-0.1817n}$. As can be seen in Figure 7, numerical calculations yield $L_1 = 2^{-0.161}$ and, ${}^{100}\sqrt{L_{100}} = 2^{-0.1817}$. A plot of $\frac{1}{k}\log_2 L_k$ as a function of k for q = 3 and $f_0(z) = (0.26x^2 + 1) x^{0.8}(1 - x)^{0.6}$ shows a convex decreasing function, similar to [8, Fig. 3], suggesting that it is reasonable to expect that for this particular $f_0(z)$, using k = 100 is already a good choice for (12) (i.e., we cannot improve much by using an higher value of k). Similarly to [8, Lemma 4] we have the following. If $P[\omega \in \Omega : I_n(\omega) \notin (\delta, 1 - \delta) \forall n \ge m_0] \ge 1 - \epsilon$ for some integer $m_0, 0 < \epsilon < 1$ and $\delta < 1/3$. Then

$$P(\omega \in \Omega : I_n(\omega) \ge 1 - \delta \ \forall n \ge m_0) \ge I(W) - \epsilon$$
$$P(\omega \in \Omega : I_n(\omega) \le \delta \ \forall n \ge m_0) \ge 1 - I(W) - \epsilon$$

The proof is essentially the same as the proof of [8, Lemma 4], with I_n replacing $1 - Z_n$. Finally, we can obtain a result similar to [8, Theorem 1]. We use essentially the same proof but with the following modification. First we obtain a result similar to [8, Equation (25)] using the same approach: $P(\omega \in \Omega : I_n(\omega) \ge 1 - \delta \ \forall n \ge m_0) \ge I(W) - \left(\frac{\alpha_1}{2\delta}\right) \cdot \frac{2^{-\rho m_0}}{1-2^{-\rho}}$. Then we combine it with [1, Equation (2)] to obtain, $P(\omega \in \Omega : Z_n(\omega) \le \zeta \ \forall n \ge m_0) \ge I(W) - \left(\frac{\alpha_1}{\zeta^2}\right) \cdot \frac{2^{-\rho m_0}}{1-2^{-\rho}}$ and proceed with the derivation in [8, Theorem 1]. Since $\rho = 0.1817, 1 + 1/\rho = 6.504$, we claim the following result

Proposition 2. Suppose that we wish to use a polar code with rate R and blocklength N to transmit over a binary-input channel, W, with block error probability at most P_e^0 . Then it is sufficient to set $N = \frac{\beta}{(I(W) - R)^{6.504}}$ (or larger) where β is a constant that depends only on P_e^0 .

IV. FUTURE RESEARCH

In this paper we showed numerically that for the case where q = 3 we can obtain an improved lower bound on $I(W) - I(W^-)$ compared to the binary (q = 2 case). Consequently we can predict a much better scaling law of the blocklength with respect to I(W) - R compared to the results in [9]. It is interesting to continue this study for other values of prime q.

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APPENDIX: SUPPLEMENTARY MATERIAL

A. Proof of Lemma 1

Assume that W_a and W_b are QSC channels with error probabilities p_a and p_b , respectively. Then, for all $y = 0, 1, \ldots, q-1$, $\mathbf{v}_a(y)$ and $\mathbf{v}_a(y)$ are circular shifts of $\tilde{\mathbf{v}}_a$ and $\tilde{\mathbf{v}}_b$, respectively, where

$$\tilde{\mathbf{v}}_{a} \triangleq [1 - p_{a}, p_{a}/(q - 1), p_{a}/(q - 1), \dots, p_{a}/(q - 1)]^{T}$$

 $\tilde{\mathbf{v}}_{b} \triangleq [1 - p_{b}, p_{b}/(q - 1), p_{b}/(q - 1), \dots, p_{b}/(q - 1)]^{T}$.

Since for the QSC case, all $\mathbf{v}(y)$ vectors are shifts of some $\tilde{\mathbf{v}}$, if W is a QSC channel, $I(W) = 1 - H(\tilde{\mathbf{v}})$. This means

$$I\left(W_a\right) = 1 - h_q\left(p_a\right) \tag{13}$$

$$I(W_b) = 1 - h_q(p_b)$$
 . (14)

Using (5), we see that $\mathbf{v}_{a \otimes b} (y_1, y_2)$ are circular shifts of $\tilde{\mathbf{v}}_{a \otimes b} \triangleq [1 - p_t, p_t/(q-1), p_t/(q-1), \dots, p_t/(q-1)]^T$, where

$$p_t = p_a + p_b - qp_a p_b / (q - 1)$$
(15)

so $W_{a \boxtimes b}$ is a QSC channel with error probability p_t , and $I(W_{a \boxtimes b}) = 1 - h_q(p_t)$. Combined with (13),(14) and (15), this means that for the QSC case, (7) becomes an equality if $g(\cdot, \cdot)$ is defined as in (8).

B. Proof of Lemma 2

Assume $\mathbf{v}_{a}(y_{1}) = [v_{a,0}, v_{a,1}, \dots, v_{a,q-1}]^{T}$ and $\mathbf{v}_{b}(y_{2}) = [v_{b,0}, v_{b,1}, \dots, v_{b,q-1}]^{T}$. Using (5) yields $\mathbf{v}_{a \boxtimes b}(y_{1}, y_{2}) = [v_{t,0}, v_{t,1}, \dots, v_{t,q-1}]^{T}$ where

$$v_{t,i} = \sum_{j=0}^{q-1} v_{a,j} v_{b,j-i}$$
 for $i = 0, 1, \dots, q-1.$ (16)

The Lagrangian related to solving the minimization in (6) is

$$L = H \left[\mathbf{v}_{a \otimes b} \left(y_{1}, y_{2} \right) \right] - \lambda_{1} \left\{ H \left[\mathbf{v}_{a} \left(y_{1} \right) \right] - 1 + G_{1} \right\} \\ - \lambda_{2} \left\{ H \left[\mathbf{v}_{b} \left(y_{2} \right) \right] - 1 + G_{2} \right\} - \lambda_{3} \left(\sum_{i=0}^{q-1} v_{a,i} - 1 \right) \\ - \lambda_{4} \left(\sum_{i=0}^{q-1} v_{b,i} - 1 \right) \\ = -\sum_{i=0}^{q-1} v_{t,i} \log_{q} v_{t,i} + \lambda_{1} \left[1 + \sum_{i=0}^{q-1} v_{a,i} \log_{q} v_{a,i} - G_{1} \right] \\ + \lambda_{2} \left[1 + \sum_{i=0}^{q-1} v_{b,i} \log_{q} v_{b,i} - G_{2} \right] \\ - \lambda_{3} \left[\sum_{i=0}^{q-1} v_{a,i} - 1 \right] - \lambda_{4} \left[\sum_{i=0}^{q-1} v_{b,i} - 1 \right]$$
(17)

and we want to achieve $\partial L / \partial v_{a,i} = \partial L / \partial v_{b,i} = 0$ for $i = 0, 1, \ldots, q-1$. By (16), $\partial v_{t,i} / \partial v_{a,j} = v_{b,j-i}$ and combining it with (17) and $\sum_{i=0}^{q-1} v_{b,i} = 1$ yields

$$\frac{\partial L}{\partial v_{a,j}} = -\frac{1}{\ln q} - \sum_{i=0}^{q-1} v_{b,j-i} \log_q v_{t,i} + \lambda_1 \left(\log_q v_{a,j} + \frac{1}{\ln q} \right) - \lambda_3 = 0 \quad \forall j \in \{0, 1, \dots, q-1\} .$$
(18)

If W_a and W_b are QSC channels, $v_{a,i} = p_a/(q-1)$ and $v_{b,i} = p_b/(q-1)$ for $i \neq 0$, $v_{a,0} = 1 - p_a$ and $v_{b,0} = 1 - p_b$. By (16), $v_{t,i} = p_t/(q-1)$ for $i \neq 0$ and $v_{t,0} = 1 - p_t$, where p_t is defined in (15). For $j \neq 0$, (18) yields

$$-\frac{1}{\ln q} - \frac{p_b}{q-1} \log_q [1-p_t] - \left(1 - \frac{p_b}{q-1}\right) \log_q \frac{p_t}{q-1} + \lambda_1 \left(\log_q \frac{p_a}{q-1} + \frac{1}{\ln q}\right) - \lambda_3 = 0$$

and for j = 0, (18) yields

$$\frac{1}{\ln q} - (1 - p_b) \log_q (1 - p_t) - p_b \log_q \frac{p_t}{q - 1} + \lambda_1 \left[\log_q (1 - p_a) + \frac{1}{\ln q} \right] - \lambda_3 = 0$$

Now, if $p_a \neq \frac{q-1}{q}$, i.e. $G_1 > 0$, we have two independent equations, so we have a single possible value for λ_1 and λ_3 . Combining these equations yields

$$\begin{split} \lambda_1 &= \left(1 - \frac{qp_b}{q - 1}\right) \\ &\cdot \log_q \frac{p_t}{(q - 1)(1 - p_t)} \left/ \log_q \frac{p_a}{(q - 1)(1 - p_a)} \right. \\ \lambda_3 &= \frac{\frac{p_b}{q - 1} \log_q (1 - p_t) + \left(1 - \frac{p_b}{q - 1}\right) \log_q \frac{p_t}{q - 1}}{\log_q \frac{p_a}{q - 1} - \log_q (1 - p_a)} \\ &\cdot \log_q (1 - p_a) \\ &- \frac{\log_q \frac{p_a}{q - 1} \left[(1 - p_b) \log_q (1 - p_t) + p_b \log_q \frac{p_t}{q - 1} \right]}{\log_q \frac{p_a}{q - 1} - \log_q (1 - p_a)} \\ &+ \frac{\lambda_1 - 1}{\ln q} \,. \end{split}$$

Similarly, by (16), $\partial v_{t,i} / \partial v_{b,j} = v_{a,j+i}$ and combining it with (17) and $\sum_{i=0}^{q-1} v_{a,i} = 1$ yields

$$\frac{\partial L}{\partial v_{b,j}} = -\frac{1}{\ln q} - \sum_{i=0}^{q-1} v_{a,j+i} \log_q v_{t,i} + \lambda_2 \left(\log_q v_{b,j} + \frac{1}{q} \right) - \lambda_4 = 0 \quad \forall j \in \{0, 1, \dots, q-1\} \quad (19)$$

If W_a and W_b are QSC channels for $j \neq 0$, (19) yields

$$-\frac{1}{\ln q} - \frac{p_a}{q-1}\log_q\left(1-p_t\right) - \left(1-\frac{p_a}{q-1}\right)\log_q\frac{p_t}{q-1} + \lambda_2\left(\log_q\frac{p_b}{q-1} + \frac{1}{\ln q}\right) - \lambda_4 = 0$$

and for j = 0, (19) yields

$$-\frac{1}{\ln q} - [1 - p_a] \log_q [1 - p_t] - p_a \log_q \frac{p_t}{q - 1} + \lambda_2 \left[\log_q (1 - p_b) + \frac{1}{\ln q} \right] - \lambda_4 = 0$$

Now, if $p_b \neq \frac{q-1}{q}$, i.e. $G_2 > 0$, we have two independent equations, so we have a single possible value for λ_2 and λ_4 .

Combining these equations yields

$$\begin{split} \lambda_2 &= \left(1 - \frac{qp_a}{q-1}\right) \\ &\quad \cdot \log_q \frac{p_t}{(q-1)(1-p_t)} \left/ \log_q \frac{p_b}{(q-1)(1-p_b)} \right. \\ \lambda_4 &= \frac{\frac{p_a}{q-1}\log_q (1-p_t) + \left(1 - \frac{p_a}{q-1}\right)\log_q \frac{p_t}{q-1}}{\log_q \frac{p_b}{q-1} - \log_q (1-p_b)} \\ &\quad \cdot \log_q (1-p_b) \\ &\quad - \frac{\log_q \frac{p_b}{q-1} \left\{ [1-p_a]\log_q [1-p_t] + p_a\log_q \frac{p_t}{q-1} \right\}}{\log_q \frac{p_b}{q-1} - \log_q [1-p_b]} \\ &\quad + \frac{\lambda_2 - 1}{\ln q} \,. \end{split}$$

Since we have found $\lambda_1, \ldots, \lambda_4$ that solve (18) and (19) for the case of W_a and W_b being QSC channels, we proved that the QSC case yields a critical point in the Lagrangian related to (6) for any value of q.

C. Properties of g_{QSC} used in the proof of Lemma 4

By (8),

$$g_{QSC}(1,G_2) = 1 - h_q \left[h_q^{-1}(0) + h_q^{-1}(1 - G_2) - \frac{q}{q-1} h_q^{-1}(0) h_q^{-1}(1 - G_2) \right]$$
$$= 1 - h_q \left[h_q^{-1}(1 - G_2) \right] = G_2$$

Straightforward calculations show that

$$\frac{\partial g_{QSC}\left(G_{1},G_{2}\right)}{\partial G_{1}} = \frac{\log_{q}\left[\left(q-1\right)\left(\frac{1}{z}-1\right)\right]\left[1-\frac{q}{q-1}v\right]}{\log_{q}\left[\left(q-1\right)\left(\frac{1}{y}-1\right)\right]}$$
(20)

where $y = h_q^{-1}(1-G_1)$, $v = h_q^{-1}(1-G_2)$ and $z = y(1-v)+v\left(1-\frac{y}{q-1}\right)$. These functions are plotted in Figure 8. By (20), $\lim_{x\to 1} \frac{\partial g_{QSC}(x,G_2)}{\partial x} = 0$ (since in this case y = 0 and z = v).

D. A proof that $(x + y) \ln(x + y) \approx x \ln x + y \ln y$ for small positive x, y

We are going to prove that

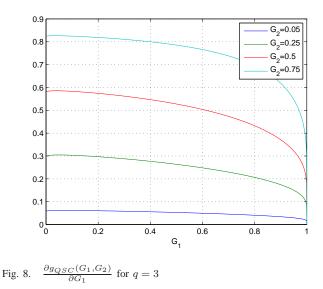
$$1 \le \frac{x \ln x + y \ln y}{(x+y) \ln(x+y)} \le 1 - \frac{\ln 2}{\ln(x+y)}$$

so

$$\lim_{x,y \to 0} \frac{x \ln x + y \ln y}{(x+y) \ln(x+y)} = 1 \; .$$

First, since $-x \ln x$ is concave,

$$\frac{-x\ln x - y\ln y}{2} \le -\left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right)$$
$$= -\left(\frac{x+y}{2}\right)\ln(x+y) + \left(\frac{x+y}{2}\right)\ln 2$$



so, dividing both sides by $-0.5(x+y)\ln(x+y)$ yields

$$\frac{x\ln x + y\ln y}{(x+y)\ln(x+y)} \le 1 - \frac{\ln 2}{\ln(x+y)}$$

For the other direction we must prove that

$$-x\ln x - y\ln y \ge -(x+y)\ln(x+y) .$$

It is equivalent to

$$x\left[\ln(x+y) - \ln x\right] \ge y\left[\ln y - \ln(x+y)\right]$$

Since ln is an increasing function, the left hand side of the inequality above is positive, and the right hand side is negative, so it is a true statement.

Note that for q variables (instead of 2) the first half of the proof is similar, using q instead of 2, and the second half is modified using q - 1 induction steps, one for each sum.

E. Proof of Lemma 7

Define $f(x) \triangleq x \cdot \frac{\partial g(x,G_2)}{\partial x} - g(x,G_2)$. We wish to prove that f(x) = 0 has exactly one solution that satisfies $\frac{\partial^2 g(x,G_2)}{\partial x^2} < 0$. First, $f'(x) = x \cdot \frac{\partial^2 g(x,G_2)}{\partial x^2}$. Since there exists x^* s.t. $\frac{\partial^2 g(x,G_2)}{\partial x^2}$ is positive for $x < x^*$ and negative for $x > x^*$ (See Property 1), f(x) is increasing for $x < x^*$ and decreasing for $x > x^*$. Combining this with f(0) = 0 yields that f(x) > 0 for $0 < x \le x^*$. Lemma 4 shows that $\lim_{x \to 1} \frac{\partial g(x,G_2)}{\partial x} = 0$ and $g(1,G_2) = G_2$, so $\lim_{x \to 1} f(x) = -G_2$. Since $f(x^*) > 0$, $\lim_{x \to 1} f(x) < 0$, and f(x) is decreasing for $x^* \le x \le 1$, f(x) = 0 has exactly one solution for $x^* < x \le 1$. The only other solution to f(x) = 0 is x = 0, and in this point $\frac{\partial^2 g(x,G_2)}{\partial x^2} > 0$.