# Exponent Function for One Helper Source Coding Problem at Rates outside the Rate Region 

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#### Abstract

We consider the one helper source coding problem posed and investigated by Ahlswede, Körner and Wyner. Two correlated sources are separately encoded and are sent to a destination where the decoder wishes to decode one of the two sources with an arbitrary small error probability of decoding. In this system, the error probability of decoding goes to one as the source block length $n$ goes to infinity. This implies that we have a strong converse theorem for the one helper source coding problem. In this paper we provide the much stronger version of this strong converse theorem for the one helper source coding problem. We prove that the error probability of decoding tends to one exponentially and derive an explicit lower bound of this exponent function.


Index Terms-One helper source coding problem, strong converse theorem, exponent of correct probability of decoding

## I. Introduction

We consider the one helper source coding problem posed and investigated by Ahlswede, Körner and Wyner. Two correlated sources are separately encoded and are sent to a destination where the decoder wishes to decode one of the two sources with an arbitrary small error probability of decoding. In this system, the error probability of decoding goes to one as the source block length $n$ goes to infinity. This implies that we have a strong converse theorem for the one helper source coding problem. In this paper we provide the much stronger version of this strong converse theorem for the one helper source coding problem. We prove that the error probability of decoding tends to one exponentially and derive an explicit lower bound of this exponent function.

## II. Problem Formulation

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets and $\left\{\left(X_{t}, Y_{t}\right)\right\}_{t=1}^{\infty}$ be a stationary discrete memoryless source. For each $t=1,2, \cdots$, the random pair $\left(X_{t}, Y_{t}\right)$ takes values in $\mathcal{X} \times \mathcal{Y}$, and has a probability distribution

$$
p_{X Y}=\left\{p_{X Y}(x, y)\right\}_{(x, y) \in \mathcal{X} \times \mathcal{Y}}
$$

We write $n$ independent copies of $\left\{X_{t}\right\}_{t=1}^{\infty}$ and $\left\{Y_{t}\right\}_{t=1}^{\infty}$, respectively as

$$
X^{n}=X_{1}, X_{2}, \cdots, X_{n} \text { and } Y^{n}=Y_{1}, Y_{2}, \cdots, Y_{n} .
$$

We consider a communication system depicted in Fig. 1. Data sequences $X^{n}$ and $Y^{n}$ are separately encoded to $\varphi_{1}^{(n)}\left(X^{n}\right)$ and $\varphi_{2}^{(n)}\left(Y^{n}\right)$ and those are sent to the information processing center. At the center the decoder function $\psi^{(n)}$ observes
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Fig. 1. One helper source coding system.
$\left(\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{2}^{(n)}\left(Y^{n}\right)\right)$ to output the estimation $\hat{Y}^{n}$ of $Y^{n}$. The encoder functions $\varphi_{1}^{(n)}$ and $\varphi_{2}^{(n)}$ are defined by

$$
\left.\begin{array}{l}
\varphi_{1}^{(n)}: \mathcal{X}^{n} \rightarrow \mathcal{M}_{1}=\left\{1,2, \cdots, M_{1}\right\},  \tag{1}\\
\varphi_{2}^{(n)}: \mathcal{Y}^{n} \rightarrow \mathcal{M}_{2}=\left\{1,2, \cdots, M_{2}\right\},
\end{array}\right\}
$$

where for each $i=1,2,\left\|\varphi_{i}^{(n)}\right\|\left(=M_{i}\right)$ stands for the range of cardinality of $\varphi_{i}^{(n)}$. The decoder function $\psi^{(n)}$ is defined by

$$
\begin{equation*}
\psi^{(n)}: \mathcal{M}_{1} \times \mathcal{M}_{2} \rightarrow \mathcal{Y}^{n} \tag{2}
\end{equation*}
$$

The error probability of decoding is

$$
\begin{equation*}
\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)=\operatorname{Pr}\left\{\hat{Y}^{n} \neq Y^{n}\right\}, \tag{3}
\end{equation*}
$$

where $\hat{Y}^{n}=\psi^{(n)}\left(\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{2}^{(n)}\left(Y^{n}\right)\right)$. A rate pair $\left(R_{1}, R_{2}\right)$ is $\varepsilon$-achievable if for any $\delta>0$, there exist a positive integer $n_{0}=n_{0}(\varepsilon, \delta)$ and a sequence of triples $\left\{\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}\right.\right.$, $\left.\left.\psi^{(n)}\right)\right\}_{n \geq n_{0}}$ such that for $n \geq n_{0}$,

$$
\begin{aligned}
& \frac{1}{n} \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}+\delta \text { for } i=1,2 \\
& \mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq \varepsilon
\end{aligned}
$$

For $\varepsilon \in(0,1)$, the rate region $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ is defined by

$$
\begin{aligned}
& \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right) \\
& :=\left\{\left(R_{1}, R_{2}\right):\left(R_{1}, R_{2}\right) \text { is } \varepsilon \text {-achievable for } p_{X Y}\right\} .
\end{aligned}
$$

Furthermore, define

$$
\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right):=\bigcap_{\varepsilon \in(0,1)} \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)
$$

We can show that the two rate regions $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right), \varepsilon \in$ $(0,1)$ and $\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)$ satisfy the following property.
Property 1:
a) The regions $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right), \varepsilon \in(0,1)$, and $\mathcal{R}_{\mathrm{AKW}}($ $\left.p_{X Y}\right)$ are closed convex sets of $\mathbb{R}_{+}^{2}$, where

$$
\mathbb{R}_{+}^{2}:=\left\{\left(R_{1}, R_{2}\right): R_{1} \geq 0, R_{2} \geq 0\right\}
$$

b) $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ has another form using $(n, \varepsilon)$-rate region $\mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)$, the definition of which is as follows. We set

$$
\begin{aligned}
& \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)=\left\{\left(R_{1}, R_{2}\right):\right. \\
& \text { There exists }\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \text { such that } \\
& \frac{1}{n} \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2, \\
& \left.\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq \varepsilon\right\} .
\end{aligned}
$$

Using $\mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right), \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ can be expressed as

$$
\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)=\mathrm{cl}\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)\right) .
$$

Proof of this property is given in Appendix A
It is well known that $\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)$ was determined by Ahlswede, Körner and Wyner. To describe their result we introduce an auxiliary random variable $U$ taking values in a finite set $\mathcal{U}$. We assume that the joint distribution of $(U, X, Y)$ is

$$
p_{U X Y}(u, x, y)=p_{U}(u) p_{X \mid U}(x \mid u) p_{Y \mid X}(y \mid x)
$$

The above condition is equivalent to $U \leftrightarrow X \leftrightarrow Y$. Define the set of probability distribution $p=p_{U X Y}$ by

$$
\mathcal{P}\left(p_{X Y}\right):=\left\{p_{U X Y}:|\mathcal{U}| \leq|\mathcal{X}|+1, U \leftrightarrow X \leftrightarrow Y\right\}
$$

Set

$$
\begin{aligned}
\mathcal{R}(p):= & \left\{\left(R_{1}, R_{2}\right): R_{1}, R_{2} \geq 0\right. \\
& \left.R_{1} \geq I_{p}(X ; U), R_{2} \geq H_{p}(Y \mid U)\right\} \\
\mathcal{R}\left(p_{X Y}\right):= & \bigcup_{p \in \mathcal{P}\left(p_{X Y}\right)} \mathcal{R}(p) .
\end{aligned}
$$

We can show that the region $\mathcal{R}\left(p_{X Y}\right)$ satisfies the following property.

## Property 2:

a) The region $\mathcal{R}\left(p_{X Y}\right)$ is a closed convex subset of $\mathbb{R}_{+}^{2}$.
b) For any $p_{X Y}$, we have

$$
\begin{equation*}
\min _{\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)}\left(R_{1}+R_{2}\right)=H_{p}(Y) \tag{4}
\end{equation*}
$$

The minimun is attained by $\left(R_{1}, R_{2}\right)=\left(0, H_{p}(Y)\right)$. This result implies that

$$
\mathcal{R}\left(p_{X Y}\right) \subseteq\left\{\left(R_{1}, R_{2}\right): R_{1}+R_{2} \geq H_{p}(Y)\right\} \cap \mathbb{R}_{+}^{2}
$$

Furthermore, the point $\left(0, H_{p}(Y)\right)$ always belongs to $\mathcal{R}\left(p_{X Y}\right)$.
Property 2 part a) is a well known property. Proof of Property 2 part b) is easy. Proofs of Property 2 parts a) and b) are omitted. A typical shape of the rate region $\mathcal{R}\left(p_{X Y}\right)$ is shown in Fig. 2

The rate region $\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)$ was determined by Ahlswede and Körner [1] and Wyner [2]. Their result is the following.

Theorem 1 (Ahlswede, Körner [1] and Wyner [2]):

$$
\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)=\mathcal{R}\left(p_{X Y}\right)
$$



Fig. 2. A typical shape of $\mathcal{R}\left(p_{X Y}\right)$.

On the converse coding theorem Ahlswede et al. [3] obtained the following.

Theorem 2 (Ahlswede et al. [3]): For each fixed $\varepsilon \in(0,1)$, we have

$$
\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)=\mathcal{R}\left(p_{X Y}\right)
$$

Gu and Effors [5] examined a speed of convergence for $\mathrm{P}_{\mathrm{e}}^{(n)}$ to tend to 1 as $n \rightarrow \infty$ by carefully checking the proof of Ahlswede et al. [3]. However they could not obtain a result on an explicit form of the exponent function with respect to the code length $n$.

Our aim is to find an explicit form of the exponent function for the error probability of decoding to tend to one as $n \rightarrow \infty$ when $\left(R_{1}, R_{2}\right) \notin \mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)$. To examine this quantity, we define the following quantity. Set

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right):=1-\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& :=\quad \min _{\substack{\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right):}}^{(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2} \\
& \quad\left(-\frac{1}{n}\right) \log \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& G\left(R_{1}, R_{2} \mid p_{X Y}\right):=\lim _{n \rightarrow \infty} G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& \mathcal{G}\left(p_{X Y}\right):=\left\{\left(R_{1}, R_{2}, G\right): G \geq G\left(R_{1}, R_{2} \mid p_{X Y}\right)\right\}
\end{aligned}
$$

By time sharing we have that

$$
\begin{align*}
& G^{(n+m)}\left(\frac{n R_{1}+m R_{1}^{\prime}}{n+m}, \left.\frac{n R_{2}+m R_{2}^{\prime}}{n+m} \right\rvert\, p_{X Y}\right) \\
& \leq \frac{n G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)+m G^{(m)}\left(R_{1}^{\prime}, R_{2}^{\prime} \mid p_{X Y}\right)}{n+m} \tag{5}
\end{align*}
$$

Choosing $R=R^{\prime}$ in (5), we obtain the following subadditivity property on $\left\{G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)\right\}_{n \geq 1}$ :

$$
\begin{aligned}
& G^{(n+m)}\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& \leq \frac{n G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)+m G^{(m)}\left(R_{1}, R_{2} \mid p_{X Y}\right)}{n+m}
\end{aligned}
$$

from which we have that $G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)$ exists and satisfies the following:

$$
\lim _{n \rightarrow \infty} G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)=\inf _{n \geq 1} G^{(n)}\left(R_{1}, R_{2} \mid p_{X Y}\right)
$$

The exponent function $G\left(R_{1}, R_{2} \mid p_{X Y}\right)$ is a convex function of $\left(R_{1}, R_{2}\right)$. In fact, from (5), we have that for any $\alpha \in[0,1]$

$$
\begin{aligned}
& G\left(\alpha R_{1}+\bar{\alpha} R_{1}^{\prime}, \alpha R_{2}+\bar{\alpha} R_{2}^{\prime} \mid p_{X Y}\right) \\
& \leq \alpha G\left(R_{1}, R_{2} \mid p_{X Y}\right)+\bar{\alpha} G\left(R_{1}^{\prime}, R_{2}^{\prime} \mid p_{X Y}\right)
\end{aligned}
$$

The region $\mathcal{G}\left(p_{X Y}\right)$ is also a closed convex set. Our main aim is to find an explicit characterization of $\mathcal{G}\left(p_{X Y}\right)$. In this paper we derive an explicit outer bound of $\mathcal{G}\left(p_{X Y}\right)$ whose section by the plane $G=0$ coincides with $\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)$.

## III. Main Result

In this section we state our main result. We first explain that the region $\mathcal{R}\left(p_{X Y}\right)$ can be expressed with a family of supporting hyperplanes. To describe this result we define a set of probability distributions on $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$ by

$$
\mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right):=\left\{p=p_{U X Y}:|\mathcal{U}| \leq|\mathcal{X}|, U \leftrightarrow X \leftrightarrow Y\right\} .
$$

For $\mu \geq 0$, define

$$
R^{(\mu)}\left(p_{X Y}\right):=\min _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)}\left\{\mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)\right\}
$$

Furthermore, define

$$
\begin{aligned}
& \mathcal{R}_{\mathrm{sh}}\left(p_{X Y}\right):=\bigcap_{\mu \in[0,1]}\left\{\left(R_{1}, R_{2}\right):\right. \mu R_{1}+\bar{\mu} R_{2} \\
&\left.\geq R^{(\mu)}\left(p_{X Y}\right)\right\}
\end{aligned}
$$

Then we have the following property.
Property 3:
a) The bound $|\mathcal{U}| \leq|\mathcal{X}|$ is sufficient to describe $R^{(\mu)}($ $\left.p_{X Y}\right)$.
b) For every $\mu \in[0,1]$, we have

$$
\begin{equation*}
\min _{\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)}\left\{\mu R_{1}+\bar{\mu} R_{2}\right\}=R^{(\mu)}\left(p_{X Y}\right) \tag{6}
\end{equation*}
$$

c) For any $p_{X Y}$ we have

$$
\begin{equation*}
\mathcal{R}_{\mathrm{sh}}\left(p_{X Y}\right)=\mathcal{R}\left(p_{X Y}\right) \tag{7}
\end{equation*}
$$

Property 3 part a) is stated as Lemma 8 in Appendix B Proof of this lemma is given in this appendix. Proofs of Property 3 parts b) and c) are given in Appendix C Set

$$
\begin{aligned}
& \mathcal{Q}\left(p_{Y \mid X}\right):=\left\{q=q_{U X Y}:|\mathcal{U}| \leq|\mathcal{X}|, U \leftrightarrow X \leftrightarrow Y,\right. \\
& \left.p_{Y \mid X}=q_{Y \mid X}\right\}
\end{aligned}
$$

For $(\mu, \alpha) \in[0,1]^{2}$, and for $q=q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$, define

$$
\begin{aligned}
& \omega_{q \mid p_{X}}^{(\mu, \alpha)}(x, y \mid u) \\
& :=\bar{\alpha} \log \frac{q_{X}(x)}{p_{X}(x)}+\alpha\left[\mu \log \frac{q_{X \mid U}(x \mid u)}{p_{X}(x)}+\bar{\mu} \log \frac{1}{q_{Y \mid U}(y \mid u)}\right], \\
& f_{q \mid p_{X}}^{(\mu, \alpha)}(x, y \mid u):=\exp \left\{-\omega_{q \mid p_{X}}^{(\mu, \alpha)}(x, y \mid u)\right\}, \\
& \Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right):=-\log \mathrm{E}_{q}\left[\exp \left\{-\omega_{q \mid p_{X}}^{(\mu, \alpha)}(X, Y \mid U)\right\}\right], \\
& \Omega^{(\mu, \alpha)}\left(p_{X Y}\right):=\min _{q \in \mathcal{Q}\left(p_{Y \mid X}\right)} \Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right), \\
& F^{(\mu, \alpha)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) \\
& :=\frac{\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}}, \\
& F\left(R_{1}, R_{2} \mid p_{X Y}\right):=\sup _{(\mu, \alpha) \in[0,1]^{2}} F^{(\mu, \alpha)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) .
\end{aligned}
$$

We next define a function serving as a lower bound of $F\left(R_{1}, R_{2} \mid p_{X Y}\right)$. For $\lambda \geq 0$ and for $p_{U X Y} \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)$, define

$$
\begin{aligned}
& \tilde{\omega}_{p}^{(\mu)}(x, y \mid u):=\mu \log \frac{p_{X \mid U}(x \mid u)}{p_{X}(x)}+\bar{\mu} \log \frac{1}{p_{Y \mid U}(Y \mid U)} \\
& \tilde{\Omega}^{(\mu, \lambda)}(p):=-\log \mathrm{E}_{p}\left[\exp \left\{-\lambda \tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right\}\right]
\end{aligned}
$$

Furthermore, set

$$
\begin{aligned}
& \tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right):=\min _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)} \tilde{\Omega}^{(\mu, \lambda)}(p) \\
& \underline{F}^{(\mu, \lambda)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) \\
& :=\frac{\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)-\lambda\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\lambda(5-\mu)} \\
& \underline{F}\left(R_{1}, R_{2} \mid p_{X Y}\right):=\sup _{\lambda \geq 0, \mu \in[0,1]} \underline{F}^{(\mu, \lambda)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right)
\end{aligned}
$$

We can show that the above functions satisfy the following property.

Property 4:
a) The cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{Q}\left(p_{Y \mid X}\right)$ is sufficient to describe the quantity $\Omega^{(\mu, \beta, \alpha)}\left(p_{X Y}\right)$. Furthermore, the cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)$ is sufficient to describe the quantity $\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)$.
b) For any $R_{1}, R_{2} \geq 0$, we have

$$
F\left(R_{1}, R_{2} \mid p_{X Y}\right) \geq \underline{F}\left(R_{1}, R_{2} \mid p_{X Y}\right)
$$

c) For any $p=p_{U X Y} \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ and any $(\mu, \lambda) \in[0$, $1]^{2}$, we have

$$
\begin{equation*}
0 \leq \tilde{\Omega}^{(\mu, \lambda)}(p) \leq \mu \log |\mathcal{X}|+\bar{\mu} \log |\mathcal{Y}| \tag{8}
\end{equation*}
$$

d) Fix any $p=p_{U X Y} \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ and $\mu \in[0,1]$. For $\lambda \in$ $[0,1]$, we define a probability distribution $p^{(\lambda)}=p_{U X Y}^{(\lambda)}$ by

$$
p^{(\lambda)}(u, x, y):=\frac{p(u, x, y) \exp \left\{-\lambda \tilde{\omega}_{p}^{(\mu)}(x, y \mid u)\right\}}{\mathrm{E}_{p}\left[\exp \left\{-\lambda \tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right\}\right]}
$$

Then for $\lambda \in[0,1 / 2], \tilde{\Omega}^{(\mu, \lambda)}(p)$ is twice differentiable. Furthermore, for $\lambda \in[0,1 / 2]$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \lambda} \tilde{\Omega}^{(\mu, \lambda)}(p) & =\mathrm{E}_{p(\lambda)}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right] \\
\frac{\mathrm{d}^{2}}{\mathrm{~d} \lambda^{2}} \tilde{\Omega}^{(\mu, \lambda)}(p) & =-\operatorname{Var}_{p^{(\lambda)}}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right]
\end{aligned}
$$

The second equality implies that $\tilde{\Omega}^{(\mu, \lambda)}\left(p \mid p_{X Y}\right)$ is a concave function of $\lambda \in[0,1 / 2]$.
e) For every $(\mu, \lambda) \in[0,1] \times[0,1 / 2]$, define

$$
\begin{aligned}
& \rho^{(\mu, \lambda)}\left(p_{X Y}\right) \\
&:= \max _{\substack{(\nu, p) \in[0, \lambda] \\
\\
\times \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right):}}^{\substack{\tilde{\Omega}^{(\mu, \lambda)}(p) \\
\\
=\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)}} \operatorname{Var}_{p^{(\nu)}}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right],
\end{aligned}
$$

and set

$$
\rho=\rho\left(p_{X Y}\right):=\max _{(\mu, \lambda) \in[0,1] \times[0,1 / 2]} \rho^{(\mu, \lambda)}\left(p_{X Y}\right)
$$

Then, we have $\rho\left(p_{X Y}\right)<\infty$. Furthermore, for any $(\mu, \lambda) \in[0,1] \times[0,1 / 2]$, we have

$$
\begin{equation*}
\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right) \geq \lambda R^{(\mu)}\left(p_{X Y}\right)-\frac{\lambda^{2}}{2} \rho\left(p_{X Y}\right) \tag{9}
\end{equation*}
$$

f) For every $\tau \in\left(0,(1 / 2) \rho\left(p_{X Y}\right)\right)$, the condition $\left(R_{1}+\tau\right.$, $\left.R_{2}+\tau\right) \notin \mathcal{R}\left(p_{X Y}\right)$ implies

$$
\underline{F}\left(R_{1}, R_{2} \mid p_{X Y}\right)>\frac{\rho\left(p_{X Y}\right)}{4} \cdot g^{2}\left(\frac{\tau}{\rho\left(p_{X Y}\right)}\right)>0
$$

where $g$ is the inverse function of $\vartheta(a):=a+$ $(5 / 4) a^{2}, a \geq 0$.
Property 3 part a) is stated as Lemma 9 in Appendix B Proof of this lemma is given in this appendix. Proof of Property 4 part b) is given in Appendix D Proofs of Property 4 parts c), d), e), and f) are given in Appendix E

Our main result is the following.
Theorem 3: For any $R_{1}, R_{2} \geq 0$, any $p_{X Y}$, and for any $\left(\varphi_{1}^{(n)}, \varphi_{1}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have

$$
\begin{equation*}
\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq 5 \exp \left\{-n F\left(R_{1}, R_{2} \mid p_{X Y}\right)\right\} . \tag{10}
\end{equation*}
$$

It follows from Theorem 3 and Property 4 part d) that if $\left(R_{1}, R_{2}\right)$ is outside the capacity region, then the error probability of decoding goes to one exponentially and its exponent is not below $F\left(R_{1}, R_{2} \mid p_{X Y}\right)$. It immediately follows from Theorem 3 that we have the following corollary.

Corollary 1:

$$
\begin{aligned}
& G\left(R_{1}, R_{2} \mid p_{X Y}\right) \geq F\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& \mathcal{G}\left(p_{X Y}\right) \subseteq \overline{\mathcal{G}}\left(p_{X Y}\right) \\
& \quad=\left\{\left(R_{1}, R_{2}, G\right): G \geq F\left(R_{1}, R_{2} \mid p_{X Y}\right)\right\}
\end{aligned}
$$

Proof of Theorem 3 will be given in the next section. The exponent function at rates outside the rate region was derived by Oohama and Han [7] for the separate source coding problem for correlated sources [6]. The techniques used by them is a method of types [8], which is not useful to prove Theorem 3 Some novel techniques based on the information spectrum method introduced by Han [9] are necessary to prove this theorem.

From Theorem 3 and Property 4 part e), we can obtain an explicit outer bound of $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ with an asymptotically vanishing deviation from $\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)=\mathcal{R}\left(p_{X Y}\right)$. The strong converse theorem established by Ahlswede et al. [3] immediately follows from this corollary. To discribe this outer bound, for $\kappa>0$, we set

$$
\begin{aligned}
& \mathcal{R}\left(p_{X Y}\right)-\kappa(1,1) \\
& :=\left\{\left(R_{1}-\kappa, R_{2}-\kappa\right):\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)\right\}
\end{aligned}
$$

which serves as an outer bound of $\mathcal{R}\left(p_{X Y}\right)$. For each fixed $\varepsilon \in(0,1)$, we define $\kappa_{n}=\kappa_{n}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$ by

$$
\begin{aligned}
\kappa_{n} & :=\rho\left(p_{X Y}\right) \vartheta\left(\sqrt{\frac{4}{n \rho\left(p_{X Y}\right)} \log \left(\frac{5}{1-\varepsilon}\right)}\right) \\
& \stackrel{(\mathrm{a})}{=} 2 \sqrt{\frac{\rho\left(p_{X Y}\right)}{n} \log \left(\frac{5}{1-\varepsilon}\right)}+\frac{5}{n} \log \left(\frac{5}{1-\varepsilon}\right)
\end{aligned}
$$

Step (a) follows from $\vartheta(a)=a+(5 / 4) a^{2}$. Since $\kappa_{n} \rightarrow 0$ as $n \rightarrow \infty$, we have the smallest positive integer $n_{0}=$ $n_{0}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$ such that $\kappa_{n} \leq(1 / 2) \rho\left(p_{X Y}\right)$ for $n \geq n_{0}$. From Theorem 3 and Property 4 part e), we have the following corollary.

Corollary 2: For each fixed $\varepsilon \in(0,1)$, we choose the above positive integer $n_{0}=n_{0}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$. Then, for any $n \geq n_{0}$, we have

$$
\mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right) \subseteq \mathcal{R}\left(p_{X Y}\right)-\kappa_{n}(1,1)
$$

The above result together with

$$
\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)=\operatorname{cl}\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)\right)
$$

yields that for each fixed $\varepsilon \in(0,1)$, we have

$$
\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)=\mathcal{R}_{\mathrm{AKW}}\left(p_{X Y}\right)=\mathcal{R}\left(p_{X Y}\right)
$$

This recovers the strong converse theorem proved by Ahlswede et al. [3].

Proof of this corollary will be given in the next section.

## IV. Proof of the Main Result

Let $\left(X^{n}, Y^{n}\right)$ be a pair of random variables from the information source. We set $S=\varphi_{1}^{(n)}\left(X^{n}\right)$. Joint distribution $p_{S X^{n} Y^{n}}$ of $\left(S, X^{n}, Y^{n}\right)$ is given by

$$
p_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right)=p_{S \mid X^{n}}\left(s \mid x^{n}\right) \prod_{t=1}^{n} p_{X_{t} Y_{t}}\left(x_{t}, y_{t}\right)
$$

It is obvious that $S \leftrightarrow X^{n} \leftrightarrow Y^{n}$. Then we have the following.

Lemma 1: For any $\eta>0$ and for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq p_{S X^{n} Y^{n}}\{ \\
& 0 \geq \frac{1}{n} \log \frac{\hat{q}_{S X^{n} Y^{n}}\left(S, X^{n}, Y^{n}\right)}{p_{S X^{n} Y^{n}}\left(S, X^{n}, Y^{n}\right)}-\eta,  \tag{12}\\
& 0 \geq \frac{1}{n} \log \frac{Q_{X^{n}}\left(X^{n}\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta,  \tag{13}\\
& R_{1} \geq \frac{1}{n} \log \frac{\tilde{Q}_{X^{n} \mid S}\left(X^{n} \mid S\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta,  \tag{14}\\
& R_{2}\left.\geq \frac{1}{n} \log \frac{1}{p_{Y^{n} \mid S}\left(Y^{n} \mid S\right)}-\eta\right\}+4 \mathrm{e}^{-n \eta} . \tag{15}
\end{align*}
$$

The probability distributions appearing in the three inequalities (12), (13), and (14) in the right members of (15) have a property that we can select them arbitrary. In (12), we can choose any probability distribution $\hat{q}_{S X^{n} Y^{n}}$ on $\mathcal{S} \times \mathcal{X}^{n} \times \mathcal{Y}^{n}$. In (13), we can choose any distribution $Q_{X^{n}}$ on $\mathcal{X}^{n}$. In (14), we can choose any stochastic matrix $\tilde{Q}_{X^{n} \mid U^{n}}: \mathcal{X}^{n} \rightarrow \mathcal{U}^{n}$.

Proof of this lemma is given in Appendix From Lemma (1) we obtain the following lemma.

Lemma 2: For any $\eta>0$ and for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& \leq p_{S X^{n} Y^{n}}\left\{0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{Q_{X_{t}}\left(X_{t}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta,\right. \\
& R_{1} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{\tilde{Q}_{X_{t} \mid S X^{t-1}}\left(X_{t} \mid S, X^{t-1}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta, \\
& \left.R_{2} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_{t} \mid S X^{t-1} Y^{t-1}}\left(Y_{t} \mid S, X^{t-1}, Y^{t-1}\right)}-2 \eta\right\} \\
& \quad+4 \mathrm{e}^{-n \eta}
\end{aligned}
$$

where for each $t=1,2, \cdots, n$, the probability distribution $Q_{X_{t}}$ on $\mathcal{X}$ appearing in 16 and the stochastic matrix $\tilde{Q}_{X_{t} \mid S X^{t-1}}: \mathcal{M}_{1} \times \mathcal{X}^{t-1} \rightarrow \mathcal{X}$ appearing in 17) have a property that we can choose their values arbitrary.

Proof: In (12) in Lemma 1 we choose $\hat{q}_{S X^{n} Y^{n}}$ having the form

$$
\begin{aligned}
& \hat{q}_{S X^{n} Y^{n}}\left(S, X^{n}, Y^{n}\right) \\
& =p_{S}(S) \prod_{t=1}^{n}\left\{p_{X_{t} \mid S X^{t-1} Y^{t}}\left(X_{t} \mid S, X^{t-1}, Y^{t}\right)\right. \\
& \left.\quad \times p_{Y_{t} \mid S Y^{t-1}}\left(Y_{t} \mid S, Y^{t-1}\right)\right\}
\end{aligned}
$$

In (13) in Lemma 1 we choose $Q_{X^{n}}$ having the form

$$
Q_{X^{n}}\left(X^{n}\right)=\prod_{t=1}^{n} Q_{X_{t}}\left(X_{t}\right)
$$

We further note that

$$
\begin{aligned}
& \frac{\tilde{Q}_{X^{n} \mid S}\left(X^{n} \mid S\right)}{p_{X^{n}}\left(X^{n}\right)}=\prod_{t=1}^{n} \frac{\tilde{Q}_{X_{t} \mid S X^{t-1}}\left(X_{t} \mid S, X^{t-1}\right)}{p_{X_{t}}\left(X_{t}\right)} \\
& p_{Y^{n} \mid S}\left(Y^{n} \mid S\right)=\prod_{t=1}^{n} p_{Y_{t} \mid S Y^{t-1}}\left(Y_{t} \mid S, Y^{t-1}\right)
\end{aligned}
$$

Then the bound (15) in Lemma 1 becomes

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq p_{S X^{n} Y^{n}}\{ \\
& 0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{p_{Y_{t} \mid S Y^{t-1}}\left(Y_{t} \mid S, Y^{t-1}\right)}{p_{Y_{t} \mid S X^{t-1} Y^{t-1}}\left(Y_{t} \mid S, X^{t-1}, Y^{t-1}\right)}-\eta, \\
& 0
\end{aligned} \begin{array}{rl}
n & 1 \\
t=1 \\
n & \log \frac{Q_{X_{t}}\left(X_{t}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta, \\
R_{1} & \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{\tilde{Q}_{X_{t} \mid S X^{t-1}}\left(X_{t} \mid S, X^{t-1}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta, \\
R_{2} & \left.\geq \frac{1}{n} \sum_{t=1}^{n} \frac{1}{p_{Y_{t} \mid S Y^{t-1}}\left(Y_{t} \mid S, Y^{t-1}\right)}-\eta\right\}+4 \mathrm{e}^{-n \eta} \\
\leq p_{S X^{n} Y^{n}}\left\{0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{Q_{X_{t}}\left(X_{t}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta,\right. \\
R_{1} & \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{\tilde{Q}_{X_{t} \mid S X^{t-1}}\left(X_{t} \mid S, X^{t-1}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta, \\
R_{2} & \left.\geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_{t} \mid S X^{t-1} Y^{t-1}}\left(Y_{t} \mid S, X^{t-1}, Y^{t-1}\right)}-2 \eta\right\} \\
\quad+4 \mathrm{e}^{-n \eta}
\end{array}
$$

completing the proof.
Lemma 3: Suppose that for each $t=1,2, \cdots, n$, the joint distribution $p_{S X^{t} Y^{t}}$ of the random vector $S X^{t} Y^{t}$ is a marginal distribution of $p_{S X^{n} Y^{n}}$. Then we have the following Markov chain:

$$
\begin{equation*}
S X^{t-1} \leftrightarrow X_{t} \leftrightarrow Y_{t} \tag{18}
\end{equation*}
$$

or equivalently that $I\left(Y_{t} ; S X^{t-1} \mid X_{t}\right)=0$. Furthermore, we have the following Markov chain:

$$
\begin{equation*}
Y^{t-1} \leftrightarrow S X^{t-1} \leftrightarrow\left(X_{t}, Y_{t}\right) \tag{19}
\end{equation*}
$$

or equivalently that $I\left(X_{t} Y_{t} ; Y^{t-1} \mid S X^{t-1}\right)=0$. The above two Markov chains are equivalent to the following one long Markov chain:

$$
\begin{equation*}
Y^{t-1} \leftrightarrow S X^{t-1} \leftrightarrow X_{t} \leftrightarrow Y_{t} . \tag{20}
\end{equation*}
$$

Proof of this lemma is given in Appendix G For $t=$ $1,2, \cdots, n$, set $\mathcal{U}_{t}:=\mathcal{M}_{1} \times \mathcal{X}^{t-1}$. Define a random variable $U_{t} \in \mathcal{U}_{t}$ by $U_{t}:=\left(S, X^{t-1}\right)$. From Lemmas 2 and 3 we have the following.

Lemma 4: For any $\eta>0$ and for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& \leq p_{S X^{n} Y^{n}}\left\{0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{Q_{X_{t}}\left(X_{t}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta\right.  \tag{21}\\
& R_{1} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{\tilde{Q}_{X_{t} \mid U_{t}}\left(X_{t} \mid U_{t}\right)}{p_{X_{t}}\left(X_{t}\right)}-\eta  \tag{22}\\
& \left.R_{2} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_{t} \mid U_{t}}\left(Y_{t} \mid U_{t}\right)}-2 \eta\right\}+4 \mathrm{e}^{-n \eta}
\end{align*}
$$

where for each $t=1,2, \cdots, n$, the probability distribution $Q_{X_{t}}$ on $\mathcal{X}$ appearing in (21) and the stochastic matrix $\tilde{Q}_{X_{t} \mid U_{t}}$ :
$\mathcal{U}_{t} \rightarrow \mathcal{X}$ appearing in (22) have a property that we can choose their values arbitrary.

For each $t=1,2, \cdots, n$, set $\underline{Q}_{t}:=\left(Q_{X_{t}}, \tilde{Q}_{X_{t} \mid U_{t}}\right)$. Let $\underline{\mathcal{Q}}_{t}$ be a set of all $\underline{Q}_{t}$. We define a quantity which serves as an exponential upper bound of $\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$. Let $\mathcal{P}^{(n)}\left(p_{X Y}\right)$ be a set of all probability distributions $p_{S X^{n} Y^{n}}$ on $\mathcal{M}_{1} \times \mathcal{X}^{n} \times \mathcal{Y}^{n}$ having a form:

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right)=p_{S \mid X^{n}}\left(s \mid x^{n}\right) \prod_{t=1}^{n} p_{X Y}\left(x_{t}, y_{t}\right) \\
& \text { for }\left(s, x^{n}, y^{n}\right) \in \mathcal{M}_{1} \times \mathcal{X}^{n} \times \mathcal{Y}^{n}
\end{aligned}
$$

For simplicity of notation we use the notation $p^{(n)}$ for $p_{S X^{n} Y^{n}}$ $\in \mathcal{P}^{(n)}\left(p_{X Y}\right)$. For each $t=1,2, \cdots, n, p_{U_{t} X_{t} Y_{t}}=p_{S X^{t} Y_{t}}$ is a marginal distribution of $p^{(n)}$. For $t=1,2, \cdots, n$, we simply write $p_{t}=p_{U_{t} X_{t} Y_{t}}$. For $\mu \in[0,1], \alpha \in[0,1), p^{(n)}$ $\in \mathcal{P}^{(n)}\left(p_{X Y}\right)$, and $\underline{Q}^{n} \in \mathcal{Q}^{n}$, we define

$$
\begin{aligned}
& \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) \\
& :=-\log \mathrm{E}_{p^{(n)}}\left[\prod_{t=1}^{n} \frac{p_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}{Q_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)} \frac{p_{X_{t}}^{\mu \alpha}\left(X_{t}\right) p_{Y_{t} \mid U_{t}}^{\mu \alpha}\left(Y_{t} \mid U_{t}\right)}{\tilde{Q}_{X_{t} \mid U_{t}}^{\mu \alpha}\left(X_{t} \mid U_{t}\right)}\right]
\end{aligned}
$$

where for each $t=1,2, \cdots, n$, the probability distribution $Q_{X_{t}}$ and the conditional probability distribution $\tilde{Q}_{X_{t} \mid U_{t}}$ appearing in the definition of $\Omega^{(\mu, \theta)}\left(p^{(n)}, \underline{Q}^{n}\right)$ can be chosen arbitrary.

The following is well known as the Cramèr's bound in the large deviation principle.

Lemma 5: For any real valued random variable $Z$ and any $\alpha \geq 0$, we have

$$
\operatorname{Pr}\{Z \geq a\} \leq \exp [-(\alpha a-\log \mathrm{E}[\exp (\alpha Z)])]
$$

By Lemmas 4 and 5, we have the following proposition.
Proposition 1: For any $(\mu, \alpha) \in[0,1]^{2}$ any $\underline{Q}^{n} \in \underline{\mathcal{Q}}^{n}$, and any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=$ 1,2 , there exists $p^{(n)} \in \mathcal{P}^{(n)}\left(W_{1}, W_{2}\right)$ such that

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq 5 \exp \left\{-n[2+\alpha \bar{\mu}]^{-1}\right. \\
& \left.\times\left[\frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)\right]\right\}
\end{aligned}
$$

Proof: By Lemma 4, for $(\mu, \alpha) \in[0,1]^{2}$, we have the
following chain of inequalities:

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& \leq p_{S X^{n} Y^{n}}\left\{0 \geq\left[\frac{1}{n} \sum_{t=1}^{n} \log \frac{Q_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}{p_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}-\bar{\alpha} \eta\right],\right. \\
& \alpha \mu R_{1} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{\tilde{Q}_{X_{t} \mid U_{t}}^{\alpha \mu}\left(X_{t} \mid U_{t}\right)}{p_{X_{t}}^{\alpha \mu}\left(X_{t}\right)}-\alpha \mu \eta, \\
& \left.\alpha \bar{\mu} R_{2} \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_{t} \mid U_{t}}^{\alpha \bar{\mu}}\left(Y_{t} \mid U_{t}\right)}-2 \alpha \bar{\mu} \eta\right\}+4 \mathrm{e}^{-n \eta} \\
& \leq p_{S X^{n} Y^{n}}\left\{\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)+(1+\alpha \bar{\mu}) \eta\right. \\
& \left.\geq-\frac{1}{n} \sum_{t=1}^{n} \log \left[\frac{p_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}{Q_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)} \frac{p_{X_{t}}^{\mu \alpha}\left(X_{t}\right) p_{Y_{t} \mid U_{t}}^{\bar{\mu} \alpha}\left(Y_{t} \mid U_{t}\right)}{\tilde{Q}_{X_{t} \mid U_{t}}^{\mu \alpha}\left(X_{t} \mid U_{t}\right)}\right]\right\} \\
& +4 \mathrm{e}^{-n \eta} \\
& =p_{S X^{n} Y^{n}}\left\{\frac { 1 } { n } \sum _ { t = 1 } ^ { n } \operatorname { l o g } \left[\frac{p_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}{Q_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}\right.\right. \\
& \left.\times \frac{p_{X_{t}}^{\mu \alpha}\left(X_{t}\right) p_{Y_{t} \mid U_{t}}^{\alpha}\left(Y_{t} \mid U_{t}\right)}{\tilde{Q}_{X_{t} \mid U_{t}}^{\mu \alpha}\left(X_{t} \mid U_{t}\right)}\right] \\
& \left.\geq-\left[\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)+(1+\alpha \bar{\mu}) \eta\right]\right\}+4 \mathrm{e}^{-n \eta} \\
& \stackrel{(\mathrm{a})}{\leq} \exp \left[n \left\{\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)+(1+\alpha \bar{\mu}) \eta\right.\right. \\
& \left.\left.-\frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)\right\}\right]+4 \mathrm{e}^{-n \eta} . \tag{23}
\end{align*}
$$

Step (a) follows from Lemma 5. When $\Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) \leq$ $n \alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)$, the bound we wish to prove is obvious. In the following argument we assume that $\Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)>$ $n \alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)$. We choose $\eta$ so that

$$
\begin{align*}
-\eta= & \alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)+(1+\alpha \bar{\mu}) \eta \\
& -\frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) \tag{24}
\end{align*}
$$

Solving (24) with respect to $\eta$, we have

$$
\eta=\frac{(1 / n) \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}}
$$

For this choice of $\eta$ and (23), we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \leq 5 \mathrm{e}^{-n \eta}=5 \exp \left\{-n[2+\alpha \bar{\mu}]^{-1}\right. \\
& \left.\quad \times\left[\frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)\right]\right\}
\end{aligned}
$$

completing the proof.
Set

$$
\begin{aligned}
& \underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right) \\
& :=\inf _{n \geq 1} \min _{p^{(n)} \in \mathcal{P}^{(n)}} \max ^{n} \in \underline{\mathcal{Q}}^{n} \\
& \frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) .
\end{aligned}
$$

By Proposition 1 we have the following corollary.

Corollary 3: For any $(\mu, \alpha) \in[0,1]^{2}$ and any $\left(\varphi_{1}^{(n)}\right.$, $\left.\varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& \leq 5 \exp \left\{-n\left[\frac{\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}}\right]\right\} .
\end{aligned}
$$

We shall call $\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)$ the communication potential. The above corollary implies that the analysis of $\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)$ leads to an establishment of a strong converse theorem for the one helper source coding problem. In the following argument we drive an explicit lower bound of $\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)$. For each $t=1,2, \cdots, n$, set $u_{t}=\left(s, x^{t-1}\right) \in \mathcal{U}_{t}$ and

$$
\mathcal{F}_{t}:=\left(p_{X_{t}}, p_{X_{t} Y_{t} \mid U_{t}}, \underline{Q}_{t}\right), \quad \mathcal{F}^{t}:=\left\{\mathcal{F}_{i}\right\}_{i=1}^{t}
$$

For $t=1,2, \cdots, n$, define a function of $\left(u_{t}, x_{t}, y_{t}\right) \in \mathcal{U}_{t} \times \mathcal{X}$ $\times \mathcal{Y}$ by

$$
f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right):=\frac{p_{X_{t}}^{\bar{\alpha}}\left(x_{t}\right)}{Q_{X_{t}}^{\bar{\alpha}}\left(x_{t}\right)} \frac{p_{X_{t}}^{\mu \alpha}\left(x_{t}\right) p_{Y_{t} \mid U_{t}}^{\alpha}\left(y_{t} \mid u_{t}\right)}{\tilde{Q}_{X_{t} \mid U_{t}}^{\mu \alpha}\left(x_{t} \mid u_{t}\right)}
$$

By definition we have

$$
\begin{aligned}
& \exp \left\{-\Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)\right\} \\
& =\sum_{s, x^{n}, y^{n}} p_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right) \prod_{t=1}^{n} f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)
\end{aligned}
$$

For each $t=1,2, \cdots, n$, we define the probability distribution

$$
p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}:=\left\{p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right)\right\}_{\left(s, x^{t}, y^{t}\right) \in \mathcal{M}_{1} \times \mathcal{X}^{t} \times \mathcal{Y}^{t}}
$$

by

$$
\begin{aligned}
& p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right):=C_{t}^{-1} p_{S X^{t} Y^{t}}\left(s, x^{t}, y^{t}\right) \\
& \quad \times \prod_{i=1}^{t} f_{\mathcal{F}_{i}}^{(\mu, \alpha)}\left(x_{i}, y_{i} \mid u_{i}\right)
\end{aligned}
$$

where

$$
C_{t}:=\sum_{s, x^{t}, y^{t}} p_{S X^{t} Y^{t}}\left(s, x^{t}, y^{t}\right) \prod_{i=1}^{t} f_{\mathcal{F}_{i}}^{(\mu, \alpha)}\left(x_{i}, y_{i}\right)
$$

are constants for normalization. For $t=1,2, \cdots, n$, define

$$
\begin{equation*}
\Phi_{t}^{(\mu, \alpha)}:=C_{t} C_{t-1}^{-1} \tag{25}
\end{equation*}
$$

where we define $C_{0}=1$. Then we have the following lemma.
Lemma 6: For each $t=1,2, \cdots, n$, and for any $\left(s, x^{t}, y^{t}\right) \in$ $\mathcal{M}_{1} \times \mathcal{X}^{t} \times \mathcal{Y}^{t}$, we have

$$
\begin{align*}
& p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right) \\
& =\left(\Phi_{t}^{(\mu, \alpha)}\right)^{-1} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{\mid} \mid S X^{t-1} Y^{t-1}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right)}^{\quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)}
\end{align*}
$$

Furthermore, we have

$$
\begin{align*}
& \Phi_{t}^{(\mu, \alpha)}=\sum_{s, x^{t}, y^{t}} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \tag{27}
\end{align*}
$$

Proof of this lemma is given in Appendix H . Define

$$
\begin{aligned}
& p_{U_{t} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(u_{t}\right)=p_{S X^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}\right) \\
& :=\sum_{y^{t-1}} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right)
\end{aligned}
$$

Then we have the following lemma, which is a key result to derive a single-letterized lower bound of $\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)$.

Lemma 7: For any $p^{(n)} \in \mathcal{P}^{(n)}\left(p_{X Y}\right)$ and any $\underline{Q}^{n} \in \underline{\mathcal{Q}}^{n}$, we have

$$
\begin{align*}
& \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)=(-1) \sum_{t=1}^{n} \log \Phi_{t}^{(\mu, \alpha)}  \tag{28}\\
& \Phi_{t}^{(\mu, \alpha)}=\sum_{u_{t}, x_{t}, y_{t}} p_{U_{t} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(u_{t}\right) p_{X_{t} \mid U_{t}}\left(x_{t} \mid u_{t}\right) p_{Y_{t} \mid X_{t}}\left(y_{t} \mid x_{t}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \tag{29}
\end{align*}
$$

Proof: We first prove (28). From (25) we have

$$
\begin{equation*}
\log \Phi_{t}^{(\mu, \alpha)}=-\log C_{t}+\log C_{t-1} \tag{30}
\end{equation*}
$$

Furthermore, by definition we have

$$
\begin{equation*}
\Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right)=-\log C_{n}, C_{0}=1 \tag{31}
\end{equation*}
$$

From (30) and (31), (28) is obvious. We next prove (29). We first observe that for $\left(s, x^{t}, y^{t}\right) \in \mathcal{S} \times \mathcal{X}^{t} \times \mathcal{Y}^{t}$ and for $t=$ $1,2, \cdots, n$,

$$
\begin{aligned}
& p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& =p_{X_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{Y_{t} \mid S X^{t} Y^{t-1}}\left(y_{t} \mid s, x^{t}, y^{t-1}\right) \\
& \stackrel{(\mathrm{a})}{=} p_{X_{t} \mid S X^{t-1}}\left(x_{t} \mid s, x^{t-1}\right) p_{Y_{t} \mid X_{t}}\left(y_{t} \mid x_{t}\right) .
\end{aligned}
$$

Step (a) follows from Lemma 3 Then by Lemma 6, we have

$$
\begin{aligned}
& \Phi_{t}^{(\mu, \alpha)}=\sum_{s, x^{t}, y^{t}} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \\
& =\sum_{s, x^{t}, y^{t}} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} \mid S X^{t-1}}\left(x_{t} \mid s, x^{t-1}\right) p_{Y_{t} \mid X_{t}}\left(y_{t} \mid x_{t}\right) f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \\
& =\sum_{s, x^{t}, y_{t}} p_{S X^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}\right) \\
& \quad \times p_{X_{t} \mid S X^{t-1}}\left(x_{t} \mid s, x^{t-1}\right) p_{Y_{t} \mid X_{t}}\left(y_{t} \mid x_{t}\right) f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)
\end{aligned}
$$

completing the proof.
The following proposition is a mathematical core to prove our main result.

Proposition 2: For any $\mu \in[0,1]$ and any $\alpha \geq 0$, we have

$$
\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right) \geq \Omega^{(\mu, \alpha)}\left(p_{X Y}\right)
$$

Proof: Set

$$
\begin{aligned}
& \mathcal{Q}_{n}\left(p_{Y \mid X}\right):=\left\{q=q_{U X Y}:|\mathcal{U}| \leq\left|\mathcal{M}_{1}\right|\left|\mathcal{X}^{n-1}\right|\left|\mathcal{Y}^{n-1}\right|,\right. \\
&\left.q_{Y \mid X}=p_{Y \mid X}, U \leftrightarrow X \leftrightarrow Y\right\} \\
& \hat{\Omega}_{n}^{(\mu, \alpha)}\left(p_{X Y}\right):=\min _{q \in \mathcal{Q}_{n}\left(p_{Y \mid X}\right)} \Omega^{(\mu, \alpha)}\left(q \mid p_{X Y}\right) .
\end{aligned}
$$

For each $t=1,2, \cdots, n$, we define $q_{t}=q_{U_{t} X_{t} Y_{t} Z_{t}}$ by

$$
\left.\begin{array}{l}
q_{U_{t}}\left(u_{t}\right)=p_{U_{t} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(u_{t}\right)  \tag{32}\\
q_{X_{t} Y_{t} \mid U_{t}}\left(x_{t}, y_{t} \mid u_{t}\right)=p_{X_{t} \mid U_{t}}\left(x_{t} \mid u_{t}\right) p_{Y \mid X}\left(y_{t} \mid x_{t}\right)
\end{array}\right\}
$$

The equation (32) imply that $q_{t}=q_{U_{t} X_{t} Y_{t}} \in \mathcal{Q}_{n}\left(p_{Y \mid X}\right)$. Furthermore, for each $t=1,2, \cdots, n$, we choose $\underline{Q}_{t}=$ $\left(Q_{X_{t}}, \tilde{Q}_{X_{t} \mid U_{t}}\right)$ appearing in

$$
f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)=\frac{p_{X_{t}}^{\bar{\alpha}}\left(x_{t}\right)}{Q_{X_{t}}^{\bar{\alpha}}\left(x_{t}\right)} \frac{p_{X_{t}}^{\mu \alpha}\left(x_{t}\right) p_{Y_{t} \mid U_{t}}^{\alpha}\left(y_{t} \mid u_{t}\right)}{\tilde{Q}_{X_{t} \mid U_{t}}^{\mu \alpha}\left(x_{t} \mid u_{t}\right)}
$$

such that $\underline{Q}_{t}=\left(Q_{X_{t}}, \tilde{Q}_{X_{t} \mid U_{t}}\right)=\left(q_{X_{t}}, q_{X_{t} \mid U_{t}}\right)$. For this choice of $\underline{Q}_{t}$, we have the following chain of inequalities:

$$
\begin{align*}
& \Phi_{t}^{(\mu, \alpha)} \stackrel{(\mathrm{a})}{=} \mathrm{E}_{q_{t}}\left[f_{\mathcal{F}_{t}}^{(\mu, \theta)}\left(X_{t}, Y_{t} \mid U_{t}\right)\right] \\
& \stackrel{(\mathrm{b})}{=} \mathrm{E}_{q_{t}}\left[\frac{p_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)}{q_{X_{t}}^{\bar{\alpha}}\left(X_{t}\right)} \frac{p_{X_{t}}^{\mu \alpha}\left(X_{t}\right) p_{Y_{t} \mid U_{t}}^{\alpha}\left(Y_{t} \mid U_{t}\right)}{q_{X_{t} \mid U_{t}}^{\mu \alpha}\left(X_{t} \mid U_{t}\right)}\right] \\
& =\mathrm{E}_{q_{t}}\left[f_{q_{t} \mid p_{X_{t}}}^{(\mu, \alpha)}\left(X_{t}, Y_{t} \mid U_{t}\right)\right]=\exp \left\{-\Omega^{(\mu, \alpha)}\left(q_{t} \mid p_{X_{t}}\right)\right\} \\
& \stackrel{(\mathrm{c})}{=} \exp \left\{-\Omega^{(\mu, \alpha)}\left(q_{t} \mid p_{X}\right)\right\} \stackrel{(\mathrm{d})}{\leq} \exp \left\{-\hat{\Omega}_{n}^{(\mu, \alpha)}\left(p_{X Y}\right)\right\} \\
& \stackrel{(\mathrm{e})}{=} \exp \left\{-\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)\right\} . \tag{33}
\end{align*}
$$

Step (a) follows from Lemma7 and (32). Step (b) follows from the choice $\left(Q_{X_{t}}, \tilde{Q}_{X_{t} \mid U_{t}}\right)=\left(q_{X_{t}}, q_{X_{t} \mid U_{t}}\right)$ of $\left(Q_{X_{t}}, \tilde{Q}_{X_{t} \mid U_{t}}\right)$ for $t=1,2, \cdots, n$. Step (c) follows from $p_{X_{t}}=p_{X}$ for $t=1,2, \cdots, n$. Step (d) follows from $q_{t} \in \mathcal{Q}_{n}\left(p_{Y \mid X}\right)$ and the definition of $\hat{\Omega}_{n}^{(\mu, \alpha)}\left(p_{X Y}\right)$. Step (e) follows from Property 4 part a). Hence we have the following:

$$
\begin{align*}
& \max _{\underline{Q}^{n} \in \underline{\mathcal{Q}}^{n}} \frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) \geq \frac{1}{n} \Omega^{(\mu, \alpha)}\left(p^{(n)}, \underline{Q}^{n}\right) \\
& \stackrel{(\mathrm{a})}{=}-\frac{1}{n} \sum_{t=1}^{n} \log \Phi_{t}^{(\mu, \alpha)} \stackrel{(\mathrm{b})}{\geq} \Omega^{(\mu, \alpha)}\left(p_{X Y}\right) \tag{34}
\end{align*}
$$

Step (a) follows from Lemma 7 Step (b) follows from 33). Since (34) holds fo any $n \geq 1$ and any $p_{S X^{n} Y^{n}}$ satisfying $S \leftrightarrow X^{n} \leftrightarrow Y^{n}$, we have that for any $(\mu, \alpha) \in[0,1]^{2}$,

$$
\underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right) \geq \Omega^{(\mu, \alpha)}\left(p_{X Y}\right)
$$

Thus, Proposition 2 is proved.

Proof of Theorem 3. For any $(\mu, \alpha) \in[0,1]^{2}$, for any $R_{1}, R_{2} \geq 0$ and for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2$, we have the following:

$$
\begin{aligned}
& \frac{1}{n} \log \left\{\frac{5}{\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)}\right\} \\
& \stackrel{(\mathrm{a})}{\geq} \underline{\Omega}^{(\mu, \alpha)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right) \\
& 2+\alpha \bar{\mu} \\
& \geq \frac{\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}} \\
& =F^{(\mu, \alpha)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) .
\end{aligned}
$$

Step (a) follows from Corollary 3 Step (b) follows from Proposition 2. Since the above bound holds for any $\mu \in[0,1]$ and any $\alpha \geq 0$, we have

$$
\frac{1}{n} \log \left\{\frac{5}{\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)}\right\} \geq F\left(R_{1}, R_{2} \mid p_{X Y}\right)
$$

Thus (10) in Theorem 3 is proved.
Proof of Corollary 2. Since $g$ is an inverse function of $\vartheta$, the definition (11) of $\kappa_{n}$ is equivalent to

$$
\begin{equation*}
g\left(\frac{\kappa_{n}}{\rho\left(p_{X Y}\right)}\right)=\sqrt{\frac{4}{n \rho\left(p_{X Y}\right)} \log \left(\frac{5}{1-\varepsilon}\right)} \tag{35}
\end{equation*}
$$

By the definition of $n_{0}=n_{0}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$, we have that $\kappa_{n} \leq(1 / 2) \rho\left(p_{X Y}\right)$ for $n \geq n_{0}$. We assume that for $n \geq n_{0}$, $\left(R_{1}, R_{2}\right) \in \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)$. Then there exists a sequence $\left\{\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)\right\}_{n \geq n_{0}}$ such that for $n \geq n_{0}$, we have

$$
\begin{align*}
& \frac{1}{n} \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2 \\
& 1-\varepsilon \leq \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \tag{36}
\end{align*}
$$

Then by Theorem 3 we have

$$
\begin{align*}
1-\varepsilon & \leq \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) \\
& \leq 5 \exp \left\{-n F\left(R_{1}, R_{2} \mid p_{X Y}\right)\right\} \tag{37}
\end{align*}
$$

for any $n \geq n_{0}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$. From (37), we have that for $n \geq$ $n_{0}\left(\varepsilon, \rho\left(p_{X Y}\right)\right)$,

$$
\begin{align*}
& F\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& \leq \frac{1}{n} \log \left(\frac{5}{1-\varepsilon}\right) \stackrel{(\text { a) }}{=} \frac{\rho\left(p_{X Y}\right)}{4} \cdot g^{2}\left(\frac{\kappa_{n}}{\rho\left(p_{X Y}\right)}\right) \tag{38}
\end{align*}
$$

Step (a) follows from (35). Hence by Property 4 part e), we have that under $\kappa_{n} \leq(1 / 2) \rho\left(p_{X Y}\right)$, the inequality 38) implies

$$
\begin{equation*}
\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)+\kappa_{n}(1,1) \tag{39}
\end{equation*}
$$

Since (39) holds for any $n \geq n_{0}$ and $\left(R_{1}, R_{2}\right) \in \mathcal{R}_{\mathrm{AKW}}($ $\left.n, \varepsilon \mid p_{X Y}\right)$, we have

$$
\mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right) \subseteq \mathcal{R}\left(p_{X Y}\right)+\kappa_{n}(1,1) \text { for } n \geq n_{0}
$$

completing the proof.


Fig. 3. One helper source coding system investigated by Wyner.

## V. One Helper Problem Studied by Wyner

We consider a communication system depicted in Fig. 2. Data sequences $X^{n}, Y^{n}$, and $Z^{n}$, respectively are separately encoded to $\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{2}^{(n)}\left(Y^{n}\right)$, and $\varphi_{3}^{(n)}\left(Z^{n}\right)$. The encoded data $\varphi_{1}^{(n)}\left(X^{n}\right)$ and $\varphi_{2}^{(n)}\left(Y^{n}\right)$ and are sent to the information processing center 1 . The encoded data $\varphi_{1}^{(n)}\left(X^{n}\right)$ and $\varphi_{3}^{(n)}\left(Z^{n}\right)$ and are sent to the information processing center 2. At the center 1 the decoder function $\psi^{(n)}$ observes $\left(\varphi_{1}^{(n)}\left(X^{n}\right)\right.$, $\left.\varphi_{2}^{(n)}\left(Y^{n}\right)\right)$ to output the estimation $\hat{Y}^{n}$ of $Y^{n}$. At the center 2 the decoder function $\phi^{(n)}$ observes $\left(\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{3}^{(n)}\left(Z^{n}\right)\right)$ to output the estimation $\hat{Z}^{n}$ of $Z^{n}$. The error probability of decoding is

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right) \\
& =\operatorname{Pr}\left\{\hat{Y}^{n} \neq Y^{n} \text { or } \hat{Z}^{n} \neq Z^{n}\right\},
\end{aligned}
$$

where $\hat{Y}^{n}=\psi^{(n)}\left(\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{2}^{(n)}\left(Y^{n}\right)\right)$ and $\hat{Z}^{n}=\psi^{(n)}($ $\left.\varphi_{1}^{(n)}\left(X^{n}\right), \varphi_{3}^{(n)}\left(Z^{n}\right)\right)$.

A rate triple $\left(R_{1}, R_{2}, R_{3}\right)$ is $\varepsilon$-achievable if for any $\delta>0$, there exist a positve interger $n_{0}=n_{0}(\varepsilon, \delta)$ and a sequence of three encoders and two decoders functions $\left\{\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}\right.\right.$, $\left.\left.\varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right)\right\}_{n \geq n_{0}}$ such that for $n \geq n_{0}(\varepsilon, \delta)$,

$$
\begin{aligned}
& \frac{1}{n} \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}+\delta \text { for } i=1,2,3 \\
& \mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right) \leq \varepsilon
\end{aligned}
$$

The rate region $\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)$ is defined by

$$
\begin{aligned}
& \mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right):=\left\{\left(R_{1}, R_{2}, R_{3}\right):\right. \\
& \left.\left(R_{1}, R_{2}, R_{3}\right) \text { is } \varepsilon \text {-achievable for } p_{X Y Z}\right\}
\end{aligned}
$$

Furthermore, define

$$
\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right):=\bigcap_{\varepsilon \in(0,1)} \mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)
$$

We can show that the two rate regions $\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right), \varepsilon \in$ $(0,1)$ and $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$ satisfy the following property.

Property 5:
a) The regions $\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right), \varepsilon \in(0,1)$, and $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$ are closed convex sets of $\mathbb{R}_{+}^{3}$.
b) We set

$$
\mathcal{R}_{\mathrm{W}}\left(n, \varepsilon \mid p_{X Y Z}\right)=\left\{\left(R_{1}, R_{2}, R_{3}\right):\right.
$$

There exists $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}\right)$ such that $\frac{1}{n} \log \left\|\varphi_{i}^{(n)}\right\| \leq R_{i}, i=1,2,3$

$$
\left.\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}\right) \leq \varepsilon\right\}
$$

which is called the $(n, \varepsilon)$-rate region. Using $\mathcal{R}_{\mathrm{W}}(n$, $\left.\varepsilon \mid p_{X Y Z}\right), \mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)$ can be expressed as

$$
\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)=\mathrm{cl}\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\mathrm{W}}\left(n, \varepsilon \mid p_{X Y Z}\right)\right)
$$

It is well known that $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$ was determined by Wyner. To describe his result we introduce an auxiliary random variable $U$ taking values in a finite set $\mathcal{U}$. We assume that the joint distribution of $(U, X, Y, Z)$ is

$$
p_{U X Y}(u, x, y, z)=p_{U}(u) p_{X \mid U}(x \mid u) p_{Y Z \mid X}(y, z \mid x)
$$

The above condition is equivalent to $U \leftrightarrow X \leftrightarrow Y Z$. Define the set of probability distribution on $\mathcal{U} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ by

$$
\begin{aligned}
\mathcal{P}\left(p_{X Y Z}\right):= & \left\{p=p_{U X Y Z}:|\mathcal{U}| \leq|\mathcal{X}|+2\right. \\
& U \leftrightarrow X \leftrightarrow Y Z\}
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathcal{R}(p):= & \left\{\left(R_{1}, R_{2}, R_{3}\right): R_{1}, R_{2}, R_{3} \geq 0\right. \\
& R_{1} \geq I_{p}(X ; U), R_{2} \geq H_{p}(Y \mid U), \\
& \left.R_{3} \geq H_{p}(Z \mid U)\right\} \\
\mathcal{R}\left(p_{X Y Z}\right):= & \bigcup_{p \in \mathcal{P}\left(p_{X Y Z}\right)} \mathcal{R}(p) .
\end{aligned}
$$

We can show that the region $\mathcal{R}\left(p_{X Y Z}\right)$ satisfies the following property.

## Property 6:

a) The region $\mathcal{R}\left(p_{X Y Z}\right)$ is a closed convex subset of $\mathbb{R}_{+}^{3}$.
b) For any $p_{X Y Z}$, and any $\gamma \in[0,1]$, we have

$$
\begin{align*}
& \min _{\left(R_{1}, R_{2}, R_{3}\right) \in \mathcal{R}\left(p_{X Y}\right)}\left(R_{1}+\bar{\gamma} R_{2}+\gamma R_{3}\right) \\
& =\bar{\gamma} H_{p}(Y)+\gamma H_{p}(Z) \tag{40}
\end{align*}
$$

The minimun is attained by $\left(R_{1}, R_{2}, R_{3}\right)=\left(0, H_{p}(Y)\right.$, $\left.H_{p}(Z)\right)$. This result implies that

$$
\begin{gathered}
\mathcal{R}\left(p_{X Y Z}\right) \subseteq\left[\bigcap _ { \gamma \in [ 0 , 1 ] } \left\{\left(R_{1}, R_{2}, R_{3}\right): R_{1}+\bar{\gamma} R_{2}+\gamma R_{3}\right.\right. \\
\left.\left.\geq \bar{\gamma} H_{p}(Y)+\gamma H_{p}(Z)\right\}\right] \cap \mathbb{R}_{+}^{3}
\end{gathered}
$$

Furthermore, the point $\left(0, H_{p}(Y), H_{p}(Z)\right)$ always belongs to $\mathcal{R}\left(p_{X Y Z}\right)$.
The rate region $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$ was determined by Wyner [2]. His result is the following.

Theorem 4 (Wyner [2]):

$$
\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)=\mathcal{R}\left(p_{X Y Z}\right)
$$

On the strong converse theorem Csiszár and Körner [8] obtained the following.

Theorem 5 (Csiszár and Körner [8]): For each fixed $\varepsilon \in$ $(0,1)$, we have

$$
\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)=\mathcal{R}\left(p_{X Y Z}\right)
$$

To examine a rate of convergence for the error probability of decoding to tend to one as $n \rightarrow \infty$ for $\left(R_{1}, R_{2}, R_{3}\right) \notin$ $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$, we define the following quantity. Set

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right) \\
& :=1-\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right) \\
& G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& :=\min _{\substack{\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)} \\
\psi^{(n)}, \phi^{(n)}\right) \\
(1 / n) \log \left\|\varphi_{2}^{(n)}\right\| \\
\\
\leq R_{i}, i=1,2,3}}\left(-\frac{1}{n}\right) \log \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right), \\
& G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right):=\lim _{n \rightarrow \infty} G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& \mathcal{G}\left(p_{X Y Z}\right) \\
& :=\left\{\left(R_{1}, R_{2}, R_{3}, G\right): G \geq G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)\right\}
\end{aligned}
$$

By time sharing we have that

$$
\begin{align*}
& G^{(n+m)}\left(\frac{n R_{1}+m R_{1}^{\prime}}{n+m}, \frac{n R_{2}+m R_{2}^{\prime}}{n+m}, \left.\frac{n R_{2}+m R_{2}^{\prime}}{n+m} \right\rvert\, p_{X Y Z}\right) \\
& \leq \frac{n G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)+m G^{(m)}\left(R_{1}^{\prime}, R_{2}^{\prime}, R_{3}^{\prime} \mid p_{X Y Z}\right)}{n+m} \tag{41}
\end{align*}
$$

Choosing $R=R^{\prime}$ in (41), we obtain the following subadditivity property on $\left\{G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)\right\}_{n \geq 1}$ :

$$
\begin{aligned}
& G^{(n+m)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& \leq \frac{n G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)+m G^{(m)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)}{n+m}
\end{aligned}
$$

from which we have that $G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)$ exists and satisfies the following:

$$
\begin{aligned}
& G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& =\inf _{n \geq 1} G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)
\end{aligned}
$$

The exponent function $G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)$ is a convex function of $\left(R_{1}, R_{2}, R_{3}\right)$. In fact, by time sharing we have that

$$
\begin{aligned}
& G^{(n+m)}\left(\frac{n R_{1}+m R_{1}^{\prime}}{n+m}, \frac{n R_{2}+m R_{2}^{\prime}}{n+m}, \left.\frac{n R_{2}+m R_{2}^{\prime}}{n+m} \right\rvert\, p_{X Y Z}\right) \\
& \leq \frac{n G^{(n)}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)+m G^{(m)}\left(R_{1}^{\prime}, R_{2}^{\prime}, R_{3}^{\prime} \mid p_{X Y Z}\right)}{n+m}
\end{aligned}
$$

from which we have that for any $\alpha \in[0,1]$

$$
\begin{aligned}
& G\left(\alpha R_{1}+\bar{\alpha} R_{1}^{\prime}, \alpha R_{2}+\bar{\alpha} R_{2}^{\prime}, \alpha R_{3}+\bar{\alpha} R_{3}^{\prime} \mid p_{X Y Z}\right) \\
& \leq \alpha G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)+\bar{\alpha} G\left(R_{1}^{\prime}, R_{2}^{\prime}, R_{3}^{\prime} \mid p_{X Y Z}\right)
\end{aligned}
$$

The region $\mathcal{G}\left(p_{X Y Z}\right)$ is also a closed convex set. Our main aim is to find an explicit characterization of $\mathcal{G}\left(p_{X Y Z}\right)$. In this paper we derive an explicit outer bound of $\mathcal{G}\left(p_{X Y Z}\right)$ whose section by the plane $G=0$ coincides with $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)$. We first explain that the region $\mathcal{R}\left(p_{X Y Z}\right)$ has another expression using
the supporting hyperplane. We define two sets of probability distributions on $\mathcal{U} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ by

$$
\begin{gathered}
\mathcal{P}_{\operatorname{sh}}\left(p_{X Y Z}\right):=\left\{p=p_{U X Y Z}:|\mathcal{U}| \leq|\mathcal{X}|\right. \\
U \leftrightarrow X \leftrightarrow Y Z\} \\
\mathcal{Q}\left(p_{Y Z \mid X}\right):=\left\{q=q_{U X Y Z}:|\mathcal{U}| \leq|\mathcal{X}|\right. \\
\left.p_{Y Z \mid X}=q_{Y Z \mid X}, U \leftrightarrow X \leftrightarrow Y Z\right\}
\end{gathered}
$$

For $(\mu, \gamma) \in[0,1]^{2}$, set

$$
\begin{aligned}
R^{(\mu, \gamma)}\left(p_{X Y Z}\right):= & \max _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y Z}\right)}\left\{\mu I_{p}(X ; U)\right. \\
& \left.+\bar{\mu}\left(\bar{\gamma} H_{p}(Y \mid U)+\gamma H_{p}(Z \mid U)\right)\right\}
\end{aligned}
$$

Furthermore, define

$$
\begin{aligned}
& \underline{\mathcal{R}}_{\mathrm{sh}}\left(p_{X Y Z}\right) \\
& =\bigcap_{(\mu, \gamma) \in[0,1]^{2}}\left\{\left(R_{1}, R_{2}, R_{3}\right): \mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right)\right. \\
& \mathcal{R}_{\mathrm{sh}}\left(p_{X Y Z}\right) \\
& =\bigcap_{(\mu, \gamma) \in[0,1]^{2}}\left\{\left(R_{1}, R_{2}, R_{3}\right): \mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right)\right. \\
& \left.\left.=p_{X Y Z}\right)\right\} .
\end{aligned}
$$

Then we have the following property.
Property 7:
a) The bound $|\mathcal{U}| \leq|\mathcal{X}|$ is sufficient to describe $R^{(\mu)}($ $\left.p_{X Y Z}\right)$.
b) For every $(\mu, \gamma) \in[0,1]^{2}$, we have

$$
\begin{aligned}
& \min _{\left(R_{1}, R_{2}, R_{3}\right) \in \mathcal{R}\left(p_{X Y Z}\right)}\left\{\mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right)\right\} \\
& =R^{(\mu, \gamma)}\left(p_{X Y Z}\right)
\end{aligned}
$$

c) For any $p_{X Y Z}$ we have

$$
\begin{equation*}
\mathcal{R}_{\mathrm{sh}}\left(p_{X Y Z}\right)=\mathcal{R}\left(p_{X Y Z}\right) \tag{42}
\end{equation*}
$$

For $(\mu, \gamma, \alpha) \in[0,1]^{3}$, and for $q=q_{U X Y Z} \in \mathcal{Q}\left(p_{Y Z \mid X}\right)$, define

$$
\begin{aligned}
& \omega_{q \mid p_{X}}^{(\mu, \gamma, \alpha)}(x, y, z \mid u):=\bar{\alpha} \log \frac{q_{X}(x)}{p_{X}(x)}+\alpha\left[\mu \log \frac{q_{X \mid U}(x \mid u)}{p_{X}(x)}\right. \\
& \left.\quad+\bar{\mu}\left(\bar{\gamma} \log \frac{1}{q_{Y \mid U}(y \mid u)}+\gamma \log \frac{1}{q_{Z \mid U}(z \mid u)}\right)\right] \\
& f_{q \mid p_{X}}^{(\mu, \gamma, \alpha)}(x, y, z \mid u):=\exp \left\{-\omega_{q \mid p_{X}}^{(\mu, \gamma, \alpha)}(x, y, z \mid u)\right\}, \\
& \Omega^{(\mu, \gamma, \alpha)}\left(q \mid p_{X}\right):=-\log \mathrm{E}_{q}\left[f_{q \mid p_{X}}^{(\mu, \gamma, \alpha)}(X, Y, Z \mid U)\right] \\
& \Omega^{(\mu, \gamma, \alpha)}\left(p_{X Y Z}\right):=\min _{q \in \mathcal{Q}\left(p_{Y Z \mid X}\right)} \Omega^{(\mu, \gamma, \alpha)}\left(q \mid p_{X}\right), \\
& F^{(\mu, \gamma, \alpha)}\left(\mu R_{1}+\bar{\gamma} R_{2}+\gamma R_{3}\right) \\
& :=\frac{\Omega^{(\mu, \gamma, \alpha)}\left(p_{X Y Z}\right)-\alpha\left[\mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right)\right]}{2+\alpha \bar{\mu}} \\
& F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& :=\sup _{(\mu, \gamma, \alpha) \in[0,1]^{3},} F^{(\mu, \gamma, \alpha)}\left(\mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right) \mid p_{X Y Z}\right)
\end{aligned}
$$

We next define a function serving as a lower bound of $F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)$. For each $p=p_{U X Y Z} \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y Z}\right)$, define

$$
\begin{aligned}
& \tilde{\omega}_{p}^{(\mu, \gamma)}(x, y, z \mid u):=\mu \log \frac{p_{X \mid U}(x \mid u)}{p_{X}(x)} \\
& \quad+\bar{\mu}\left(\bar{\gamma} \log \frac{1}{p_{Y \mid U}(y \mid u)}+\gamma \log \frac{1}{p_{Z \mid U}(z \mid u)}\right) \\
& \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p):=-\log \mathrm{E}_{p}\left[\exp \left\{-\lambda \omega_{p}^{(\mu, \gamma)}(X, Y, Z \mid U)\right\}\right] .
\end{aligned}
$$

Furthermore, set

$$
\begin{aligned}
& \tilde{\Omega}^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right):=\min _{p \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y Z}\right)} \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p), \\
& \underline{F}^{(\mu, \gamma, \lambda)}\left(\mu R_{1}+\bar{\gamma} R_{2}+\gamma R_{3} \mid p_{X Y Z}\right) \\
& :=\frac{\tilde{\Omega}^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right)-\lambda\left[\mu R_{1}+\bar{\mu}\left(\bar{\gamma} R_{2}+\gamma R_{3}\right)\right]}{2+\lambda(5-\mu)} \\
& \underline{F}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& :=\sup _{\substack{(\mu, \gamma) \in[0,1]^{2}, \lambda \geq 0}} \underline{F}^{(\mu, \gamma, \lambda)}\left(\mu R_{1}+\bar{\mu} \bar{\gamma} R_{2}+\gamma R_{3} \mid p_{X Y Z}\right) .
\end{aligned}
$$

We can show that the above functions and sets satisfy the following property.

## Property 8:

a) The cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{Q}\left(p_{Y \mid X}\right)$ is sufficient to describe the quantity $\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)$. Furthermore, the cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{Q}\left(p_{Y Z \mid X}\right)$ is sufficient to describe the quantity $\tilde{\Omega}^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right)$.
b) For any $R_{1}, R_{2}, R_{3} \geq 0$, we have

$$
F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \geq \underline{F}\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)
$$

c) For any $p=p_{U X Y} \in \mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$ and any $(\mu, \gamma, \lambda) \in[0$, $1]^{3}$, we have

$$
\begin{equation*}
0 \leq \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p) \leq \mu \log |\mathcal{X}|+\bar{\mu} \log \left(|\mathcal{Y}|^{\bar{\gamma}}|\mathcal{Z}|^{\gamma}\right) \tag{43}
\end{equation*}
$$

d) Fix any $p=p_{U X Y Z} \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y Z}\right)$ and $(\mu, \gamma) \in[0,1]^{2}$. We define a probability distribution $p^{(\lambda)}=p_{U X Y Z}^{(\lambda)}$ by

$$
\begin{aligned}
& p^{(\lambda)}(u, x, y, z) \\
& :=\frac{p(u, x, y, z) \exp \left\{-\lambda \omega_{p}^{(\mu, \gamma)}(x, y, z \mid u)\right\}}{\mathrm{E}_{p}\left[\exp \left\{-\lambda \omega_{p}^{(\mu, \gamma)}(X, Y, Z \mid U)\right\}\right]} .
\end{aligned}
$$

Then for $\lambda \in[0,1 / 2], \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p)$ is twice differentiable. Furthermore, for $\lambda \in[0,1 / 2]$, we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} \lambda} \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p) \\
& \quad=\mathrm{E}_{p}(\lambda)\left[\omega_{p}^{(\mu, \gamma)}(X, Y, Z \mid U)\right] \\
& \frac{\mathrm{d}^{2}}{\mathrm{~d} \lambda^{2}} \tilde{\Omega}^{(\mu, \gamma, \lambda)}(p) \\
& \quad=-\operatorname{Var}_{p(\lambda)}\left[\omega_{p}^{(\mu, \gamma)}(X, Y, Z \mid U)\right]
\end{aligned}
$$

The second equality implies that $\tilde{\Omega}^{(\mu, \gamma, \lambda)}(p)$ is a concave function of $\lambda \in[0,1 / 2]$.
e) For $(\mu, \gamma, \lambda) \in[0,1]^{2} \times[0,1 / 2]$, define

$$
\begin{aligned}
& \rho^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right) \\
& :=\max _{\substack{\left.(\nu, p) \in[0, \lambda] \\
\times \mathcal{P}_{\operatorname{sh}}(p X Y Z): \\
\tilde{\Omega}^{s h}, \gamma, \lambda\right) \\
=\tilde{\Omega}^{(\mu, \gamma, \lambda)}(p) \\
=}} \operatorname{Var}_{\left.p^{(\nu Y Z}\right)}\left[\tilde{\omega}_{p}^{(\mu, \gamma)}(X, Y, Z \mid U)\right],
\end{aligned}
$$

and set

$$
\rho=\rho\left(p_{X Y Z}\right):=\max _{(\mu, \gamma, \lambda) \in[0,1]^{2} \times[0,1 / 2]} \rho^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right)
$$

Then we have $\rho\left(p_{X Y Z}\right)<\infty$. Furthermore, for any $(\mu, \gamma, \lambda) \in[0,1]^{2} \times[0,1 / 2]$, we have

$$
\tilde{\Omega}^{(\mu, \gamma, \lambda)}\left(p_{X Y Z}\right) \geq \lambda R^{(\mu, \gamma)}\left(p_{X Y Z}\right)-\frac{\lambda^{2}}{2} \rho\left(p_{X Y Z}\right)
$$

f) For every $\tau \in\left(0,(1 / 2) \rho\left(p_{X Y Z}\right)\right)$, the condition $\left(R_{1}+\tau\right.$, $\left.R_{2}+\tau, R_{3}+\tau\right) \notin \mathcal{R}\left(p_{X Y Z}\right)$ implies

$$
\begin{aligned}
& F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& >\frac{\rho\left(p_{X Y Z}\right)}{4} \cdot g^{2}\left(\frac{\tau}{\rho\left(p_{X Y Z}\right)}\right)>0
\end{aligned}
$$

Since proofs of the results stated in Property 8 are quite parallel with those of the results stated in Property 4 we omit them. Our main result is the following.

Theorem 6: For any $R_{1}, R_{2}, R_{3} \geq 0$, any $p_{X Y Z}$, and for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\|$ $\leq R_{i}, i=1,2,3$, we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \varphi_{3}^{(n)}, \psi^{(n)}, \phi^{(n)}\right) \\
& \leq 7 \exp \left\{-n F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)\right\}
\end{aligned}
$$

It follows from Theorem 6 and Property 8 part d) that if ( $R_{1}, R_{2}, R_{3}$ ) is outside the capacity region, then the error probability of decoding goes to one exponentially and its exponent is not below $F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)$. It immediately follows from Theorem 3 that we have the following corollary.

Corollary 4:

$$
\begin{aligned}
& G\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \geq F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right) \\
& \mathcal{G}\left(p_{X Y Z}\right) \subseteq \overline{\mathcal{G}}\left(p_{X Y Z}\right) \\
& =\left\{\left(R_{1}, R_{2}, R_{3}, G\right): G \geq F\left(R_{1}, R_{2}, R_{3} \mid p_{X Y Z}\right)\right\}
\end{aligned}
$$

Proof of Theorem 6 will be given in the next section. The exponent function at rates outside the rate region was derived by Oohama and Han [7] for the separate source coding problem for correlated sources [6]. The techniques used by them is a method of types [8], which is not useful to prove Theorem 3 Some novel techniques based on the information spectrum method introduced by Han [9] are necessary to prove this theorem.

From Theorem 6 and Property 8 part e), we can obtain an explicit outer bound of $\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)$ with an asymptotically vanishing deviation from $\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)=\mathcal{R}\left(p_{X Y Z}\right)$. The strong converse theorem established by Csiszár and Körner
[8] immediately follows from this corollary. To discribe this outer bound, for $\kappa>0$, we set

$$
\begin{aligned}
& \mathcal{R}\left(p_{X Y Z}\right)-\kappa(1,1,1) \\
& :=\left\{\left(R_{1}-\kappa, R_{2}-\kappa, R_{3}-\kappa\right):\left(R_{1}, R_{2}, R_{3}\right) \in \mathcal{R}\left(p_{X Y Z}\right)\right\},
\end{aligned}
$$

which serves as an outer bound of $\mathcal{R}\left(p_{X Y Z}\right)$. For each fixed $\varepsilon \in(0,1)$, we define $\tilde{\kappa}_{n}=\tilde{\kappa}_{n}\left(\varepsilon, \rho\left(p_{X Y Z}\right)\right)$ by

$$
\begin{align*}
\tilde{\kappa}_{n} & :=\rho\left(p_{X Y}\right) \vartheta\left(\sqrt{\frac{4}{n \rho\left(p_{X Y}\right)} \log \left(\frac{7}{1-\varepsilon}\right)}\right)  \tag{44}\\
& \stackrel{(\mathrm{a})}{=} 2 \sqrt{\frac{\rho\left(p_{X Y}\right)}{n} \log \left(\frac{7}{1-\varepsilon}\right)}+\frac{5}{n} \log \left(\frac{7}{1-\varepsilon}\right)
\end{align*}
$$

Step (a) follows from $\vartheta(a)=a+(5 / 4) a^{2}$. Since $\tilde{\kappa}_{n} \rightarrow 0$ as $n \rightarrow \infty$, we have the smallest positive integer $n_{1}=$ $n_{1}\left(\varepsilon, \rho\left(p_{X Y Z}\right)\right)$ such that $\tilde{\kappa}_{n} \leq(1 / 2) \rho\left(p_{X Y Z}\right)$ for $n \geq n_{1}$. From Theorem 6 and Property 8 part e), we have the following corollary.

Corollary 5: For each fixed $\varepsilon \in(0,1)$, we choose the above positive integer $n_{1}=n_{1}\left(\varepsilon, \rho\left(p_{X Y Z}\right)\right)$. Then, for any $n \geq n_{1}$, we have

$$
\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right) \subseteq \mathcal{R}\left(p_{X Y Z}\right)-\tilde{\kappa}_{n}(0,1,1)
$$

The above result together with

$$
\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)=\mathrm{cl}\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\mathrm{W}}\left(n, \varepsilon \mid p_{X Y Z}\right)\right)
$$

yields that for each fixed $\varepsilon \in(0,1)$, we have

$$
\mathcal{R}_{\mathrm{W}}\left(\varepsilon \mid p_{X Y Z}\right)=\mathcal{R}_{\mathrm{W}}\left(p_{X Y Z}\right)=\mathcal{R}\left(p_{X Y Z}\right)
$$

This recovers the strong converse theorem proved by Csiszár and Körner [8].

Proof of this corollary is quite parallel with that of Corollary (2) We omit the detail.

## Appendix

## A. Properties of the Rate Regions

In this appendix we prove Property 1 Property 1 part a) can easily be proved by the definitions of the rate distortion regions. We omit the proofs of this part. In the following argument we prove the part b ).

Proof of Property $\square$ part b: We set

$$
\underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)=\bigcap_{n \geq m} \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right) .
$$

By the definitions of $\underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)$ and $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$, we have that $\underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right) \subseteq \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ for $m \geq 1$. Hence we have that

$$
\begin{equation*}
\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right) \subseteq \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right) \tag{45}
\end{equation*}
$$

We next assume that $\left(R_{1}, R_{2}\right) \in \mathcal{R}_{\text {AKW }}\left(\varepsilon \mid p_{X Y}\right)$. Set

$$
\begin{aligned}
& \mathcal{R}_{\mathrm{AKW}}^{(\delta)}\left(\varepsilon \mid p_{X Y}\right) \\
& :=\left\{\left(R_{1}+\delta, R_{2}+\delta\right):\left(R_{1}, R_{2}\right) \in \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)\right\}
\end{aligned}
$$

Then, by the definitions of $\mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)$ and $\mathcal{R}_{\mathrm{AKW}}($ $\left.\varepsilon \mid p_{X Y}\right)$, we have that for any $\delta>0$, there exists $n_{0}(\varepsilon, \delta)$ such that for any $n \geq n_{0}(\varepsilon, \delta)$,

$$
\left(R_{1}+\delta, R_{2}+\delta\right) \in \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)
$$

which implies that

$$
\begin{align*}
& \mathcal{R}_{\mathrm{AKW}}^{(\delta)}\left(\varepsilon \mid p_{X Y}\right) \subseteq \bigcup_{n \geq n_{0}(\varepsilon, \delta)} \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right) \\
= & \underline{\mathcal{R}}_{\mathrm{AKW}}\left(n_{0}(\delta), \varepsilon \mid p_{X Y}\right) \\
\subseteq & \mathrm{cl}\left(\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)\right) . \tag{46}
\end{align*}
$$

Here we assume that there exists a pair $\left(R_{1}, R_{2}\right)$ belonging to $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ such that

$$
\begin{equation*}
\left(R_{1}, R_{2}\right) \notin \operatorname{cl}\left(\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)\right) \tag{47}
\end{equation*}
$$

Since the set in the right hand side of is a closed set, we have

$$
\begin{equation*}
\left(R_{1}+\delta, R_{2}+\delta\right) \notin \mathrm{cl}\left(\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)\right) \tag{48}
\end{equation*}
$$

for some small $\delta>0$. On the other hand we have $\left(R_{1}+\right.$ $\left.\delta, R_{2}+\delta\right) \in \mathcal{R}_{\text {AKW }}^{(\delta)}\left(\varepsilon \mid p_{X Y}\right)$, which contradicts (46). Thus we have

$$
\begin{align*}
& \bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right) \\
& \subseteq \mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right) \subseteq \mathrm{cl}\left(\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)\right) \tag{49}
\end{align*}
$$

Note here that $\mathcal{R}_{\mathrm{AKW}}\left(\varepsilon \mid p_{X Y}\right)$ is a closed set. Then from 49), we conclude that

$$
\begin{aligned}
\mathcal{R}_{\mathrm{AKW}}(\varepsilon \mid W) & =\mathrm{cl}\left(\bigcup_{m \geq 1} \underline{\mathcal{R}}_{\mathrm{AKW}}\left(m, \varepsilon \mid p_{X Y}\right)\right) \\
& =\mathrm{cl}\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\mathrm{AKW}}\left(n, \varepsilon \mid p_{X Y}\right)\right)
\end{aligned}
$$

completing the proof.

## B. Cardinality Bound on Auxiliary Random Variables

We first prove the following lemma.
Lemma 8:

$$
\begin{aligned}
& \underline{R}^{(\mu)}\left(p_{X Y}\right):=\min _{p \in \mathcal{P}\left(p_{X Y}\right)}\left\{\mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)\right\} \\
& =R^{(\mu)}\left(p_{X Y}\right):=\min _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)}\left\{\mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)\right\} .
\end{aligned}
$$

Proof: We bound the cardinality $|\mathcal{U}|$ of $U$ to show that the bound $|\mathcal{U}| \leq|\mathcal{X}|$ is sufficient to describe $\underline{R}^{(\mu)}\left(p_{X Y}\right)$. Observe that

$$
\begin{align*}
& p_{X}(x)=\sum_{u \in \mathcal{U}} p_{U}(u) p_{X \mid U}(x \mid u)  \tag{50}\\
& \mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)=\sum_{u \in \mathcal{U}} p_{U}(u) \pi\left(p_{X \mid U}(\cdot \mid u)\right), \tag{51}
\end{align*}
$$

where

$$
\begin{aligned}
& \pi\left(p_{X \mid U}(\cdot \mid u)\right):=\sum_{(x, y) \in \mathcal{X} \times \mathcal{Y}} p_{X \mid U}(x \mid u) p_{Y \mid X}(y \mid x) \\
& \quad \times \log \left\{\frac{p_{X \mid U}^{\mu}(x \mid u)}{p_{X}^{\mu}(x)}\left[\sum_{\tilde{x} \in \mathcal{X}} p_{Y \mid X}(y \mid \tilde{x}) p_{X \mid U}(\tilde{x} \mid u)\right]^{-\bar{\mu}}\right\}
\end{aligned}
$$

For each $u \in \mathcal{U}, \pi\left(p_{X \mid U}(\cdot \mid u)\right)$ is a continuous function of $p_{X \mid U}(\cdot \mid u)$. Then by the support lemma,

$$
|\mathcal{U}| \leq|\mathcal{X}|-1+1=|\mathcal{X}|
$$

is sufficient to express $|\mathcal{X}|-1$ values of (50) and one value of (51).

Next we prove the following lemma.
Lemma 9: The cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{Q}\left(p_{Y \mid X}\right)$ is sufficient to describe the quantity $\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)$. The cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|$ in $\mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ is sufficient to describe the quantity $\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)$.

Proof: We first bound the cardinality $|\mathcal{U}|$ of $U$ in $\mathcal{Q}\left(p_{Y \mid X}\right)$ to show that the bound $|\mathcal{U}| \leq|\mathcal{X}|$ is sufficient to describe $\Omega^{(\mu, \alpha)}\left(p_{X Y}\right)$. Observe that

$$
\begin{align*}
& q_{X}(x)=\sum_{u \in \mathcal{U}} q_{U}(u) q_{X \mid U}(x \mid u)  \tag{52}\\
& \exp \left\{-\Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right)\right\} \\
&=\sum_{u \in \mathcal{U}} q_{U}(u) \Pi^{(\mu, \alpha)}\left(q_{X}, q_{X Y \mid U}(\cdot, \cdot \mid u)\right) \tag{53}
\end{align*}
$$

where

$$
\begin{aligned}
& \Pi^{(\mu, \alpha)}\left(q_{X}, q_{X Y \mid U}(\cdot, \cdot \mid u)\right) \\
& :=\sum_{\substack{(x, y) \\
\in \mathcal{X} \times \mathcal{Y}}} q_{X Y \mid U}(x, y \mid u) \exp \left\{-\omega_{q \mid p_{X}}^{(\mu, \alpha)}(x, y \mid u)\right\} .
\end{aligned}
$$

The value of $q_{X}$ included in $\Pi^{(\mu, \alpha)}\left(q_{X}, q_{X Y \mid U}(\cdot, \cdot \mid u)\right)$ must be preserved under the reduction of $\mathcal{U}$. For each $u \in \mathcal{U}$, $\Pi^{(\mu, \alpha)}\left(q_{X}, q_{X Y \mid U}(\cdot, \cdot \mid u)\right)$ is a contineous function of $q_{X Y \mid U}($ $\cdot, \cdot \mid u)$. Then by the support lemma,

$$
|\mathcal{U}| \leq|\mathcal{X}|-1+1=|\mathcal{X}|
$$

is sufficient to express $|\mathcal{X}|-1$ values of (52) and one value of (53). We next bound the cardinality $|\mathcal{U}|$ of $U$ in $\mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ to show that the bound $|\mathcal{U}| \leq|\mathcal{X}|$ is sufficient to describe $\Omega^{(\mu, \lambda)}$ ( $p_{X Y}$ ). Observe that

$$
\begin{align*}
& p_{X}(x)=\sum_{u \in \mathcal{U}} p_{U}(u) p_{X \mid U}(x \mid u)  \tag{54}\\
& \exp \left\{-\Omega^{(\mu, \lambda)}(p)\right\} \\
&=\sum_{u \in \mathcal{U}} p_{U}(u) \tilde{\Pi}^{(\mu, \lambda)}\left(p_{X}, p_{X Y \mid U}(\cdot, \cdot \mid u)\right) \tag{55}
\end{align*}
$$

where

$$
\begin{aligned}
& \tilde{\Pi}^{(\mu, \lambda)}\left(p_{X}, p_{X Y \mid U}(\cdot, \cdot \mid u)\right) \\
& :=\sum_{\substack{(x, y) \\
\in \mathcal{X} \times \mathcal{Y}}} p_{X Y \mid U}(x, y \mid u) \exp \left\{-\lambda \tilde{\omega}_{p}^{(\mu)}(x, y \mid u)\right\} .
\end{aligned}
$$

The value of $p_{X}$ included in $\tilde{\Pi}^{(\mu, \lambda)}\left(p_{X}, p_{X Y \mid U}(\cdot, \cdot \mid u)\right)$ must be preserved under the reduction of $\mathcal{U}$. For each $u \in$ $\mathcal{U}, \tilde{\Pi}^{(\mu, \lambda)}\left(p_{X}, p_{X Y \mid U}(\cdot, \cdot \mid u)\right)$ is a contineous function of $p_{X Y \mid U}(\cdot, \cdot \mid u)$. Then by the support lemma,

$$
|\mathcal{U}| \leq|\mathcal{X}|-1+1=|\mathcal{X}|
$$

is sufficient to express $|\mathcal{X}|-1$ values of (54) and one value of (55).

## C. Supporting Hyperplain Expressions of $\mathcal{R}\left(p_{X Y}\right)$

In this appendix we prove Property 3 parts b), c). We first prove the part b).

Proof of Property 3 part b): For any $\mu \geq 0$, we have the following chain of inequalities:

$$
\begin{aligned}
& \min _{\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)}\left\{\mu R_{1}+\bar{\mu} R_{2}\right\} \\
& =\min _{p \in \mathcal{P}\left(p_{X Y}\right)}\left\{\mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)\right\} \\
& \stackrel{(a)}{=} \min _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)}\left\{I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)\right\} \\
& =R^{(\mu)}\left(p_{X Y}\right)
\end{aligned}
$$

Step (a) follows from Lemma 8 stating that the cardinality bound $|\mathcal{U}| \leq|\mathcal{X}|+1$ in $\mathcal{P}\left(p_{X Y}\right)$ can be reduced to that $|\mathcal{U}| \leq$ $|\mathcal{X}|$ in $\mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)$.

We next prove the part c). We first prepare a lemma useful to prove this property. From the convex property of the region $\mathcal{R}\left(p_{X Y}\right)$, we have the following lemma.

Lemma 10: Suppose that $\left(\hat{R}_{1}, \hat{R}_{2}\right)$ does not belong to $\mathcal{R}\left(p_{X Y}\right)$. Then there exist $\epsilon>0$ and $\mu_{0} \geq 0$ such that for any $\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)$ we have

$$
\mu_{0}\left(R_{1}-\hat{R}_{1}\right)+\overline{\mu_{0}}\left(R_{2}-\hat{R}_{2}\right)-\epsilon \geq 0
$$

Proof of this lemma is omitted here. Lemma 10 is equivalent to the fact that if the region $\mathcal{R}\left(p_{X Y}\right)$ is a convex set, then for any point $\left(\hat{R}_{1}, \hat{R}_{2}\right)$ outside the region $\mathcal{R}\left(p_{X Y}\right)$, there exists a line which separates the point $\left(\hat{R}_{1}, \hat{R}_{2}\right)$ from the region $\mathcal{R}\left(p_{X Y}\right)$.

Proof of Property 3 part $c$ ): We first prove $\underline{\mathcal{R}}_{\mathrm{sh}}\left(p_{X Y}\right)$ $\subseteq \mathcal{R}\left(p_{X Y}\right)$. We assume that $\left(\hat{R}_{1}, \hat{R}_{2}\right) \notin \mathcal{R}\left(p_{X Y}\right)$. Then by Lemma 10, there exist $\epsilon>0$ and $\mu_{0} \geq 0$ such that for any $\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)$, we have

$$
\mu_{0} \hat{R}_{1}+\overline{\mu_{0}} \hat{R}_{2} \leq \mu_{0} R_{1}+\overline{\mu_{0}} R_{2}-\epsilon
$$

Then we have

$$
\begin{align*}
& \mu_{0} \hat{R}_{1}+\overline{\mu_{0}} \hat{R}_{2} \leq \min _{\left(R_{1}, R_{2}\right) \in \mathcal{R}\left(p_{X Y}\right)}\left\{\mu_{0} R_{1}+\overline{\mu_{0}} R_{2}\right\}-\epsilon \\
& \stackrel{(a)}{=} \min _{p \in \mathcal{P}\left(p_{X Y}\right)}\left\{\mu_{0} I_{p}(U ; X)+\overline{\mu_{0}} H_{p}(Y \mid U)\right\}-\epsilon \\
& \leq \min _{p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)}\left\{\mu_{0} I_{p}(U ; X)+\overline{\mu_{0}} H_{p}(Y \mid U)\right\}-\epsilon \\
& =R^{\left(\mu_{0}\right)}\left(p_{X Y}\right)-\epsilon \tag{56}
\end{align*}
$$

Step (a) follows from the definition of $\mathcal{R}\left(p_{X Y}\right)$. The inequality (56) implies that $\left(\hat{R}_{1}, \hat{R}_{2}\right) \notin \mathcal{R}_{\mathrm{sh}}\left(p_{X Y}\right)$. Thus $\mathcal{R}_{\mathrm{sh}}\left(p_{X Y}\right) \subseteq$ $\mathcal{R}\left(p_{X Y}\right)$ is concluded.

## D. Proof of Property 4 Part b)

In this appendix we prove Property 4 part b). Fix $q=$ $q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$ and $p=p_{U X Y}=\left(p_{U \mid X}, p_{X Y}\right) \in \mathcal{P}_{\operatorname{sh}}($ $p_{X Y}$ ) arbitrary. For $\beta \geq 0, p \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$, and $q_{Y \mid U}$ induced by $q$, define

$$
\begin{aligned}
& \hat{\omega}_{p, q_{Y \mid U}}^{(\mu)}(x, y \mid u) \\
& :=\left[\mu \log \frac{p_{X \mid U}(x \mid u)}{p_{X}(x)}+\bar{\mu} \log \frac{1}{q_{Y \mid U}(y \mid u)}\right] \\
& \hat{\Omega}^{(\mu, \beta)}\left(p, q_{Y \mid U}\right):=-\log \mathrm{E}_{p}\left[\exp \left\{-\beta \hat{\omega}_{p, q_{Y \mid U}}^{(\mu)}(X, Y \mid U)\right\}\right]
\end{aligned}
$$

Then we have the following two lemmas.
Lemma 11: For any $\mu \in[0,1], \alpha \in[0,1)$, and any $q=$ $q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$, there exists $p=p_{U X Y} \in \mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$ such that

$$
\begin{equation*}
\Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right) \geq \bar{\alpha} \hat{\Omega}^{\left(\mu, \frac{\alpha}{\alpha}\right)}\left(p, q_{Y \mid U}\right) \tag{57}
\end{equation*}
$$

Lemma 12: For any $\mu, \alpha$ satisfying $\mu \in[0,1], \alpha \in[0,1 / 2)$, any $p=p_{U X Y} \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)$, and any stochastic matrix $q_{Y \mid U}$ induced by $q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$, we have

$$
\begin{equation*}
\hat{\Omega}^{\left(\mu, \frac{\alpha}{\alpha}\right)}\left(p, q_{Y \mid U}\right) \geq \frac{1-2 \alpha}{\bar{\alpha}} \tilde{\Omega}^{\left(\mu, \frac{\alpha}{1-2 \alpha}\right)}(p) \tag{58}
\end{equation*}
$$

From Lemmas 11 and 12 we have the following corollary.
Corollary 6: For any $\mu, \alpha$ satisfying $\mu \in[0,1], \alpha \in[0,1 / 2)$, and any $q=q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$, there exists $p=p_{U X Y} \in$ $\mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$ such that

$$
\begin{equation*}
\Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right) \geq(1-2 \alpha) \tilde{\Omega}^{\left(\mu, \frac{\alpha}{1-2 \alpha}\right)}(p) \tag{59}
\end{equation*}
$$

From (59), we have that for any $\mu \in[0,1], \alpha \in[0,1 / 2)$, we have

$$
\begin{equation*}
\Omega^{(\mu, \alpha)}\left(p_{X Y}\right) \geq(1-2 \alpha) \tilde{\Omega}^{\left(\mu, \frac{\alpha}{1-2 \alpha}\right)}\left(p_{X Y}\right) \tag{60}
\end{equation*}
$$

Proof of Lemma 11. We fix $(\mu, \alpha) \in[0,1]^{2}$ arbitrary. For each $q=q_{U X Y} \in \mathcal{Q}\left(p_{Y \mid X}\right)$, we choose $p=p_{U X Y} \in$ $\mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ so that $p_{U \mid X}=q_{U \mid X}$. Then we have the following:

$$
\begin{align*}
& \exp \left\{-\Omega^{(\mu, \alpha)}\left(q \mid p_{X}\right)\right\} \\
& =\mathrm{E}_{q}\left[\frac{p_{X}^{\bar{\alpha}}(X)}{q_{X}^{\bar{\alpha}}(X)}\left\{\frac{p_{X}^{\mu \alpha}(X) q_{Y \mid U}^{\bar{\mu} \alpha}(Y \mid U)}{q_{X \mid U}^{\mu \alpha}(X \mid U)}\right\}\right] \\
& =\mathrm{E}_{q}\left[\left\{\frac{p_{U X}(U, X)}{q_{U X}(U, X)}\right\}^{\bar{\alpha}}\left\{\frac{p_{X}^{\mu \frac{\alpha}{\alpha}}(X) q_{Y \mid U}^{\bar{\mu} \frac{\alpha}{\alpha}}(Y \mid U)}{p_{X \mid U}^{\mu \frac{\alpha}{\alpha}}(X \mid U)}\right\}^{\bar{\alpha}}\right. \\
& \left.\times\left\{\frac{p_{X \mid U}^{\mu}(X \mid U)}{q_{X \mid U}^{\mu}(X \mid U)}\right\}^{\alpha}\right] \\
& \stackrel{(a)}{\leq}\left(\mathrm{E}_{q}\left[\frac{p_{U X}(U, X)}{q_{U X}(U, X)} \frac{p_{X}^{\mu \frac{\alpha}{\bar{\alpha}}}(X) q_{Y \mid U}^{\bar{\mu} \frac{\alpha}{\bar{\alpha}}}(Y \mid U)}{p_{X \mid U}^{\mu \frac{\alpha}{\alpha}}(X \mid U)}\right]\right)^{\bar{\alpha}} \\
& \times\left(\mathrm{E}_{q}\left[\frac{p_{X \mid U}^{\mu}(X \mid U)}{q_{X \mid U}^{\mu}(X \mid U)}\right]\right)^{\alpha} \\
& =\exp \left\{-\bar{\alpha} \hat{\Omega}^{\left(\mu, \frac{\alpha}{\alpha}\right)}\left(p, q_{Y \mid U}\right)\right\} A^{\alpha}, \tag{61}
\end{align*}
$$

where we set

$$
A:=\mathrm{E}_{q}\left[\frac{p_{X \mid U}^{\mu}(X \mid U)}{q_{X \mid U}^{\mu}(X \mid U)}\right]
$$

Step (a) follows from Hölder's inequality. From (61), we can see that it suffices to show $A \leq 1$ to complete the proof. When $\mu=1$, we have $A=1$. When $\mu \in[0,1)$, we apply Hölder's inequality to $A$ to obtain

$$
A=\mathrm{E}_{q}\left[\frac{p_{X \mid U}^{\mu}(X \mid U)}{q_{X \mid U}^{\mu}(X \mid U)}\right] \leq\left(\mathrm{E}_{q}\left[\frac{p_{X \mid U}(X \mid U)}{q_{X \mid U}(X \mid U)}\right]\right)^{\mu}=1
$$

Hence we have (57) in Lemma 11
Proof of Lemma [12. We fix $\mu \in[0,1], \alpha \in[0,1 / 2)$, arbitrary. For any $p=p_{U X Y} \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$, and any $q=q_{U X Y} \in$ $\mathcal{Q}\left(p_{Y \mid X}\right)$, we have the following chain of inequalities:

$$
\begin{aligned}
& \exp \left\{-\hat{\Omega}^{\left(\mu, \frac{\alpha}{\alpha}\right)}\left(p, q_{Y \mid U}\right)\right\} \\
&=\mathrm{E}_{p}\left[\left\{\frac{p_{X \mid U}^{\mu \frac{\alpha}{1-2 \alpha}}(X \mid U) p_{Y \mid U}^{\bar{\mu} 1 \frac{\alpha}{1-2 \alpha}}(Y \mid U)}{p_{X}^{\mu \frac{\alpha}{1-2 \alpha}}(X)}\right\}^{\frac{1-2 \alpha}{\bar{\alpha}}}\right. \\
&\left.\times\left\{\frac{q_{Y \mid U}^{\bar{\mu}}(Y \mid U)}{p_{Y \mid U}^{\bar{\mu}}(Y \mid U)}\right\}^{\frac{\alpha}{\alpha}}\right] \\
&= \exp \left\{-\frac{1-2 \alpha}{\bar{\alpha}} \tilde{\Omega}^{\left(\mu, \frac{\alpha}{1-2 \alpha}\right)}(p)\right\} B,
\end{aligned}
$$

where we set

$$
B:=\mathrm{E}_{q}\left[\frac{q_{Y \mid U}^{\bar{\mu}}(Y \mid U)}{p_{Y \mid U}^{\bar{\mu}}(Y \mid U)}\right]
$$

Step (a) follows from Hölder's inequality. From (62), we can see that it suffices to show $B \leq 1$ to complete the proof. In a manner quite smilar to the proof of $A \leq 1$ in the proof of (57) in Lemma 11 we can show that $B \leq 1$. Thus we have (58) in Lemma 12

Proof of Property 4 part b): We evaluate lower bounds of $F\left(R_{1}, R_{2} \mid p_{X Y}\right)$ to obtain the following chain of inequalities:

$$
\begin{align*}
& F\left(R_{1}, R_{2} \mid p_{X Y}\right) \\
& \stackrel{(\mathrm{a})}{\geq} \sup _{\substack{\mu \in[0,1], \alpha \in[0,1 / 2)}} \frac{(1-2 \alpha) \tilde{\Omega}^{\left(\mu, \frac{\alpha}{1-2 \alpha}\right)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}} \\
& =\sup _{\substack{\mu \in[0,1], \alpha \in[0,1 / 2), \lambda=\frac{\alpha}{1-2 \alpha}}} \frac{(1-2 \alpha) \tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}} \\
& \stackrel{(\mathrm{b})}{=} \sup _{\substack{\mu \in[0,1], \alpha=\frac{\lambda}{1+2 \lambda}, \lambda \geq 0}} \frac{(1-2 \alpha) \tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)-\alpha\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\alpha \bar{\mu}} \\
& \stackrel{(\text { c) })}{=} \sup _{\mu \in[0,1], \lambda \geq 0} \frac{\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)-\lambda\left(\mu R_{1}+\bar{\mu} R_{2}\right)}{2+\lambda(5-\mu)} \\
& =\sup _{\mu \in[0,1], \lambda \geq 0} \underline{F}^{(\mu, \beta)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) . \tag{63}
\end{align*}
$$

Step (a) follows from the definition of $F\left(R_{1}, R_{2} \mid p_{X Y}\right)$ and (60) in Corollary 6 Steps (b) and (c) follow from that

$$
\alpha \in[0,1 / 2), \lambda=\frac{\alpha}{1-2 \alpha} \Leftrightarrow \lambda \geq 0, \alpha=\frac{\lambda}{1+2 \lambda}
$$

From (63), we have

$$
\begin{aligned}
& F\left(R_{1}, R_{2} \mid p_{X Y}\right) \geq \sup _{\mu \in[0,1], \lambda \geq 0} \underline{F}^{(\mu, \lambda)}\left(\mu R_{1}+\bar{\mu} R_{2} \mid p_{X Y}\right) \\
& =\underline{F}\left(R_{1}, R_{2} \mid p_{X Y}\right)
\end{aligned}
$$

completing the proof.

## E. Proof of Property 4 parts $c$ ), $d$ ), $e$, and $f$ )

In this appendix we prove Property 4 parts c), d), e), and f). We first prove the part c). and next prove the parts d) and e). We finally prove the part f).

Proof of Property 4 part c): We first prove the second inequality in (8) in the part c). We frist observe that

$$
\begin{equation*}
\exp \left[-\tilde{\Omega}^{(\mu, \lambda)}(p)\right]=\mathrm{E}_{p}\left[\frac{p_{X}^{\mu \lambda}(X) p_{Y \mid U}^{\bar{\mu} \lambda}(Y \mid U)}{p_{X \mid U}^{\mu \lambda}(X \mid U)}\right] \tag{64}
\end{equation*}
$$

Let $\bar{p}_{X}$ be the uniform distribution on $\mathcal{X}$ and let $\bar{p}_{\tilde{\sim}}$ be the uniform distribution on $\mathcal{Y}$. On lower bound of $\exp \left[-\tilde{\Omega}^{(\mu, \lambda)}(p)\right]$ for $p \in \mathcal{P}_{\mathrm{sh}}\left(p_{X Y}\right)$ and $(\mu, \lambda) \in[0,1]^{2}$, we have the following chain of inequalities:

$$
\begin{align*}
& \exp \left[-\tilde{\Omega}^{(\mu, \lambda)}(p)\right]=\frac{1}{|\mathcal{X}|^{\mu \lambda}|\mathcal{Y}|^{\bar{\mu} \lambda}} \mathrm{E}_{p}\left[p_{X \mid U}^{-\mu \lambda}(X \mid U)\right. \\
& \left.\quad \times\left\{\frac{p_{X}(X)}{\bar{p}_{X}(X)}\right\}^{\mu \lambda}\left\{\frac{p_{Y \mid U}(Y \mid U)}{\bar{p}_{Y}(Y)}\right\}^{\bar{\mu} \lambda}\right] \\
& \stackrel{(\mathrm{a})}{\geq} \frac{1}{|\mathcal{X}|^{\mu}|\mathcal{Y}|^{\bar{\mu}}} \mathrm{E}_{p}\left[\left\{\frac{\bar{p}_{X}(X)}{p_{X}(X)}\right\}^{-\mu \lambda}\left\{\frac{\bar{p}_{Y}(Y)}{p_{Y \mid U}(Y \mid U)}\right\}^{-\bar{\mu} \lambda}\right] \\
& \stackrel{(\mathrm{b})}{\geq} \frac{1}{|\mathcal{X}|^{\mu}|\mathcal{Y}|^{\bar{\mu}}}\left(\mathrm{E}_{p}\left[\frac{\bar{p}_{X}(X)}{p_{X}(X)}\right]\right)^{-\mu \lambda} \\
& \quad \times\left(\mathrm{E}_{p}\left[\frac{\bar{p}_{Y}(Y)}{p_{Y \mid U}(Y \mid U)}\right]\right)^{-\bar{\mu} \lambda}=\frac{1}{|\mathcal{X}|^{\mu}|\mathcal{Y}|^{\bar{\mu}}} \tag{65}
\end{align*}
$$

Step (a) follows from that $\lambda \in[0,1]$ and $p_{X \mid U}(x \mid u) \leq 1$ for any $(u, x) \in \mathcal{U} \times \mathcal{X}$. Step (b) follows from the reverse Hölder's inequality. The bound (65) implies the second inequality in (8). We next show that $\tilde{\Omega}^{(\mu, \lambda)}(p) \geq 0$ for $\lambda \in[0,1]$. On upper bounds of $\exp \left[-\tilde{\Omega}^{(\mu, \lambda)}(p)\right]$ for $p \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ and $\lambda \in[0,1]$, we have the following chain of inequalities:

$$
\begin{align*}
& \exp \left[-\tilde{\Omega}^{(\mu, \lambda)}(p)\right] \stackrel{(\mathrm{a})}{\leq} \mathrm{E}_{p}\left[\left\{\frac{p_{X}(X)}{p_{X \mid U}(X \mid U)}\right\}^{\mu \lambda}\right] \\
& \stackrel{\text { (b) }}{\leq}\left\{\mathrm{E}_{p}\left[\frac{p_{X}(X)}{p_{X \mid U}(X \mid U)}\right]\right\}^{\mu \lambda}=1 \tag{66}
\end{align*}
$$

Step (a) follows from 64) and $p_{Y \mid U}(y \mid u) \leq 1$ for any $(u, y) \in \mathcal{U} \times \mathcal{Y}$. Step (b) follows from $\mu \lambda \in[0,1]$ and Hölder's inequality.

Proof of Property 4 parts d) and e): We first prove that for each $p \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$ and $\mu \in[0,1], \tilde{\Omega}^{(\mu, \lambda)}(p)$ is twice differentiable for $\lambda \in[0,1 / 2]$. For simplicity of notations, set

$$
\begin{aligned}
& \underline{a}:=(u, x, y), \underline{A}:=(U, X, Y), \underline{\mathcal{A}}:=\mathcal{U} \times \mathcal{X} \times \mathcal{Y}, \\
& \tilde{\omega}_{p}^{(\mu)}(x, y \mid u):=\varsigma(\underline{a}), \tilde{\Omega}^{(\mu, \lambda)}(p):=\xi(\lambda) .
\end{aligned}
$$

Then we have

$$
\begin{equation*}
\tilde{\Omega}^{(\mu, \lambda)}(p)=\xi(\lambda)=-\log \left[\sum_{\underline{a} \in \underline{\mathcal{A}}} p_{\underline{A}}(\underline{a}) \mathrm{e}^{-\lambda \varsigma(\underline{a})}\right] . \tag{67}
\end{equation*}
$$

The quantity $p^{(\lambda)}(\underline{a})=p_{\underline{A}}^{(\lambda)}(\underline{a}), \underline{a} \in \mathcal{A}$ has the following form:

$$
\begin{equation*}
p^{(\lambda)}(\underline{a})=\mathrm{e}^{\xi(\lambda)} p(\underline{a}) \mathrm{e}^{-\lambda \varsigma(\underline{a})} \tag{68}
\end{equation*}
$$

By simple computations we have

$$
\begin{align*}
& \xi^{\prime}(\lambda)=\mathrm{e}^{\xi(\lambda)}\left[\sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \varsigma(\underline{a}) \mathrm{e}^{-\lambda \varsigma(\underline{a})}\right]=\sum_{\underline{a} \in \underline{\mathcal{A}}} p^{(\lambda)}(\underline{a}) \varsigma(\underline{a}), \\
& \xi^{\prime \prime}(\lambda)=-\mathrm{e}^{2 \xi(\lambda)} \\
& \quad \times\left[\sum_{\underline{a}, \underline{b} \in \underline{\mathcal{A}}} p(\underline{a}) p_{\underline{A}}(\underline{b}) \frac{\{\varsigma(\underline{a})-\varsigma(\underline{b})\}^{2}}{2} \mathrm{e}^{-\lambda\{\varsigma(\underline{a})+\varsigma(\underline{b})\}}\right] \\
& =-\sum_{\underline{a}, \underline{b} \in \underline{\mathcal{A}}} p^{(\lambda)}(\underline{a}) p^{(\lambda)}(\underline{b}) \frac{\{\varsigma(\underline{a})-\varsigma(\underline{b})\}^{2}}{2} \\
& =-\sum_{\underline{a} \in \underline{\mathcal{A}}} p^{(\lambda)}(\underline{a}) \varsigma^{2}(\underline{a})+\left[\sum_{\underline{a} \in \underline{\mathcal{A}}} p^{(\lambda)}(\underline{a}) \varsigma(\underline{a})\right]^{2} \leq 0 . \tag{69}
\end{align*}
$$

On upper bound of $-\xi^{\prime \prime}(\lambda) \geq 0$ for $\lambda \in[0,1 / 2]$, we have the following chain of inequalities:

$$
\begin{align*}
& -\xi^{\prime \prime}(\lambda) \stackrel{(\mathrm{a})}{\leq} \sum_{\underline{a} \in \underline{\mathcal{A}}} p^{(\lambda)}(\underline{a}) \varsigma^{2}(\underline{a}) \stackrel{(\mathrm{b})}{=} \sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \varsigma^{2}(\underline{a}) \mathrm{e}^{-\lambda \varsigma(\underline{a})+\xi(\lambda)} \\
& =\mathrm{e}^{\xi(\lambda)} \sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \sqrt{\mathrm{e}^{-2 \lambda \varsigma(\underline{a})}} \sqrt{\varsigma^{4}(\underline{a})} \\
& \leq \sqrt{(\mathrm{c})} \\
& \leq \sqrt{\mathrm{e}^{2 \xi(\lambda)-\xi(2 \lambda)}} \sqrt{\sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \varsigma^{4}(\underline{a})}  \tag{70}\\
& \leq \sqrt{\mathrm{e}^{2 \xi(\lambda)}} \sqrt{\sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \varsigma^{4}(\underline{a})}
\end{align*}
$$

Step (a) follows from (69). Step (b) follows from 68). Step (c) follows from Cauchy-Schwarz inequality and 67). Step (d) follows from that $\xi(2 \lambda) \geq 0$ for $2 \lambda \in[0,1]$. Note that $\xi(\lambda)$ exists for $\lambda \in[0,1 / 2]$. Furtheomore we have the following:

$$
\sum_{\underline{a} \in \underline{\mathcal{A}}} p(\underline{a}) \varsigma^{4}(\underline{a})<\infty
$$

Hence, by (70), $\xi^{\prime \prime}(\lambda)$ exists for $\lambda \in[0,1 / 2]$. We next prove the part e). We derive the lower bound (9) of $\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right)$. Fix any $(\mu, \lambda) \in[0,1] \times[0,1 / 2]$ and any $p \in \mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$. By the Taylor expansion of $\xi(\lambda)=\tilde{\Omega}^{(\mu, \lambda)}(p)$ with respect to $\lambda$
around $\lambda=0$, we have that for any $p \in \mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$ and for some $\nu \in[0, \lambda]$

$$
\begin{aligned}
& \tilde{\Omega}^{(\mu, \lambda)}(p)=\xi(0)+\xi^{\prime}(0) \lambda+\frac{1}{2} \xi^{\prime \prime}(\nu) \lambda^{2} \\
& =\lambda \mathrm{E}_{p}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right]-\frac{\lambda^{2}}{2} \operatorname{Var}_{p^{(\nu)}}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right]
\end{aligned}
$$

$$
\begin{equation*}
\stackrel{(\mathrm{a})}{\geq} \lambda R^{(\mu)}\left(p_{X Y}\right)-\frac{\lambda^{2}}{2} \operatorname{Var}_{p^{(\nu)}}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y, Z \mid U)\right] . \tag{71}
\end{equation*}
$$

Step (a) follows from $p \in \mathcal{P}_{\operatorname{sh}}\left(p_{X Y}\right)$,

$$
\mathrm{E}_{p}\left[\tilde{\omega}_{p}^{(\mu)}(X, Y \mid U)\right]=\mu I_{p}(X ; U)+\bar{\mu} H_{p}(Y \mid U)
$$

and the definition of $R^{(\mu)}\left(p_{X Y}\right)$. Let $\left(\nu_{\mathrm{opt}}, p_{\mathrm{opt}}\right) \in[0, \lambda] \times$ $\mathcal{P}_{\text {sh }}\left(p_{X Y}\right)$ be a pair which attains $\rho^{(\mu, \lambda)}\left(p_{X Y}\right)$. By this definition we have that

$$
\begin{equation*}
\tilde{\Omega}^{(\mu, \lambda)}\left(p_{\mathrm{opt}}\right)=\tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right) \tag{72}
\end{equation*}
$$

and that for any $\nu \in[0, \lambda]$,

$$
\begin{align*}
& \operatorname{Var}_{p_{\mathrm{opt}}^{(\nu)}}\left[\omega_{p_{\mathrm{opt}}}^{(\mu)}(X, Y \mid U)\right] \\
& \leq \operatorname{Var}_{p_{\mathrm{opt}}^{(\nu \mathrm{opt})}}\left[\omega_{p_{\mathrm{opt}}}^{(\mu)}(X, Y \mid U)\right]=\rho^{(\mu, \lambda)}\left(p_{X Y}\right) \tag{73}
\end{align*}
$$

On lower bounds of $\Omega^{(\mu, \lambda)}\left(p_{X Y}\right)$, we have the following chain of inequalities:

$$
\begin{aligned}
& \tilde{\Omega}^{(\mu, \lambda)}\left(p_{X Y}\right) \stackrel{(\mathrm{a})}{=} \tilde{\Omega}^{(\mu, \lambda)}\left(p_{\mathrm{opt}}\right) \\
& \stackrel{(\mathrm{b})}{\geq} \lambda R^{(\mu)}\left(p_{X Y}\right)-\frac{\lambda^{2}}{2} \operatorname{Var}_{p_{\mathrm{opt}}^{(\nu)}}\left[\tilde{\omega}_{p_{\mathrm{opt}}^{(\mu)}}(X, Y \mid U)\right] \\
& \stackrel{(\mathrm{c})}{\geq} \lambda R^{(\mu)}\left(p_{X Y}\right)-\frac{\lambda^{2}}{2} \rho^{(\mu, \lambda)}\left(p_{X Y}\right) \\
& \stackrel{(\mathrm{d})}{\geq} \lambda R^{(\mu)}\left(p_{X Y}\right)-\frac{\lambda^{2}}{2} \rho\left(p_{X Y}\right)
\end{aligned}
$$

Step (a) follows from (72). Step (b) follows from (71). Step (c) follows from (73). Step (d) follows from the definition of $\rho\left(p_{X Y}\right)$.

To prove the part f ), we use the following lemma.
Lemma 13: When $\tau \in(0,(1 / 2) \rho]$, the maximum of

$$
\frac{1}{2+5 \lambda}\left\{-\frac{\rho}{2} \lambda^{2}+\tau \lambda\right\}
$$

for $\lambda \in(0,1 / 2]$ is attained by the positive $\lambda_{0}$ satisfying

$$
\begin{equation*}
\vartheta\left(\lambda_{0}\right):=\lambda_{0}+\frac{5}{4} \lambda_{0}^{2}=\frac{\tau}{\rho} \tag{74}
\end{equation*}
$$

Let $g(a)$ be the inverse function of $\vartheta(a)$ for $a \geq 0$. Then the condition of (74) is equivalent to $\lambda_{0}=g\left(\frac{\tau}{\rho}\right)$. The maximum is given by

$$
\frac{1}{2+5 \lambda_{0}}\left\{-\frac{\rho}{2} \lambda_{0}^{2}+\tau \lambda_{0}\right\}=\frac{\rho}{4} \lambda_{0}^{2}=\frac{\rho}{4} g^{2}\left(\frac{\tau}{\rho}\right)
$$

By an elementary computation we can prove this lemma. We omit the detail.

Proof of Property 4 part $f$ ): By the hyperplane expression $\mathcal{R}_{\mathrm{sh}}\left(p_{X Y}\right)$ of $\mathcal{R}\left(p_{X Y}\right)$ stated Property 3 part b) we have that when $\left(R_{1}+\tau, R_{2}+\tau\right) \notin \mathcal{R}\left(p_{X Y}\right)$, we have

$$
\begin{equation*}
R^{\left(\mu_{0}\right)}\left(p_{X Y}\right)-\left(\mu_{0} R_{1}+\overline{\mu_{0}} R_{2}\right)>\tau \tag{75}
\end{equation*}
$$

for some $\mu_{0} \in[0,1]$. Then for each positive $\tau$, we have the following chain of inequalities:

$$
\begin{aligned}
& \underline{F}\left(R_{1}, R_{2} \mid p_{X Y}\right) \geq \sup _{\lambda \in(0,1 / 2]} \underline{F}^{\left(\mu_{0}, \lambda\right)}\left(\mu_{0} R_{1}+\overline{\mu_{0}} R_{2} \mid p_{X Y}\right) \\
& =\sup _{\lambda \in(0,1 / 2]} \frac{\tilde{\Omega}^{\left(\mu_{0}, \lambda\right)}\left(p_{X Y}\right)-\lambda\left(\mu_{0} R_{1}+\overline{\mu_{0}} R_{2}\right)}{2+\lambda\left(5-\mu_{0}\right)} \\
& \stackrel{(\text { a) }}{\geq} \sup _{\lambda \in(0,1 / 2]} \frac{1}{2+5 \lambda}\left\{-\frac{\rho}{2} \lambda^{2}\right. \\
& \left.\quad+\lambda R^{\left(\mu_{0}\right)}\left(p_{X Y}\right)-\lambda\left(\mu_{0} R_{1}+\overline{\mu_{0}} R_{2}\right)\right\} \\
& \stackrel{\text { (b) }}{>} \sup _{\lambda \in(0,1 / 2]} \frac{1}{2+5 \lambda}\left\{-\frac{\rho}{2} \lambda^{2}+\tau \lambda\right\} \stackrel{(\text { c) })}{=} \frac{\rho}{4} g^{2}\left(\frac{\tau}{\rho}\right) .
\end{aligned}
$$

Step (a) follows from Property 4 part d). Step (b) follows from (75). Step (c) follows from Lemma 13

## F. Proof of Lemma $\square$

To prove Lemma we prepare a lemma. Set

$$
\mathcal{A}_{n}:=\left\{\left(s, x^{n}, y^{n}\right): \frac{1}{n} \log \frac{p_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right)}{\hat{q}_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right)} \geq-\eta\right\}
$$

Furthermore, set

$$
\begin{aligned}
& \tilde{\mathcal{B}}_{n}:=\left\{x^{n}: \frac{1}{n} \log \frac{p_{X^{n}}\left(x^{n}\right)}{Q_{X^{n}}\left(x^{n}\right)} \geq-\eta\right\} \\
& \mathcal{B}_{n}:=\tilde{\mathcal{B}}_{n} \times \mathcal{M}_{1} \times \mathcal{Y}^{n}, \mathcal{B}_{n}^{c}:=\tilde{\mathcal{B}}_{n}^{c} \times \mathcal{M}_{1} \times \mathcal{Y}^{n} \\
& \tilde{\mathcal{C}}_{n}:=\left\{\left(s, x^{n}\right): s=\varphi_{1}^{(n)}\left(x^{n}\right)\right. \\
&\left.p_{X^{n} \mid S}\left(x^{n} \mid s\right) \leq M_{1} \mathrm{e}^{n \eta} p_{X^{n}}\left(x^{n}\right)\right\} \\
& \mathcal{C}_{n}:=\tilde{\mathcal{C}}_{n} \times \mathcal{Y}^{n}, \mathcal{C}_{n}^{c}:=\tilde{\mathcal{C}}_{n}^{\mathrm{c}} \times \mathcal{Y}^{n} \\
& \mathcal{D}_{n}:=\left\{\left(s, x^{n}, y^{n}\right): s=\varphi_{1}^{(n)}\left(x^{n}\right),\right. \\
&\left.p_{Y^{n} \mid S}\left(y^{n} \mid s\right) \geq\left(1 / M_{2}\right) \mathrm{e}^{-n \eta}\right\} \\
& \mathcal{E}_{n}:=\left\{\left(s, x^{n}, y^{n}\right): \begin{array}{rl}
s=\varphi_{1}^{(n)}\left(x^{n}\right), \\
& \left.\psi^{(n)}\left(\varphi_{1}^{(n)}\left(x^{n}\right), \varphi_{2}^{(n)}\left(y^{n}\right)\right)=y^{n}\right\}
\end{array}\right.
\end{aligned}
$$

Then we have the following lemma.
Lemma 14:

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n}^{\mathrm{c}}\right) \leq \mathrm{e}^{-n \eta}, p_{S X^{n} Y^{n}}\left(\mathcal{B}_{n}^{\mathrm{c}}\right) \leq \mathrm{e}^{-n \eta} \\
& p_{S X^{n} Y^{n}}\left(\mathcal{C}_{n}^{\mathrm{c}}\right) \leq \mathrm{e}^{-n \eta}, p_{S X^{n} Y^{n}}\left(\mathcal{D}_{n}^{\mathrm{c}} \cap \mathcal{E}_{n}\right) \leq \mathrm{e}^{-n \eta}
\end{aligned}
$$

Proof: We first prove the first inequality.

$$
p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n}^{\mathrm{c}}\right)=\sum_{\left(s, x^{n}, y^{n}\right) \in \mathcal{A}_{n}^{\mathrm{c}}} p_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right)
$$

$$
\begin{aligned}
& \stackrel{\text { (a) }}{\leq} \sum_{\left(s, x^{n}, y^{n}\right) \in \mathcal{A}_{n}^{c}} \mathrm{e}^{-n \eta} \hat{q}_{S X^{n} Y^{n}}\left(s, x^{n}, y^{n}\right) \\
& =\mathrm{e}^{-n \eta} \hat{q}_{S X^{n} Y^{n}}\left(\mathcal{A}_{n}^{\mathrm{c}}\right) \leq \mathrm{e}^{-n \eta}
\end{aligned}
$$

Step (a) follows from the definition of $\mathcal{A}_{n}$. On the second inequality we have

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{B}_{n}^{\mathrm{c}}\right)=p_{X^{n}}\left(\tilde{\mathcal{B}}_{n}^{\mathrm{c}}\right)=\sum_{x^{n} \in \tilde{\mathcal{B}}_{n}^{c}} p_{X_{n}}\left(x^{n}\right) \\
& \stackrel{(\mathrm{a})}{\leq} \sum_{x^{n} \in \tilde{\mathcal{B}}_{n}^{c}} \mathrm{e}^{-n \eta} Q_{X^{n}}\left(x^{n}\right)=\mathrm{e}^{-n \eta} Q_{X^{n}}\left(\tilde{\mathcal{B}}_{n}^{\mathrm{c}}\right) \leq \mathrm{e}^{-n \eta} .
\end{aligned}
$$

Step (a) follows from the definition of $\mathcal{B}_{n}$. We next prove the third inequality.

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{C}_{n}^{\mathrm{c}}\right)=p_{S X^{n}}\left(\tilde{\mathcal{C}}_{n}^{\mathrm{c}}\right) \\
& =\sum_{s \in \mathcal{M}_{1}} \sum_{\substack{x^{n}: \varphi_{1}^{(n)}\left(x^{n}\right)=s \\
p_{X^{n}}\left(x^{n}\right) \leq\left(1 / M_{1}\right) \mathrm{e}^{-n \eta} \\
\times \tilde{Q}_{X^{n} \mid S}\left(x^{n} \mid s\right)}} p_{X^{n}}\left(x^{n}\right) \\
& \leq \frac{1}{M_{1}} \mathrm{e}^{-n \eta} \sum_{\substack{s \in \mathcal{M}_{1}}}^{\substack{\begin{subarray}{c}{x^{n}: \varphi_{1}^{(n)}\left(x^{n}\right)=s \\
p_{X^{n}}\left(x^{n}\right) \leq\left(1 / M_{1}\right) \mathrm{e}^{-n \eta} \\
\times \tilde{Q}_{X^{n} \mid S}\left(x^{n} \mid s\right)} }}\end{subarray}} \tilde{Q}_{X^{n} \mid S}\left(x^{n} \mid s\right) \\
& \leq \frac{1}{M_{1}} \mathrm{e}^{-n \eta}\left|\mathcal{M}_{1}\right|=\mathrm{e}^{-n \eta} .
\end{aligned}
$$

Finally we prove the fourth inequality. We first observe that

$$
p_{S}(s)=\sum_{x^{n}: \varphi_{1}^{(n)}\left(x^{n}\right)=s} p_{X^{n}}\left(x^{n}\right), p_{X^{n} \mid S}\left(x^{n} \mid s\right)=\frac{p_{X^{n}}\left(x^{n}\right)}{p_{S}(s)} .
$$

We have the following chain of inequalities:

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{D}_{n}^{\mathrm{c}} \cap \mathcal{E}_{n}\right) \\
& =\sum_{s \in \mathcal{M}_{1}} p_{S}(s) \sum_{\substack{x^{n}: \varphi_{1}^{(n)}\left(x^{n}\right)=s}} p_{X^{n} \mid S}\left(x^{n} \mid s\right) \\
& \times \sum_{\substack{y^{n}: \psi^{(n)}\left(s, \varphi_{2}^{(n)}\left(y^{n}\right)\right)=y^{n} \\
p_{Y}{ }^{n} \mid S\left(y^{n} \mid s\right) \leq\left(1 / M_{2}\right) \mathrm{e}^{-n \eta}}} p_{Y^{n} \mid X^{n}}\left(y^{n} \mid x^{n}\right) \\
& =\sum_{s \in \mathcal{M}_{1}} p_{S}(s) \sum_{\substack{y^{n}: \psi^{(n)}\left(s, \varphi_{2}^{(n)}\left(y^{n}\right)\right)=y^{n} \\
p_{Y^{n} \mid S}\left(y^{n} \mid s\right) \leq\left(1 / M_{2}\right) \mathrm{e}^{-n \eta}}} p_{Y^{n} \mid S}\left(y^{n} \mid s\right) \\
& \leq \sum_{s \in \mathcal{M}_{1}} p_{S}(s) \frac{1}{M_{2}} \mathrm{e}^{-n \eta\left|\left\{y^{n}: \psi^{(n)}\left(s, \varphi_{2}^{(n)}\left(y^{n}\right)\right)=y^{n}\right\}\right|} \\
& \text { (a)} \leq \sum_{s \in \mathcal{M}_{1}} p_{S}(s) \frac{1}{M_{2}} \mathrm{e}^{-n \eta} M_{2}=\mathrm{e}^{-n \eta} .
\end{aligned}
$$

Step (a) follows from that the number of $y^{n}$ correctly decoded does not exceed $M_{2}$.

Proof of Lemma 7. By definition we have

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right) \\
& =p_{S X^{n} Y^{n}}\left\{\frac{1}{n} \log \frac{p_{S X^{n} Y^{n}}\left(S, X^{n}, Y^{n}\right)}{\hat{q}_{S X^{n} Y^{n}\left(S, X^{n}, Y^{n}\right)}^{2} \geq-\eta}\right. \\
& 0 \geq \frac{1}{n} \log \frac{q_{X^{n}}\left(X^{n}\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta \\
& \\
& \quad \frac{1}{n} \log M_{1} \geq \frac{1}{n} \log \frac{p_{X^{n} \mid S}\left(X^{n} \mid S\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta \\
& \\
& \left.\frac{1}{n} \log M_{2} \geq \frac{1}{n} \log \frac{1}{p_{Y^{n} \mid S}\left(Y^{n} \mid S\right)}-\eta\right\}
\end{aligned}
$$

Then for any $\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left\|\varphi_{i}^{(n)}\right\| \leq$
$R_{i}, i=1,2$, we have

$$
\begin{aligned}
& p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right) \\
& \leq p_{S X^{n} Y^{n}}\left\{\frac{1}{n} \log \frac{p_{S X^{n} Y^{n}}\left(S, X^{n}, Y^{n}\right)}{\hat{q}_{S X^{n} Y^{n}\left(S, X^{n}, Y^{n}\right)} \geq-\eta}\right. \\
& 0 \geq \frac{1}{n} \log \frac{q_{X^{n}}\left(X^{n}\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta \\
& R_{1} \geq \frac{1}{n} \log \frac{p_{X^{n} \mid S}\left(X^{n} \mid S\right)}{p_{X^{n}}\left(X^{n}\right)}-\eta, \\
& \left.R_{2} \geq \frac{1}{n} \log \frac{1}{p_{Y^{n} \mid S}\left(Y^{n} \mid S\right)}-\eta\right\}
\end{aligned}
$$

Hence, it suffices to show

$$
\begin{aligned}
\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right) & \leq p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right) \\
& +4 \mathrm{e}^{-n \eta}
\end{aligned}
$$

to prove Lemma 1. By definition we have

$$
\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)=p_{S X^{n} Y^{n}}\left(\mathcal{E}_{n}\right)
$$

Then we have the following.

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \psi^{(n)}\right)=p_{S X^{n} Y^{n}}\left(\mathcal{E}_{n}\right) \\
& =p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n} \cap \mathcal{E}_{n}\right) \\
& \quad+p_{S X^{n} Y^{n}}\left(\left[\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right]^{\mathrm{c}} \cap \mathcal{E}_{n}\right) \\
& \leq p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right) \\
& \quad+p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n}^{\mathrm{c}}\right)+p_{S X^{n} Y^{n}}\left(\mathcal{B}_{n}^{\mathrm{c}}\right) \\
& \quad+p_{S X^{n} Y^{n}}\left(\mathcal{C}_{n}^{\mathrm{c}}\right)+p_{S X^{n} Y^{n}}\left(\mathcal{D}_{n}^{\mathrm{c}} \cap \mathcal{E}_{n}\right) \\
& \quad \leq p_{S X^{n} Y^{n}}\left(\mathcal{A}_{n} \cap \mathcal{B}_{n} \cap \mathcal{C}_{n} \cap \mathcal{D}_{n}\right)+4 \mathrm{e}^{-n \eta}
\end{aligned}
$$

Step (a) follows from Lemma 14 ,

## G. Proof of Lemma 3

In this appendix we prove Lemma 3,
Proof of Lemma 3. We first prove the following Markov chain (18) in Lemma 3 .

$$
S X^{t-1} \leftrightarrow X_{t} \leftrightarrow Y_{t}
$$

We have the following chain of inequalities:

$$
\begin{aligned}
& I\left(Y_{t} ; S X^{t-1} \mid X_{t}\right)=H\left(Y_{t} \mid X_{t}\right)-H\left(Y_{t} \mid S X^{t-1} X_{t}\right) \\
& \leq H\left(Y_{t} \mid X_{t}\right)-H\left(Y_{t} \mid S X^{n}\right) \stackrel{(\mathrm{a})}{=} H\left(Y_{t} \mid X_{t}\right)-H\left(Y_{t} \mid X^{n}\right) \\
& \stackrel{(\mathrm{b})}{=} H\left(Y_{t} \mid X_{t}\right)-H\left(Y_{t} \mid X_{t}\right)=0
\end{aligned}
$$

Step (a) follows from that $S=\varphi_{1}^{(n)}\left(X^{n}\right)$ is a function of $X^{n}$. Step (b) follows from the memoryless property of the information source $\left\{\left(X_{t}, Y_{t}\right)\right\}_{t=1}^{\infty}$. Next we prove the following Markov chain (19) in Lemma 3.

$$
Y^{t-1} \leftrightarrow S X^{t-1} \leftrightarrow\left(X_{t}, Y_{t}\right)
$$

We have the following chain of inequalities:

$$
\begin{aligned}
& I\left(X_{t} Y_{t} ; Y^{t-1} \mid S X^{t-1}\right) \\
& =H\left(Y^{t-1} \mid S X^{t-1}\right)-H\left(Y^{t-1} \mid S X^{t-1} X_{t} Y_{t}\right) \\
& \leq H\left(Y^{t-1} \mid X^{t-1}\right)-H\left(Y^{t-1} \mid X^{n} S Y_{t}\right) \\
& \stackrel{(\mathrm{a})}{=} H\left(Y^{t-1} \mid X^{t-1}\right)-H\left(Y^{t-1} \mid X^{n} Y_{t}\right) \\
& \stackrel{(\mathrm{b})}{=} H\left(Y^{t-1} \mid X^{t-1}\right)-H\left(Y^{t-1} \mid X^{t-1} Y_{t}\right)=0
\end{aligned}
$$

Step (a) follows from that $S=\varphi_{1}^{(n)}\left(X^{n}\right)$ is a function of $X^{n}$. Step (b) follows from the memoryless property of the information source $\left\{\left(X_{t}, Y_{t}\right)\right\}_{t=1}^{\infty}$.

## H. Proof of Lemma 6

In this appendix we prove Lemma 6
Proof of Lemma 6 : By the definition of $p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}(s$, $\left.x^{t}, y^{t}\right)$, for $t=1,2, \cdots, n$, we have

$$
\begin{align*}
& p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right)=C_{t}^{-1} p_{S X^{t} Y^{t}}\left(s, x^{t}, y^{t}\right) \\
& \quad \times \prod_{i=1}^{t} f_{\mathcal{F}_{i}}^{(\mu, \alpha)}\left(x_{i}, y_{i} \mid u_{i}\right) \tag{76}
\end{align*}
$$

Then we have the following chain of equalities:

$$
\begin{align*}
& p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right) \stackrel{(\mathrm{a})}{=} C_{t}^{-1} p_{S X^{t} Y^{t}}\left(s, x^{t}, y^{t}\right) \\
& \times \prod_{i=1}^{t} f_{\mathcal{F}_{i}}^{(\mu, \alpha)}\left(x_{i}, y_{i} \mid u_{i}\right) \\
&= C_{t}^{-1} p_{S X^{t-1}} Y^{t-1}\left(s, x^{t-1}, y^{t-1}\right) \\
& \times \prod_{i=1}^{t-1} f_{\mathcal{F}_{i}}^{(\mu, \alpha)}\left(x_{i}, y_{i} \mid u_{i}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \\
& \stackrel{(\mathrm{b})}{=} C_{t}^{-1} C_{t-1} p_{S X^{t-1} Y^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right)} \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \\
&=\left(\Phi_{t}^{(\mu, \alpha)}\right)^{-1} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)
\end{align*}
$$

Steps (a) and (b) follow from (76). From (77), we have

$$
\begin{align*}
& \Phi_{t}^{(\mu, \alpha)} p_{S X^{t} Y^{t} ; \mathcal{F}^{t}}^{(\mu, \alpha)}\left(s, x^{t}, y^{t}\right)  \tag{78}\\
& =p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) \\
& \quad \times f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right) \tag{79}
\end{align*}
$$

Taking summations of (78) and (79) with respect to $s, x^{t}, y^{t}$, we obtain

$$
\begin{aligned}
& \Phi_{t}^{(\mu, \alpha)}=\sum_{s, x^{t}, y^{t}} p_{S X^{t-1} Y^{t-1} ; \mathcal{F}^{t-1}}^{(\mu, \alpha)}\left(s, x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} Y_{t} \mid S X^{t-1} Y^{t-1}}\left(x_{t}, y_{t} \mid s, x^{t-1}, y^{t-1}\right) f_{\mathcal{F}_{t}}^{(\mu, \alpha)}\left(x_{t}, y_{t} \mid u_{t}\right)
\end{aligned}
$$

completing the proof.

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