

# NON-ADAPTIVE POLICIES FOR 20 QUESTIONS TARGET LOCALIZATION

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## ABSTRACT

The problem of target localization with noise is addressed. The target is a sample from a continuous random variable with known distribution and the goal is to locate it with minimum mean squared error distortion. The localization scheme or policy proceeds by queries, or questions, whether or not the target belongs to some subset as it is addressed in the 20-question framework. These subsets are not constrained to be intervals and the answers to the queries are noisy. While this situation is well studied for adaptive querying, this paper is focused on the non adaptive querying policies based on dyadic questions. The asymptotic minimum achievable distortion under such policies is derived. Furthermore, a policy named the *Aurelian*<sup>1</sup> is exhibited which achieves asymptotically this distortion.

**Index Terms**— target localization, 20 questions, dyadic policy, non adaptive policies

## 1. INTRODUCTION

Consider the following problem of localizing a one dimensional deterministic target  $X \in [0; 1]$  with a question/answer process. At each step, a subset of  $[0, 1]$  is chosen and is questioned: “whether or not  $X$  belongs to this subset”. Assume for now that the answers are truthful, i.e., there is no noise. By repeating this querying, an estimator of  $X$  notated  $\hat{X}$  is derived. The performance of such policy is measured by the supremum distance between the target and the estimator,  $\sup_X |X - \hat{X}|$ . Note that

$$\sup_X |X - \hat{X}| \geq \frac{1}{2^{n+1}} \quad (1)$$

since at most  $n$  bits of information can be learned using  $n$  binary questions. Assume now that the subsets are intervals and the policy is non-adaptive. Then,

$$\sup_X |X - \hat{X}| \geq \frac{1}{2(n+1)} \quad (2)$$

which is achieved by choosing intervals of the form  $[0, \frac{i}{n+1}]$ , for  $1 \leq i \leq n$ . Among adaptive policies, the lowest achievable error in (1) is achieved using the dichotomy policy, consisting in splitting in two intervals of equal size the current interval containing the target. This performance can also be achieved using a non adaptive policy with question sets which are not constrained to be intervals. The dyadic policy, consisting in querying the bits of  $X$  in its expansion

<sup>1</sup>The policy is named Aurelian to acknowledge Aurelien Garivier who provided the intuition for it.

in base 2 is an example of a non-adaptive policy which achieves optimum performance.

Consider now the situation when the answers are noisy according to a memoryless channel and where  $X$  is a sample from a known distribution. The performance is then measured using the mean square error. In this case, as shown in [?],

$$E[(X - \hat{X}_n)^2] \geq A_1 \exp(-A_2 n) \quad A_1, A_2 > 0 \quad (3)$$

where the expectation is taken over  $X$  as well as the noise.  $A_1$  and  $A_2$  are explicit functions of the entropy of the distribution of  $X$  and the characteristics of the noise, respectively. Similarly to the noiseless case, this result is derived by considering the maximum amount of uncertainty reduced in average by the question/answering process. Now, is this lower bound achievable?

The bisection policy consists in choosing the interval  $[0, a_n]$  where  $a_n$  is the median of the posterior distribution of  $X$  after observing the answer of the  $n - 1$  first questions. It is an adaptive policy which achieves the bound in (3) albeit with different constants. A weaker statement of (3) is that the bisection policy achieves

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln E[(X - \hat{X}_n)^2] = -C \quad (4)$$

for some  $C > 0$ . A natural question consists in asking if there exists a non adaptive policy achieving (4) while allowing for querying arbitrary sets, that is, which are not necessarily intervals. The answer to this question is unknown up to our knowledge. In order to progress in answering it, we consider a family of non-adaptive policies. We start with the dyadic policy. Indeed this policy is optimal in the noiseless case. We also know that the dyadic policy is optimal for a different loss function which is the differential entropy of the posterior distribution, see [?] and that the differential entropy of the posterior need to be small for the mean square loss to be small. However, the reciprocal is not true. We extend the dyadic policy as follows: In the case where the prior distribution is Uniform, the dyadic questions correspond to the coefficients of  $X$  in its expansion in base 2. We allow each bit to be queried not only once but an arbitrary number of times under the constraint of a fixed total number of questions. This construction is extended to non Uniform priors by first transforming  $X$  into a Uniform random variable using the transformation  $X \mapsto F(X)$  where  $F$  is the cumulative distribution of  $X$  and then considering the dyadic questions for  $F(X)$ .

## 2. INFORMATION TRANSMISSION EQUIVALENT PROBLEM

The problem of interest can be mapped to the well-know point to point communication system [?] which transmits the location of the

target through a memoryless channel. Consider the transmission of a message  $x$  in the interval  $[0, 1]$  through a binary input, arbitrary output memoryless channel. The transmission aims at minimizing the mean square error between the input message and the decoded message. The analysis is presented in the case where the source is Uniformly distributed. This analysis is then extended to other distributions in Section 5. It is assumed that the source provides the binary representation of the message  $x$ :

$$x = \sum_{k=1}^{\infty} x_k 2^{-k} \quad (5)$$

Since the transmission of the infinite bit sequence  $x_1, x_2, \dots$ , is not feasible in practice, the message should be quantized by selecting a finite number of bits, notate by  $l$ .

The encoding, shown in Figure 1, consists in removing the source redundant information in order to gain compression efficiency. This is achieved by a *quantization* scheme. For the uniform source class, the scalar quantizer where the quantized levels are explicitly represented via a linear combination of the  $x_k$ 's has been proved to be optimal [?, ?] for the mean square error distortion. A source encoder with fixed rate  $l$  is assumed in this paper. The output of the source encoder is the first  $l$  bits of the infinite binary representation of  $x$ . This quantized version is denoted as  $\tilde{X}_l$  in Figure 1. The truncated message is denoted  $\tilde{x}_l = \sum_{k=1}^l x_k 2^{-k}$ .

The channel encoder is designed to add redundancy to the source bit stream to protect the source information from the channel noise. The bits in the binary representation are of different importance due to the different weights in the sum 5. This motivates the coding policy which transmits each bit at a different rate. [?] proposed to transmit each  $l$  bits through a single binary symmetric channel with different transmission rate. This paper considers the general transmission scheme for which each bit is allowed to be transmitted a different number of times while the total number of bits to be transmitted is fixed. In Figure 1, the channel encoder sends the  $n$  bits codeword  $C_n = (x_{i_1}, x_{i_2}, \dots, x_{i_n})$  for which  $1 \leq i_j \leq l$ . We denote  $q = \max_j i_j$  to be the maximum bit index selected for transmission. Note that  $q \leq l$ . For any bit  $k$ , we define

$$t_{n,k} = \sum_{j=1}^n \mathbb{1}_k(i_j) \quad (6)$$

to be the number of times that bit  $k$  is transmitted in the codeword of length  $n$ . A transmission policy is then denoted by a vector  $t_n = (t_{n,1}, t_{n,2}, \dots)$  such that  $t_{n,k} \geq 0$  for any  $k \geq 1$  and  $\sum_{k \geq 1} t_{n,k} = n$ .

The received message  $Y_n = (y_{i_1}, y_{i_2}, \dots, y_{i_n})$  is different from the transmitted message due to the channel noise. Here, we use a *binary input channel* with the following channel posterior probability:

$$P(Y = y|X = x) = \begin{cases} f_1(y) & x = 1 \\ f_0(y) & x = 0 \end{cases} \quad (7)$$

where  $f_0$  and  $f_1$  are point mass functions or densities.

The decoder is designed to minimize the end-to-end mean square error (MSE) between the decoded message  $\hat{X}_n$  and the original message  $X$ , notated as  $E[(\hat{X}_n - X)^2]$ .

The communication system of interest is shown in Figure 1. Here, the source encoder and the channel encoder are two separate entities, but there is a tight connection between them. This type of *joint source-channel* coding has been addressed in [?]. This

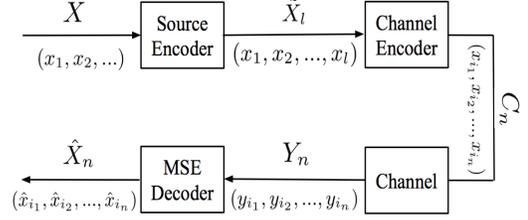


Fig. 1. The general transmission interface of a continuous message.

paper considers only *non-adaptive* transmission policies. The non-adaptive policies are the policies that uses a one way communication between sender and receiver. On the other hand the adaptive policies are corresponding to the system for which there is a noiseless feedback channel which provides sender the results of the previous transmission [?, ?].

### 3. END-TO-END DISTORTION

After transmission of  $n$  bits, the decoded message  $\hat{X}_n$  is different from the source message  $X$  due to the quantization and transmission distortion. The MSE distortion at step  $n$  is defined as:

$$E[(\hat{X}_n - X)^2] = E[E[(\hat{X}_n - X)^2|B_n]] \quad (8)$$

here,  $E[\cdot]$  denotes statistical expectation and  $B_n$  is the history of the transmissions of the previous bits.

The square error distortion of Eq 8 is minimized when  $\hat{X}_n = E[X|B_n]$  which lead to the minimum end-to-end distortion:

$$\begin{aligned} D_n &= E[(E[X|B_n] - X)^2|B_n] \\ &= E[\text{var}(X|B_n)] \end{aligned} \quad (9)$$

For a given history  $B_n$  the transmitted bits  $X_i$  and  $X_j$ ,  $i, j \leq n$  and  $i \neq j$  are independent and the minimum end-to-end distortion is therefore, using (5), simplified as:

$$D_n = \sum_{k \geq 1} 2^{-2k} E[\text{var}(X_k|B_n)] \quad (10)$$

where  $X_k$  is the random variable corresponding to the  $k$ 'th bit of  $X$  in the binary representation of Eq 5. Since each bit  $k$  in the above sum is weighted differently, an unequal bit error protection transmission pattern is desired. The variable  $t_{n,k}$  denotes the number of times that bit  $k$  has been transmitted within the first  $n$  transmissions.

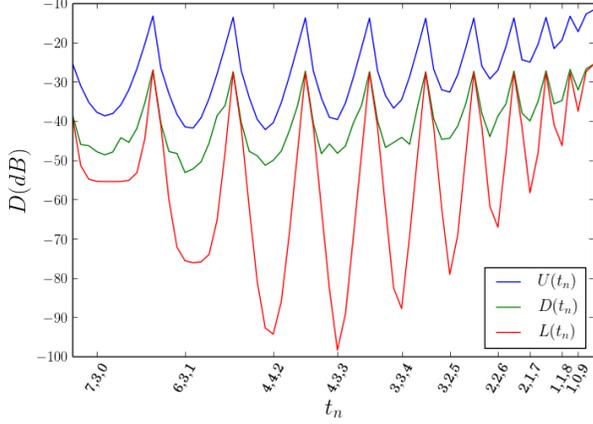
We now provide bounds on the distortion achieved by any such transmission scheme or equivalently on any non adaptive policy using the dyadic questions. The proof involves classical large deviation inequalities and is provided in the Appendix.

**Theorem 1** For any transmission pattern  $t_n$ , the minimum end-to-end distortion  $D_n$  is bounded as follows:

$$L(t_n) \leq D_n \leq U(t_n) \quad (11)$$

where,

$$\begin{aligned} L(t_n) &= \frac{1}{4} \sum_{k \geq 1} 2^{-2k} \exp(-t_{n,k} B(f_1, f_0)) \\ U(t_n) &= \sum_{k \geq 1} 2^{-2k} \exp(-t_{n,k} C(f_1, f_0)) \end{aligned} \quad (12)$$



**Fig. 2.** The distortion is bounded from above and below for all transmission policy  $\bar{t}_n$  for  $n = 10$  and the maximum depth of 3.

$C(f_1, f_0)$  is the chernoff information bound [?] and

$$B(f_1, f_0) = E \left[ \exp \left( - \left| \ln \frac{f_1}{f_0}(Y) \right| \right) \right] \quad (13)$$

Figure 2 provides an illustration of theorem 1. We show the distortion as well as the lower and the upper bound for  $n = 10$  questions and  $l = 3$  bits source. The channel is binary with transitions probabilities  $p_{0 \rightarrow 0} = 0.9, p_{1 \rightarrow 1} = 0.8$ . For this setting,  $C = 0.77$  and  $B = 2.08$ . There are only 66 possible transmissions patterns for which the distortion and the bounds are plotted in the figure. The experiments are done over 100 iterations. These policies are shown in the x axis. The policy  $t$  which minimizes  $D_{10}(t)$  is  $(6, 3, 1)$  consisting in sending the first bit 6 times, the second, 3 times and the third once.

#### 4. AURELIAN CODING SCHEME

The optimal non-adaptive policies are the ones which minimize the end-to-end distortion  $D_n$ . However, it is difficult to find such optimum transmission patterns from Eq 10. Instead, we define the *efficient* policies as follows:

**Definition 1** An *efficient policy* is a policy  $t_n = (t_{n,1}, t_{n,2}, \dots)$  that minimizes the upper bound  $U$  of the end-to-end distortion.

$$\begin{aligned} \underset{t_n}{\operatorname{argmin}} \quad & \sum_{k \geq 1} 2^{-2k} \exp(-t_{n,k} C(f_1, f_0)) \\ \text{subject to} \quad & \sum_{k=1}^{\infty} t_{n,k} = n, t_{n,k} \in \mathbb{N} \cup \{0\} \end{aligned}$$

As Figure 2 shows, the upper distortion bound can have many local minima. Solving such an integer minimization problem might be hard directly. However, we can characterize some properties of an *efficient* transmission policy.

**Lemma 1** For any efficient transmission policy  $t_n$ ,

- there is no gap between bit indices selected for transmission, i.e.  $t_{n,k} \geq 1$  for  $1 \leq k \leq q$ , where  $q$  is the last non-zero transmission bit index.
- for any  $k_1, k_2 \geq 1$ , the difference between transmission values of the corresponding bits is bounded from above and below:

$$(k_2 - k_1)r - 1 \leq t_{n,k_1} - t_{n,k_2} \leq (k_2 - k_1)r + 1 \quad (14)$$

where  $r = \ln(4)/C(f_1, f_0)$ .

These properties are essential to derive the following lower bound on the minimum end-to-end distortion for all efficient policies.

**Theorem 2** The logarithm of the minimum end-to-end distortion of any efficient non-adaptive transmission policy goes to  $-\infty$  with a rate in  $O(-\sqrt{n})$ , more specifically:

$$-A_1 \leq \lim_{n \rightarrow \infty} \frac{\ln(D_n(t_n))}{\sqrt{n}}$$

for each sequence  $\{t_n\}_{n \geq 1}$  of efficient policies, where

$$A_1 = \min \left( \sqrt{2} \left( \frac{\ln(4)}{C(f_1, f_0)} + 1 \right) B(f_1, f_0), \ln(4) \right)$$

Even though we derived some *necessary* important properties for policies that achieve the infimum of the upper bound, we did not provide an explicit efficient non-adaptive transmission policy. Our strategy is the following: We propose a policy, called the *Aurelian* policy, derived from the minimization of (12) for sequences of real numbers  $t_n$  (Of course we then adapt the  $t$ 's such that we obtain a sequence of integers).

**Definition 2 Aurelian Policy:**

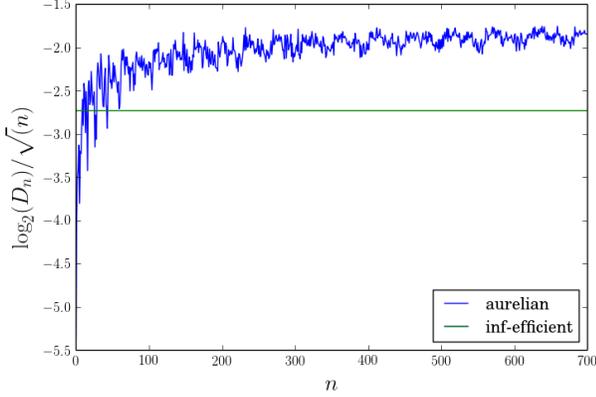
$$\begin{aligned} q &= \left\lfloor \sqrt{\frac{2n}{r} + \frac{1}{4} - \frac{1}{2}} \right\rfloor \\ t_{n,1} &= qr \\ t_{n,k} &= t_{n,1} - (k-1)r \\ r &= \lfloor \ln(4)/C(f_1, f_0) \rfloor \end{aligned} \quad (15)$$

At this stage, even if the choice of the *Aurelian* policy is based on some "good" intuitive reasons (the minimization of the continuous problem), we do not have any insurance on the rate of convergence to 0 of the distortion rate when this policy is used, compared to the rate of convergence of an efficient non-adaptive transmission policy. Theorem 2 address this this problem. First of all, we will derive a upper bound for the convergence rate to  $-\infty$  of the distortion rate of the *Aurelian* policy. Having this rate, we will show that it is comparable to the distortion rate of an efficient non-adaptive transmission policy. In other words, we can conclude that we don't loose "too much" by following an *Aurelian* policy instead of deriving an explicit non-adaptive transmission policy.

**Theorem 3** The logarithmic rate of convergence of the *Aurelian* policy is no more than  $O(-\sqrt{n})$ . More precisely,

$$\limsup_{n \rightarrow \infty} \frac{\ln(D_n)}{\sqrt{n}} \leq -A_2 \quad (16)$$

where  $A_2 = \sqrt{2r}C(f_1, f_0)$ .



**Fig. 3.** The normalized distortion versus the number of transmissions as well as the lower bound, i.e the constant  $-A_1$  provided in Theorem 2.

And finally,

**Corollary 1** *If we denote by  $D_n$  the distortion rate of the Aurelian policy, and  $D_n^*$  the distortion rate of any efficient non-adaptive transmission policy, then we have:*

$$\liminf_{n \rightarrow \infty} \frac{\ln(D_n)}{\ln(D_n^*)} \geq \frac{A_1}{A_2} > 0 \quad (17)$$

for  $A_1$  and  $A_2$  given in Theorem 2 and Theorem 3, respectively.

The blue curve in Figure 3 shows the asymptotic behavior of the aurelian policy. As this figure shows, after about 300 transmissions, the receiver is able to decode the message with very high confidence. It is also worth to mention that the rate of convergence is of order  $1/\sqrt{n}$ .

## 5. NON-UNIFORM DISTRIBUTION

Note that the lower bound of Theorem 2 generalizes to random variables for which the cumulative distribution is Lipschitz continuous. Indeed in this case, by definition, there exists a constant  $k > 0$  such that

$$(F(u_1) - F(u_2))^2 \leq k(u_1 - u_2)^2 \quad (18)$$

The cumulative distribution of any random variable is a uniform random variable in interval  $[0, 1]$ . Instead of locating target  $X$ , we can search for the target  $F(X)$ . Lets  $F_n$  be the estimated location after  $n$  steps and let  $\hat{X}_n = F^{-1}(F_n)$ .

$$(F_n - F(X))^2 \leq k(\hat{X}_n - X)^2$$

The lower bound of the distortion is then derived using Theorem 2.

## 6. CONCLUSION

The problem of noisy target localization under mean square distortion is addressed through an iterative binary question/answering process. Among non-adaptive policies, the policies involving dyadic questions were considered. It was shown that the logarithm of the mean square distortion is  $O(-\sqrt{n})$  for  $n$  large enough. An explicit policy achieving this rate was exhibited. We finally conjecture that this policy is optimal among all non-adaptive policies.

## 7. APPENDIX

### 7.1. Proof of Theorem 1:

#### 7.1.1. Upper bound

*Proof:* We first prove the following inequality:

$$E[\text{var}(X_k|B_n)] \leq \exp(-t_{n,k}C(f_1, f_0)) \quad (19)$$

the upper bound is then derived by considering the fact that for any random variable  $A$  and scalar  $b$ , if  $A \leq b$ , then  $E[A] \leq b$ . Note:

$$\begin{aligned} \text{Var}[X_k|B_n] &= P(X_k = 1|B_n)(1 - P(X_k = 1|B_n)) \\ &\leq \min(P(X_k = 0|B_n), P(X_k = 1|B_n)) \\ &= P(\tilde{X}_k \neq X_k|B_n) \end{aligned}$$

where  $\tilde{X}_k$  is the mode of  $p(X_k|B_n)$ . The above equation can be simplified further as:

$$\begin{aligned} P(\tilde{X}_k \neq X_k|B_n) &= P(\tilde{x}_k = 1|B_n, X_k = 0)P(X_k = 0) \\ &\quad + P(\tilde{X}_k = 0|B_n, X_k = 1)P(X_k = 1) \end{aligned}$$

where,

$$\begin{aligned} P(\tilde{X}_k = 1|B_n, X_k = 0) &= P\left(\sum_{j=1}^n \mathbb{1}_k(i_j) \log\left(\frac{f_1}{f_0}(y_{i_j}) > 0\right)\right) \\ &= P\left(\exp\left(s \sum_{j=1}^n \mathbb{1}_k(i_j) \log\left(\frac{f_1}{f_0}(y_{i_j})\right)\right) > 1\right) \\ &\leq E\left[\exp\left(s \sum_{j=1}^n \mathbb{1}_k(i_j) \log\left(\frac{f_1}{f_0}(y_{i_j})\right)\right)\right] \\ &= \exp(-t_{n,k}C(f_1, f_0)) \end{aligned}$$

here,  $\mathbb{1}_k(i_j)$  is the indicator of sending bit  $k$  at transmission step  $i_j$ . The second equality is hold for any positive value  $s$ . The third inequality is derived using Markov inequality and the final equality holds based on the fact that  $\sum_{j=1}^n \mathbb{1}_k(i_j) = t_{n,k}$  by definition. The fact that each transmission is independent of the rest and have same i.i.d distribution is also used for deriving the expectation of the last inequality.

#### 7.1.2. Lower bound

Next, Lets define the posterior function  $p_{n,k} = P(X_k = 1|B_n)$ .

**Lemma 2** *The posterior density on bit  $k$  at step  $n$  is  $p_{n,k} = p_{n-1,k}$  whenever  $\mathbb{1}_k(i_n) = 0$ . Otherwise the posterior is a function of the posterior at previous transmission step:*

$$p_{n,k} = \frac{f_1(y_{i_n})p_{n-1,k}}{f_1(y_{i_n})p_{n-1,k} + f_0(y_{i_n})(1 - p_{n-1,k})} \quad (20)$$

*Prrof:* First note that since bits are independent random variables, sending a bit different from  $k$  will not change the posterior at step  $n$ , thus  $p_{n,k} = p_{n-1,k}$  when  $i_n \neq k$ . In the other hand, whenever the bit  $k$  was sent through the channel at the  $n$ 'th transmission:

$$\begin{aligned} p_{n,k} &= P(X_k = 1|B_n) \\ &= \frac{P(y_{i_n}|X_k = 1, B_{n-1})P(X_k = 1|B_{n-1})}{P(y_{i_n}|B_{n-1})} \end{aligned}$$

where,

$$\begin{aligned}
P(y_{i_n}|B_{n-1}) &= \sum_{u \in \{0,1\}} P(y_{i_n}, X_k = u|B_{n-1}) \\
&= P(y_{i_n}|X_k = 1, B_{n-1})P(X_k = 1|B_{n-1}) \\
&+ P(y_{i_n}|X_k = 0, B_{n-1})P(X_k = 0|B_{n-1}) \\
&= f_1(y_{i_n})p_{n-1,k} + f_0(y_{i_n})(1 - p_{n-1,k}) \quad (21)
\end{aligned}$$

**Lemma 3** If  $i_n = k$

$$p_{n,k}(1 - p_{n,k}) \geq p_{n-1,k}(1 - p_{n-1,k})e^{-\mathbb{1}_k(i_n)|\log(\frac{f_1}{f_0}(y_{i_n}))} \quad (22)$$

*Proof:* For  $i_n = k$ .

$$\begin{aligned}
p_{n,k}(1 - p_{n,k}) &= \frac{f_1(y_{i_n})f_0(y_{i_n})p_{n-1,k}(1 - p_{n-1,k})}{[f_1(y_{i_n})p_{n-1,k} + f_0(y_{i_n})(1 - p_{n-1,k})]^2} \\
&\geq \frac{f_1(y_{i_n})f_0(y_{i_n})p_{n-1,k}(1 - p_{n-1,k})}{f_1(y_{i_n})\mathbb{1}_{\frac{f_1(y_{i_n})}{f_0(y_{i_n})} \geq 1} + f_0(y_{i_n})\mathbb{1}_{\frac{f_0(y_{i_n})}{f_1(y_{i_n})} \geq 1}} \\
&= \frac{p_{n-1,k}(1 - p_{n-1,k})}{\frac{f_0(y_{i_n})}{f_1(y_{i_n})}\mathbb{1}_{\frac{f_1(y_{i_n})}{f_0(y_{i_n})} \geq 1} + \frac{f_1(y_{i_n})}{f_0(y_{i_n})}\mathbb{1}_{\frac{f_0(y_{i_n})}{f_1(y_{i_n})} \geq 1}} \\
&= p_{n-1,k}(1 - p_{n-1,k}) \exp(-|\log \frac{f_1}{f_0}(y_{i_n})|)
\end{aligned}$$

**Corollary 2** After transmission of  $n$  bits,

$$p_{n,k}(1 - p_{n,k}) \geq p_{0,k}(1 - p_{0,k}) \exp(-t_{n,k}B(f_1, f_0)) \quad (23)$$

where,  $B(f_1, f_0) = E[|\log(\frac{f_1(y)}{f_0(y)})|]$ .

*Proof:* Using Lemma 3 recursively:

$$\begin{aligned}
p_{n,k}(1 - p_{n,k}) &\geq p_{0,k}(1 - p_{0,k}) \prod_{j=1}^n \exp(-\mathbb{1}_k(i_j)|\log(\frac{f_1(y_{i_j})}{f_0(y_{i_j})})|) \\
&= p_{0,k}(1 - p_{0,k}) \exp(-\sum_{j=1}^n \mathbb{1}_k(i_j)|\log(\frac{f_1(y_{i_j})}{f_0(y_{i_j})})|)
\end{aligned}$$

we claim  $U(t'_n) \leq U(t_n)$ :

$$\begin{aligned}
U(t'_n) - U(t_n) &= \sum_{k \geq 1} 2^{-2k} (\exp(t'_{n,k}C) - \exp(t_{n,k}C)) \\
&= 2^{-2k_1} (e^{-t_{n,k_2}C} - 1) + 2^{-2k_2} (1 - e^{-t_{n,k_2}C}) \\
&= (e^{-t_{n,k_2}C} - 1)(2^{-2k_1} - 2^{-2k_2}) \\
&\leq 0
\end{aligned}$$

Since  $0 \leq C \leq 1$  and  $k_1 < k_2$ .

Let us call the transmission policy  $t'_n$  derivable from  $t_n$  by a  $(k_1, k_2)$ -move when

$$t'_{k,n} = \begin{cases} t_{k,n} + 1 & \text{if } k = k_1 \\ t_{k,n} - 1 & \text{if } k = k_2 \\ t_{k,n} & \text{Otherwise} \end{cases}$$

here it is assumed that  $t_{n,k_2} \geq 1$ , otherwise the above move can not be defined. An optimal transmission policy is such that no further  $(k_1, k_2)$ -move can lead to a transmission policy with lower upper bound distortion. This is the key idea to prove the following lemma: **Property 2:** Lets  $t_n$  be a coding policy and  $t'_n$  be the policy derived by a  $(k_1, k_2)$ -move between two bit indices  $k_1$  and  $k_2$  such that  $t_{n,k_2} \geq 1$ . This move reduce the upper bound values if  $U(t_n) \geq U(t'_n)$ , simplification of both side prove the following lower bound inequality

$$(k_2 - k_1)r - 1 \leq t_{n,k_1} - t_{n,k_2}$$

Similarly, the other bound can be derived by considering situations that a  $(k_2, k_1)$ -move can reduce the upper bound.

**Corollary 3** Using the above properties, we can derive the following bounds for the last transmitted bit index  $q$  and the transmission number of first bit,  $t_{n,1}$ :

$$\begin{aligned}
t_{n,1} &\leq q(r + 1) \\
q &\leq \sqrt{2n + 1}/2 - 1/4
\end{aligned}$$

The proof is very straightforward using the properties and note  $\sum_{k=1}^q t_{n,k} = n$ .

Taking expectation from both side and using Jensen inequality:

$$\begin{aligned}
E[\text{var}(X_k|B_n)] &= E[p_{n,k}(1 - p_{n,k})] \\
&\geq p_{0,k}(1 - p_{0,k}) \exp(-t_{n,k}E[|\log(\frac{f_1(y)}{f_0(y)})|])
\end{aligned}$$

where the last equation derived based on the fact that there is only  $t_{n,k}$  steps that bit  $k$  has been sent and these steps are i.i.d.

## 7.2. Optimal Properties

**Property 1:** Lets assume there is a bit  $k_1$  such that  $t_{n,k_1} = 0$  and let  $k_2$  be the first index greater  $k_1$  such that  $t_{n,k_2} > 0$ . Lets  $t'_n$  be a new policy made from the earlier policy  $t_n$ , such that

$$t'_{n,k} = \begin{cases} t_{n,k_2} & k = k_1 \\ 0 & k = k_2 \\ t_{n,k} & k \notin \{k_1, k_2\} \end{cases} \quad (24)$$

## 7.3. Asymptotic Behavior of Infimum of the Efficient Policies

Lets  $t_n$  be an efficient policy. For such policy

$$\begin{aligned}
L(t_n) &= \frac{1}{4} [\sum_{k \geq 1} 2^{-2k} \exp(-t_{n,k}B)] \\
&= \frac{1}{4} [\sum_{k=1}^q \exp(\underbrace{(-k \log(4) - t_{n,k}B)}_{v_k}) + \sum_{k > q} 2^{-2k}] \\
&= \frac{1}{4} [e^{v_1} \sum_{k=1}^q \exp(v_k - v_1) + 2^{-2(q+1)} \frac{4}{3}] \\
&\geq \frac{1}{4} [e^{v_1 - B} \frac{e^{-q\alpha} - 1}{e^{-\alpha}} + \frac{1}{3} e^{-q \log(4)}] \quad (25)
\end{aligned}$$

where  $\alpha = \log(4)(1 + \frac{B}{C}) > \log(4)$ , since  $B, C \geq 0$ . Using the Corollary 3:

$$v_1 \geq \log(4) - q(\frac{\log(4)}{C} + 1)B \quad (26)$$

let  $n \rightarrow \infty$  and using the upper bound of  $q$ ,

$$\lim_{n \rightarrow \infty} \inf_{t_n \in \mathbb{T}^*} \frac{\log(L(t_n))}{\sqrt{n}} \geq -A_1 \quad (27)$$

where  $A_1 = \min(\sqrt{2}(\frac{\log(4)}{C} + 1)B, \log(4))$  which complete the proof.

#### 7.4. Asymptotic Behavior of Supremum of the Aurelian Policies

For an Aurelian policy  $t_n$ : Lets  $t_n$  be an efficient policy. For such policy

$$\begin{aligned} U(t_n) &= \sum_{k \geq 1} 2^{-2k} \exp(-t_{n,k}C) \\ &= \sum_{k=1}^q \exp((-k \log(4) - t_{n,k}C)) + \sum_{k>q} 2^{-2k} \\ &= \sum_{k=1}^q \exp(-\log(4) - t_{n,1}C) + 2^{-2(q+1)} \frac{4}{3} \\ &= q \exp(-\log(4) - qrC) + \frac{1}{3} \exp(-q \log(4)) \end{aligned} \quad (28)$$

For  $n \rightarrow \infty$ , since  $rC \leq \log(4)$ :

$$\lim_{n \rightarrow \infty} \sup_{t_n \in \mathbb{T}^*} \frac{\log(U(t_n))}{\sqrt{n}} \leq -A_2 \quad (29)$$

where  $A_2 = \sqrt{(2r)C}$ .