Guessing Attacks on Distributed-Storage Systems

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Abstract

The secrecy of a distributed-storage system for passwords is studied. The encoder, Alice, observes a length-n password and describes it using two hints, which she stores in different locations. The legitimate receiver, Bob, observes both hints. In one scenario the requirement is that the expected number of guesses it takes Bob to guess the password approach one as n tends to infinity, and in the other that the expected size of the shortest list that Bob must form to guarantee that it contain the password approach one. The eavesdropper, Eve, sees only one of the hints. Assuming that Alice cannot control which hints Eve observes, the largest normalized (by n) exponent that can be guaranteed for the expected number of guesses it takes Eve to guess the password is characterized for each scenario. Key to the proof are new results on Arikan's guessing and Bunte and Lapidoth's task-encoding problem; in particular, the paper establishes a close relation between the two problems. A rate-distortion version of the model is also discussed, as is a generalization that allows for Alice to produce δ (not necessarily two) hints, for Bob to observe ν (not necessarily two) of the hints, and for Eve to observe η (not necessarily one) of the hints. The generalized model is robust against $\delta - \nu$ disk failures.

1 Introduction

Suppose that some sensitive information X (e.g. password) is drawn from a finite set \mathcal{X} according to some probability mass function (PMF) P_X . A (stochastic) encoder, Alice, maps (possibly using randomization) X to two hints M_1 and M_2 and stores them on different disks in different locations. The hints are intended for a legitimate receiver, Bob, who knows where they are stored and sees both. An eavesdropper, Eve, sees one of the hints but not both;

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we do not know which. Which hint is revealed to Eve is a subtle question. We adopt a conservative approach and assume that, after observing X, an adversarial "genie" reveals to Eve the hint that minimizes her ambiguity. Not allowing the genie to observe X would lead to a weaker form of secrecy (Example 1). Given some notion of ambiguity, we would ideally like Bob's ambiguity about X to be small and Eve's large.

There are several ways to define ambiguity. One approach would be to require that Bob be able to reconstruct X whenever X is "typical" and that the conditional entropy of X given Eve's observation be large. For some scenarios, such an approach might be unsuitable. First, it may not properly address Bob's needs when X is not typical. For example, if Bob must guess X, this approach does not guarantee that the expected number of guesses be small: it only guarantees that the probability of success after one guess be large. It does not indicate the number of guesses that Bob might need when X is atypical. Second, conditional entropy need not be an adequate measure of Eve's ambiguity. For example, if X is some password that Eve wishes to uncover, then we may care more about the number of guesses that Eve needs than about the conditional entropy [1].

In this paper, we assume that Eve wants to guess X with the minimal number of guesses of the form "Is X = x?". We quantify Eve's ambiguity about X by the expected number of guesses that she needs to uncover X. In this sense, Eve faces an instance of the Massey-Arikan guessing problem [2, 3]: When faced with the problem of guessing X after observing that Z = z, where Z denotes Eve's observation, Eve must come up with a guessing order for the elements of \mathcal{X} . Such an order can be specified using a bijective function $G(\cdot|z)$ from \mathcal{X} onto the set $\{1, \ldots, |\mathcal{X}|\}$ —a guessing function with the understanding that if Eve observes z, then the question "Is X = x?" will be her G(x|z)-th question. Eve's expected number of guesses is $\mathbb{E}[G(X|Z)]$. This expectation is minimized if for each $z \in \mathcal{Z}$ the guessing function $G(\cdot|z)$ orders the elements of \mathcal{X} in decreasing order of their posterior probabilities given Z = z.

As to Bob, we will consider two different criteria: In the "guessing version" the criterion is the expected number of guesses it takes Bob to guess X, and in the "list version" the criterion is the expected size of the list that Bob must form to guarantee that it contain X.

The former criterion is natural when Bob can check whether a guess is correct: if X is some password, then Bob can stop guessing as soon as he has gained access to the account that is secured by X. The latter criterion is appropriate if Bob does not know whether a guess is correct. For example, if X is a task that Bob must perform, then the only way for Bob to make sure that he performs X is to perform all the tasks in the list \mathcal{L}_{M_1,M_2} comprising the tasks that have positive posterior probabilities given his observation. In this scenario, a good measure for Bob's ambiguity about X is the expected number of tasks that he must perform, i.e., $\mathbb{E}[|\mathcal{L}_{M_1,M_2}|]$, and this will be small whenever Alice is a good task-encoder for Bob [4].

Alternatively, the list-size criterion can also be viewed as a worst-case version of the guessing criterion: Even if Bob is incognizant of the PMF of X, the number of guesses it takes him to guess X can be guaranteed not to exceed the size of the smallest list that is

guaranteed to contain X.

The guessing and the list-size criterion for Bob lead to similar results in the following sense: Clearly, every guessing function $G(\cdot|M_1, M_2)$ for X that guesses the elements of \mathcal{X} of zero posterior probability only after those of positive posterior probabilities satisfies $\mathbb{E}[G(X|M_1, M_2)] \leq \mathbb{E}[|\mathcal{L}_{M_1,M_2}|]$. Conversely, one can prove that every pair of ambiguities for Bob and Eve that is achievable in the guessing version is—up to polylogarithmic factors of $|\mathcal{X}|$ —also achievable in the list version (Remark 18). These polylogarithmic factors wash out in the asymptotic regime where the sensitive information is an *n*-tuple and *n* tends to infinity.

Things are different for Eve: applying the list-size criterion for Eve would lead to results that markedly differ from those that apply under the guessing criterion; see Theorem 19 and the subsequent discussion.

To derive our results, we establish new results on guessing and task-encoding: we relate task-encoders to guessing functions (Theorem 8), and we quantify how additional side information can help guessing (Lemma 5). These results may be of interest in their own right. For example, the former result leads to alternative proofs of Bunte and Lapidoth's asymptotic task-encoding results [4, Theorems I.2 and VI.2] as well as the direct part of [5, Theorem I.1], which states that, in the presence of feedback, the listsize capacity of a discrete-memoryless channel (DMC) with positive zero-error capacity equals the cutoff rate with feedback (which is in fact equal to that without feedback [5, Corollary I.4]). The latter result on how additional side information can help guessing is related to [6]: To quantify how additional side information can help guessing, we establish how an encoder must describe X to minimize the expected number of guesses that a decoder needs to guess X. The list-size analog is Lapidoth and Pfister's optimal task-encoder [6], which describes X to minimize the expected size of the decoder's list. Despite the close relation between task-encoding and guessing, an optimal encoder for a guessing decoder is typically quite different from an optimal task-encoder.

We also generalize our problem in two different directions. The first, along the lines of [7, 4], is a rate-distortion version of the model where Bob and Eve are content with reconstructing the sensitive information to within some given allowed distortion. The second considers the case where Alice produces δ s-bit hints, Bob sees $\nu \leq \delta$ hints, and Eve sees $\eta < \nu$ hints (not necessarily a subset of those that Bob sees). This may model a scenario where the hints are stored on different disks and we want to guarantee robustness against the failure of $\delta - \nu$ disks and the compromise of η disks. We adopt again a conservative approach and assume that, after observing X, an adversarial genie reveals to Bob the ν hints that maximize his ambiguity and to Eve the η hints that minimize her ambiguity. This guarantees that—no matter which disks fail—the model be robust against the failure of $\delta - \nu$ disks and the compromise of η disks. The generalized model is a distributed-storage system, which is static in the sense that failed disks are not replaced.

The case where X is drawn uniformly, Bob must reconstruct X, and Eve's observation must satisfy some information-theoretic security criterion (e.g., the mutual information between Eve's observation and X must be null) corresponds to the erasure-erasure wiretap channel studied in [8] and is a special case of the wiretap networks in [9, 10]. In the literature, this setting is also known as "secret sharing." In traditional secret sharing, each set of hints either reveals X or reveals no information about X [11, 12]. More general are ramp schemes, where any ν hints reveal X and the amount of information that fewer-than- ν hints reveal is controlled (see e.g. [13]). Our setting is different in that we assume $X \sim P_X$ and in that, using some notion of ambiguity, we quantify how difficult it is for Bob and Eve to reconstruct X.

To better bring out the role of Rényi entropy, we generalize the models and replace expectations with ρ -th moments. (The generalization comes with no extra effort.) For an arbitrary $\rho > 0$, we thus study the ρ -th (instead of the first) moment of the list-size and of the number of guesses. Moreover, we shall allow some side information Y that is available to all parties.

The connection between Rényi entropy and the ρ -th moment of the minimum number of guesses has been studied extensively in the literature [3, 14, 15, 16]. The connection with encoding tasks was studied in [4].

The idea to quantify Eve's ambiguity by the ρ -th moment of the number of guesses she needs to uncover X is due to Arikan and Merhav, who studied the Shannon cipher system with a guessing wiretapper [1]. Their approach was later adopted in [17, 18]. The current setting differs from the ones in [1, 17, 18] in the following sense: Instead of mapping X to a public message using a secret key, which is available to Bob but not to Eve, here Alice produces two hints and stores them so that Bob sees both but Eve sees only one. Moreover, unlike [1, 17, 18] we do not measure Bob's ambiguity in terms of the probability that X is not his first guess.

The rest of this paper is structured as follows. Section 2 briefly describes our notation and summarizes some notions and results pertaining to the guessing problem and the problem of encoding tasks. In Section 3, we quantify how additional side information can help guessing and relate task-encoders to guessing functions, thereby establishing the prerequisites for the proofs of our main results. Section 4 contains the problem statement and the main results (both finite-blocklength and asymptotic). The results are discussed in Section 5 and proved in Section 6. Section 7 generalizes the model to allow for a limited number of disk failures. Section 8 considers the rate-distortion version of the problem stated in Section 4 and extends the results on guessing and task-encoding of Section 3 accordingly. Section 9 concludes the paper.

2 Notation and Preliminaries

In this paper (X, Y) is a pair of chance variables that is drawn from the finite set $\mathcal{X} \times \mathcal{Y}$ according to the PMF $P_{X,Y}$, and $\rho > 0$ is fixed. We denote by P_X the marginal PMF of X

and by P_Y the marginal PMF of Y, e.g.,

$$P_X(x) = \sum_{y \in \mathcal{Y}} P_{X,Y}(x,y), \ \forall x \in \mathcal{X}.$$

For every positive integer $n \in \mathbb{N}$ we denote by $P_{X,Y}^n$ the *n*-fold product of $P_{X,Y}$, i.e.,

$$P_{X,Y}^{n}(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{n} P_{X,Y}(x_{i},y_{i}), \ \forall (\mathbf{x},\mathbf{y}) \in \mathcal{X}^{n} \times \mathcal{Y}^{n}$$

A generic probability measure on a measurable space (Ω, \mathcal{F}) is denoted \mathbb{P} , i.e., whenever we introduce a set of chance variables (e.g., X and Y), we denote by \mathbb{P} the probability measure associated with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the chance variables live.

For some positive integer k, we denote by \oplus_k addition modulo k, so $\alpha \oplus_k \beta$ is for any pair of integers (α, β) the unique element $\gamma \in \{0, \ldots, k-1\}$ satisfying

$$\gamma \equiv \alpha + \beta \mod k$$

We denote by \mathbb{F}_q the Galois field with q elements.

By default $\log(\cdot)$ denotes base-2 logarithm, and $\ln(\cdot)$ denotes natural logarithm. We denote by $\alpha \lor \beta$ the maximum of two real numbers α and β and by $\alpha \land \beta$ their minimum. For some real number α , we denote by $[\alpha]^+$ the maximum of α and zero

$$[\alpha]^+ = \alpha \vee 0$$

by $\lceil \alpha \rceil$ the smallest integer that is at least as large as α , and by $\lfloor \alpha \rfloor$ the largest integer that is at most as large as α . We sometimes use the identity

$$\left[\xi\right]^{\rho} < 1 + 2^{\rho}\xi^{\rho}, \quad \xi \in \mathbb{R}_{0}^{+}, \tag{1}$$

which is easily checked by considering separately the cases $0 \le \xi \le 1$ and $\xi > 1$ [4].

2.1 The Conditional Rényi Entropy

To describe our results, we shall need the conditional version of Rényi entropy (originally proposed by Arimoto [19] and also studied in [4, 20])

$$H_{\alpha}(X|Y) = \frac{\alpha}{1-\alpha} \log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{X,Y}(x,y)^{\alpha} \right)^{1/\alpha},$$
(2)

where $\alpha \in [0, \infty]$ is the order and where the cases where α is 0, 1, or ∞ are treated by a limiting argument. Let $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ be a discrete-time stochastic process with finite alphabet $\mathcal{X} \times \mathcal{Y}$. Whenever the limit as *n* tends to infinity of $H_{\alpha}(X^n|Y^n)/n$ exists, we denote it by $H_{\alpha}(\mathbf{X}|\mathbf{Y})$ and call it conditional Rényi entropy-rate. In this paper α will equal $1/(1 + \rho)$, and thus, since $\rho > 0$, will take values in the set (0, 1). To simplify notation, we henceforth write $\tilde{\rho}$ for $1/(1 + \rho)$

$$\tilde{\rho} \triangleq \frac{1}{1+\rho}.$$
(3)

The conditional Rényi entropy satisfies the following properties (see, e.g. [20, Theorem 2]):

Lemma 1. Let (X, Y, Z) be a triple of chance variables taking values in the finite set $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ according to the joint PMF $P_{X,Y,Z}$. For every $\alpha \in [0, \infty]$

$$H_{\alpha}(X|Y) \le H_{\alpha}(X,Z|Y). \tag{4}$$

Lemma 2. [20, Theorem 3] Let (X, Y, Z) be a triple of chance variables taking values in the finite set $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ according to the joint PMF $P_{X,Y,Z}$. For every $\alpha \in [0, \infty]$

$$H_{\alpha}(X|Y,Z) \ge H_{\alpha}(X,Z|Y) - \log|\mathcal{Z}|.$$
(5)

2.2 Optimal Guessing Functions and Task-Encoders

Suppose we want to guess X with guesses of the form "Is X = x?" Following the notation of [3], we call a bijection $G: \mathcal{X} \to \{1, \ldots, |\mathcal{X}|\}$ a guessing function for X. The guessing function determines the guessing order: If we use $G(\cdot)$ to guess X, then the question "Is X = x?" will be our G(x)-th question. With a slight abuse of the term "function," we call $G(\cdot|Y)$ a guessing function for X given Y if the mapping $G(\cdot|y): \mathcal{X} \to \{1, \ldots, |\mathcal{X}|\}$ is for every $y \in \mathcal{Y}$ a guessing function for X. If we use $G(\cdot|Y)$ to guess X from the observation Y and observe that Y = y, then the question "Is X = x?" will be our G(x|y)-th question.

In the following we shall consider guessing functions for X given Y. Since every guessing function for X can be viewed as a guessing function for X given Y for the case where Y is null, the results also apply to guessing functions for X.

The performance of a guessing function is studied in terms of the ρ -th moment of the number of guesses that we need to guess X when we use that function. That is, the expectation $\mathbb{E}[G(X|Y)^{\rho}]$ is the performance of $G(\cdot|Y)$. We say that a guessing function $G(\cdot|Y)$ is optimal if its performance is optimal, i.e., $G(\cdot|Y)$ is optimal if, and only if, (iff) it minimizes $\mathbb{E}[G(X|Y)^{\rho}]$ among all the guessing functions for X given Y. It is easy to see that a guessing function $G(\cdot|Y)$ is optimal iff for every $y \in \mathcal{Y}$, the function $G(\cdot|y)$ orders the possible realizations of X in decreasing order of their posterior probabilities given Y = y. We can use Arikan's results on guessing [3] to bound the performance of optimal guessing functions:

Theorem 3 (On the Performance of Optimal Guessing Functions). [3, Theorem 1 and Proposition 4] There exists some guessing function $G(\cdot|Y)$ for which

$$\mathbb{E}\left[G(X|Y)^{\rho}\right] \le 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(6)

Conversely, for every guessing function $G(\cdot|Y)$

$$\mathbb{E}\left[G(X|Y)^{\rho}\right] \ge \left(1 + \ln|\mathcal{X}|\right)^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \lor 1.$$
(7)

For task-encoders we adopt the terminology of [4]. Given some finite set of descriptions \mathcal{Z} , we call a mapping $f: \mathcal{X} \to \mathcal{Z}$ a *task-encoder* for X. We associate every task-encoder with a decoder of the form

$$f^{-1} \colon \mathcal{Z} \to 2^{\mathcal{X}}$$
$$z \mapsto \Big\{ x \in \mathcal{X} \colon \big\{ P_X(x) > 0 \big\} \cap \big\{ f(x) = z \big\} \Big\}.$$
(8)

If the encoder describes X by $Z \triangleq f(X)$, then the list $\mathcal{L}_Z \triangleq f^{-1}(Z)$ produced by the decoder is the list containing all the realizations of X of positive a priori probability that the encoder could have described by Z. (This is the shortest list that is almost-surely guaranteed to contain X given its description Z.)

Consider now the scenario where some side information Y is revealed to the encoder and decoder [4, Section VI]. In this scenario we call $f(\cdot|Y)$ a task-encoder for X given Y if the mapping $f(\cdot|y): \mathcal{X} \to \mathcal{Z}$ is for every $y \in \mathcal{Y}$ a task-encoder for X. We associate every taskencoder with a decoder $f^{-1}(\cdot|Y)$ satisfying for every $y \in \mathcal{Y}$ that $f^{-1}(\cdot|y)$ is of the form (8), i.e., that

$$f^{-1}(\cdot|y) \colon \mathcal{Z} \to 2^{\mathcal{X}}$$

$$z \mapsto \left\{ x \in \mathcal{X} \colon \left\{ P_{X|Y}(x|y) > 0 \right\} \cap \left\{ f(x|y) = z \right\} \right\}.$$
(9)

If, upon observing Y, the encoder describes X by $Z \triangleq f(X|Y)$, then the list $\mathcal{L}_Z^Y \triangleq f^{-1}(Z|Y)$ produced by the decoder is the list containing all the realizations of X that—given the side information Y—have a positive posterior probability under $P_{X|Y}$ and that the encoder could have described by Z.

In the following we shall consider task-encoders for X given Y. Since every task-encoder for X can be viewed as a task-encoder for X given Y for the case where Y is null, the results also apply to task-encoders for X.

We shall also need the notion of a *stochastic task-encoder*. Such an encoder associates with every possible realization $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of the pair (X, Y) a PMF on \mathcal{Z} and, upon observing the side information y, describes x by drawing Z from \mathcal{Z} according to the PMF associated with (x, y). The conditional probability that Z = z given (X, Y) = (x, y) is thus determined by the stochastic encoder, and we denote it by

$$\mathbb{P}[Z = z | X = x, Y = y], \quad (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}.$$
(10)

Based on (Y, Z) the decoder associated with the encoder (10) produces the smallest list \mathcal{L}_Z^Y that is guaranteed to contain X, i.e., if (Y, Z) = (y, z), then the decoder produces the list

$$\mathcal{L}_{z}^{y} = \left\{ x \in \mathcal{X} : \mathbb{P}[X = x | Y = y, Z = z] > 0 \right\}, \quad (y, z) \in \mathcal{Y} \times \mathcal{Z}$$
(11)

of all the possible realizations $x \in \mathcal{X}$ of X of positive posterior probability

$$\mathbb{P}[X = x | Y = y, Z = z] = \frac{P_{X,Y}(x,y) \mathbb{P}[Z = z | X = x, Y = y]}{\sum_{\tilde{x} \in \mathcal{X}} P_{X,Y}(\tilde{x},y) \mathbb{P}[Z = z | X = \tilde{x}, Y = y]}.$$
(12)

We assess the performance of a task-encoder in terms of the ρ -th moment $\mathbb{E}\left[|\mathcal{L}_Z^Y|^{\rho}\right]$ of the size of the list that the associated decoder must form. As we argue shortly, deterministic task-encoders are optimal in the sense that for every stochastic task-encoder there exists a deterministic task-encoder that performs at least as well. Therefore, we can use Bunte and Lapidoth's results on deterministic task-encoders [4] to bound the performance of optimal stochastic task-encoders: **Theorem 4** (On the Performance of the Optimal Task-Encoders). [4, Theorem VI.1] Let \mathcal{Z} be a finite set. If $|\mathcal{Z}| > \log |\mathcal{X}| + 2$, then there exists a deterministic task-encoder $f(\cdot|Y)$ for which

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] = \mathbb{E}\left[f^{-1}\left(f(X|Y)\big|Y\right)^{\rho}\right] < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{Z}| - \log|\mathcal{X}| - 2) + 2)}.$$
(13)

Conversely, given any stochastic task-encoder (10), the associated decoding lists $\{\mathcal{L}_z^y\}$ (11) satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{Z}|)} \vee 1.$$
(14)

We conclude this section by showing that for every stochastic task-encoder there exists a deterministic task-encoder that performs at least as well. Given a stochastic task-encoder (10) with associated decoding lists (11), we can construct a deterministic task-encoder $f(\cdot|Y)$ as follows. If $(x, y) \in \mathcal{X} \times \mathcal{Y}$ satisfies $P_{X|Y}(x|y) > 0$, then we choose f(x|y) as one that among all elements of $\{z \in \mathcal{Z} : x \in \mathcal{L}_z^y\}$ —minimizes $|\mathcal{L}_z^y|$, so

$$f(x|y) \in \arg\min_{z \in \mathcal{Z}: \ x \in \mathcal{L}_z^y} |\mathcal{L}_z^y|.$$
(15)

Otherwise, we choose f(x|y) to be an arbitrary element of \mathcal{Z} . It then follows from (9) that the deterministic task-encoder performs at least as well as the stochastic task-encoder:

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\sum_{z\in\mathcal{Z}}P_{X,Y}(x,y)\mathbb{P}[Z=z|X=x,Y=y]|\mathcal{L}_{z}^{y}|^{\rho}$$
(16)

$$\geq \sum_{\substack{(x,y)\in\mathcal{X}\times\mathcal{Y}:\\P_{X,Y}(x,y)>0}} \sum_{z\in\mathcal{Z}} P_{X,Y}(x,y) \mathbb{P}[Z=z|X=x,Y=y] \min_{z'\in\mathcal{Z}:\ x\in\mathcal{L}_{z'}^y} |\mathcal{L}_{z'}^y|^{\rho}$$
(17)

$$= \sum_{\substack{(x,y)\in\mathcal{X}\times\mathcal{Y}:\\P_{X,Y}(x,y)>0}} P_{X,Y}(x,y) \min_{z'\in\mathcal{Z}:\ x\in\mathcal{L}_{z'}^y} |\mathcal{L}_{z'}^y|^{\rho}$$
(18)

$$=\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}^{n,y+n,y} P_{X,Y}(x,y) \left|\mathcal{L}_{f(x|y)}^{y}\right|^{\rho}$$
(19)

$$\stackrel{a)}{\geq} \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} P_{X,Y}(x,y) \left| f^{-1} \big(f(x|y) \big| y \big) \right|^{\rho}$$

$$\tag{20}$$

$$= \mathbb{E}\Big[\Big|f^{-1}\big(f(X|Y)\big|Y\big)\Big|^{\rho}\Big],\tag{21}$$

where (a) holds because (9) and (15) imply that $f^{-1}(f(x|y)|y) \subseteq \mathcal{L}_{f(x|y)}^{y}$.

3 Lists and Guesses

In this section we relate task-encoders to guessing functions and explain why the performance guarantees for optimal guessing functions (Theorem 3) and task-encoders (Theorem 4) are remarkably similar. Moreover, we quantify how additional side information can help guessing.

We shall need these results to characterize the secrecy of the distributed-storage systems we study in the present paper, but they may also be of independent interest.

We start by quantifying how some additional information Z (e.g., some description produced by an encoder) can help guessing. As the following lemma shows, Z can reduce the ρ -th moment of the number of guesses by at most a factor of $|\mathcal{Z}|^{-\rho}$:

Lemma 5. Given a finite set Z, draw Z from Z according to some conditional PMF $P_{Z|X,Y}$, so $(X, Y, Z) \sim P_{X,Y} \times P_{Z|X,Y}$. For optimal guessing functions $G^*(\cdot|Y, Z)$ and $G^*(\cdot|Y)$ (which minimize $\mathbb{E}[G(X|Y, Z)^{\rho}]$ and $\mathbb{E}[G(X|Y)^{\rho}]$, respectively)

$$\mathbb{E}\left[G^{\star}(X|Y,Z)^{\rho}\right] \ge \mathbb{E}\left[\left[G^{\star}(X|Y)/|\mathcal{Z}|\right]^{\rho}\right].$$
(22)

Equality holds whenever Z = f(X, Y) for some mapping $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for which $f(x, y) = f(\tilde{x}, y)$ implies either $[G^*(x|y)/|\mathcal{Z}|] \neq [G^*(\tilde{x}|y)/|\mathcal{Z}|]$ or $x = \tilde{x}$. Such a mapping always exists, because for all $l \in \mathbb{N}$ at most $|\mathcal{Z}|$ different $x \in \mathcal{X}$ satisfy $[G^*(x|y)/|\mathcal{Z}|] = l$.

Proof. To prove (22) we first show that

$$\mathbb{E}\left[G^{\star}(X|Y,Z)^{\rho}\right]$$

is minimum if Z is deterministic given (X, Y). Indeed, define the function $g: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ so that $g(x, y) \in \arg\min_{z \in \mathcal{Z}} G^{\star}(x|y, z)$ holds for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. This implies that

$$G^{\star}(X|Y,Z) \ge G^{\star}(X|Y,g(X,Y))$$
(23)

and consequently that

$$\mathbb{E}\left[G^{\star}(X|Y,Z)^{\rho}\right] \ge \min_{G(\cdot|Y)} \mathbb{E}\left[G\left(X|Y,g(X,Y)\right)^{\rho}\right].$$
(24)

It thus suffices to prove (22) for the case where Z is deterministic given (X, Y), and we thus assume w.l.g. that Z = g(X, Y) for some function $g: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. For every guessing function $G(\cdot|Y, g(X, Y))$ we have

$$\mathbb{E}\left[G\left(X|Y,g(X,Y)\right)^{\rho}\right] = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} P_{X,Y}(x,y) G\left(x|y,g(x,y)\right)^{\rho}.$$
(25)

Moreover, for every distinct $x, \tilde{x} \in \mathcal{X}$ and every $y \in \mathcal{Y}$ the equality

$$G(x|y,g(x,y)) = G(\tilde{x}|y,g(\tilde{x},y))$$

implies that $g(x, y) \neq g(\tilde{x}, y)$, because $G(\cdot|y, z) \colon \mathcal{X} \to \{1, \ldots, |\mathcal{X}|\}$ is for every $z \in \mathcal{Z}$ oneto-one. Consequently, for every $\ell \in \mathbb{N}$ there are at most $|\mathcal{Z}|$ different $x \in \mathcal{X}$ for which $G(x|y,g(x,y)) = \ell$. For every $y \in \mathcal{Y}$ order the possible realizations of X in decreasing order of $P_{X,Y}(x,y)$ or, equivalently, in decreasing order of their posterior probabilities given Y = y, and let x_j^y denote the *j*-th element. Clearly, (25) is minimum over $g(\cdot, \cdot)$ and $G(\cdot|Y, g(X,Y))$ if for every $\ell \in \mathbb{N}$ and every $y \in \mathcal{Y}$ we have $G(x|y,g(x,y)) = \ell$ whenever $x = x_j^y$ for some *j* satisfying $(\ell - 1)|\mathcal{Z}| + 1 \leq j \leq \ell|\mathcal{Z}|$ or, equivalently, $[j/|\mathcal{Z}|] = \ell$. Since $G^*(\cdot|Y)$ minimizes $\mathbb{E}[G(X|Y)^{\rho}], \text{ it orders the elements of } \mathcal{X} \text{ in decreasing order of their posterior probabilities} \\ \text{given } Y, \text{ and consequently we can choose } x_j^y \text{ to be the unique } x \in \mathcal{X} \text{ for which } G^{\star}(x|y) = j. \\ \text{Hence, (25) is minimized if } f(\cdot, \cdot) \text{ satisfies the specifications in the lemma, } g(\cdot, \cdot) = f(\cdot, \cdot), \text{ and} \\ G(x|y, f(x, y)) = \left[G^{\star}(x|y)/|\mathcal{Z}|\right] \text{ (see Figure 1). Moreover, the minimum equals the RHS of (22).} \\ \Box$

One can infer from Lemma 5 how to construct an optimal encoder $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for a guessing decoder, i.e., an encoder Z = f(X, Y) that minimizes $\min_{G(\cdot|Y,Z)} \mathbb{E}[G(X|Y,Z)^{\rho}]$ among all the possible descriptions Z that are drawn from \mathcal{Z} according to some conditional PMF $P_{Z|X,Y}$. To that end recall that a guessing function $G(\cdot|Y)$ is optimal, i.e., minimizes $\mathbb{E}[G(X|Y)^{\rho}]$, iff for every $y \in \mathcal{Y} G(\cdot|y)$ orders the possible realizations of X in decreasing order of their posterior probabilities given Y = y. An optimal encoder $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for a guessing decoder can be constructed as follows: For every $y \in \mathcal{Y}$ we first order the possible realizations of X in decreasing order of $P_{X,Y}(x,y)$ or, equivalently, in decreasing order of their posterior probabilities given Y = y, and we let x_j^y denote the j-th element. (Ties are resolved at will.) We then choose some mapping $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for which $f(x_j^y, y) = f(x_{j'}^y, y)$ implies either $[j/|\mathcal{Z}|] \neq [j'/|\mathcal{Z}|]$ or j = j', e.g., by indexing the elements of \mathcal{Z} by the elements of $\{0, \ldots, |\mathcal{Z}| - 1\}$ and choosing $f(x_j^y, y)$ as the element of \mathcal{Z} indexed by the remainder of the Euclidean division of j - 1 by $|\mathcal{Z}|$ (see Figure 1).



Figure 1: How to construct an optimal encoder $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for a guessing decoder when $\mathcal{Z} = \{\star, \bullet, \diamond\}$. Light background tones indicate small values of $P(\cdot|y)$ or $G^{\star}(\cdot|y)$.

Lemma 5 and (1) imply the following corollary:

Corollary 6. Given a finite set \mathcal{Z} , there exists some mapping $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ such that

$$\min_{G(\cdot|Y,Z)} \mathbb{E}\left[G(X|Y,Z)^{\rho}\right] < 1 + 2^{\rho}|\mathcal{Z}|^{-\rho} \min_{G(\cdot|Y)} \mathbb{E}\left[G(X|Y)^{\rho}\right],\tag{26}$$

where Z denotes f(X,Y). Conversely, for every chance variable Z that takes values in \mathcal{Z}

$$\min_{G(\cdot|Y,Z)} \mathbb{E}\left[G(X|Y,Z)^{\rho}\right] \ge |\mathcal{Z}|^{-\rho} \min_{G(\cdot|Y)} \mathbb{E}\left[G(X|Y)^{\rho}\right] \lor 1.$$
(27)

From Corollary 6 and Theorem 3, which characterizes the performance of optimal guessing functions $G(\cdot|Y)$, we obtain the following upper and lower bounds on the smallest ambiguity $\min_{G(\cdot|Y,Z)} \mathbb{E}[G(X|Y,Z)^{\rho}]$ that is achievable for a given $|\mathcal{Z}|$. The bounds are tight up to polylogarithmic factors of $|\mathcal{X}|$.

Corollary 7. Given a finite set \mathcal{Z} , there exists some mapping $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ for which

$$\min_{G(\cdot|Y,Z)} \mathbb{E}\left[G(X|Y,Z)^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{Z}| + 1)},\tag{28}$$

where Z denotes f(X,Y). Conversely, for every chance variable Z that takes values in \mathcal{Z}

$$\min_{G(\cdot|Y,Z)} \mathbb{E}\left[G(X|Y,Z)^{\rho}\right] \ge \left(1 + \ln|\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{Z}|)} \vee 1.$$
(29)

Note that (29) also follows from (7) in Theorem 3 and the properties of conditional Rényi entropy in Lemmas 1 and 2.

The performance guarantees for optimal guessing functions (Theorem 3 and Corollary 7) and task-encoders (Theorem 4) are remarkably similar. To provide some intuition on this, we relate task-encoders to guessing functions. As the following theorem shows, a "good" guessing function "induces" a "good" task-encoder and vice versa:¹

Theorem 8. Let \mathcal{Z} be a finite set.

 Given any stochastic task-encoder (10), the associated decoding lists {L^y_z} (11) induce a guessing function G(·|Y) that satisfies

$$\mathbb{E}\left[G(X|Y)^{\rho}\right] \le |\mathcal{Z}|^{\rho} \mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right].$$
(30)

2. Every guessing function $G(\cdot|Y)$ and every positive integer $\omega \leq |\mathcal{X}|$ satisfying

$$|\mathcal{Z}| \ge \omega \left(1 + \left\lfloor \log \left\lceil |\mathcal{X}| / \omega \right\rceil \right\rfloor \right)$$
(31)

induce a deterministic task-encoder, i.e., a stochastic task-encoder whose conditional PMF (10) is $\{0,1\}$ -valued, whose associated decoding lists $\{\mathcal{L}_z^y\}$ (11) satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] \leq \mathbb{E}\left[\left[G(X|Y)/\omega\right]^{\rho}\right].$$
(32)

To prove Theorem 8, we need the following fact:

Fact 9. For every $k \in \mathbb{N}$

$$\left|\left\{\tilde{k} \in \mathbb{N} \colon \left\lfloor \log \tilde{k} \right\rfloor = \left\lfloor \log k \right\rfloor\right\}\right| \le k.$$
(33)

¹We call a guessing function or task-encoder "good" if its performance is nearly optimal, and "induce" means here that—without knowing the PMF $P_{X,Y}$ —we can construct from a guessing function a task-encoder and vice versa.

Proof of Fact 9. If $k, \tilde{k} \in \mathbb{N}$ are such that $\lfloor \log \tilde{k} \rfloor = \lfloor \log k \rfloor$, then

$$2^{\lfloor \log k \rfloor} \le \tilde{k} < 2^{\lfloor \log k \rfloor + 1}.$$
(34)

Hence,

$$\left|\left\{\tilde{k} \in \mathbb{N} \colon \left\lfloor \log \tilde{k} \right\rfloor = \left\lfloor \log k \right\rfloor\right\}\right| \le 2^{\left\lfloor \log k \right\rfloor} \le k.$$
(35)

Proof of Theorem 8. As to the first part, suppose we are given a stochastic task-encoder (10) with associated decoding-lists $\{\mathcal{L}_z^y\}$ (11). For every $y \in \mathcal{Y}$ order the lists $\{\mathcal{L}_z^y\}_{z \in \mathcal{Z}}$ in increasing order of their cardinalities, and order the elements in each list in some arbitrary way. Now consider the guessing order where we first guess the elements of the first (and smallest) list in their respective order followed by those elements in the second list that have not yet been guessed (i.e., that are not contained in the first list), and where we continue until concluding by guessing those elements of the last (and longest) list that have not been previously guessed. Let $G(\cdot|Y)$ be the corresponding guessing function, and observe that

$$\mathbb{E}\left[G(X|Y)^{\rho}\right] = \sum_{x,y} P_{X,Y}(x,y) \left|\left\{\tilde{x} \colon G(\tilde{x}|y) \le G(x|y)\right\}\right|^{\rho}$$
(36)

$$\stackrel{(a)}{\leq} \sum_{x,y} P_{X,Y}(x,y) \left| \mathcal{Z} \right|^{\rho} \min_{z \colon x \in \mathcal{L}_z^y} \left| \mathcal{L}_z^y \right|^{\rho} \tag{37}$$

$$\stackrel{(b)}{\leq} |\mathcal{Z}|^{\rho} \mathbb{E}\Big[|\mathcal{L}_{Z}^{Y}|^{\rho} \Big], \tag{38}$$

where (a) holds because for every $x, \tilde{x} \in \mathcal{X}$ and $y \in \mathcal{Y}$ a necessary condition for $G(\tilde{x}|y) \leq G(x|y)$ is that $\tilde{x} \in \mathcal{L}_{\tilde{z}}^{y}$ for some $\tilde{z} \in \mathcal{Z}$ satisfying

$$|\mathcal{L}_{\tilde{z}}^{y}| \leq \min_{z \colon x \in \mathcal{L}_{z}^{y}} |\mathcal{L}_{z}^{y}|$$

and because the number of lists whose size does not exceed $\min_{z: x \in \mathcal{L}_z^y} |\mathcal{L}_z^y|$ is at most $|\mathcal{Z}|$; and (b) holds because the list \mathcal{L}_Z^Y contains X (11).

As to the second part, suppose we are given a guessing function $G(\cdot|Y)$ and a positive integer $\omega \leq |\mathcal{X}|$ satisfying (31). Let $\mathcal{O} = \{0, \dots, \omega - 1\}$ and

$$S = \left\{0, \dots, \left\lfloor \log \left\lceil |\mathcal{X}| / \omega \right\rceil \right\rfloor \right\}.$$

From (31) it follows that $|\mathcal{Z}| \geq |\mathcal{O}| |\mathcal{S}|$. It thus suffices to prove the existence of a task-encoder that uses only $|\mathcal{O}| |\mathcal{S}|$ possible descriptions, and we thus assume w.l.g. that $\mathcal{Z} = \mathcal{O} \times \mathcal{S}$. That is, using the side-information y the task-encoder (deterministically) describes x by z = (o, s). The encoding involves two steps:

Step 1: In Step 1 the encoder first computes $O \in \mathcal{O}$ as the remainder of the Euclidean division of G(X|Y) - 1 by $|\mathcal{O}|$. This guarantees that if (Y, O) = (y, o), then X be in the set

$$\mathcal{X}_{y,o} \triangleq \Big\{ x \in \mathcal{X} : \big(G(x|y) - 1 \big) \equiv o \mod |\mathcal{O}| \Big\}.$$

It then constructs from $G(\cdot|Y)$ a guessing function $G(\cdot|Y,O)$ as follows. The encoder constructs the guessing function $G(\cdot|y,o)$ so that—in the corresponding guessing order—we first guess the elements of $\mathcal{X}_{y,o}$ in increasing order of G(x|y). Our first $|\mathcal{X}_{y,o}|$ guesses are thus the elements of $\mathcal{X}_{y,o}$ with $x \in \mathcal{X}_{y,o}$ being guessed before $\tilde{x} \in \mathcal{X}_{y,o}$ whenever $G(x|y) < G(\tilde{x}|y)$. Once we have guessed all the elements of $\mathcal{X}_{y,o}$, we guess the remaining elements of \mathcal{X} in some arbitrary order. This order is immaterial, because X is guaranteed to be in the set $\mathcal{X}_{y,o}$. As we argue next, the guessing function $G(\cdot|Y,O)$ for X satisfies

$$G(X|Y,O) = \left\lceil G(X|Y) / |\mathcal{O}| \right\rceil.$$
(39)

Indeed, observe that for every $(y, o) \in \mathcal{Y} \times \mathcal{O}$ and $l \in \{1, \ldots, |\mathcal{X}_{y,o}|\}$ our *l*-th guess x_l is the element of $\mathcal{X}_{y,o}$ for which $G(x_l|y) = o + 1 + (l-1)|\mathcal{O}|$. Since $o + 1 \in \{1, \ldots, |\mathcal{O}|\}$, we find that $G(x|y, o) = \lceil G(x|y)/|\mathcal{O}| \rceil$ whenever $x \in \mathcal{X}_{y,o}$. But X is guaranteed to be in the set $\mathcal{X}_{y,o}$. This proves that the guessing function $G(\cdot|Y, O)$ for X satisfies (39). By (39) and because $|\mathcal{O}| = \omega$,

$$G(X|Y,O) = \left[G(X|Y)/\omega\right].$$
(40)

Step 2: In Step 2 the encoder first computes $S = \lfloor \log G(X|Y, O) \rfloor$ and then describes X by $Z \triangleq (O, S)$. By (40)

$$1 \le G(X|Y, O) \le \left[|\mathcal{X}| / \omega \right]$$

and consequently $S \in S$. Since O and S are deterministic given (X, Y), the conditional PMF (10) corresponding to the description Z = (O, S) is $\{0, 1\}$ -valued. It remains to show that the decoding lists $\{\mathcal{L}_z^y\}$ (11) satisfy (32). To this end note that if (Y, O, S) = (y, o, s), then X is in the set

$$\mathcal{X}_{y,o,s} \triangleq \Big\{ x \in \mathcal{X} : \big\lfloor \log G(x|y,o) \big\rfloor = s \Big\}.$$

Because every pair $x, \tilde{x} \in \mathcal{X}_{y,o,s}$ satisfies $\lfloor \log G(x|y,o) \rfloor = \lfloor \log G(\tilde{x}|y,o) \rfloor$, Fact 9 and the fact that the guessing function $G(\cdot|y,o)$ is a bijection imply that

$$|\mathcal{X}_{y,o,s}| \le G(x|y,o), \ \forall x \in \mathcal{X}_{y,o,s}.$$
(41)

Recalling that

$$\left((Y,O,S) = (y,o,s)\right) \implies X \in \mathcal{X}_{y,o,s},\tag{42}$$

we obtain from (41) that

$$|\mathcal{X}_{Y,O,S}| \le G(X|Y,O). \tag{43}$$

By (42) and because Z = (O, S), the list \mathcal{L}_Z^Y (11) is contained in the set $\mathcal{X}_{Y,O,S}$ and consequently satisfies $|\mathcal{L}_Z^Y| \leq |\mathcal{X}_{Y,O,S}|$. Hence, (43) implies that

$$\left|\mathcal{L}_{Z}^{Y}\right| \le G(X|Y,O). \tag{44}$$

From (40) and (44) we conclude that

$$\mathbb{E}\Big[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\Big] \leq \mathbb{E}\big[G(X|Y,O)^{\rho}\big] = \mathbb{E}\Big[\big[G(X|Y)/\omega\big]^{\rho}\Big].$$
(45)

To better understand the second part of Theorem 8, we briefly discuss the construction of a deterministic task-encoder from an optimal guessing function $G^*(\cdot|Y)$ (which minimizes $\mathbb{E}[G(X|Y)^{\rho}]$). If $G^*(\cdot|Y)$ is an optimal guessing function, then the two-step construction in the proof of Theorem 8 can be alternatively described as follows. We construct a task-encoder that describes X by

$$Z = (O, S)$$

where O takes values in some set \mathcal{O} of size ω , where

$$1 \le \omega \le |\mathcal{X}|,$$

and S takes values in some set ${\mathcal S}$ of size

$$1 + \left\lfloor \log \left\lceil |\mathcal{X}| / \omega \right\rceil \right\rfloor \le 1 + \log |\mathcal{X}|.$$

(Note that the description Z assumes at most $|\mathcal{O}| |\mathcal{S}|$ different values, and by (31) $|\mathcal{O}| |\mathcal{S}| \leq |\mathcal{Z}|$.) In the first step of the construction, we choose the first part of the description, O. We choose O as one that—among all O's that are drawn from \mathcal{O} according to some conditional PMF $P_{O|X,Y}$ —minimizes $\min_{G(\cdot|Y,O)} \mathbb{E}[G(X|Y,O)^{\rho}]$. From Lemma 5 (and the subsequent paragraph) we already know how to construct O. Indeed, from Lemma 5 it follows that

$$\min_{G(\cdot|Y,O)} \mathbb{E}\left[G(X|Y,O)^{\rho}\right] \ge \mathbb{E}\left[\left[G^{\star}(X|Y)/|\mathcal{O}|\right]^{\rho}\right],$$

where equality is achieved by choosing $O = f_1(X, Y)$ for some mapping $f_1: \mathcal{X} \times \mathcal{Y} \to \mathcal{O}$ for which $f_1(x, y) = f_1(\tilde{x}, y)$ implies either $\left[G^*(x|y)/|\mathcal{O}|\right] \neq \left[G^*(\tilde{x}|y)/|\mathcal{O}|\right]$ or $x = \tilde{x}$. For example, in the case where $\mathcal{O} = \{0, \ldots, \omega - 1\}$ we can choose O as the remainder of the Euclidean division of G(X|Y) - 1 by $|\mathcal{O}|$. Based on the optimal guessing function $G^*(\cdot|Y)$ and the first part of the description, O, we can construct an optimal guessing function $G^*(\cdot|Y, O)$ (which minimizes $\mathbb{E}\left[G(X|Y, O)^{\rho}\right]$) by choosing some $G^*(\cdot|Y, O)$ for which

$$G^{\star}(x|y, f_1(x, y)) = \left[G^{\star}(x|y) / |\mathcal{O}| \right], \ \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

In the second step of the construction we choose the second part of the description, S. We choose $S = f_2(x, y)$, where

$$f_2(x,y) = \left\lfloor \log G^*(x|y, f_1(x,y)) \right\rfloor, \ \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

This will guarantee that the decoding lists satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] \leq \mathbb{E}\left[G^{\star}(X|Y,O)^{\rho}\right] = \mathbb{E}\left[\left[G^{\star}(X|Y)/|\mathcal{O}|\right]^{\rho}\right],$$

where

$$Z = (O, S) = (f_1(X, Y), f_2(X, Y)).$$

Note that the size of the support S of S is only logarithmic in $|\mathcal{X}|$ and thus negligible in asymptotic settings, i.e., in asymptotic settings $|\mathcal{Z}| \approx |\mathcal{O}|$.

The following corollary results from Theorem 8 and (1) by setting

$$\omega = \left\lfloor |\mathcal{Z}| / \left(1 + \left\lfloor \log |\mathcal{X}| \right\rfloor\right) \right\rfloor$$

in Theorem 8.

Corollary 10. Given a set \mathcal{Z} of cardinality $|\mathcal{Z}| \ge 1 + \lfloor \log |\mathcal{X}| \rfloor$, any guessing function $G(\cdot|Y)$ induces a deterministic task-encoder, i.e., a stochastic task-encoder whose conditional PMF (10) is $\{0,1\}$ -valued, whose associated decoding lists $\{\mathcal{L}_z^y\}$ (11) satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\right] \leq 1 + 2^{\rho} \mathbb{E}\left[G(X|Y)^{\rho}\right] \left(\frac{|\mathcal{Z}|}{1 + \log|\mathcal{X}|} - 1\right)^{-\rho}.$$
(46)

Combined with Theorem 3, which bounds the performance of an optimal guessing function, Equations (30) and (46) provide an upper and a lower bound on the smallest $\mathbb{E}[|\mathcal{L}_Z^Y|^{\rho}]$ that is achievable for a given $|\mathcal{Z}|$. These bounds are weaker than [4, Theorem I.1 and Theorem VI.1] (see Theorem 4) in the finite blocklength regime but tight enough to prove the asymptotic results [4, Theorem I.2 and Theorem VI.2].

Another interesting corollary to Theorem 8 results from the choice $\omega = 1$ in Theorem 8:

Corollary 11. Given a set \mathcal{Z} of cardinality $|\mathcal{Z}| = 1 + \lfloor \log |\mathcal{X}| \rfloor$, any guessing function $G(\cdot|Y)$ induces a deterministic task-encoder, i.e., a stochastic task-encoder whose conditional PMF (10) is $\{0,1\}$ -valued, whose associated decoding lists $\{\mathcal{L}_z^y\}$ (11) satisfy

$$\mathbb{E}\Big[\left|\mathcal{L}_{Z}^{Y}\right|^{\rho}\Big] \leq \mathbb{E}\big[G(X|Y)^{\rho}\big].$$
(47)

E.g., if

$$\mathcal{Z} = \Big\{0, \dots, \lfloor \log |\mathcal{X}| \rfloor \Big\},$$

then the task-encoder $f(\cdot|Y)$ defined by

$$f(\cdot|y) = \lfloor \log G(\cdot|y) \rfloor, \ \forall y \in \mathcal{Y}$$
(48a)

satisfies (47) or, equivalently,

$$\mathbb{E}\left[f^{-1}\left(f(X|Y)\big|Y\right)^{\rho}\right] \le \mathbb{E}\left[G(X|Y)^{\rho}\right].$$
(48b)

An implication of Corollary 11 for the problems studied in this paper is discussed in Remark 18. Another example where Corollary 11 is useful is in determining the feedback listsize capacity of a DMC W(y|x) with positive zero-error capacity. Corollary 11 can be used to give an elegant proof of the direct part of [5, Theorem I.1], which states that in the presence of perfect feedback the listsize capacity of W(y|x) equals the cutoff rate $R_{\text{cutoff}}(\rho)$ with feedback (which is in fact equal to the cutoff rate without feedback [5, Corollary I.4]). To see this, suppose that we are given a sequence of (feedback) codes of rate R for which the ρ -th moment of the number of guesses $G^*(M|Y^n)$ a decoder needs to guess the transmitted message M based on the channel-outputs Y^n approaches one as the blocklength n tends to infinity. (Recall that $R_{\text{cutoff}}(\rho)$ is the supremum of all rates for which such a sequence exists.) Suppose now that the transmission does not stop after n channel uses. Instead, the encoder computes

$$Z \triangleq \left\lfloor \log G^{\star}(M|Y^{n}) \right\rfloor \in \left\{0, \dots, \lfloor nR \rfloor\right\}$$

from the feedback Y^n and uses another n' channel uses to transmit Z at a positive rate while guaranteeing that the receiver can decode it with probability one. Since a positive zero-error (feedback) capacity cannot be smaller than one [21], it is enough to take $n' \leq \lceil \log(nR) \rceil$. Hence, (n + n')/n converges to one as n tends to infinity, and the rate of the code thus converges to R. At the same time, when we substitute (M, Y^n, Z) for (X, Y, Z) in Corollary 11, Corollary 11 implies that the size of the smallest decoding-list $\mathcal{L}^{Y^{n+n'}}$ that is guaranteed to contain M satisfies $|\mathcal{L}^{Y^{n+n'}}| = |\mathcal{L}_Z^{Y^n}| \leq G^*(M|Y^n)$, and consequently that the ρ -th moment of $|\mathcal{L}^{Y^{n+n'}}|$ converges to one as n tends to infinity. This proves that in the presence of perfect feedback the listsize capacity of W(y|x) is lower-bounded by $R_{\text{cutoff}}(\rho)$.

4 Problem Statement and Main Results

We consider two problems: the "guessing version" and the "list version." The two differ in the definition of Bob's ambiguity. In both versions a pair (X, Y) is drawn from the finite set $\mathcal{X} \times \mathcal{Y}$ according to the PMF $P_{X,Y}$, and $\rho > 0$ is fixed. Upon observing (X, Y) = (x, y), Alice draws the hints M_1 and M_2 from some finite set $\mathcal{M}_1 \times \mathcal{M}_2$ according to some conditional PMF

$$\mathbb{P}[M_1 = m_1, M_2 = m_2 | X = x, Y = y].$$
(49)

Bob sees both hints and the side information Y. In the guessing version Bob's ambiguity about X is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) = \min_{G(\cdot|M_1,M_2)} \mathbb{E} \big[G(X|Y,M_1,M_2)^{\rho} \big].$$
(50)

In the list version Bob's ambiguity about X is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) = \mathbb{E}\Big[\big|\mathcal{L}_{M_{1},M_{2}}^{Y}\big|^{\rho}\Big],\tag{51}$$

where for all $y \in \mathcal{Y}$ and $(m_1, m_2) \in \mathcal{M}_1 \times \mathcal{M}_2$

$$\mathcal{L}_{m_1,m_2}^y = \left\{ x \colon \mathbb{P}[X = x | Y = y, M_1 = m_1, M_2 = m_2] > 0 \right\}$$
(52)

is the list of all the realizations of X of positive posterior probability

$$\mathbb{P}[X = x | Y = y, M_1 = m_1, M_2 = m_2] = \frac{P_{X,Y}(x, y) \mathbb{P}[M_1 = m_1, M_2 = m_2 | X = x, Y = y]}{\sum_{\tilde{x}} P_{X,Y}(\tilde{x}, y) \mathbb{P}[M_1 = m_1, M_2 = m_2 | X = \tilde{x}, Y = y]}.$$
(53)

Eve sees one of the hints and guesses X based on this hint and the side information Y. Which of the hints is revealed to her is determined by an accomplice of hers to minimize her guessing efforts. In both versions Eve's ambiguity about X is

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = \min_{G_1(\cdot|Y,M_1), \, G_2(\cdot|Y,M_2)} \mathbb{E} \big[G_1(X|Y,M_1)^{\rho} \wedge G_2(X|Y,M_2)^{\rho} \big].$$
(54)

Optimizing over Alice's mapping, i.e., the choice of the conditional PMF in (49), we wish to characterize the largest ambiguity that we can guarantee that Eve will have subject to a given upper bound on the ambiguity that Bob may have.

Note that by quantifying Eve's ambiguity using (54), we are implicitly assuming that Eve's accomplice observes X and Y before determining the hint that minimizes Eve's guessing efforts. Less conservative is the ambiguity

$$\tilde{\mathscr{A}}_{\mathrm{E}}(P_{X,Y}) = \min_{k \in \{1,2\}} \min_{G_k(\cdot|Y,M_k)} \mathbb{E}\left[G_k(X|Y,M_k)^{\rho}\right],\tag{55}$$

which applies if the accomplice does not observe (X, Y) and reveals to Eve the hint that in expectation over (X, Y) minimizes her guessing efforts. Definition (55) is less conservative than (55) in the sense that

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le \mathscr{\tilde{A}}_{\mathrm{E}}(P_{X,Y}).$$
(56)

Why we prefer (54) over (55) is explained in Section 5.

Of special interest to us is the asymptotic regime where (X, Y) is an *n*-tuple (not necessarily drawn IID), and where

$$\mathcal{M}_1 = \{1, \dots, 2^{nR_1}\}, \quad \mathcal{M}_2 = \{1, \dots, 2^{nR_2}\},\$$

where (R_1, R_2) is a nonnegative pair corresponding to the rate.² For both versions of the problem, we shall characterize the largest exponential growth that we can guarantee for Eve's ambiguity subject to the constraint that Bob's ambiguity tend to one.³ This asymptote turns out not to depend on the version of the problem, and in the asymptotic analysis $\mathscr{A}_{\rm B}$ can stand for either $\mathscr{A}_{\rm B}^{({\rm g})}$ or $\mathscr{A}_{\rm B}^{({\rm l})}$.

The following definition phrases mathematically what we mean by the "largest exponential growth that we can guarantee for Eve's ambiguity:"

Definition 1 (Privacy-Exponent). Let $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ be a stochastic process over the finite alphabet $\mathcal{X} \times \mathcal{Y}$, and denote by P_{X^n, Y^n} the PMF of (X^n, Y^n) . Given a nonnegative rate-pair (R_1, R_2) , we call $E_{\mathbf{E}}$ an *achievable ambiguity-exponent* if there exists a sequence of stochastic encoders such that Bob's ambiguity (which is always at least one) satisfies

$$\lim_{n \to \infty} \mathscr{A}_{\mathcal{B}}(P_{X^n, Y^n}) = 1, \tag{57}$$

and such that Eve's ambiguity satisfies

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \ge E_{\mathrm{E}}.$$
(58)

The privacy-exponent $\overline{E_{\rm E}}$ is the supremum of all achievable ambiguity-exponents. If (57) cannot be satisfied, then the set of achievable ambiguity-exponents is empty, and we define the privacy-exponent as negative infinity.

²When we say that a positive integer $k \in \mathbb{N}$ assumes the value 2^{nR} , where R > 0 corresponds to a rate, we mean that $k = \lfloor 2^{nR} \rfloor$.

³Note that in the guessing version $G(X|Y, M_1, M_2)^{\rho}$ is one iff Bob's first guess is X^n , and in the list version $|\mathcal{L}_{M_1,M_2}^Y|^{\rho}$ is one iff Bob forms the "perfect" list comprising only X^n .

We also consider a scenario where we impose only a modest requirement on Bob's ambiguity and allow it to grow exponentially with a given normalized (by n) exponent $E_{\rm B}$. For this scenario the following definition introduces the mathematical quantity by which we characterize the largest exponential growth that we can guarantee for Eve's ambiguity:

Definition 2 (Modest Privacy-Exponent). Let $E_{\rm B} \ge 0$. We call $E_{\rm E}^{\rm m}(E_{\rm B})$ an *achievable* modest-ambiguity-exponent if there is a sequence of stochastic encoders such that Bob's ambiguity satisfies

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathcal{B}}(P_{X^n, Y^n}))}{n} \le E_{\mathcal{B}},\tag{59}$$

and such that Eve's ambiguity satisfies

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \ge E_{\mathrm{E}}^{\mathrm{m}}(E_{\mathrm{B}}).$$
(60)

For every $E_{\rm B} \geq 0$, the modest privacy-exponent $\overline{E_{\rm E}^{\rm m}(E_{\rm B})}$ is the supremum of all achievable modest-ambiguity-exponents. If (59) cannot be satisfied, then the set of achievable modest-ambiguity-exponents is empty, and we define the modest privacy-exponent as negative infinity.

We next present our results to the stated problems in the finite-blocklength regime (Section 4.1) and in the asymptotic regime (Section 4.2).

4.1 Finite-Blocklength Results

In the next two theorems c_s is related to how much information can be gleaned about the secret X from the pair of hints (M_1, M_2) but not from one hint alone; c_1 is related to how much can be gleaned from M_1 ; and c_2 is related to how much can be gleaned from M_2 . More precisely, in the proof of the two theorems (see Section 6 ahead) we shall see that Alice first maps (X, Y) to the triple (V_s, V_1, V_2) , which takes value in a set $\mathcal{V}_s \times \mathcal{V}_1 \times \mathcal{V}_2$, whose marginal cardinalities satisfy $|\mathcal{V}_{\nu}| = c_{\nu}, \ \nu \in \{s, 1, 2\}$. Independently of (X, Y) she then draws a (one-time-pad like) random variable U uniformly over \mathcal{V}_s and maps (U, V_s) to a variable \widetilde{V}_s choosing the (XOR like) mapping so that V_s can be recovered from (\widetilde{V}_s, U) while \widetilde{V}_s alone is independent of (X, Y). The hints are $M_1 = (\widetilde{V}_s, V_1)$ and $M_2 = (U, V_2)$. Since the tuple (\widetilde{V}_s, V_1) takes value in the set $\mathcal{V}_s \times \mathcal{V}_1$ of size $c_s c_1$, we must have that $c_s c_1 \leq |\mathcal{M}_1|$. Likewise, we must have that $c_s c_2 \leq |\mathcal{M}_2|$. Because c_s, c_1 , and c_2 are positive integers, they thus satisfy (61) ahead. Alice does not use randomization if $c_s = 1$.

Theorem 12 (Finite-Blocklength Guessing-Version). For every triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ satisfying

$$c_{\rm s} \le |\mathcal{M}_1| \land |\mathcal{M}_2|, \quad c_1 \le \lfloor |\mathcal{M}_1|/c_s \rfloor, \quad c_2 \le \lfloor |\mathcal{M}_2|/c_s \rfloor,$$
(61a)

there is a choice of the conditional PMF in (49) for which Bob's ambiguity about X is upperbounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_{\mathrm{s}}c_{1}c_{2}) + 1)},\tag{62}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\rm E}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_1 + c_2))}.$$
(63)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_1| |\mathcal{M}_2|))} \vee 1, \tag{64}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)},\tag{65}$$

where (65) holds even if we replace (54) by (55), i.e.,

$$\tilde{\mathscr{A}}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \wedge 2^{\rho H_{\bar{\rho}}(X|Y)},\tag{66}$$

Proof. See Section 6.1.

Theorem 13 (Finite-Blocklength List-Version). If $|\mathcal{M}_1| |\mathcal{M}_2| > \log |\mathcal{X}| + 2$, then for every triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ satisfying

$$c_{\rm s} \le |\mathcal{M}_1| \land |\mathcal{M}_2|, \quad c_1 \le \lfloor |\mathcal{M}_1|/c_{\rm s} \rfloor, \quad c_2 \le \lfloor |\mathcal{M}_2|/c_{\rm s} \rfloor,$$
 (67a)

$$c_{\rm s}c_1c_2 > \log|\mathcal{X}| + 2,\tag{67b}$$

there is a choice of the conditional PMF in (49) for which Bob's ambiguity about X is upperbounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(c_{\rm s}c_1c_2 - \log|\mathcal{X}| - 2) + 2)},\tag{68}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\rm E}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_1 + c_2))}.$$
(69)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_1| |\mathcal{M}_2|))} \vee 1, \tag{70}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{A}_{\mathrm{B}}^{(1)}(P_{X,Y}) \wedge 2^{\rho H_{\bar{\rho}}(X|Y)},\tag{71}$$

where (71) holds even if we replace (54) by (55), i.e.,

$$\tilde{\mathscr{A}}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{A}_{\mathrm{B}}^{(1)}(P_{X,Y}) \wedge 2^{\rho H_{\bar{\rho}}(X|Y)},\tag{72}$$

Proof. See Section 6.1.

We next present the finite-blocklength results (Theorems 12 and 13) in a simplified and more accessible form:

Corollary 14 (Simplified Finite-Blocklength Guessing-Version). For any constant $\mathscr{U}_{\rm B}$ satisfying

$$\mathscr{U}_{\mathrm{B}} \ge 1 + 2^{\rho} \left(\left| \mathcal{M}_1 \right| \left| \mathcal{M}_2 \right| \right)^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)},\tag{73}$$

there is a choice of the conditional PMF in (49) for which Bob's ambiguity about X is upperbounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < \mathscr{U}_{\mathrm{B}},\tag{74}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge 2^{-\rho} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} \left[2^{-4\rho} \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} (\mathscr{U}_{\mathrm{B}} - 1) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}\right].$$
(75)

Conversely, (74) cannot hold for

$$\mathscr{U}_{\mathrm{B}} < \left(1 + \ln |\mathcal{X}|\right)^{-\rho} \left(|\mathcal{M}_{1}| |\mathcal{M}_{2}|\right)^{-\rho} 2^{\rho H_{\bar{\rho}}(X|Y)} \vee 1, \tag{76}$$

and if Bob's ambiguity satisfies (74) for some \mathscr{U}_B , then Eve's ambiguity about X is upperbounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{U}_{\mathrm{B}} \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(77)

Proof. The result is a corollary to Theorem 12 (see Appendix A for a proof). \Box

Corollary 15 (Simplified Finite-Blocklength List-Version). For $|\mathcal{M}_1| |\mathcal{M}_2| > \log |\mathcal{X}| + 2$ and any constant \mathscr{U}_B satisfying

$$\mathscr{U}_{\rm B} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_1| |\mathcal{M}_2| - \log|\mathcal{X}| - 2) + 2)},\tag{78}$$

there is a choice of the conditional PMF in (49) for which Bob's ambiguity about X is upperbounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < \mathscr{U}_{\mathrm{B}},\tag{79}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \geq 2^{-\rho} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} \left[2^{-6\rho} \left(|\mathcal{M}_{1}| \wedge |\mathcal{M}_{2}|\right)^{\rho} (\mathscr{U}_{\mathrm{B}} - 1) \right. \\ \left. \wedge 2^{-4\rho} \left(2 + \log |\mathcal{X}|\right)^{-\rho} \left(|\mathcal{M}_{1}| \wedge |\mathcal{M}_{2}|\right)^{\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \right. \\ \left. \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)} \right].$$

$$(80)$$

Conversely, (79) cannot hold for

$$\mathscr{U}_{\mathrm{B}} < \left(|\mathcal{M}_1| \, |\mathcal{M}_2| \right)^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \lor 1, \tag{81}$$

and if Bob's ambiguity satisfies (79) for some \mathscr{U}_{B} , then Eve's ambiguity about X is upperbounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} \mathscr{U}_{\mathrm{B}} \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(82)

Proof. The result is a corollary to Theorem 13 (see Appendix B for a proof).

Note that the simplified achievability results (namely (73)-(75) in the guessing version and (78)-(80) in the list version) match the corresponding converse results (namely (76)-(77)in the guessing version and (81)-(82) in the list version) up to polylogarithmic factors of $|\mathcal{X}|$.

4.2 Asymptotic Results

Suppose now that (X, Y) is an *n*-tuple. We study the asymptotic regime where *n* tends to infinity. Recall that in this regime we refer to both $\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}$ and $\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}$ by \mathscr{A}_{B} , because the results are the same for both versions of the problem. Theorems 12 and 13 imply the following asymptotic result:

Theorem 16 (Privacy-Exponent). Let $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ be a discrete-time stochastic process with finite alphabet $\mathcal{X} \times \mathcal{Y}$, and suppose its conditional Rényi entropy-rate $H_{\tilde{\rho}}(\mathbf{X}|\mathbf{Y})$ is welldefined. Given any positive rate-pair (R_1, R_2) , the privacy-exponent is

$$\overline{E_{\rm E}} = \begin{cases} \rho \big(R_1 \wedge R_2 \wedge H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}) \big) & R_1 + R_2 > H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}), \\ -\infty, & R_1 + R_2 < H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}). \end{cases}$$
(83)

Proof. See Section 6.2.

Suppose now that Bob's ambiguity need not tend to one but can grow exponentially with a given normalized (by n) exponent $E_{\rm B}$. For this case Theorems 12 and 13 imply the following asymptotic result:

Theorem 17 (Modest Privacy-Exponent). Let $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ be a discrete-time stochastic process with finite alphabet $\mathcal{X} \times \mathcal{Y}$, and suppose its conditional Rényi entropy-rate $H_{\tilde{\rho}}(\mathbf{X}|\mathbf{Y})$ is well-defined. Given any positive rate-pair (R_1, R_2) , the modest privacy-exponent for $E_{\rm B} \geq 0$ is

$$\overline{E_{\mathrm{E}}^{\mathrm{m}}(E_{\mathrm{B}})} = \begin{cases} \left(\rho(R_{1} \wedge R_{2}) + E_{\mathrm{B}}\right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) & R_{1} + R_{2} \geq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}}, \\ -\infty & R_{1} + R_{2} < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}}. \end{cases}$$
(84)

Proof. See Section 6.3.

This section provides some intuition and discusses some of the models and their underlying assumptions. We begin with some intuition as to why the guessing and list-size criteria for Bob lead to similar results. Then, we explain why we quantify Eve's ambiguity by (54). We show that if—rather than guessing—Eve were required to form a list, then perfect secrecy would come almost for free. Finally, we explain how our results change in the following two scenarios: 1) Alice knows which hint Eve observes; or 2) Alice describes X using only one hint, but Alice and Bob see a secret key, which is not revealed to Eve.

The following remark explains why the results for the guessing and the list version differ only by polylogarithmic factors of $|\mathcal{X}|$ (and are consequently the same in the asymptotic regime):

Remark 18 (Why Do the Two Criteria for Bob Lead to Similar Results?). Consider any choice of the conditional PMF in (49). In the guessing version Bob uses an optimal guessing function $G^*(\cdot|Y, M_1, M_2)$ (which minimizes $\mathbb{E}[G(X|Y, M_1, M_2)^{\rho}]$) to guess X based on the side information Y and the hints M_1 and M_2 , and his ambiguity is $\mathbb{E}\left[G^*(X|Y, M_1, M_2)^{\rho}\right]$. By Corollary 11 we can construct from $G^*(\cdot|Y, M_1, M_2)$ an additional hint M that takes values in a set of size at most $1 + \lfloor \log |\mathcal{X}| \rfloor$ such that

$$\mathbb{E}\left[\left|\mathcal{L}_{M_{1},M_{2},M}^{Y}\right|^{\rho}\right] \leq \mathbb{E}\left[G^{\star}(X|Y,M_{1},M_{2})^{\rho}\right],\tag{85}$$

where $\mathcal{L}_{M_1,M_2,M}^Y$ is the smallest list that is guaranteed to contain X given (Y, M_1, M_2, M) . Suppose now that Alice maps X to the hints $M'_1 \triangleq (M_1, M)$ and $M'_2 \triangleq M_2$. This implies that Bob's ambiguity in the list version is

$$\mathbb{E}\left[\left|\mathcal{L}_{M_{1}^{\prime},M_{2}^{\prime}}^{Y}\right|^{\rho}\right] = \mathbb{E}\left[\left|\mathcal{L}_{M_{1},M_{2},M}^{Y}\right|^{\rho}\right]$$

and consequently no larger than $\mathbb{E}\left[G^{\star}(X|Y, M_1, M_2)^{\rho}\right]$. Moreover, because M takes values in a set of size at most $1 + \lfloor \log |\mathcal{X}| \rfloor$, we can use Lemma 5 to show that—compared to the case where the hints are M_1 and M_2 —Eve's ambiguity decreases by at most a polylogarithmic factor of $|\mathcal{X}|$.

We next explain why we choose to quantify Eve's ambiguity by (54) and not by (55). As we have seen, (54) is more conservative than (55) in the sense that (56) holds. Consequently, it follows from (66) and (72) that the results of Theorems 12 and 13 hold irrespective of whether we quantify Eve's ambiguity by (54) or by (55). We prefer to quantify Eve's ambiguity by (54), because—as the following example shows—(55) leads to a weaker notion of secrecy than (54):

Example 1. Suppose that Y is null, X is uniform over \mathcal{X} , and Alice produces the hints at random: they are equally likely to be $(M_1 = X, M_2 = *)$ or $(M_1 = *, M_2 = X)$, where the symbol * is not in \mathcal{X} . Since Bob can recover X from (M_1, M_2) (by producing the hint that is not *),

$$\min_{G(\cdot|M_1,M_2)} \mathbb{E}\left[G(X|M_1,M_2)^{\rho}\right] = \mathbb{E}\left[|\mathcal{L}_{M_1,M_2}|^{\rho}\right] = 1.$$

The system is clearly insecure, because one of the hints always reveals X, and $\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = 1$. However, as we next argue, this weakness is not captured by $\mathscr{\tilde{A}}_{\mathrm{E}}(P_{X,Y})$. The probability of M_1 being * is 1/2, so the ρ -th moment of $G_1(X|M_1)$ is at least $\min_{G(\cdot)} \mathbb{E}[G(X)^{\rho}]/2$. Likewise, by symmetry, for $G_2(X|M_2)$. Thus $\mathscr{\tilde{A}}_{\mathrm{E}}(P_{X,Y})$ differs from $\min_{G(\cdot)} \mathbb{E}[G(X)^{\rho}]$ by a factor of at most 1/2.

So far, we have explained why we prefer (54) over (55). But why do we allow Eve to guess even in the list version of our problem? That is, why do we prefer (54) over

$$\mathscr{A}_{\mathrm{E}}^{(1)} = \mathbb{E}\left[\left|\mathcal{L}_{M_{1}}^{Y}\right|^{\rho} \wedge \left|\mathcal{L}_{M_{2}}^{Y}\right|^{\rho}\right] \tag{86}$$

even when Bob must form a list?

We prefer (54) over (86) because, as Theorem 19 ahead will show, forcing Eve to produce a short list would severely handicap her and make it trivial to defeat her: when Eve must form a list, perfect secrecy is almost free. Theorem 19 (Eve Must Form a List). If

$$|\mathcal{M}_1| \wedge |\mathcal{M}_2| \ge 1 + \lfloor \log |\mathcal{X}| \rfloor, \tag{87}$$

then there exists a conditional PMF as in (49) for which Bob's ambiguity about X is upperbounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) \le 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_1||\mathcal{M}_2|) + 2\log(1 + \lfloor \log|\mathcal{X}\rfloor) + 3)},\tag{88}$$

and Eve's ambiguity about X is

$$\mathscr{A}_{\mathrm{E}}^{(1)}(P_{X,Y}) = \mathbb{E}\big[|\mathcal{L}_Y|^\rho\big],\tag{89}$$

where

$$\mathbb{E}\left[|\mathcal{L}_Y|^{\rho}\right] = \sum_y P_Y(y) \left| \left\{ x \in \mathcal{X} \colon P_{X|Y}(x|y) > 0 \right\} \right|^{\rho}.$$
(90)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) \ge 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{M}_1| |\mathcal{M}_2|))} \vee 1, \tag{91}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}^{(1)}(P_{X,Y}) \le \mathbb{E}\big[|\mathcal{L}_Y|^\rho\big]. \tag{92}$$

Proof. See Appendix C.

To see why perfect secrecy is almost free when Eve is required to form a list, note that the RHS of (89) would also be Eve's list size if she only saw Y and did not get to see *any* hint, so in this sense achieving (89) is tantamount to achieving perfect secrecy. And the cost is very small: Condition (87) is satisfied in the large-blocklength regime whenever the rates of the two hints are positive; and the RHS of (88) will tend to one in this regime whenever the sum of the rates exceeds the conditional Rényi entropy rate—a condition that is necessary even in the absence of an adversay (Theorem 4).

That perfect secrecy is (almost) free when we quantify Eve's ambiguity by (86) is highly intuitive: By forcing Eve to form a list that is guaranteed to contain X, we force her to include in her list all the realizations of X that have a positive posterior probability, no matter how small. This implies that, if Eve were to form a list, then perfect secrecy could be attained by hiding very little information from Eve. The situation is different in case Eve guesses X, because allowing Eve to guess X, i.e., quantifying Eve's ambiguity by (54), is tantamount to first indexing the elements of the list in (86)—which she would otherwise have to form—in decreasing order of their posterior probability, and to then downweigh the large indices of the realizations at the bottom of the list by their small posterior probabilities.

To conclude the discussion of how to quantify Eve's ambiguity, we relate Eve's ambiguity (54) to the concept of *equivocation*. In the classical Shannon cipher system [22], a popular way to measure imperfect secrecy is in terms of equivocation, i.e., in terms of the conditional

entropy H(X|Z), where X denotes some sensitive information and Z Eve's observation. In the settings where Bob is a list-decoder or a guessing decoder, Rényi entropy plays the role of Shannon entropy in the sense that the minimum required rate to encode an *n*-tuple $X = X^n$ is the Rényi entropy rate $H_{\tilde{\rho}}(X)$ rather than the Shannon entropy rate $H(X) = H_1(X)$ (this follows from Theorems 4 and Corollary 7). Consequently, in these settings the conditional Rényi entropy $H_{\tilde{\rho}}(X|Z)$ qualifies as a "natural" equivalent for equivocation. But $H_{\tilde{\rho}}(X|Z)$ has a nice operational characterization: $2^{\rho H_{\tilde{\rho}}(X|Z)}$ is (up to polylogarithmic factors of |X|) the ρ -th moment of the number of guesses that Eve needs to guess X from her observation Z (see Theorem 3). This is another reason why it makes sense to quantify Eve's ambiguity in terms of the ρ -th moment of the number of guesses that she needs to guess X.

In the remainder of this section we briefly discuss how the results of Theorems 12 and 13 change in the following two scenarios: 1) Alice knows which hint Eve observes; or 2) Alice describes X using only one hint, but Alice and Bob share a secret key, which is unknown to Eve. We begin with Scenario 1. In this scenario Alice draws the public hint M_p and the secret hint M_s from some finite set $\mathcal{M}_p \times \mathcal{M}_s$ according to some conditional PMF

$$\mathbb{P}[M_{\rm p} = m_{\rm p}, M_{\rm s} = m_{\rm s} | X = x, Y = y].$$
(93)

Bob sees both hints. In the guessing version his ambiguity about X is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) = \min_{G(\cdot|Y,M_{\mathrm{p}},M_{\mathrm{s}})} \mathbb{E}\left[G(X|Y,M_{\mathrm{p}},M_{\mathrm{s}})^{\rho}\right]$$
(94)

and in the list version

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) = \mathbb{E}\Big[\big|\mathcal{L}_{M_{\mathrm{p}},M_{\mathrm{s}}}^{Y}\big|^{\rho}\Big].$$
(95)

Eve sees only the public hint. In both versions her ambiguity about X is

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = \min_{G(\cdot|Y,M_{\mathrm{p}})} \mathbb{E}\left[G(X|Y,M_{\mathrm{p}})^{\rho}\right].$$
(96)

The next two theorems characterize the largest ambiguity that we can guarantee that Eve will have subject to a given upper bound on the ambiguity that Bob may have (see Appendix D for a proof). As in the case where the hints are not secret and public, the guessing and the list version lead to similar results (cf. Remark 18). In the next two theorems c is related to how much can be gleaned about X from M_p .

Theorem 20 (Secret Hint Guessing-Version). For every $c \in \mathbb{N}$ satisfying

$$c \le |\mathcal{M}_{\rm p}|,\tag{97}$$

there is a $\{0,1\}$ -valued choice of the conditional PMF in (93) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\rm B}^{(\rm g)}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c |\mathcal{M}_{\rm s}|) + 1)},\tag{98}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log c)}.$$
(99)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{\rm (g)}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{M}_{\rm p}|\,|\mathcal{M}_{\rm s}|))} \vee 1,\tag{100}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le |\mathcal{M}_{\mathrm{s}}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \wedge 2^{\rho H_{\bar{\rho}}(X|Y)}.$$
(101)

Theorem 21 (Secret Hint List-Version). If $|\mathcal{M}_p| |\mathcal{M}_s| > \log |\mathcal{X}| + 2$, then for every $c \in \mathbb{N}$ satisfying

$$c \le |\mathcal{M}_{\mathrm{p}}|, \quad c |\mathcal{M}_{\mathrm{s}}| > \log |\mathcal{X}| + 2,$$
(102)

there is a $\{0,1\}$ -valued choice of the conditional PMF in (93) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c\,|\mathcal{M}_{\rm s}| - \log|\mathcal{X}| - 2) + 2)},\tag{103}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log c)}.$$
(104)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_{\rm p}| \, |\mathcal{M}_{\rm s}|))} \vee 1, \tag{105}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le |\mathcal{M}_{\mathrm{s}}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) \wedge 2^{\rho H_{\bar{\rho}}(X|Y)}.$$
(106)

We next contrast Theorems 20 and 21 to their counterparts in the previous scenario, i.e., to Theorems 12 and 13. By comparing the respective upper and lower bounds on Eve's ambiguity, we see that c and $|\mathcal{M}_{\rm s}|$ in the current scenario, which relate to how much information can be gleaned about X from $M_{\rm p}$ and $M_{\rm s}$, play the roles of $c_1 + c_2 \approx c_1 \vee c_2$ and $|\mathcal{M}_1| \wedge |\mathcal{M}_2|$ in the previous scenario, which relate to how much information can be gleaned about X from the hint that—among M_1 and M_2 —reveals more information about X and the one that—among M_1 and M_2 —reveals less information about X. This reflects the fact that in the current scenario Eve always sees $M_{\rm p}$, whereas in the previous scenario she sees the hint that reveals more information about X and hence minimizes her ambiguity.

Unlike Theorems 12 and 13, Theorems 20 and 21 imply that in the current scenario Alice can describe X deterministically by choosing a $\{0, 1\}$ -valued conditional PMF (93). To see why, recall that in the current scenario Eve sees only the public hint M_p , and hence there is no need to encrypt information that can be gleaned from the secret hint M_s . Consequently, Alice need not draw a one-time-pad like random variable and ensure that some information can be gleaned about X from $(M_{\rm p}, M_{\rm s})$ but not from one hint alone. Instead, she can store that information on $M_{\rm s}$ without prior encryption.

We now proceed to Scenario 2, where Alice describes X using only one hint, but Alice and Bob share a secret key, which is unknown to Eve. The secret key K is drawn independently of the pair (X, Y) and uniformly over some finite set \mathcal{K} . Upon observing (X, Y) = (x, y) and K = k, Alice draws the hint M from some finite set \mathcal{M} according to some conditional PMF

$$\mathbb{P}[M=m|X=x, Y=y, K=k].$$
(107)

Throughout, we assume that $|\mathcal{K}| \leq |\mathcal{M}|$. Bob sees the secret key and the hint. In the guessing version his ambiguity about X is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) = \min_{G(\cdot|Y,K,M)} \mathbb{E}\left[G(X|Y,K,M)^{\rho}\right]$$
(108)

and in the list version

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) = \mathbb{E}\Big[\big|\mathcal{L}_{M}^{Y,K}\big|^{\rho}\Big].$$
(109)

Eve sees sees only the hint. In both versions her ambiguity about X is

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = \min_{G(\cdot|Y,M)} \mathbb{E}\left[G(X|Y,M)^{\rho}\right].$$
(110)

The next two theorems characterize the largest ambiguity that we can guarantee that Eve will have subject to a given upper bound on the ambiguity that Bob may have (see Appendix E for a proof). Again, the guessing and the list version lead to similar results. Here $|\mathcal{K}|$ is related to how much information can be gleaned about X from (K, M) but not from M alone, i.e., to the "encrypted" information stored on M, and c is related to how much information can be gleaned about X from M, i.e., to the "unencrypted" information stored on M.

Theorem 22 (Secret Key Guessing-Version). For every $c \in \mathbb{N}$ satisfying

$$c\left|\mathcal{K}\right| \le |\mathcal{M}|,\tag{111}$$

there is a $\{0,1\}$ -valued choice of the conditional PMF in (107) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c\,|\mathcal{K}|) + 1)},\tag{112}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log c)}.$$
(113)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}|)} \vee 1, \tag{114}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le |\mathcal{K}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(115)

Theorem 23 (Secret Key List-Version). If $\lfloor |\mathcal{M}|/|\mathcal{K}| \rfloor |\mathcal{K}| > \log |\mathcal{X}| + 2$, then for every $c \in \mathbb{N}$ satisfying

$$c|\mathcal{K}| \le |\mathcal{M}|, \quad c|\mathcal{K}| > \log|\mathcal{X}| + 2,$$
(116)

there is a $\{0,1\}$ -valued choice of the conditional PMF in (107) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c\,|\mathcal{K}| - \log|\mathcal{X}| - 2) + 2)},\tag{117}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log c)}.$$
(118)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\mathcal{B}}^{(l)}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}|)} \vee 1, \tag{119}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le |\mathcal{K}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(120)

Theorems 22 and 23 are reminiscent of their counterparts for the scenario with a public and a secret hint, i.e., of Theorems 20 and 21. The main difference is that in the current scenario c and $|\mathcal{K}|$, which relate to the "unencrypted" and the "encrypted" information stored on M, respectively, play the roles of c and $|\mathcal{M}_{\rm s}|$, which in the previous scenario relate to the information stored on the public and the secret hint, respectively. Like Theorems 20 and 21, Theorems 22 and 23 imply that in the current scenario Alice can describe X deterministically by choosing a $\{0, 1\}$ -valued conditional PMF (107); there is no need for Alice to draw a onetime-pad like random variable, because she can use the secret key K as a one-time-pad.

6 Proofs

6.1 A Proof of Theorems 12 and 13

We first establish the achievability results, i.e., (62)-(63) in the guessing version and (68)-(69) in the list version. To this end fix $(c_s, c_1, c_2) \in \mathbb{N}^3$ satisfying (61) in the guessing version and (67) in the list version. For every $\nu \in \{s, 1, 2\}$ let V_{ν} be a chance variable taking values in the set $\mathcal{V}_{\nu} = \{0, \ldots, c_{\nu} - 1\}$. Corollary 7 implies that there exists some $\{0, 1\}$ -valued conditional PMF $\mathbb{P}[(V_s, V_1, V_2) = (v_s, v_1, v_2)|X = x, Y = y]$ for which

$$\min_{G(\cdot|Y,V_{\rm s},V_1,V_2)} \mathbb{E}\left[G(X|Y,V_{\rm s},V_1,V_2)^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_{\rm s}c_1c_2) + 1)}.$$
(121)

Moreover, Theorem 4 implies that there exists some deterministic task-encoder $f(\cdot|Y) \colon \mathcal{X} \to \mathcal{V}_s \times \mathcal{V}_1 \times \mathcal{V}_2$ for which

$$\mathbb{E}\left[\left|\mathcal{L}_{V_{s},V_{1},V_{2}}^{Y}\right|^{\rho}\right] < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(c_{s}c_{1}c_{2} - \log|\mathcal{X}| - 2) + 2)},\tag{122}$$

where $(V_s, V_1, V_2) = f(X|Y)$. Both (61) and (67) imply that $|\mathcal{M}_1| \ge c_s c_1$ and $|\mathcal{M}_2| \ge c_s c_2$. It thus suffices to prove (62)–(63) and (68)–(69) for a conditional PMF (49) that assigns positive probability only to $c_s c_1$ elements of \mathcal{M}_1 and $c_s c_2$ elements of \mathcal{M}_2 . Therefore, we can assume w.l.g. that $\mathcal{M}_1 = \mathcal{V}_s \times \mathcal{V}_1$ and $\mathcal{M}_2 = \mathcal{V}_s \times \mathcal{V}_2$. That is, we can choose $M_1 = (V_s \oplus_{c_s} U, V_1)$ and $M_2 = (U, V_2)$, where (V_s, V_1, V_2) is drawn according to one of the above conditional PMFs depending on the version, and where U is independent of (X, Y, V_s, V_1, V_2) and uniform over \mathcal{V}_s . Bob observes both hints and can thus recover (V_s, V_1, V_2) . Hence, in the guessing version (62) follows from (121) and in the list version (68) follows from (122).

The proof of (63) and (69) is more involved. It builds on the following two intermediate claims, which we prove next:

- 1. We can assume w.l.g. that Eve must guess not only X but the pair (X, U).
- 2. Given any pair of guessing functions $G_1(\cdot, \cdot|Y, M_1)$ and $G_2(\cdot, \cdot|Y, M_2)$ for (X, U), there exist a chance variable Z that takes values in a set of size at most $c_s(c_1 + c_2)$ and a guessing function $G(\cdot, \cdot|Y, Z)$ for (X, U) for which

$$G(X, U|Y, Z) = G_1(X, U|Y, M_1) \land G_2(X, U|Y, M_2).$$
(123)

We first prove the first intermediate claim. To this end note that in both versions (guessing and list) there exist some mappings $g_1: \mathcal{X} \times \mathcal{Y} \times \mathcal{M}_1 \to \mathcal{V}_s$ and $g_2: \mathcal{X} \times \mathcal{Y} \times \mathcal{M}_2 \to \mathcal{V}_s$ for which

$$U = g_1(X, Y, M_1) = g_2(X, Y, M_2).$$
(124)

Given any guessing functions $G_1(\cdot|Y, M_1)$ and $G_2(\cdot|Y, M_2)$ for X, introduce some guessing functions $G_1(\cdot, \cdot|Y, M_1)$ and $G_2(\cdot, \cdot|Y, M_2)$ for (X, U) satisfying, for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $m_1 \in \mathcal{M}_1$, and $m_2 \in \mathcal{M}_2$, that

$$G_k(x, g_k(x, y, m_k) | y, m_k) = G_k(x | y, m_k), \ \forall k \in \{1, 2\}.$$
(125)

From (124) it follows that

$$G_k(X, U|Y, M_k) = G_k(X|Y, M_k), \ \forall k \in \{1, 2\}.$$
(126)

Consequently, Eve can guess X and the pair (X, U) with the same number of guesses. This proves the first intermediate claim.

We next prove the second intermediate claim. Given any pair of guessing functions $G_1(\cdot, \cdot|Y, M_1)$ and $G_2(\cdot, \cdot|Y, M_2)$ for (X, U), define the triple of chance variables

$$(I, \hat{U}, \hat{V}) \triangleq \begin{cases} (1, V_{\rm s} \oplus_{c_{\rm s}} U, V_1) & \text{if } G_1(X, U|Y, M_1) \le G_2(X, U|Y, M_2), \\ (2, U, V_2) & \text{otherwise} \end{cases}$$
(127)

over the alphabet $\mathcal{I} \times \mathcal{V}_{s} \times \hat{\mathcal{V}}$, where $\mathcal{I} = \{1, 2\}$ and $\hat{\mathcal{V}} = \{0, 1, \dots, c_{1} \lor c_{2} - 1\}$. Observing (Y, I, \hat{U}, \hat{V}) , Eve can guess (X, U) using either G_{1} or G_{2} depending on the value of I. That is, Eve can guess (X, U) using some guessing function $G(\cdot, \cdot|Y, I, \hat{U}, \hat{V})$ satisfying, for every $y \in \mathcal{Y}, i \in \mathcal{I}, \hat{u} \in \mathcal{V}_{s}$, and $\hat{v} \in \{0, 1, \dots, c_{i} - 1\}$, that

$$G(\cdot, \cdot|y, i, \hat{u}, \hat{v}) = G_i(\cdot, \cdot|y, (\hat{u}, \hat{v})).$$
(128)

By (127) the number of guesses that she needs to do so is given by

$$G(X, U|Y, I, \hat{U}, \hat{V})$$

$$G(X, U|Y, (\hat{U}, \hat{V}))$$
(120)

$$=G_I(X, U|Y, (U, V))$$
(129)

$$=G_I(X, U|Y, M_I) \tag{130}$$

$$= G_1(X, U|Y, M_1) \wedge G_2(X, U|Y, M_2).$$
(131)

Consequently, (123) holds when we set $Z = (I, \hat{U}, \hat{V})$. To conclude the proof of the second intermediate claim, note that the triple (I, \hat{U}, \hat{V}) takes values in the set

$$\left\{(1,\hat{u},\hat{v})\colon (\hat{u},\,\hat{v})\in\mathcal{V}_{\mathrm{s}}\times\mathcal{V}_{1}\right\}\cup\left\{(2,\hat{u},\hat{v})\colon (\hat{u},\,\hat{v})\in\mathcal{V}_{\mathrm{s}}\times\mathcal{V}_{2}\right\},\$$

whose cardinality is given by

$$|\mathcal{V}_{\mathrm{s}} \times \mathcal{V}_{1}| + |\mathcal{V}_{\mathrm{s}} \times \mathcal{V}_{2}| = c_{\mathrm{s}}(c_{1} + c_{2}).$$

We are now ready to prove (63) and (69):

$$\mathbb{E} \left[G_1(X|Y, M_1)^{\rho} \wedge G_2(X|Y, M_2)^{\rho} \right]$$

$$\stackrel{(a)}{=} \mathbb{E} \left[G_1(X, U|Y, M_1)^{\rho} \wedge G_2(X, U|Y, M_2)^{\rho} \right]$$
(132)
$$\stackrel{(b)}{=} \mathbb{E} \left[G_1(X, U|Y, M_1)^{\rho} \wedge G_2(X, U|Y, M_2)^{\rho} \right]$$
(132)

$$\stackrel{(b)}{=} \mathbb{E}\left[G(X, U|Y, I, \hat{U}, \hat{V})^{\rho}\right]$$
(133)

$$\stackrel{(c)}{\geq} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\bar{\rho}}(X, U|Y) - \log(c_{s}(c_{1} + c_{2})))}$$
(134)

$$\stackrel{(d)}{=} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_1 + c_2))},\tag{135}$$

where (a) holds by (126); (b) holds by (131); (c) follows from Corollary 7 and the fact that (I, \hat{U}, \hat{V}) takes values in a set of size $c_s(c_1 + c_2)$; and (d) holds because

$$H_{\tilde{\rho}}(X, U|Y) = \frac{1}{\rho} \log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} \sum_{u \in \mathcal{V}_{s}} \left(P_{X,Y}(x, y) / |\mathcal{V}_{s}| \right)^{\tilde{\rho}} \right)^{1+\rho}$$
(136)

$$= \frac{1}{\rho} \log \left(\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{X,Y}(x,y)^{\tilde{\rho}} \right)^{1+\rho} |\mathcal{V}_{s}|^{\rho} \right)$$
(137)

$$=H_{\tilde{\rho}}(X|Y) + \log c_{\rm s}.$$
(138)

The equality in (136) holds because U is independent of (X, Y) and uniform over the set \mathcal{V}_{s} of size $|\mathcal{V}_{s}| = c_{s}$. This concludes the proof of the achievability results.

It remains to establish the converse results, i.e., (64)-(66) in the guessing version and (70)-(72) in the list version. In the guessing version (64) follows from Corollary 7, and in the list version (70) follows from Theorem 4. From (56) we see that (65) and (71) follow from (66) and (72), respectively, and hence it only remains to establish (66) and (72). By Corollary 6, it holds for every $k \in \{1, 2\}$ and $l \in \{1, 2\} \setminus \{k\}$ that

$$\min_{G(\cdot|Y,M_1,M_2)} \mathbb{E}\left[G(X|Y,M_1,M_2)^{\rho}\right] \ge |\mathcal{M}_l|^{-\rho} \min_{G_k(\cdot|Y,M_k)} \mathbb{E}\left[G_k(X|Y,M_k)^{\rho}\right].$$
(139)

Since

$$\min_{G(\cdot|Y,M_1,M_2)} \mathbb{E}\left[G(X|Y,M_1,M_2)^{\rho}\right] \le \mathbb{E}\left[\left|\mathcal{L}_{M_1,M_2}^{Y}\right|^{\rho}\right],$$

(139) implies that in both versions the ambiguity $\tilde{\mathscr{A}}_{E}(P_{X,Y})$ exceeds Bob's ambiguity by at most a factor of $(|\mathcal{M}_1| \wedge |\mathcal{M}_2|)^{\rho}$. That is, $\tilde{\mathscr{A}}_{E}(P_{X,Y}) \leq (|\mathcal{M}_1| \wedge |\mathcal{M}_2|)^{\rho} \mathscr{A}_{B}^{(g)}(P_{X,Y})$ and $\tilde{\mathscr{A}}_{E}(P_{X,Y}) \leq (|\mathcal{M}_1| \wedge |\mathcal{M}_2|)^{\rho} \mathscr{A}_{B}^{(1)}(P_{X,Y})$. Another upper bound on $\tilde{\mathscr{A}}_{E}(P_{X,Y})$ is obtained by considering the case where Eve ignores the hint that she observes and guesses X based on Y alone. In this case it follows from Theorem 3 that

$$\min_{G_k(\cdot|Y,M_k)} \mathbb{E}\left[G_k(X|Y,M_k)^{\rho}\right] \le 2^{\rho H_{\bar{\rho}}(X|Y)}, \ \forall k \in \{1,2\}.$$
(140)

From (140) we obtain that in both versions the ambiguity $\tilde{\mathscr{A}}_{E}(P_{X,Y})$ cannot exceed $2^{\rho H_{\bar{\rho}}(X|Y)}$, i.e., $\tilde{\mathscr{A}}_{E}(P_{X,Y}) \leq 2^{\rho H_{\bar{\rho}}(X|Y)}$. This concludes the proof of (66) and (72) and consequently that of the converse results.

6.2 A Proof of Theorem 16

If $R_1 + R_2 < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$, then (64) in the guessing version and (70) in the list version imply that the privacy-exponent is negative infinity. We hence assume that $R_1 + R_2 > H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$.

We first show that the privacy-exponent cannot exceed the RHS of (83). To this end suppose that (57) holds and consequently

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathcal{B}}(P_{X^n, Y^n}))}{n} = 0.$$
(141)

This, combined with (65) in the guessing version and (71) in the list version, implies that

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \le \rho(R_1 \wedge R_2 \wedge H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})).$$
(142)

Hence, the privacy-exponent cannot exceed the RHS of (83).

We next show that the privacy-exponent cannot be smaller than the RHS of (83). By possibly relabeling the hints, we can assume w.l.g. that $R_2 = R_1 \wedge R_2$. Fix some $\epsilon > 0$ satisfying

$$\epsilon \le R_1 + R_2 - H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}). \tag{143}$$

Choose a nonnegative rate-triple $(R_s, \tilde{R}_1, \tilde{R}_2) \in (\mathbb{R}_0^+)^3$ as follows:

1. If $R_2 \leq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})/2$, then choose

$$R_{\rm s} = 0, \quad \tilde{R}_1 = H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - R_2 + \epsilon, \quad \tilde{R}_2 = R_2. \tag{144}$$

2. Else if $H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})/2 < R_2 \leq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$, then choose

$$R_{\rm s} = 2R_2 - H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \epsilon, \quad \tilde{R}_1 = \tilde{R}_2 = H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - R_2 + \epsilon.$$
(145)

(To guarantee that $R_s \ge 0$, we assume in this case that $\epsilon > 0$ is sufficiently small so that, in addition to (143), also

$$\epsilon < 2R_2 - H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) \tag{146}$$

holds.)

3. Else if $H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) < R_2$, then choose

$$R_{\rm s} = R_2, \quad R_1 = R_2 = 0. \tag{147}$$

Having chosen $(R_s, \tilde{R}_1, \tilde{R}_2)$, choose the triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ to be $(2^{nR_s}, 2^{n\tilde{R}_1}, 2^{n\tilde{R}_2})$. For every sufficiently-large n, this choice implies (61) and (67), and by Theorems 12 and Theorem 13 we can thus guarantee (62)–(63) in the guessing version and (68)–(69) in the list version. Note that

$$R_{\rm s} + \tilde{R}_1 + \tilde{R}_2 > H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}). \tag{148}$$

Combining (148) with (62) in the guessing version and with (68) in the list version yields (57). Moreover, combining (148) with (63) in the guessing version and with (69) in the list version implies that

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \ge \rho(H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - (\tilde{R}_1 \vee \tilde{R}_2))$$
(149)

$$\geq \rho \big((R_1 \wedge R_2 - \epsilon) \wedge H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}) \big).$$
(150)

Letting ϵ tend to zero proves that the privacy-exponent cannot be smaller than the RHS of (83).

6.3 A Proof of Theorem 17

If $R_1 + R_2 < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\rm B}$, then (64) in the guessing version and (70) in the list version imply that the modest privacy-exponent is negative infinity. We hence assume that $R_1 + R_2 \geq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\rm B}$.

We first show that the modest privacy-exponent cannot exceed the RHS of (84). To this end suppose that (59) holds. This, combined with (65) in the guessing version and (71) in the list version, implies that

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \le \left(\rho(R_1 \wedge R_2) + E_{\mathrm{B}}\right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}).$$
(151)

Hence, the modest privacy-exponent cannot exceed the RHS of (84).

We next show that the modest privacy-exponent cannot be smaller than the RHS of (84). By possibly relabeling the hints, we can assume w.l.g. that $R_2 = R_1 \wedge R_2$. Choose a nonnegative rate-triple $(R_s, \tilde{R}_1, \tilde{R}_2) \in (\mathbb{R}_0^+)^3$ as follows:

1. If $R_2 \leq \left(H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}}\right)/2$, then choose

$$R_{\rm s} = 0, \quad \tilde{R}_1 = H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\rm B} - R_2, \quad \tilde{R}_2 = R_2.$$
 (152)

2. Else if $(H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}})/2 < R_2 \leq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}}$, then choose

$$R_{\rm s} = 2R_2 - H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) + \rho^{-1}E_{\rm B}, \quad \tilde{R}_1 = \tilde{R}_2 = H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\rm B} - R_2.$$
(153)

3. Else if $H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\mathrm{B}} < R_2$, then choose

$$R_{\rm s} = R_2, \quad \tilde{R}_1 = \tilde{R}_2 = 0.$$
 (154)

Having chosen $(R_s, \tilde{R}_1, \tilde{R}_2)$, choose the triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ to be $(2^{nR_s}, 2^{n\bar{R}_1}, 2^{n\bar{R}_2})$. For every sufficiently-large n, this choice implies (61) and (67), and by Theorems 12 and Theorem 13 we can thus guarantee (62)–(63) in the guessing version and (68)–(69) in the list version. Note that

$$R_{\rm s} + \tilde{R}_1 + \tilde{R}_2 \ge H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1} E_{\rm B}.$$
(155)

Combining (155) with (62) in the guessing version and with (68) in the list version yields (59). Moreover, combining (155) with (63) in the guessing version and with (69) in the list version implies that

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \ge \rho(H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - (\tilde{R}_1 \vee \tilde{R}_2))$$
(156)

$$\geq \left(\rho(R_1 \wedge R_2) + E_{\rm B}\right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}). \tag{157}$$

Consequently, the modest privacy-exponent cannot be smaller than the RHS of (84), which concludes the proof.

7 Resilience against Disk Failures

In this section we generalize the model of Section 4 to allow for Alice to produce δ hints (not necessarily two) and store them on different disks, for Bob to see $\nu \leq \delta$ (not necessarily 2) of those hints, and for Eve to see $\eta < \nu$ (not necessarily one) of the hints. We assume that, after observing X and Y, an adversarial "genie" reveals to Bob the ν hints that maximize his ambiguity and to Eve the η hints that minimize her ambiguity. The former guarantees that the system be robust against $\delta - \nu$ disk failures, no matter which disks fail; and the latter guarantees that Eve's ambiguity be "large" no matter which η hints she sees. We allow the genie to observe (X, Y), because, as we have seen, not allowing the genie to observe (X, Y) would lead to a weaker form of secrecy (see Example 1).

The current network can be described as follows. As in Section 4, we consider two problems, the "guessing version" and the "list version," which differ in the definition of Bob's ambiguity. Upon observing (X, Y) = (x, y), Alice draws the δ -tuple $\mathbf{M} = (M_1, \ldots, M_{\delta})$ from the finite set $\mathbb{F}_{2^s}^{\delta}$ according to some conditional PMF

$$\mathbb{P}[\boldsymbol{M} = \mathbf{m} | \boldsymbol{X} = \boldsymbol{x}, \boldsymbol{Y} = \boldsymbol{y}], \quad \mathbf{m} \in \mathbb{F}_{2^s}^{\delta}.$$
(158)

We assume here that each hint comprises s bits (i.e., that M takes values in $\mathbb{F}_{2^s}^{\delta}$); why this assumption is reasonable will be explained shorty (see Theorem 27 and Remark 28 ahead). Bob gets to see a size- ν set $\mathcal{B} \subseteq \{1, \ldots, \delta\}$, the components $M_{\mathcal{B}}$ of M indexed by \mathcal{B} , and the side information Y. As already mentioned, the index set \mathcal{B} is chosen by an adversary of his. In the guessing version Bob guesses X using an optimal guessing function $G_{\mathcal{B}}(\cdot|Y, M_{\mathcal{B}})$, which minimizes the ρ -th moment of the number of guesses that he needs. (As indicated by the subscript, the guessing function $G_{\mathcal{B}}(\cdot|Y, M_{\mathcal{B}})$ can depend on \mathcal{B} .) His min-max ambiguity about X is thus given by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) = \min_{G_{\mathcal{B}}(\cdot|Y, M_{\mathcal{B}})} \mathbb{E}\Big[\max_{\mathcal{B}} G_{\mathcal{B}}(X|Y, M_{\mathcal{B}})^{\rho}\Big].$$
(159)

In the list version Bob's ambiguity about X is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) = \mathbb{E}\Big[\max_{\mathcal{B}} \left|\mathcal{L}_{M_{\mathcal{B}}}^{Y}\right|^{\rho}\Big],\tag{160}$$

where for all $y \in \mathcal{Y}$ and $\mathbf{m}_{\mathcal{B}} \in \mathbb{F}_{2^s}^{\delta}$

$$\mathcal{L}_{\mathbf{m}_{\mathcal{B}}}^{y} = \left\{ x \colon \mathbb{P}[X = x | Y = y, \mathbf{M}_{\mathcal{B}} = \mathbf{m}_{\mathcal{B}}] > 0 \right\}$$
(161)

is the list of all the realizations of X of positive posterior probability

$$\mathbb{P}[X = x | Y = y, \mathbf{M}_{\mathcal{B}} = \mathbf{m}_{\mathcal{B}}] = \frac{P_{X,Y}(x, y) \mathbb{P}[\mathbf{M}_{\mathcal{B}} = \mathbf{m}_{\mathcal{B}} | X = x, Y = y]}{\sum_{\tilde{x}} P_{X,Y}(\tilde{x}, y) \mathbb{P}[\mathbf{M}_{\mathcal{B}} = \mathbf{m}_{\mathcal{B}} | X = \tilde{x}, Y = y]}.$$
(162)

Note that for $\mathcal{B}^c \triangleq \{1, \ldots, \delta\} \setminus \mathcal{B}$ we have

$$\mathbb{P}[\boldsymbol{M}_{\mathcal{B}} = \mathbf{m}_{\mathcal{B}} | X = x, Y = y] = \sum_{\mathbf{m}_{\mathcal{B}^{c}}} \mathbb{P}[\boldsymbol{M} = \mathbf{m} | X = x, Y = y]$$

Eve observes a size- η set $\mathcal{E} \subseteq \{1, \ldots, \delta\}$, the components $M_{\mathcal{E}}$ of M indexed by \mathcal{E} , and the side information Y. The index set \mathcal{E} is chosen by an accomplice of hers. Eve guesses X using an optimal guessing function $G_{\mathcal{E}}(\cdot|X, M_{\mathcal{E}})$, which minimizes the ρ -th moment of the number of guesses that she needs. (The guessing function $G_{\mathcal{E}}(\cdot|X, M_{\mathcal{E}})$ can depend on \mathcal{E} .) In both versions her ambiguity about X is thus given by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = \min_{G_{\mathcal{E}}(\cdot|X, M_{\mathcal{E}})} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, M_{\mathcal{E}})^{\rho}\right].$$
(163)

Optimizing over Alice's choice of the conditional PMF in (158), we wish to characterize the largest ambiguity that we can guarantee that Eve will have subject to a given upper bound on the ambiguity that Bob may have.

Of special interest to us is the asymptotic regime where (X, Y) is an *n*-tuple (not necessarily drawn IID), and where each hint stores

$$s = nR_s$$

bits, where R_s is nonnegative and corresponds to the per-hint storage-rate. (We assume that δ , ν , and η are fixed.) For both versions of the problem, we shall characterize the largest exponential growth that we can guarantee for Eve's ambiguity subject to the constraint that Bob's ambiguity tend to one, i.e., we shall characterize the privacy-exponent $\overline{E_{\rm E}}$ defined in Definition 1. In addition, we shall also characterize the largest exponential growth that we can guarantee for Eve's ambiguity in case Bob's ambiguity is allowed to grow exponentially with a given normalized (by n) exponent $E_{\rm B} \ge 0$, i.e., we shall characterize the modest privacy-exponent $\overline{E_{\rm E}^{\rm m}(E_{\rm B})}$ defined in Definition 2. As for the model of Section 4, the privacy-exponent and the modest privacy-exponent turn out not to depend on the version of the problem, and in the asymptotic analysis $\mathscr{A}_{\rm B}$ can thus stand for either $\mathscr{A}_{\rm B}^{\rm (g)}$ or $\mathscr{A}_{\rm R}^{\rm (l)}$.

7.1 Finite-Blocklength Results

In the next two theorems $(\nu - \eta)r$ should be viewed as the number of information-bits that can be gleaned about X from ν but not from η hints. Moreover, for every $\gamma \in {\{\eta, \nu\}}$, γp should be viewed as the number of information-bits that any γ hints reveal about X. By adapting the proof of Theorems 24 and 25 to the case at hand (see Appendix F), we obtain the following results:

Theorem 24 (Finite-Blocklength Guessing-Version). For every pair $(p, r) \in \{0, ..., s\}^2$ satisfying

$$p + r = s, \tag{164a}$$

$$p, r \in \{0\} \cup \{ \lceil \log \delta \rceil, \lceil \log \delta \rceil + 1, \ldots \},$$
(164b)

there is a choice of the conditional PMF in (158) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\rm B}^{\rm (g)}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \nu s + \eta r + 1)},\tag{165}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \eta(s-r) - \eta \log \delta - \log(1 + \ln |\mathcal{X}|))}.$$
(166)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{(\rm g)}(P_{X,Y}) \ge 2^{\rho(H_{\bar{\rho}}(X|Y) - \nu s - \log(1 + \ln|\mathcal{X}|))} \vee 1, \tag{167}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le 2^{\rho(\nu-\eta)s} \mathscr{A}_{\mathrm{B}}^{\mathrm{(g)}}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(168)

Proof. See Appendix F.2.

Theorem 25 (Finite-Blocklength List-Version). If $2^{\nu s} > \log |\mathcal{X}| + 2$, then for every pair $(p, r) \in \{0, \dots, s\}$ satisfying

$$p + r = s, \tag{169a}$$

$$p, r \in \{0\} \cup \{ \lceil \log \delta \rceil, \lceil \log \delta \rceil + 1, \ldots \},$$
(169b)

$$2^{\nu s - \eta r} > \log |\mathcal{X}| + 2, \tag{169c}$$

there is a choice of the conditional PMF in (158) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(2^{\nu s - \eta r} - \log|\mathcal{X}| - 2) + 2)},\tag{170}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \eta(s-r) - \eta \log \delta - \log(1 + \ln |\mathcal{X}|))}.$$
(171)

Conversely, for every conditional PMF, Bob's ambiguity is lower-bounded by

$$\mathscr{A}_{\rm B}^{(1)}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \nu s)} \vee 1, \tag{172}$$

and Eve's ambiguity is upper-bounded by

$$\mathscr{A}_{\rm E}(P_{X,Y}) \le 2^{\rho(\nu-\eta)s} \mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(173)

Proof. See Appendix F.2.

The bounds in Theorems 24 and 25 are tight in the sense that, with a judicious choice of p and r, the achievability results (namely (165)–(166) in the guessing version and (170)– (171) in the list version) match the corresponding converse results (namely (167)–(168) in the guessing version and (172)–(173) in the list version) up to polynomial factors of δ^{η} and of $\ln |\mathcal{X}|$. This can be seen from the following corollary to Theorems 24 and 25, which states the achievability results in a simplified and more accessible form:

Corollary 26 (Simplified Finite-Blocklength Achievability-Results). In the guessing version, for any constant \mathcal{U}_B satisfying

$$\mathscr{U}_{\rm B} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \nu s + 1)},\tag{174}$$

there is a choice of the conditional PMF in (158) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < \mathscr{U}_{\mathrm{B}},\tag{175}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(\delta^{\eta}(1+\ln|\mathcal{X}|)\right)^{-\rho} \left(\left((2\delta)^{-\rho\eta} 2^{\rho(\nu-\eta)s}(\mathscr{U}_{\mathrm{B}}-1)\right) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}\right).$$
(176)

In the list version, for any constant \mathscr{U}_{B} satisfying

$$\mathscr{U}_{\rm B} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(2^{\nu_s} - \log|\mathcal{X}| - 2) + 2)},\tag{177}$$

there is a choice of the conditional PMF in (158) for which Bob's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < \mathscr{U}_{\mathrm{B}},\tag{178}$$

and Eve's ambiguity about X is lower-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \geq \left(\delta^{\eta}(1+\ln|\mathcal{X}|)\right)^{-\rho} \left(\left(2^{-3\rho}(2\delta)^{-\rho\eta}2^{\rho(\nu-\eta)s}(\mathscr{U}_{\mathrm{B}}-1)\right) \right) \\ \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)} \\ \wedge \left(\left(2(2\delta)^{\eta}(2+\log|\mathcal{X}|)\right)^{-\rho}2^{\rho((\nu-\eta)s+H_{\tilde{\rho}}(X|Y))} \right) \right).$$
(179)

Proof. The result is a corollary to Theorems 24 and 25. See Appendix G for a detailed proof. $\hfill \Box$

We conclude this section by explaining why it is a good idea to store an equal number of bits on each disk. This can be seen from the next theorem:

Theorem 27 (Converse Results: Disk ℓ stores s_{ℓ} Bits). Suppose that for every $\ell \in \{1, \ldots, \delta\}$ Disk ℓ stores s_{ℓ} bits, where $s_1 \leq \ldots \leq s_{\delta}$. For every conditional PMF in (158), Bob's ambiguity about X is—depending on the version of the problem—lower-bounded by

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \ge 2^{\rho(H_{\bar{\rho}}(X|Y) - \sum_{\ell=1}^{\nu} s_{\ell} - \log(1 + \ln|\mathcal{X}|))} \vee 1, \tag{180a}$$

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \sum_{\ell=1}^{\nu} s_{\ell})} \vee 1, \tag{180b}$$

and Eve's ambiguity about X is upper-bounded by

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le 2^{\rho \sum_{\ell=1}^{\nu-\eta} s_{\ell}} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}, \tag{181a}$$

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le 2^{\rho \sum_{\ell=1}^{\nu-\eta} s_{\ell}} \mathscr{A}_{\mathrm{B}}^{(1)}(P_{X,Y}) \wedge 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(181b)

Proof. See Appendix H.

Remark 28 (Why Store *s* Bits on Each Disk?). Compare a scenario where for every $\ell \in \{1, \ldots, \delta\}$ Disk ℓ stores s_{ℓ} bits, where $s_1 \leq \ldots \leq s_{\delta}$, with a scenario where each disk stores $\lfloor (s_1 + \ldots + s_{\delta})/\delta \rfloor$ bits. Based on Theorem 27 and Corollary 26, neglecting polynomial factors of δ^{η} and of $\ln |\mathcal{X}|$, every pair of ambiguities for Bob and Eve that is achievable in the former scenario is also achievable in the latter scenario.

7.2 Asymptotic Results

Suppose now that (X, Y) is an *n*-tuple. We study the asymptotic regime where *n* tends to infinity. Recall that in this regime we refer to both $\mathscr{A}_{\rm B}^{({\rm g})}$ and $\mathscr{A}_{\rm B}^{({\rm l})}$ by $\mathscr{A}_{\rm B}$, because the results are the same for both versions. As we prove in Appendix I, Theorems 24 and 25 and Corollary 26 imply the following asymptotic result:

Theorem 29 (Privacy-Exponent and Modest Privacy-Exponent). Let $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ be a discrete-time stochastic process with finite alphabet $\mathcal{X} \times \mathcal{Y}$, and suppose its conditional Rényi entropy-rate $H_{\tilde{\rho}}(\mathbf{X}|\mathbf{Y})$ is well-defined. Given any nonnegative rate R_s , the privacy-exponent is

$$\overline{E}_{\rm E} = \begin{cases} \rho \big(R_s(\nu - \eta) \wedge H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}) \big) & \nu R_s > H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}), \\ -\infty & \nu R_s < H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}), \end{cases}$$
(182)

and the modest privacy-exponent for $E_B \ge 0$ is

$$\overline{E_{\rm E}^{\rm m}(E_{\rm B})} = \begin{cases} \left(\rho R_s(\nu - \eta) + E_{\rm B}\right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) & \nu R_s \ge H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1} E_{\rm B}, \\ -\infty & \nu R_s < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1} E_{\rm B}. \end{cases}$$
(183)
By (182) we can achieve the maximum privacy-exponent $\rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$ if the per-hint storage-rate satisfies

$$R_s \geq H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})/(\nu - \eta),$$

where $H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$ is the minimum rate that is necessary to describe the source for Bob. This agrees with the well-known result that the optimal share-size to share a k-bit secret so that any ν shares reveal X and any η shares provide no information about X is $k/(\nu - \eta)$ (see, e.g., [8]).

8 Coding and Encryption under a Fidelity Criterion

In this section we study a rate-distortion version of the model of Section 4, where reconstructions are lossy but subject to a given fidelity criterion. We only treat the asymptotic regime where (X, Y) is an *n*-tuple, and we shall assume that the *n*-tuple is drawn IID. Throughout this section, $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ is thus a discrete-time stochastic process of IID pairs (X_i, Y_i) that are drawn from the finite set $\mathcal{X} \times \mathcal{Y}$ according to the PMF $P_{X,Y}$.

Consider some "reconstruction alphabet" $\hat{\mathcal{X}}$ and some nonnegative "distortion-function" $d: \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}^+_0$. We quantify the distortion between any pair of *n*-tuples $(\mathbf{x}, \hat{\mathbf{x}}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n$ by their average distortion

$$d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i).$$
(184)

The fidelity criterion we study is that any reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ of X^n satisfy

$$d^{(n)}(X^n, \hat{\mathbf{x}}) \le \Delta \tag{185}$$

for some nonnegative "distortion-level" $\Delta \geq 0$. Following the convention of [7], we assume that for every $x \in \mathcal{X}$ there exists some $\hat{x} \in \hat{\mathcal{X}}$ for which $d(x, \hat{x}) = 0$, i.e., that

$$\min_{\hat{x}\in\hat{\mathcal{X}}} d(x,\hat{x}) = 0, \ \forall x\in\mathcal{X}.$$
(186)

To describe the results in this section, we denote by $R_{X|Y}(Q_{X,Y}, \Delta)$ the classical ratedistortion function of X given Y under some fixed PMF $Q_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$ [23, Ch. 7]

$$R_{X|Y}(Q_{X,Y}, \Delta) = \min_{\substack{Q_{\hat{X}|X,Y}:\\ \mathbb{E}[d(X, \hat{X})] \le \Delta}} I(X, \hat{X}|Y);$$
(187)

and we denote by $D(Q_{X,Y}||P_{X,Y})$ the Kullback-Leibler divergence between two PMFs $Q_{X,Y}$ and $P_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$. By $E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$ we refer to the functional

$$E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) = \sup_{Q_{X,Y}} \left(R_{X|Y}(Q_{X,Y},\Delta) - \rho^{-1} D(Q_{X,Y}||P_{X,Y}) \right),$$
(188)

where the supremum is over all PMFs $Q_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$.

The remainder of this section is structured as follows. Section 8.1 summarizes some notions and results pertaining to the rate-distortion versions of the guessing and task-encoding problems. Section 8.2 extends the results on guessing and task-encoding of Section 3 to the case where the reconstruction is subject to the fidelity criterion (185). Finally, Section 8.3 studies a rate-distrotion version of the model of Section 4.

8.1 Optimal Guessing Functions and Task-Encoders

Suppose we want to guess a reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ of X^n that satisfies the fidelity criterion (185) with guesses of the form "Is $d^{(n)}(X^n, \hat{\mathbf{x}}) \leq \Delta$?" Similarly as in Section 2.2, we call $\hat{G}(\cdot|Y^n)$ a guessing function on $\hat{\mathcal{X}}^n$ if for every $\mathbf{y} \in \mathcal{Y}^n$ the mapping $\hat{G}(\cdot|\mathbf{y}): \hat{\mathcal{X}}^n \to \{1, \ldots, |\hat{\mathcal{X}}|^n\}$ is one-to-one.⁴ The guessing function determines the guessing order: If we use $\hat{G}(\cdot|Y^n)$ to guess a reconstruction of X^n from the observation Y^n and observe that Y^n equals \mathbf{y} , then the question "Is $d^{(n)}(X^n, \hat{\mathbf{x}}) \leq \Delta$?" will be our $\hat{G}(\hat{\mathbf{x}}|\mathbf{y})$ -th question.

Suppose we are given a guessing function $\hat{G}(\cdot|Y^n)$. For every $\mathbf{y} \in \mathcal{Y}^n$ we define

$$G_{\Delta}(\cdot|\mathbf{y})\colon \mathcal{X}^n \to \left\{1,\ldots,|\hat{\mathcal{X}}|^n\right\}$$

as the unique mapping satisfying that, if (X^n, Y^n) equals (\mathbf{x}, \mathbf{y}) , then the first question that will be answered with "Yes!" will be our $G_{\Delta}(\mathbf{x}|\mathbf{y})$ -th question.⁵ That is, for every pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ we denote by $G_{\Delta}(\mathbf{x}|\mathbf{y})$ the smallest positive integer j satisfying that $d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) \leq \Delta$ holds for the unique n-tuple $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ for which $\hat{G}(\hat{\mathbf{x}}|\mathbf{y}) = j$. The success function corresponding to $\hat{G}(\cdot|Y^n)$ is the collection $\{G_{\Delta}(\cdot|\mathbf{y})\}_{\mathbf{y}\in\mathcal{Y}^n}$ and is denoted $G_{\Delta}(\cdot|Y^n)$. For every $\mathbf{y} \in \mathcal{Y}^n$ we define

$$\psi(\cdot|\mathbf{y})\colon \mathcal{X}^n \to \hat{\mathcal{X}}^n$$

as the unique mapping satisfying that

$$\left(\psi(\mathbf{x}|\mathbf{y}) = \hat{\mathbf{x}} \iff G_{\Delta}(\mathbf{x}|\mathbf{y}) = \hat{G}(\hat{\mathbf{x}}|\mathbf{y})\right), \ \forall (\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n \times \mathcal{Y}^n,$$
(189)

so if (X^n, Y^n) equals (\mathbf{x}, \mathbf{y}) , then the question "Is $d^{(n)}(X^n, \hat{\mathbf{x}}) \leq \Delta$?" will be answered with "Yes!" for the first time when $\hat{\mathbf{x}} = \psi(\mathbf{x}|\mathbf{y})$. The *reconstruction function* corresponding to $\hat{G}(\cdot|Y^n)$ is the collection $\{\psi(\cdot|\mathbf{y})\}_{\mathbf{x}\in\mathcal{V}^n}$ and is denoted $\psi(\cdot|Y^n)$.

We assess the performance of a guessing function in terms of the ρ -th moment of the number of guesses that we need to guess a reconstruction $\hat{\mathbf{x}}$ that satisfies the fidelity criterion (185). That is, the performance of $\hat{G}(\cdot|Y^n)$ is $\mathbb{E}[G_{\Delta}(X^n|Y^n)^{\rho}]$, where $G_{\Delta}(\cdot|Y^n)$ is the success function corresponding to $\hat{G}(\cdot|Y^n)$. We say that a guessing function is optimal if its performance is optimal, i.e., $\hat{G}(\cdot|Y^n)$ is optimal iff its corresponding success function minimizes $\mathbb{E}[G_{\Delta}(X^n|Y^n)^{\rho}]$ among all success functions. We can use Arikan and Merhav's results in [7] to characterize the asymptotic performance of optimal guessing functions on $\hat{\mathcal{X}}^n$:

Theorem 30 (Asymptotic Performance of Optimal Guessing Functions on $\hat{\mathcal{X}}^n$). [7, Section VI. C.] There exist guessing functions $\hat{G}(\cdot|Y^n)$ whose corresponding success functions

⁴Unlike the guessing problem of Section 2.2, where we guess over the source-sequence alphabet \mathcal{X}^n , here we guess over the reconstruction-sequence alphabet $\hat{\mathcal{X}}^n$.

⁵By (186) and because $\Delta \ge 0$, at least one question will be answered with "Yes!".

 $G_{\Delta}(\cdot|Y^n)$ satisfy

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\mathbb{E} \left[G_{\Delta}(X^n | Y^n)^{\rho} \right] \right) \le \rho E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta).$$
(190)

Conversely, for every guessing functions $\hat{G}(\cdot|Y^n)$ with corresponding success functions $G_{\Delta}(\cdot|Y^n)$

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\mathbb{E} \left[G_{\Delta}(X^n | Y^n)^{\rho} \right] \right) \ge \rho E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta).$$
(191)

For task-encoders we adopt the terminology of [4, Section 7]. Given some finite set \mathcal{Z} , a task-encoder $f(\cdot|Y^n)$ for X^n given side-information Y^n is for every $\mathbf{y} \in \mathcal{Y}^n$ a mapping $f(\cdot|\mathbf{y}) \colon \mathcal{X}^n \to \mathcal{Z}$. A corresponding task-decoder $\phi(\cdot|Y^n)$ is, for every $\mathbf{y} \in \mathcal{Y}^n$, a mapping $\phi(\cdot|\mathbf{y}) \colon \mathcal{Z} \to 2^{\hat{\mathcal{X}}^n}$ for which

$$\forall \mathbf{x} \in \mathcal{X}^n \text{ s.t. } P_{X|Y}^n(\mathbf{x}|\mathbf{y}) > 0 \quad \exists \, \hat{\mathbf{x}} \in \phi(f(\mathbf{x}|\mathbf{y})|\mathbf{y}) \colon d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) \le \Delta.$$
(192)

If, upon observing Y^n , the task-encoder describes X^n by $Z = f(X^n|Y^n)$, then the corresponding decoder produces the list $\mathcal{L}_Z^{Y^n} \triangleq \phi(Z|Y^n)$. By (192) this list is guaranteed to contain a reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ of X^n that satisfies the fidelity criterion (185).

As in Section 2.2, a stochastic task-encoder associates with every realization $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ of the pair (X^n, Y^n) a PMF on \mathcal{Z} and, upon observing the side information \mathbf{y} , describes \mathbf{x} by drawing Z from \mathcal{Z} according to the PMF associated with (\mathbf{x}, \mathbf{y}) , so conditonal on $(X, Y) = (\mathbf{x}, \mathbf{y})$ the probability that Z = z is

$$\mathbb{P}[Z=z|X^n=\mathbf{x},Y^n=\mathbf{y}],\quad (\mathbf{x},\mathbf{y},z)\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}.$$
(193)

A corresponding task-decoder is a collection of lists $\{\mathcal{L}_{z}^{\mathbf{y}}\}$ for which

$$\forall (\mathbf{x}, \mathbf{y}, z) \in \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z} \text{ s.t. } P_{X,Y}^n(\mathbf{x}, \mathbf{y}) \mathbb{P}[Z = z | X^n = \mathbf{x}, Y^n = \mathbf{y}] > 0 \quad \exists \, \hat{\mathbf{x}} \in \mathcal{L}_z^{\mathbf{y}} :$$

$$d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) \leq \Delta.$$
(194)

If, upon observing Y^n , the task-encoder describes X^n by Z, then the corresponding decoder produces the list $\mathcal{L}_Z^{Y^n} \subseteq \hat{\mathcal{X}}^n$. By (194) this list is guaranteed to contain a reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ of X^n that satisfies the fidelity criterion (185).

We assess the performance of an encoder-decoder pair in terms of the ρ -th moment $\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right]$ of the size of the list that the decoder produces. Bunte and Lapidoth characterized the asymptotic performance of optimal encoder-decoder pairs for the case where Y^{n} is null and the task-encoder is deterministic [4, Theorem VII.1]. A generalization of the results in [4] to the case at hand where Y^{n} need not be null and the task-encoder may be stochastic is feasible but not carried out in this paper. Instead, we shall use the close connection between task-encoding and guessing to characterize the asymptotic performance of optimal encoder-decoder pairs. The performance guarantees for optimal encoder-decoder pairs are thus presented in Section 8.2 ahead (Corollary 36 ahead).

8.2 Lists and Guesses

This section extends the results of Section 3 to the case where the reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ of X^n is subject to the fidelity criterion (185). We begin with the rate-distortion version of Lemma 5, which quantifies how some additional information Z (e.g., some description produced by an encoder), can help guessing:

Lemma 31. Given a finite set Z, draw Z from Z according to some conditional PMF $P_{Z|X^n,Y^n}$, so $(X^n,Y^n,Z) \sim P_{X,Y}^n \times P_{Z|X^n,Y^n}$. For optimal guessing functions $\hat{G}^{\star}(\cdot|Y^n,Z)$ and $\hat{G}^{\star}(\cdot|Y^n)$ with corresponding success function $G^{\star}_{\Delta}(\cdot|Y^n,Z)$ and $G^{\star}_{\Delta}(\cdot|Y^n)$ (which minimize $\mathbb{E}\left[G_{\Delta}(X^n|Y^n,Z)^{\rho}\right]$ and $\mathbb{E}\left[G_{\Delta}(X^n|Y^n)^{\rho}\right]$, respectively)

$$\mathbb{E}\left[G^{\star}_{\Delta}(X^{n}|Y^{n},Z)^{\rho}\right] \ge |\mathcal{Z}|^{-\rho} \mathbb{E}\left[G^{\star}_{\Delta}(X^{n}|Y^{n})^{\rho}\right].$$
(195)

Conversely, if $\psi(\cdot|Y^n)$ is the reconstruction function corresponding to $\hat{G}^{\star}(\cdot|Y^n)$ (for which (189) holds when we substitute $\hat{G}^{\star}(\hat{\mathbf{x}}|\mathbf{y})$ for $\hat{G}(\hat{\mathbf{x}}|\mathbf{y})$ and $G^{\star}_{\Delta}(\mathbf{x}|\mathbf{y})$ for $G_{\Delta}(\mathbf{x}|\mathbf{y})$ in (189)) and $Z = f(\psi(X^n|Y^n), Y^n)$ for some mapping $f: \hat{\mathcal{X}}^n \times \mathcal{Y}^n \to \mathcal{Z}$ for which $f(\hat{\mathbf{x}}, \mathbf{y}) = f(\hat{\mathbf{x}}', \mathbf{y})$ implies either $[\hat{G}^{\star}(\hat{\mathbf{x}}|\mathbf{y})/|\mathcal{Z}|] \neq [\hat{G}^{\star}(\hat{\mathbf{x}}'|\mathbf{y})/|\mathcal{Z}|]$ or $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$, then

$$\mathbb{E}\left[G^{\star}_{\Delta}(X^{n}|Y^{n},Z)^{\rho}\right] \leq \mathbb{E}\left[\left[G^{\star}_{\Delta}(X^{n}|Y^{n})/|\mathcal{Z}|\right]^{\rho}\right].$$
(196)

Such a mapping f always exists, because for all $l \in \mathbb{N}$ at most $|\mathcal{Z}|$ different $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ satisfy $[\hat{G}^*(\hat{\mathbf{x}}|\mathbf{y})/|\mathcal{Z}|] = l.$

Proof. See Appendix J.

Lemma 31 and (1) imply the following rate-distortion version of Corollary 6:

Corollary 32. Given a finite set \mathcal{Z} , there exists some mapping $f: \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{Z}$ such that $Z = f(X^n, Y^n)$ satisfies

$$\min_{\hat{G}(\cdot|Y^n,Z)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n,Z)^{\rho}\right] < 1 + 2^{\rho}|\mathcal{Z}|^{-\rho} \min_{\hat{G}(\cdot|Y^n)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n)^{\rho}\right].$$
(197)

Conversely, for every chance variable Z that takes values in Z

$$\min_{\hat{G}(\cdot|Y^n,Z)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n,Z)^{\rho}\right] \ge |\mathcal{Z}|^{-\rho} \min_{\hat{G}(\cdot|Y^n)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n)^{\rho}\right] \lor 1.$$
(198)

(In (197) and (198) $G_{\Delta}(\cdot|Y^n, Z)$ and $G_{\Delta}(\cdot|Y^n)$ are the success functions corresponding to $\hat{G}(\cdot|Y^n, Z)$ and $\hat{G}(\cdot|Y^n)$, respectively.)

From Corollary 32 and Theorem 30, which characterizes the asymptotic performance of optimal guessing functions $\hat{G}(\cdot|Y^n)$, we obtain the following asymptotic rate-distortion version of Corollary 7:

Corollary 33. Let $\hat{G}(\cdot|Y^n, Z)$ be guessing functions and let $G_{\Delta}(\cdot|Y^n, Z)$ be the corresponding success functions. Then, given a positive rate R > 0 and finite sets \mathcal{Z}_n satisfying

$$\lim_{n \to \infty} \frac{\log |\mathcal{Z}_n|}{n} = R,\tag{199}$$

there exist mappings $f_n \colon \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{Z}_n$ for which $Z_n = f_n(X^n, Y^n)$ satisfy

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\min_{\hat{G}(\cdot|Y^n, Z)} \mathbb{E} \left[G_{\Delta}(X^n | Y^n, Z_n)^{\rho} \right] \right) \le \rho \left(E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - R \right) \lor 0.$$
(200)

Moreover, if $R > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$, then there exist mappings $f_n: \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{Z}_n$ for which $Z_n = f_n(X^n, Y^n)$ satisfy

$$\lim_{n \to \infty} \min_{\hat{G}(\cdot|Y^n, Z)} \mathbb{E} \left[G_{\Delta}(X^n | Y^n, Z_n)^{\rho} \right] = 1.$$
(201)

Conversely, for all chance variables Z_n taking values in \mathcal{Z}_n

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\min_{\hat{G}(\cdot|Y^n, Z)} \mathbb{E} \left[G_{\Delta}(X^n | Y^n, Z_n)^{\rho} \right] \right) \ge \rho \left(E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - R \right) \lor 0.$$
(202)

Our next result is a rate-distortion version of Theorem 8:

Theorem 34. Let \mathcal{Z} be a finite set.

1. Given any stochastic task-encoder (193), every decoder with lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ (194) induces a guessing function $\hat{G}(\cdot|Y^n)$ whose corresponding success function $G_{\Delta}(\cdot|Y^n)$ satisfies

$$\mathbb{E}\left[G_{\Delta}(X^{n}|Y^{n})^{\rho}\right] \leq |\mathcal{Z}|^{\rho} \mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right].$$
(203)

2. Every guessing function $\hat{G}(\cdot|Y^n)$ with corresponding success function $G_{\Delta}(\cdot|Y^n)$ and every positive integer $\omega \leq |\hat{\mathcal{X}}|^n$ satisfying

$$|\mathcal{Z}| \ge \omega \left(1 + \left\lfloor \log \left\lceil |\hat{\mathcal{X}}|^n / \omega \right\rceil \right\rfloor \right)$$
(204)

induce a deterministic task-encoder, i.e., a stochastic task-encoder whose conditional PMF (193) is $\{0,1\}$ -valued, and a decoder whose lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ (194) satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right] \leq \mathbb{E}\left[\left[G_{\Delta}(X^{n}|Y^{n})/\omega\right]^{\rho}\right].$$
(205)

Proof. See Appendix K.

The following rate-distortion version of Corollary 10 results from Theorem 34 and (1) by setting

$$\omega = \left\lfloor |\mathcal{Z}| / \left(1 + \left\lfloor \log |\hat{\mathcal{X}}|^n \right\rfloor \right) \right\rfloor$$

in Theorem 34.

Corollary 35. Given a set \mathcal{Z} of cardinality $|\mathcal{Z}| \geq 1 + \lfloor \log |\hat{\mathcal{X}}|^n \rfloor$, any guessing function $\hat{G}(\cdot|Y^n)$ with corresponding success function $G_{\Delta}(\cdot|Y^n)$ induces a deterministic task-encoder, *i.e.*, a stochastic task-encoder whose conditional PMF (193) is $\{0,1\}$ -valued, and a decoder with lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ (194) that satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right] \leq 1 + 2^{\rho} \mathbb{E}\left[G_{\Delta}(X^{n}|Y^{n})^{\rho}\right] \left(\frac{|\mathcal{Z}|}{1 + \log|\hat{\mathcal{X}}|^{n}} - 1\right)^{-\rho}.$$
(206)

We can combine (203) and (206) with Theorem 30, which characterizes the asymptotic performance of an optimal guessing function $\hat{G}(\cdot|Y^n)$, to characterize the asymptotic performance of optimal encoder-decoder pairs:

Corollary 36 (Asymptotic Performance of Optimal Encoder-Decoder Pairs). Given a positive rate R > 0 and finite sets satisfying

$$\lim_{n \to \infty} \frac{\log |\mathcal{Z}_n|}{n} = R,$$
(207)

there exist deterministic task-encoders, i.e., stochastic task-encoders whose conditional PMFs (193) (where we substitute Z_n for Z in (193)) are $\{0,1\}$ -valued, and decoders whose lists $\{\mathcal{L}_{z_n}^{\mathbf{y}}\}$ satisfy (194) (when we substitute Z_n for Z in (194)) for which

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}\left[\left|\mathcal{L}_{Z_n}^{Y^n}\right|^{\rho}\right] \le \rho\left(E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - R\right) \lor 0;$$
(208)

and if, moreover, $R > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$, then there exist encoder-decoder pairs for which

$$\lim_{n \to \infty} \mathbb{E}\left[\left| \mathcal{L}_{M_n}^{Y^n} \right|^{\rho} \right] = 1.$$
(209)

Conversely, for any stochastic task-encoders (193) (where we substitute Z_n for Z in (193)) and decoders whose lists $\{\mathcal{L}_{z_n}^{\mathbf{y}}\}\$ satisfy (194) (when we substitute Z_n for Z, z_n for z, and Z_n for Z in (194))

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}\left[\left| \mathcal{L}_{Z_n}^{Y^n} \right|^{\rho} \right] \ge \rho \left(E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - R \right) \lor 0.$$
(210)

Note that for the special case where Y^n is null Corollary 36 specializes to [4, Theorem VII. 1].

Another interesting corollary to Theorem 34, that is to say a rate-distortion version of Corollary 11, results from the choice $\omega = 1$ in Theorem 34:

Corollary 37. Given a set \mathcal{Z} of cardinality $|\mathcal{Z}| = 1 + \lfloor \log |\hat{\mathcal{X}}|^n \rfloor$, any guessing function $\hat{G}(\cdot|Y^n)$ with corresponding success function $G_{\Delta}(\cdot|Y^n)$ induces a deterministic task-encoder, *i.e.*, a stochastic task-encoder whose conditional PMF (193) is $\{0,1\}$ -valued, and a decoder with lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ (194) that satisfy

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right] \leq \mathbb{E}\left[G_{\Delta}(X^{n}|Y^{n})^{\rho}\right].$$
(211)

8.3 Distributed-Storage Systems

We consider the following rate-distortion version of the model in Section 4. Upon observing $(X^n, Y^n) = (\mathbf{x}, \mathbf{y})$, Alice draws the hints M_1 and M_2 from the finite set $\mathcal{M}_1 \times \mathcal{M}_2$ according to some conditional PMF

$$\mathbb{P}[M_1 = m_1, M_1 = m_1 | X^n = \mathbf{x}, Y^n = \mathbf{y}].$$
(212)

We assume here that

$$\mathcal{M}_1 = \{1, \dots, 2^{nR_1}\}, \quad \mathcal{M}_2 = \{1, \dots, 2^{nR_2}\},\$$

where (R_1, R_2) is a nonnegative pair corresponding to the rate. Bob sees both hints. In the guessing version he guesses a reconstruction of X^n that satisfies (185) based on the hints and the side information Y^n , and Bob's ambiguity about X^n is thus

$$\mathscr{A}_{\rm B}^{\rm (g)}(P_{X,Y}^n, \Delta) = \min_{\hat{G}(\cdot|Y^n, M_1, M_2)} \mathbb{E} \left[G_{\Delta}(X^n | Y^n, M_1, M_2)^{\rho} \right],$$
(213)

where $G_{\Delta}(\cdot|Y^n, M_1, M_2)$ is the success function corresponding to the guessing function $\hat{G}(\cdot|Y^n, M_1, M_2)$. In the list version Bob's ambiguity about X^n is

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}^{n},\Delta) = \mathbb{E}\Big[\big|\mathcal{L}_{M_{1},M_{1}}^{Y^{n}}\big|^{\rho}\Big],\tag{214}$$

where $\{\mathcal{L}_{m_1,m_1}^{\mathbf{y}}\}\$ are the lists of a decoder corresponding to the stochastic encoder (212) and thus satisfy (194) (when we substitute (M_1, M_2) for Z, (m_1, m_2) for z, and $\mathcal{M}_1 \times \mathcal{M}_2$ for Zin (194)), so

$$P_{X,Y}^{n}(\mathbf{x}, \mathbf{y}) \mathbb{P}[M_{1} = m_{1}, M_{1} = m_{1} | X^{n} = \mathbf{x}, Y^{n} = \mathbf{y}] > 0$$
$$\implies \exists \, \hat{\mathbf{x}} \in \mathcal{L}_{m_{1},m_{2}}^{\mathbf{y}} \colon d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) \leq \Delta.$$
(215)

Eve sees one of the hints and guesses a reconstruction of X^n that satisfies (185) based on this hint and the side information Y. We assume that an accomplice of hers chooses the hint so that her guessing efforts are minimum. In both versions Eve's ambiguity about X is thus

$$\mathscr{A}_{\mathbb{E}}(P_{X,Y}^{n},\Delta) = \min_{\hat{G}^{(1)}(\cdot|Y^{n},M_{1}),\,\hat{G}^{(2)}(\cdot|Y^{n},M_{2})} \mathbb{E}\Big[G_{\Delta}^{(1)}(X^{n}|Y^{n},M_{1})^{\rho} \wedge G_{\Delta}^{(2)}(X^{n}|Y^{n},M_{2})^{\rho}\Big],\,(216)$$

where $G_{\Delta}^{(1)}(\cdot|Y^n, M_1)$ and $G_{\Delta}^{(2)}(\cdot|Y^n, M_2)$ are the success functions corresponding to the guessing functions $\hat{G}^{(1)}(\cdot|Y^n, M_1)$ and $\hat{G}^{(2)}(\cdot|Y^n, M_2)$, respectively.

For both versions of the problem, we shall characterize the largest exponential growth that we can guarantee for Eve's ambiguity subject to the constraint that Bob's ambiguity tend to one, i.e., we shall characterize the privacy-exponent $\overline{E_{\rm E}}$ defined in Definition 1. In addition, we shall also characterize the largest exponential growth that we can guarantee for Eve's ambiguity in case Bob's ambiguity is allowed to grow exponentially with a given normalized (by *n*) exponent $E_{\rm B} \ge 0$, i.e., we shall characterize the modest privacy-exponent $\overline{E_{\rm E}^{\rm m}(E_{\rm B})}$ defined in Definition 2. Like the model studied in Section 4, the privacy-exponent and the modest privacy-exponent turn out not to depend on the version of the problem, and $\mathscr{A}_{\rm B}$ can thus stand for either $\mathscr{A}_{\rm B}^{\rm (g)}$ or $\mathscr{A}_{\rm B}^{\rm (l)}$.

Our results are presented in the following theorem, which generalizes Theorems 16 and 17. To prove the theorem, we combine the proofs of Theorems 12 and 13 with the proofs of Theorems 16 and 17. Thereby, we replace the results of Section 3 with their rate-distortion versions, i.e., with the results of Section 8.2. The main difficulty in adapting the proofs to the rate-distortion version of the problem is that Claim 1 in the proof of Theorems 12 and 13 need not hold, because Eve need not guess X^n but only a reconstruction of it that satisfies (185).

Theorem 38. Given any nonnegative rate-pair (R_1, R_2) and distortion-level $\Delta \ge 0$, the privacy exponent is

$$\overline{E_{\rm E}} = \begin{cases} \rho \Big(R_1 \wedge R_2 \wedge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) \Big) & R_1 + R_2 > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta), \\ -\infty & R_1 + R_2 < E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta); \end{cases}$$
(217)

and the modest privacy exponent for $E_{\rm B} \geq 0$ is

$$\overline{E_{\rm E}^{\rm m}(E_{\rm B})} = \begin{cases} \left(\rho(R_1 \wedge R_2) + E_{\rm B}\right) \wedge \rho E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) \\ R_1 + R_2 \ge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_{\rm B}, \\ -\infty & R_1 + R_2 < E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_{\rm B}. \end{cases}$$
(218)

Proof. See Appendix L.

9 Summary

This paper studies a distributed-storage system whose encoder, Alice, observes some sensitive information X (e.g., a password) that takes values in a finite set \mathcal{X} and describes it using two hints, which she stores in different locations. The legitimate receiver, Bob, sees both hints, and—depending on the version of the problem—must either guess X (the guessing version) or must form a list that is guaranteed to contain X (the list version). The eavesdropper, Alice, sees only one of the hints; an accomplice of hers controls which. Based on her observation, Eve wishes to guess X. For an arbitrary $\rho > 0$, Bob's and Eve's ambiguity about X are quantified as follows: In the guessing version we quantify Bob's ambiguity by the ρ -th moment of the number of guesses that he needs to guess X, and in the list version we quantify Bob's ambiguity by the ρ -th moment of the size of the list that he must form. In both versions we quantify Eve's ambiguity by the ρ -th moment of the number of guesses that she needs to guess X. For each version this paper characterizes—up to polylogarithmic factors of $|\mathcal{X}|$ —the largest ambiguity that we can guarantee that Eve will have subject to a given upper bound on the ambiguity that Bob may have. Our results imply that, if the hint that is available to Bob but not to Eve can assume σ realizations, then—up to polylogarithmic factors of $|\mathcal{X}|$ —the ambiguity that we can guarantee that Eve will have either exceeds the ambiguity that Bob may have by a factor of σ^{ρ} or—in case the hint that Eve observes reveals no information about X—is as large as it can be. This holds even if we require that—up to polylogarithmic factors of $|\mathcal{X}|$ —Bob's ambiguity be as small as it can be. The paper also discusses extensions to a distributed-storage system that is robust against disk failures and a rate-distortion version of the problem.

The results for the guessing and the list version are remarkably similar: every pair of ambiguities for Bob and Eve that is achievable in the guessing version is—up to polylogarithmic factors of $|\mathcal{X}|$ —also achievable in the list version and vice versa. This can be explained by the close relation between Arikan's guessing problem [3] and Bunte and Lapidoth's task-encoding problem [4] that this paper reveals. The relation can be used to give alternative proofs of [4, Theorems I.2 and VI.2] as well as the direct part of [5, Theorem I.1]. It holds also for the rate-distortion versions of the guessing and task-encoding problems, which were introduced in [7, 4]; and in this case it can be used to give an alternative proof of [4, Theorem VII.1].

A A Proof of Corollary 14

Proof. The converse results readily follow from the converse results of Theorem 12: (64) implies (76) and (65) implies (77). The proof of the achievability results (74)–(75) is more involved. Suppose that (73) holds. To show that there is a choice of the conditional PMF in (49) for which (74)–(75) hold, we will exhibit a judicious choice of the triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ for which (74) follows from (62) and (75) from (63). By possibly relabeling the hints, we can assume w.l.g. that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$. Our choice of (c_s, c_1, c_2) depends on the constant \mathscr{U}_B and the cardinalities $|\mathcal{M}_1|$ and $|\mathcal{M}_2|$. Specifically, we distinguish between three different cases.

The first case is the case where

$$\mathscr{U}_{\rm B} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 1)}.$$
(219)

In this case we choose

$$c_{\rm s} = |\mathcal{M}_2| \text{ and } c_1 = c_2 = 1.$$
 (220)

Note that this choice satisfies (61). Consequently, (62) implies that Bob's ambiguity satisfies (74):

$$\mathscr{A}_{\rm B}^{\rm (g)}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 1)}$$
(221)

$$\leq \mathscr{U}_{\mathrm{B}},$$
 (222)

where the second inequality holds by (219). Moreover, it follows from (63) that Eve's ambiguity satisfies (75):

$$\mathscr{A}_{\rm E}(P_{X,Y}) \ge (1 + \ln |\mathcal{X}|)^{-\rho} 2^{\rho(H_{\bar{\rho}}(X|Y) - \log 2)}$$
(223)

$$= 2^{-\rho} (1 + \ln |\mathcal{X}|)^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(224)

The second case is the case where

$$\mathscr{U}_{\mathrm{B}} \ge 1 + \left\lfloor |\mathcal{M}_1| / |\mathcal{M}_2| \right\rfloor^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 1)}$$
(225a)

and

$$\mathscr{U}_{\rm B} < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}_2| + 1)}.$$
 (225b)

In this case we choose

$$c_{\rm s} = |\mathcal{M}_2|, \quad c_1 = \left\lceil 2^{H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 1 - \rho^{-1}\log(\mathscr{U}_{\rm B} - 1)} \right\rceil, \quad c_2 = 1.$$
 (226)

By (225a), this choice satisfies (61). Moreover, note that

$$c_{\rm s}c_1c_2 \ge |\mathcal{M}_2| \, 2^{H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 1 - \rho^{-1}\log(\mathscr{U}_{\rm B} - 1)} \tag{227}$$

$$= 2^{H_{\tilde{\rho}}(X|Y) + 1 - \rho^{-1}\log(\mathscr{U}_{B} - 1)}.$$
(228)

Consequently, it follows from (62) that Bob's ambiguity satisfies (74):

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - (H_{\tilde{\rho}}(X|Y) + 1 - \rho^{-1}\log(\mathscr{U}_{\mathrm{B}} - 1)) + 1)}$$
(229)

$$=\mathscr{U}_{\mathrm{B}}.$$

From (225b) it follows that

$$1 < 2^{H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}_2| + 1 - \rho^{-1} \log(\mathscr{U}_{\mathrm{B}} - 1)}.$$
(231)

Note that, for every $\xi > 1$, it holds that $\lceil \xi \rceil < 2\xi$. Consequently, (226) and (231) imply that

$$c_1 + c_2 = c_1 + 1 \tag{232}$$

$$< 2c_1 \tag{233}$$

$$< 2^{H_{\tilde{\rho}}(X|Y) - \log|\mathcal{M}_2| + 3 - \rho^{-1}\log(\mathscr{U}_{\mathrm{B}} - 1)}.$$
(234)

Eve's ambiguity satisfies (75), because from (63) and (234) it follows that:

$$\mathscr{A}_{\rm E}(P_{X,Y}) > \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - (H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}_2| + 3 - \rho^{-1} \log(\mathscr{U}_{\rm B} - 1)))}$$
(235)

$$=2^{-3\rho} (1+\ln|\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho} (\mathscr{U}_{\rm B}-1)$$
(236)

$$=2^{-3\rho} \left(1+\ln|\mathcal{X}|\right)^{-\rho} \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2|\right)^{\rho} (\mathscr{U}_{\mathrm{B}}-1),\tag{237}$$

where the last equality holds by the assumption that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$.

The third and last case is the case where

$$\mathscr{U}_{\mathrm{B}} < 1 + \left\lfloor |\mathcal{M}_1| / |\mathcal{M}_2| \right\rfloor^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}_2| + 1)}.$$
(238)

In this case we let $k^{\star} \in \mathbb{N}$ be the largest positive integer k for which

$$1 + 2^{\rho} k^{-\rho} \lfloor |\mathcal{M}_1|/k \rfloor^{-\rho} \lfloor |\mathcal{M}_2|/k \rfloor^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \le \mathscr{U}_{\mathrm{B}},$$
(239)

and we choose

$$c_{\rm s} = k^{\star}, \quad c_1 = \lfloor |\mathcal{M}_1|/k^{\star} \rfloor, \quad c_2 = \lfloor |\mathcal{M}_2|/k^{\star} \rfloor.$$
 (240)

The existence of such a k^{\star} follows from (73), which implies that (239) holds when we substitute 1 for k. The choice in (240) satisfies (61). Consequently, (62) implies that Bob's ambiguity satisfies (74):

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_{\mathrm{s}} \lfloor |\mathcal{M}_{1}|/c_{\mathrm{s}} \rfloor \lfloor |\mathcal{M}_{2}|/c_{\mathrm{s}} \rfloor) + 1)}$$
(241)

$$\leq \mathscr{U}_{\mathrm{B}},$$
 (242)

where in the second inequality we used that (239) holds when we substitute c_s for k. By the choice of c_s in (240) we also have

$$2^{-\rho H_{\tilde{\rho}}(X|Y)}(\mathscr{U}_{\rm B}-1) \stackrel{(a)}{<} 2^{\rho} (c_{\rm s}+1)^{-\rho} \left\lfloor \frac{|\mathcal{M}_1|}{c_{\rm s}+1} \right\rfloor^{-\rho} \left\lfloor \frac{|\mathcal{M}_2|}{c_{\rm s}+1} \right\rfloor^{-\rho}$$
(243)

$$\stackrel{(b)}{<} 2^{3\rho} \left(\frac{c_{\rm s} + 1}{|\mathcal{M}_1| |\mathcal{M}_2|} \right)^{\rho} \tag{244}$$

$$\stackrel{(c)}{\leq} 2^{4\rho} \left(\frac{c_{\rm s}}{|\mathcal{M}_1| |\mathcal{M}_2|} \right)^{\rho},\tag{245}$$

where (a) holds because c_s is the largest positive integer k for which (239) holds and consequently

$$\mathscr{U}_{\rm B} < 1 + 2^{\rho} (c_{\rm s} + 1)^{-\rho} \left\lfloor \frac{|\mathcal{M}_1|}{c_{\rm s} + 1} \right\rfloor^{-\rho} \left\lfloor \frac{|\mathcal{M}_2|}{c_{\rm s} + 1} \right\rfloor^{-\rho} 2^{\rho H_{\bar{\rho}}(X|Y)};$$

(b) holds because (238) and the fact that (239) holds for every positive integer $k < c_s + 1$ imply that $|\mathcal{M}_2| \geq c_s + 1$ and consequently that $|\mathcal{M}_1| \wedge |\mathcal{M}_2| \geq c_s + 1$, and because

$$\xi/2 < \lfloor \xi \rfloor, \quad \xi \ge 1;$$

and (c) holds because $c_s \ge 1$ and consequently $c_s + 1 \le 2c_s$. From (245) we obtain that

$$(c_1 + c_2)^{-\rho} \stackrel{(a)}{=} \left(\lfloor |\mathcal{M}_1|/c_{\rm s} \rfloor + \lfloor |\mathcal{M}_2|/c_{\rm s} \rfloor \right)^{-\rho} \tag{246}$$

$$\stackrel{(0)}{\geq} 2^{-\rho} \left(\frac{c_{\rm s}}{|\mathcal{M}_1|} \right)^{\prime} \tag{247}$$

$$> 2^{-5\rho} |\mathcal{M}_2|^{\rho} 2^{-\rho H_{\tilde{\rho}}(X|Y)} (\mathscr{U}_{\mathrm{B}} - 1),$$
 (248)

where (a) holds by (240); (b) holds by the assumption that $|\mathcal{M}_2| \leq |\mathcal{M}_1|$; and (c) holds by (245). From (248) and (63) we obtain that Eve's ambiguity satisfies (75):

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) > 2^{-5\rho} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} |\mathcal{M}_2|^{\rho} (\mathscr{U}_{\mathrm{B}} - 1)$$

$$\tag{249}$$

$$=2^{-5\rho}\left(1+\ln|\mathcal{X}|\right)^{-\rho}\left(|\mathcal{M}_1|\wedge|\mathcal{M}_2|\right)^{\rho}(\mathscr{U}_{\mathrm{B}}-1),\tag{250}$$

where the last equality holds by the assumption that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$.

B A Proof of Corollary 15

Proof. The converse results readily follow from the converse results of Theorem 13: (70) implies (81), and (71) implies (82). The proof of the achievability results (79)–(80) is more involved. Suppose that $|\mathcal{M}_1| |\mathcal{M}_2| > \log |\mathcal{X}| + 2$ and that (78) holds. To show that there is a choice of the conditional PMF in (49) for which (79)–(80) hold, we will exhibit a judicious choice of the triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ for which (79) follows from (68) and (80) from (69). By possibly relabeling the hints, we can assume w.l.g. that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$. Our choice of (c_s, c_1, c_2) depends on \mathscr{U}_{B} , $|\mathcal{M}_1|$, and $|\mathcal{M}_2|$; specifically, we distinguish three different cases.

The first case is the case where

$$\mathscr{U}_{\rm B} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_2| - \log|\mathcal{X}| - 2) + 2)}.$$
(251)

In this case we choose

$$c_{\rm s} = |\mathcal{M}_2|, \quad c_1 = c_2 = 1.$$
 (252)

Note that this choice satisfies (67). Consequently, (68) implies that Bob's ambiguity satisfies (79), because

$$\mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_2| - \log|\mathcal{X}| - 2) + 2)}$$
(253)

$$\leq \mathscr{U}_{\mathrm{B}},$$
 (254)

where the second inequality holds by (251). Moreover, from (69) it follows that Eve's ambiguity satisfies (80):

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log 2)} \tag{255}$$

$$=2^{-\rho} (1+\ln|\mathcal{X}|)^{-\rho} 2^{\rho H_{\bar{\rho}}(X|Y)}.$$
(256)

The second case is the case where

$$\mathscr{U}_{\mathrm{B}} \ge 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_2| \lfloor |\mathcal{M}_1|/|\mathcal{M}_2| \rfloor - \log|\mathcal{X}| - 2) + 2)}$$

$$(257a)$$

and

$$\mathscr{U}_{\rm B} < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_2| - \log|\mathcal{X}| - 2) + 2)}.$$
(257b)

In this case we choose

$$c_{\rm s} = |\mathcal{M}_2|, \quad c_1 = \left[\left(2^{H_{\tilde{\rho}}(X|Y) + 2-\rho^{-1}\log(\mathscr{U}_{\rm B}-1)} + \log|\mathcal{X}| + 2 \right) / |\mathcal{M}_2| \right], \quad c_2 = 1.$$
(258)

By (257a), this choice satisfies (67). Moreover, note that

$$c_{s}c_{1}c_{2} \geq 2^{H_{\tilde{\rho}}(X|Y)+2-\rho^{-1}\log(\mathscr{U}_{B}-1)} + \log|\mathcal{X}| + 2.$$
(259)

Consequently, (68) implies that Bob's ambiguity satisfies (79), because

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < 1 + 2^{\rho \left(H_{\tilde{\rho}}(X|Y) - \log\left(2^{H_{\tilde{\rho}}(X|Y) + 2-\rho^{-1}\log(\mathscr{U}_{\mathrm{B}}-1)}\right) + 2\right)}$$
(260)

$$=\mathscr{U}_{\mathrm{B}}.$$

From (257b) it follows that

$$1 < \left(2^{H_{\tilde{\rho}}(X|Y) + 2 - \rho^{-1}\log(\mathscr{U}_{B} - 1)} + \log|\mathcal{X}| + 2\right) / |\mathcal{M}_{2}|.$$
(262)

Note that, for every $\xi > 1$, it holds that $\lceil \xi \rceil < 2\xi$. Consequently, (258) and (262) imply that

$$c_1 + c_2 = c_1 + 1 \tag{263}$$

$$< 2c_1$$
 (264)

$$< 4 \left(2^{H_{\tilde{\rho}}(X|Y) + 2 - \rho^{-1} \log(\mathscr{U}_{\mathrm{B}} - 1)} + \log |\mathcal{X}| + 2 \right) / |\mathcal{M}_2|.$$
(265)

From (69) and (265) it follows that Eve's ambiguity satisfies (80):

$$\mathscr{A}_{\rm E}(P_{X,Y}) > 2^{-2\rho} (1 + \ln |\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho} \times 2^{\rho \left(H_{\tilde{\rho}}(X|Y) - \log\left(2^{H_{\tilde{\rho}}(X|Y) + 2-\rho^{-1}\log(\mathscr{U}_{\rm B}^{-1})} + \log |\mathcal{X}| + 2\right)\right)}$$
(266)

$$= 2^{-2\rho} (1 + \ln |\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \times (2^{H_{\tilde{\rho}}(X|Y) + 2-\rho^{-1} \log(\mathscr{U}_{\mathrm{B}}-1)} + \log |\mathcal{X}| + 2)^{-\rho}$$
(267)

$$\overset{(a)}{\geq} 2^{-5\rho} (1 + \ln |\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho} (\mathscr{U}_{\mathrm{B}} - 1)$$

$$\wedge 2^{-3\rho} (1 + \ln |\mathcal{X}|)^{-\rho} (2 + \log |\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho} 2^{\rho H_{\bar{\rho}}(X|Y)}$$

$$(268)$$

$$\stackrel{(b)}{=} 2^{-5\rho} \left(1 + \ln |\mathcal{X}| \right)^{-\rho} \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2| \right)^{\rho} (\mathscr{U}_{\mathrm{B}} - 1)$$

$$\wedge 2^{-3\rho} \left(1 + \ln |\mathcal{X}| \right)^{-\rho} (2 + \log |\mathcal{X}|)^{-\rho} \left(|\mathcal{M}_1| \wedge |\mathcal{M}_2| \right)^{\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)}, \quad (269)$$

where (a) holds because

$$\frac{1}{a+b} \ge \frac{1}{2a} \land \frac{1}{2b}, \quad a, b > 0;$$

and (b) holds by the assumption that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$.

The third and last case is the case where

$$\mathscr{U}_{\rm B} < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_2| \lfloor |\mathcal{M}_1| / |\mathcal{M}_2| \rfloor - \log |\mathcal{X}| - 2) + 2)}.$$
(270)

In this case we let $k^{\star} \in \mathbb{N}$ be the largest positive integer k for which

$$1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(k\lfloor |\mathcal{M}_1|/k\rfloor \lfloor |\mathcal{M}_2|/k\rfloor - \log |\mathcal{X}| - 2) + 2)} \le \mathscr{U}_{\mathrm{B}},\tag{271}$$

and we choose

$$c_{\rm s} = k^{\star}, \quad c_1 = \lfloor |\mathcal{M}_1|/k^{\star} \rfloor, \quad c_2 = \lfloor |\mathcal{M}_2|/k^{\star} \rfloor.$$
 (272)

The existence of such a k^* follows from (78), which implies that (271) holds when we substitute 1 for k. Note that the choice in (272) satisfies (67). Consequently, (68) implies that Bob's ambiguity satisfies (79), because

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_{\mathrm{s}}\lfloor|\mathcal{M}_{1}|/c_{\mathrm{s}}\rfloor\lfloor|\mathcal{M}_{2}|/c_{\mathrm{s}}\rfloor - \log|\mathcal{X}| - 2) + 2)}$$
(273)

$$\leq \mathscr{U}_{\mathrm{B}},$$
 (274)

where in the second inequality we used that (271) holds when we substitute c_s for k. By the choice of c_s in (272) we also have

$$2^{-\rho(H_{\tilde{\rho}}(X|Y)+2)}(\mathscr{U}_{\mathrm{B}}-1) \stackrel{(a)}{\leq} \left((c_{\mathrm{s}}+1) \left\lfloor \frac{|\mathcal{M}_{1}|}{c_{\mathrm{s}}+1} \right\rfloor \left\lfloor \frac{|\mathcal{M}_{2}|}{c_{\mathrm{s}}+1} \right\rfloor - \log|\mathcal{X}| - 2 \right)^{-\rho}$$
(275)

$$\stackrel{(b)}{\leq} \left(\frac{|\mathcal{M}_1| |\mathcal{M}_2|}{4(c_{\rm s}+1)} - \log |\mathcal{X}| - 2 \right)^{-\rho} \tag{276}$$

$$\stackrel{(c)}{\leq} \left(\frac{|\mathcal{M}_1| |\mathcal{M}_2|}{8c_{\rm s}} - \log |\mathcal{X}| - 2\right)^{-\rho},\tag{277}$$

where (a) holds because c_s is the largest positive integer k for which (271) holds and consequently

$$\mathscr{U}_{B} < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log((c_{s}+1)\lfloor|\mathcal{M}_{1}|/(c_{s}+1)\rfloor\lfloor|\mathcal{M}_{2}|/(c_{s}+1)\rfloor - \log|\mathcal{X}|-2)+2)};$$

(b) holds because (270) and the fact that (271) holds for every positive integer $k < c_s + 1$ imply that $|\mathcal{M}_2| \geq c_s + 1$ and consequently that $|\mathcal{M}_1| \wedge |\mathcal{M}_2| \geq c_s + 1$, and because

$$\xi/2 < \lfloor \xi \rfloor, \quad \xi \ge 1;$$

and (c) holds because $c_s \ge 1$ and consequently $c_s + 1 \le 2c_s$. From (277) we obtain that

$$\left(\frac{c_{\rm s}}{|\mathcal{M}_1|}\right)^{\rho} > 2^{-3\rho} |\mathcal{M}_2|^{\rho} \left((\mathscr{U}_{\rm B} - 1)^{-1/\rho} 2^{H_{\tilde{\rho}}(X|Y) + 2} + \log|\mathcal{X}| + 2 \right)^{-\rho},\tag{278}$$

and consequently that

$$(c_1 + c_2)^{-\rho} \stackrel{(a)}{=} \left(\left\lfloor |\mathcal{M}_1|/c_s \right\rfloor + \left\lfloor |\mathcal{M}_2|/c_s \right\rfloor \right)^{-\rho}$$

$$(279)$$

$$\stackrel{(o)}{\geq} 2^{-\rho} \left(\frac{c_{\rm s}}{|\mathcal{M}_1|} \right) \tag{280}$$

$$\sum_{i=1}^{(c)} 2^{-4\rho} |\mathcal{M}_2|^{\rho} \Big((\mathscr{U}_{\mathrm{B}} - 1)^{-1/\rho} 2^{H_{\tilde{\rho}}(X|Y) + 2} + \log |\mathcal{X}| + 2 \Big)^{-\rho}$$
(281)

$$\overset{(d)}{\geq} 2^{-7\rho} |\mathcal{M}_2|^{\rho} (\mathscr{U}_{\mathrm{B}} - 1) 2^{-\rho H_{\bar{\rho}}(X|Y)} \wedge 2^{-5\rho} (2 + \log |\mathcal{X}|)^{-\rho} |\mathcal{M}_2|^{\rho}, \quad (282)$$

where (a) holds by (272); (b) holds by the assumption that $|\mathcal{M}_2| \leq |\mathcal{M}_1|$; (c) holds by (278); and (d) holds because

$$\frac{1}{a+b} \ge \frac{1}{2a} \land \frac{1}{2b}, \quad a, b > 0.$$

From (282) and (69) we obtain that Eve's ambiguity satisfies (80):

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) > 2^{-5\rho} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} |\mathcal{M}_{2}|^{\rho} \left(2^{-2\rho} (\mathscr{U}_{\mathrm{B}} - 1) \right)$$

$$\wedge \left(2 + \log |\mathcal{X}|\right)^{-\rho} 2^{\rho H_{\bar{\rho}}(X|Y)}$$

$$= 2^{-5\rho} \left(1 + \ln |\mathcal{X}|\right)^{-\rho} \left(|\mathcal{M}_{1}| \wedge |\mathcal{M}_{2}|\right)^{\rho} \left(2^{-2\rho} (\mathscr{U}_{\mathrm{B}} - 1)\right)$$
(283)

$$\wedge \left(2 + \log |\mathcal{X}|\right)^{-\rho} 2^{\rho H_{\tilde{\rho}}(X|Y)} \right), \tag{284}$$

where the last equality holds by the assumption that $|\mathcal{M}_2| = |\mathcal{M}_1| \wedge |\mathcal{M}_2|$.

C A Proof of Theorem 19

Proof. We first establish the achievability results, i.e., (88)–(89). To this end suppose that $|\mathcal{M}_1| \wedge |\mathcal{M}_2| \geq 1 + \lfloor \log |\mathcal{X}| \rfloor$. Let

$$c_{\rm s} = 1 + \lfloor \log |\mathcal{X}| \rfloor, \quad c_1 = \lfloor \frac{|\mathcal{M}_1|}{c_{\rm s}} \rfloor, \quad c_2 = \lfloor \frac{|\mathcal{M}_2|}{c_{\rm s}} \rfloor,$$
(285)

and for each $\nu \in \{c_s, c_1, c_2\}$ let V_{ν} be a chance variable taking values in the set $\mathcal{V}_{\nu} = \{0, \ldots, c_{\nu} - 1\}$. Corollary 7 implies that there exists some $\{0, 1\}$ -valued conditional PMF $\mathbb{P}[(V_1, V_2) = (v_1, v_2)|X = x, Y = y]$ for which

$$\min_{G(\cdot|Y,V_1,V_2)} \mathbb{E}\left[G(X|Y,V_1,V_2)^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_1c_2) + 1)}.$$
(286)

Draw (V_1, V_2) from $\mathcal{V}_1 \times \mathcal{V}_2$ according to the above conditional PMF. Fix $\epsilon > 0$ and draw (V'_1, V'_2) from $\mathcal{V}_1 \times \mathcal{V}_2$ according to the conditional PMF

$$\mathbb{P}\left[\left(V_{1}', V_{2}'\right) = (v_{1}', v_{2}') \middle| (V_{1}, V_{2}) = (v_{1}, v_{2})\right] \\
= \left(1 - 2^{-\epsilon} - \frac{2^{-\epsilon}}{|\mathcal{V}_{1}| |\mathcal{V}_{2}|}\right) \mathbb{1}_{\{(v_{1}', v_{2}') = (v_{1}, v_{2})\}} + \frac{2^{-\epsilon}}{|\mathcal{V}_{1}| |\mathcal{V}_{2}|}.$$
(287)

Note that, irrespective of the realization (v_1, v_2) of (V'_1, V'_2) , the probability that (V'_1, V'_2) equals (v_1, v_2) is $1-2^{-\epsilon}$. Let $G_{\star}(\cdot|Y, V_1, V_2)$ be an optimal guessing function, which minimizes $\mathbb{E}[G(X|Y, V_1, V_2)^{\rho}]$. Define the guessing function $G(\cdot|Y, V'_1, V'_2)$ by

$$G(x|y,v_1',v_2') = G_{\star}(x|y,v_1',v_2'), \ \forall (x,y,v_1',v_2') \in \mathcal{X} \times \mathcal{Y} \times \mathcal{V}_1 \times \mathcal{V}_2.$$

$$(288)$$

Using the trivial bound

$$G(x|y,v_1',v_2') \leq |\mathcal{X}|, \ \forall \, (x,y,v_1',v_2') \in \mathcal{X} \times \mathcal{Y} \times \mathcal{V}_1 \times \mathcal{V}_2,$$

we obtain that

$$\mathbb{E}\left[G(X|Y,V_1',V_2')^{\rho}\right] \le (1-2^{-\epsilon})\mathbb{E}\left[G_{\star}(X|Y,V_1,V_2)^{\rho}\right] + 2^{-\epsilon}|\mathcal{X}|^{\rho}.$$
(289)

Consequently,

$$\min_{G(\cdot|Y,V_1',V_2')} \mathbb{E} \left[G(X|Y,V_1',V_2')^{\rho} \right] \\
\leq (1-2^{-\epsilon}) \min_{G(\cdot|Y,V_1,V_2)} \mathbb{E} \left[G(X|Y,V_1,V_2)^{\rho} \right] + 2^{-\epsilon} |\mathcal{X}|^{\rho} \tag{290}$$

$$= e^{-(\epsilon - \epsilon) \log|\mathcal{X}|} = e^{2(H_1(X|Y) - \log(\epsilon,\epsilon)) + 1} \tag{291}$$

$$<1+2^{-(\epsilon-\rho\log|\mathcal{X}|)}+2^{\rho(H_{\bar{\rho}}(X|Y)-\log(c_1c_2)+1)},$$
(291)

where (291) follows from (286). Corollary 11 and (285) imply that there exists some $\{0, 1\}$ -valued conditional PMF

$$\mathbb{P}[V_{\rm s} = v_{\rm s} | X = x, Y = y, V_1' = v_1, V_2' = v_2]$$

for which

$$\mathbb{E}\Big[\left|\mathcal{L}_{V_{\rm s},V_{1}',V_{2}'}^{Y}\right|^{\rho}\Big] \le \min_{G(\cdot|Y,V_{1}',V_{2}')} \mathbb{E}\big[G(X|Y,V_{1}',V_{2}')^{\rho}\big]$$
(292)

$$< 1 + 2^{-(\epsilon - \rho \log|\mathcal{X}|)} + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c_1 c_2) + 1)}.$$
(293)

Draw V_s from \mathcal{V}_s according to the above conditional PMF. Using the assumption that $|\mathcal{M}_1| \wedge |\mathcal{M}_2| \geq 1 + |\log |\mathcal{X}||$ and (285), we obtain that

$$c_k > \frac{|\mathcal{M}_k|}{2\left(1 + \lfloor \log |\mathcal{X}| \rfloor\right)}, \quad k \in \{1, 2\}.$$
(294)

From (293) and (294) it follows that

$$\mathbb{E}\left[\left|\mathcal{L}_{V_{\mathrm{s}},V_{1}',V_{2}'}^{Y}\right|^{\rho}\right] < 1 + 2^{-(\epsilon-\rho\log|\mathcal{X}|)} + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_{1}||\mathcal{M}_{2}|) + 2\log(1+\lfloor\log|\mathcal{X}|\rfloor) + 3)}.$$
(295)

By (285) $|\mathcal{M}_1| \geq c_s c_1$ and $|\mathcal{M}_2| \geq c_s c_2$, and hence it suffices to prove (88)–(89) for a conditional PMF (49) that assigns positive probability only to $c_s c_1$ elements of \mathcal{M}_1 and $c_s c_2$ elements of \mathcal{M}_2 , and we thus assume w.l.g. that $\mathcal{M}_1 = \mathcal{V}_s \times \mathcal{V}_1$ and $\mathcal{M}_2 = \mathcal{V}_s \times \mathcal{V}_2$. That is, we can choose $M_1 = (V_s \oplus_{c_s} U, V_1')$ and $M_2 = (U, V_2')$, where U is independent of (X, Y, V_s, V_1', V_2') and uniform over \mathcal{V}_s . For this choice it follows from (295) that

$$\mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y}) < 1 + 2^{-(\epsilon - \rho \log |\mathcal{X}|)} + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{M}_1||\mathcal{M}_2|) + 2\log(1 + \lfloor \log |\mathcal{X}|\rfloor) + 3)}.$$
(296)

This proves that (88) holds for every sufficiently-large ϵ . As to (69), note that for every $\epsilon > 0$

$$\mathcal{L}_{M_1}^Y = \mathcal{L}_{M_2}^Y = \mathcal{L}^Y, \tag{297}$$

because

$$\mathbb{P}[M_1 = m_1, M_2 = m_2 | X = x, Y = y] > 0, \ \forall (x, y, m_1, m_2) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{M}_1 \times \mathcal{M}_2.$$
(298)

We next conclude by establishing the converse results (91)-(92). Theorem 4 implies (91); and (92) trivially holds, because the list that Eve forms based on Y and the hint that she observes cannot be larger than the list that she would have to form if she were to observe only Y.

D A Proof of Theorems 20 and 21

Proof. We first establish the achievability results, i.e., (98)-(99) in the guessing version and (103)-(104) in the list version. To this end, fix $c \in \mathbb{N}$ satisfying (97) in the guessing version and (102) in the list version. Both (97) and (102) imply that $c \leq |\mathcal{M}_{\rm p}|$. Hence it suffices to prove (98)–(99) and (103)–(104) for a {0,1}-valued conditional PMF as in (93) that assigns positive probability only to c elements of $\mathcal{M}_{\rm p}$. We can thus assume w.l.g. that $|\mathcal{M}_{\rm p}| = c$. Corollary 7 implies that there exists some {0,1}-valued conditional PMF

$$\mathbb{P}[M_{\rm p} = m_{\rm p}, M_{\rm s} = m_{\rm s}|X = x, Y = y]$$

for which

$$\min_{G(\cdot|Y,M_{\rm p},M_{\rm s})} \mathbb{E}\left[G(X|Y,M_{\rm p},M_{s})^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_{\rm p}||\mathcal{M}_{\rm s}|) + 1)}$$
(299)

$$= 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c |\mathcal{M}_{s}|) + 1)}.$$
(300)

In addition, Theorem 4 implies that there exists some deterministic task-encoder $f(\cdot|Y) \colon \mathcal{X} \to \mathcal{M}_p \times \mathcal{M}_s$ for which

$$\mathbb{E}\left[\left|\mathcal{L}_{M_{\mathrm{p}},M_{\mathrm{s}}}^{Y}\right|^{\rho}\right] < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{M}_{\mathrm{p}} \times \mathcal{M}_{\mathrm{s}}| - \log|\mathcal{X}| - 2) + 2)}$$
(301)

$$= 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c|\mathcal{M}_{s}| - \log|\mathcal{X}| - 2) + 2)}, \tag{302}$$

where $(M_p, M_s) = f(X|Y)$. Accordingly, in the guessing version (98) follows from (300) and in the list version (103) follows from (302). Moreover, Corollary 7 implies (99) in the guessing version and (104) in the list version:

$$\min_{G.(X|Y,M_{\rm p})} \mathbb{E}[G(X|Y,M_{\rm p})^{\rho}] \ge \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log |\mathcal{M}_{\rm p}|)}$$
(303)

$$= (1 + \ln |\mathcal{X}|)^{-\rho} 2^{\rho(H_{\bar{\rho}}(X|Y) - \log c)}.$$
 (304)

It remains to establish the converse results, i.e., (100)-(101) in the guessing version and (105)-(106) in the list version. In the guessing version (100) follows from Corollary 7, and in the list version (105) follows from Theorem 4. To prove (101) and (106), we first note from Corollary 6 that

$$\min_{G(\cdot|Y,M_{\rm p},M_{\rm s})} \mathbb{E}\left[G(X|Y,M_{\rm p},M_{\rm s})^{\rho}\right] \ge |\mathcal{M}_{\rm s}|^{-\rho} \min_{G(\cdot|Y,M_{\rm p})} \mathbb{E}\left[G(X|Y,M_{\rm p})^{\rho}\right].$$
(305)

Moreover, we also note that

$$\min_{G(\cdot|Y,M_{\rm p},M_{\rm s})} \mathbb{E}\left[G(X|Y,M_{\rm p},M_{\rm s})^{\rho}\right] \le \mathbb{E}\left[\left|\mathcal{L}_{M_{\rm p},M_{\rm s}}^{Y}\right|^{\rho}\right].$$
(306)

From (305) and (306) it follows that in both versions Eve's ambiguity exceeds Bob's by at most a factor of $|\mathcal{M}_{\rm s}|^{\rho}$, i.e., $\mathscr{A}_{\rm E}(P_{X,Y}) \leq |\mathcal{M}_{\rm s}|^{\rho} \mathscr{A}_{\rm B}^{({\rm g})}(P_{X,Y})$ and $\mathscr{A}_{\rm E}(P_{X,Y}) \leq |\mathcal{M}_{\rm s}|^{\rho} \mathscr{A}_{\rm B}^{({\rm l})}(P_{X,Y})$. Since Eve can ignore $\mathcal{M}_{\rm p}$ and guess X based on Y alone, we obtain from Theorem 3 that in both versions Eve's ambiguity cannot exceed $2^{\rho H_{\bar{\rho}}(X|Y)}$. That is,

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) = \min_{G(\cdot|Y,M_{\mathrm{p}})} \mathbb{E}\left[G(X|Y,M_{\mathrm{p}})^{\rho}\right] \le 2^{\rho H_{\tilde{\rho}}(X|Y)}.$$
(307)

This concludes the proof of (101) and (106) and consequently that of the converse results. \Box

E A Proof of Theorems 22 and 23

Proof. We first establish the achievability results, i.e., (112)–(113) in the guessing version and (117)–(118) in the list version. To this end fix $c \in \mathbb{N}$ satisfying (111) in the guessing version and (116) in the list version. Let $M_{\rm p}$ be a chance variable that takes values in the set $\mathcal{M}_{\rm p}$, and let $M_{\rm s}$ be a chance variable that takes values in the set \mathcal{K} . Corollary 7 implies that there exists some $\{0, 1\}$ -valued conditional PMF $\mathbb{P}[M_{\rm p} = m_{\rm p}, M_{\rm s} = m_{\rm s}|X = x, Y = y]$ for which

$$\min_{G(\cdot|Y,M_{\rm p},M_{\rm s})} \mathbb{E}\left[G(X|Y,M_{\rm p},M_{\rm s})^{\rho}\right] < 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(|\mathcal{M}_{\rm p}|\,|\mathcal{M}_{\rm s}|) - 1)}$$
(308)

$$= 1 + 2^{\rho(H_{\bar{\rho}}(X|Y) - \log(c\,|\mathcal{K}|) - 1)}.$$
(309)

Theorem 4 implies that there exists some deterministic task-encoder $f(\cdot|Y) \colon \mathcal{X} \to \mathcal{M}_p \times \mathcal{M}_s$ for which

$$\mathbb{E}\left[\left|\mathcal{L}_{M_{\mathrm{p}},M_{\mathrm{s}}}^{Y}\right|^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(|\mathcal{M}_{\mathrm{p}}||\mathcal{M}_{\mathrm{s}}| - \log|\mathcal{X}| - 2) + 2)}$$
(310)

$$= 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(c |\mathcal{K}| - \log |\mathcal{X}| - 2) + 2)}, \tag{311}$$

where $(M_{\rm p}, M_{\rm s}) = f(X|Y)$. Both (111) and (116) imply that $c|\mathcal{K}| \leq |\mathcal{M}|$. Hence it suffices to prove (112)–(113) and (117)–(118) for a {0,1}-valued conditional PMF as in (107) that assigns positive probability only to $c|\mathcal{K}|$ elements of \mathcal{M} . We can thus assume w.l.g. that $\mathcal{M} = \mathcal{K} \times \mathcal{M}_{\rm p}$, where $\mathcal{M}_{\rm p}$ is a set of cardinality c, and $\mathcal{K} = \{0, \ldots, |\mathcal{K}| - 1\}$. That is, we can choose $M = (M_{\rm s} \oplus_{|\mathcal{K}|} K, M_{\rm p})$, where $(M_{\rm s}, M_{\rm p})$ is drawn according to one of the above conditional PMFs depending on the version. Bob observes the hint M and the secret key Kand can thus recover the pair $(M_{\rm s}, M_{\rm p})$. Hence, in the guessing version (112) follows from (309), and in the list version (117) follows from (311).

The proof of (113) and (118) is more involved. Note that in both versions (guessing and list) there exists some mapping $g: \mathcal{X} \times \mathcal{Y} \times \mathcal{M} \to \mathcal{K}$ for which

$$K = g(X, Y, M). \tag{312}$$

Given any guessing function $G(\cdot|Y, M)$ for X, introduce some guessing function $G(\cdot, \cdot|Y, M)$ for (X, K) satisfying that

$$G(x, g(x, y, m)|y, m) = G(x|y, m), \ \forall (x, y, m) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{M}.$$
(313)

From (312) it then follows that

$$G(X, K|Y, M) = G(X|Y, M),$$
(314)

and consequently that Eve can guess X and the pair (X, K) with the same number of guesses. In particular,

$$\mathbb{E}\left[G(X|Y,M)^{\rho}\right] = \mathbb{E}\left[G(X,K|Y,M)^{\rho}\right].$$
(315)

Corollary 7 implies that

$$\min_{G(\cdot,\cdot|Y,M)} \mathbb{E}\left[G(X,K|Y,M)^{\rho}\right] \ge \left(1 + \ln|\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X,K|Y) - \log|\mathcal{M}|)}$$
(316)

$$= \left(1 + \ln |\mathcal{X}|\right)^{-\rho} 2^{\rho(H_{\tilde{\rho}}(X,K|Y) - \log(c |\mathcal{K}|))}.$$
(317)

Note, that

$$H_{\tilde{\rho}}(X,K|Y) = \frac{1}{\rho} \log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} \sum_{k \in \mathcal{K}} \left(\frac{P_{X,Y}(x,y)}{|\mathcal{K}|} \right)^{\tilde{\rho}} \right)^{1+\rho}$$
(318)

$$= \frac{1}{\rho} \log \left(\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{X,Y}(x,y)^{\tilde{\rho}} \right)^{1+\rho} |\mathcal{K}|^{\rho} \right)$$
(319)

$$=H_{\tilde{\rho}}(X|Y) + \log|\mathcal{K}|, \qquad (320)$$

where the first equality holds because K is independent of (X, Y) and uniform over the set \mathcal{K} . Consequently, (315) and (317) imply (113) in the guessing version and (118) in the list version.

It remains to establish the converse results, i.e., (114)-(115) in the guessing version and (119)-(120) in the list version. To this end we first note that

$$H_{\tilde{\rho}}(X|Y,K) = \frac{\alpha}{1-\alpha} \log \sum_{y \in \mathcal{Y}} \sum_{k \in \mathcal{K}} \left(\sum_{x \in \mathcal{X}} \left(\frac{P_{X,Y}(x,y)}{|\mathcal{K}|} \right)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(321)

$$= \frac{\alpha}{1-\alpha} \log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{X,Y}(x,y)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(322)

$$=H_{\tilde{\rho}}(X|Y),\tag{323}$$

where the first equality holds because K is independent of (X, Y) and uniform over the set \mathcal{K} . In the guessing version (114) follows from Corollary 7 and (323), and in the list version (119) follows from Theorem 4 and (323). To prove (115) and (120), we first note that by Corollary 6

$$\min_{G(\cdot|Y,K,M)} \mathbb{E}\left[G(X|Y,K,M)^{\rho}\right] \ge |\mathcal{K}|^{-\rho} \min_{G(\cdot|Y,M)} \mathbb{E}\left[G(X|Y,M)^{\rho}\right].$$
(324)

Because

$$\min_{G(\cdot|Y,K,M)} \mathbb{E}\left[G(X|Y,K,M)^{\rho}\right] \leq \mathbb{E}\left[\left|\mathcal{L}_{M}^{Y,K}\right|^{\rho}\right]$$

(324) implies that in both versions Eve's ambiguity exceeds Bob's by at most a factor of $|\mathcal{K}|^{\rho}$, i.e., $\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \leq |\mathcal{K}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{g})}(P_{X,Y})$ and $\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \leq |\mathcal{K}|^{\rho} \mathscr{A}_{\mathrm{B}}^{(\mathrm{l})}(P_{X,Y})$. Since Eve can ignore M and guess X based on Y alone, we obtain from Theorem 3 that in both versions Eve's ambiguity cannot exceed $2^{\rho H_{\bar{\rho}}(X|Y)}$:

$$A_{\rm E}(P_{X,Y}) = \min_{G(\cdot|Y,M)} \mathbb{E}\left[G(X|Y,M)^{\rho}\right] \le 2^{\rho H_{\bar{\rho}}(X|Y)}.$$
(325)

This concludes the proof of (115) and (120) and consequently that of the converse results. \Box

F A Proof of Theorems 24 and 25

In Section F.1 we summarize the results on maximum-distance separable (MDS) codes that we shall use in the proof of Theorems 24 and 25. Theorems 24 and 25 are proved in Section F.2.

F.1 Properties of MDS Codes

The following results on maximum-distance separable (MDS) codes can be found, e.g., in [24]. An (n, k) linear code \mathscr{C} over a finite field \mathbb{F}_q is a k-dimensional linear subspace of the vector space \mathbb{F}_q^n of all n-tuples over \mathbb{F}_q . An (n, k, d) linear code is an (n, k) linear code satisfying that the minimum Hamming distance between any two codewords (or, equivalently, the minimum Hamming weight of any nonzero codeword) is d. By the Singleton bound $k \leq n-d+1$, where equality is achieved iff the following holds for every size-k set $\mathcal{C} \subseteq [1:n]$, where k = n-d+1: if we reduce all q^k codewords to the components indexed by \mathcal{C} , then we obtain all q^k k-tuples over \mathbb{F}_q . An MDS code is a linear code that satisfies the Singleton bound with equality. In this paper we are interested in the case where $q = 2^{\ell}$, $\ell \in \mathbb{N}$, and we denote by α a primitive element of \mathbb{F}_q . If n = q, then for every $k \in \{1, \ldots, n\}$

$$G_{k,q} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \alpha & \dots & \alpha^{-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \alpha^{k-1} & \dots & \alpha^{-(k-1)} \end{pmatrix} \in \mathbb{F}_q^{k \times q}$$
(326)

is a generator matrix of a (q, k) MDS code. (More precisely, $G_{k,q}$ is a generator matrix of a Reed-Solomon (RS) code.) To see this, note that

$$\mathbf{u}G_{k,q} = \left(u(0), u(\alpha^0), u(\alpha), \dots, u(\alpha^{-1})\right), \quad \mathbf{u} \in \mathbb{F}_q^k,$$
(327)

where $u(\beta) = \sum_{j=0}^{k-1} u_j \beta^j$, $\beta \in \mathbb{F}_q$ is computed in the field \mathbb{F}_{2^ℓ} . Hence, the first component of $\mathbf{u}_{k,q}$ is zero iff zero is a root of u(z), and for every $i \in \{2, \ldots, q\}$ the *i*-th component of $\mathbf{u}_{k,q}$ is zero iff α^{i-2} is a root of u(z). Since α is a primitive element of \mathbb{F}_q , we know that $0, 1, \alpha, \ldots, \alpha^{q-1}$ are distinct elements of \mathbb{F}_q . Moreover, the polynomial u(z) has degree at most k-1, and hence the fundamental theorem of algebra asserts that if $u(z) \neq 0$, then u(z) can have at most k-1 roots in \mathbb{F}_q . Consequently, at most k-1 components of any nonzero codeword can be zero, and hence every nonzero codeword has Hamming weight at least n-k+1. This and the Singleton bound imply that d = n-k+1 and consequently that the code with generator matrix (326) is a (q, k) MDS code.

If $k \leq n \leq q$, then the matrix $G_{k,n} \in \mathbb{F}_q^{k \times n}$ that we obtain by taking the first n columns of $G_{k,q}$ is a generator matrix of an (n,k) MDS code. To see this, note that reducing $G_{k,q}$ to its first n columns is tantamount to reducing each codeword to its first n components. This implies that the Hamming weight of any codeword or, equivalently, the Hamming distance between any two codewords can decrease by at most q-n, and consequently that the minimum Hamming distance between any two codewords can decrease by at most q-n. Consequently, the new code is an (n, k, d) linear code with $d \geq q - k + 1 - (q - n) = n - k + 1$. This and the Singleton bound imply that d = n - k + 1 and consequently that the new code is an MDS code.

We also note here that, for any generator matrix $G_{k,n}$ of an (n,k) MDS code over \mathbb{F}_q , where $k \leq n \leq q$, and any k' < k, the matrix $G_{k',n}$ that we obtain by taking the first k' rows of $G_{k,n}$ is a generator matrix of an (n, k') MDS code.

F.2 A Proof of Theorems 24 and 25

Proof. We first establish the achievability results, i.e., (165)-(166) in the guessing version and (170)-(171) in the list version. We begin with an outline of the proof ideas. We shall use the following coding scheme. Upon observing (X, Y), Alice describes X deterministically by a tuple (V, W), where V takes values in the finite field $\mathbb{F}_{2^p}^{\nu}$ and W in $\mathbb{F}_{2^r}^{\nu-\eta}$. Depending on the version, she chooses the description (V, W) so that, if Bob's observation were (V, W), then his ambiguity about X would satisfy (165) in the guessing version and (170) in the list version. Then, she maps V to a length- δ codeword of a $(\delta, \nu, \delta - \nu + 1)$ MDS code over \mathbb{F}_{2^p} and stores each codeword symbol on a different disc. Since the code is MDS, any $\gamma \leq \nu$ hints reveal γp bits of V. Independently of (X, Y), Alice draws a random variable U uniformly over the field $\mathbb{F}_{2^r}^{\eta}$, maps (W, U) to a length- δ codeword of a $(\delta, \nu, \delta - \nu + 1)$ MDS code over the field \mathbb{F}_{2^r} , and stores each codeword symbol on a different disc. She chooses the mapping so that any η codeword symbols are independent of W or, equivalently, that given W it is possible to reconstruct U from any η codeword symbols. (As in [8], this is accomplished using nested MDS codes.) As a consequence, W can be recovered from any ν hints, while any η hints reveal no information about W.

Summing up, the outlined coding scheme guarantees that, upon observing ν hints, Bob can reconstruct the tuple (V, W). Hence, his ambiguity about X satisfies (165) in the guessing version and (170) in the list version. Observing η hints enables Eve to recover ηp bits of V, but it does not enable her to recover any information about W. Using the results of Section 3, we can thus show that observing η hints can decrease Eve's guessing efforts by at most a factor of $2^{-\rho\nu p}$.⁶ Since we quantify Eve's ambiguity by (163), we assume that—upon observing all the hints and (X, Y)—an adversarial genie reveals to Eve the η hints that minimize her ambiguity. In doing so, the genie can decrease Eve's ambiguity by an additional factor of at most $\delta^{-\rho\eta}$ (this is due to Corollary 6 and the fact that there are $\binom{\delta}{\eta} \leq \delta^{\eta}$ size- η subsets of $\{1, \ldots, \delta\}$).

The described MDS codes exist if each nonnegative integer p and r is either zero or at least log δ (see Appendix F.1). Recalling that each disc stores up to s bits, we can thus construct the MSD codes whenever p and r satisfy (164). In the list version the stronger requirement (169)—in addition to guaranteeing the existence of the described MDS codes allows us to use Theorem 4 in order to guarantee that Bob's ambiguity satisfy (170).

We are now ready to give a formal proof of the achievability results, i.e., (165)-(166) in the guessing version and (170)-(171) in the list version. To this end fix $p, r \in \{1, \ldots, s\}$ satisfying (164) in the guessing version and (169) in the list version, and let V and W be chance variables taking values in $\mathcal{V} = \mathbb{F}_{2^p}^{\nu}$ and $\mathcal{W} = \mathbb{F}_{2^r}^{\nu-\eta}$, respectively. Corollary 7 implies that there exists some $\{0, 1\}$ -valued conditional PMF $\mathbb{P}[(V, W) = (v, w) | X = x, Y = y]$ for which

$$\min_{G(\cdot|Y,V,W)} \mathbb{E} \left[G(X|Y,V,W)^{\rho} \right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \nu s + \eta r + 1)}.$$
(328)

Theorem 4 implies that there exists some deterministic task-encoder $f(\cdot|Y) \colon \mathcal{X} \to \mathcal{V} \times \mathcal{W}$ for which

$$\mathbb{E}\left[\left|\mathcal{L}_{V,W}^{Y}\right|^{\rho}\right] < 1 + 2^{\rho(H_{\tilde{\rho}}(X|Y) - \log(2^{\nu s - \eta r} - \log|\mathcal{X}| - 2) + 2)},\tag{329}$$

where (V, W) = f(X|Y). Draw U independently of (X, Y) and uniformly over $\mathbb{F}_{2^r}^{\eta}$. Choose

⁶The coding scheme is reminiscent of the coding scheme in the proof of Theorem 12 and 13, where after describing X Alice stores part of the description (insecurely) on the first hint, another part (insecurely) on the second hint, and the remaining portion (securely) so that it can only be computed from both hints.

 $G_{\mathcal{V}} \in \mathbb{F}_{2^{p}}^{\nu imes \delta}, \, G_{\mathcal{W}} \in \mathbb{F}_{2^{r}}^{(\nu-\eta) imes \delta}, \, \text{and} \, G_{\mathcal{U}} \in \mathbb{F}_{2^{r}}^{\eta imes \delta} \text{ so that}$

$$G_{\mathcal{V}}, \quad \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{W}} \end{pmatrix}, \quad G_{\mathcal{U}}$$

are generator matrices of MDS codes. (This is possible, because both (164) and (169) imply that

$$p > 0 \implies 2^p \ge \delta,$$
 (330a)

$$r > 0 \implies 2^r \ge \delta;$$
 (330b)

if p = 0, then V can assume but one value, and hence we do not need $G_{\mathcal{V}}$; and if r = 0, then (W, U) can assume but one value, and hence we do not need $G_{\mathcal{W}}$ and $G_{\mathcal{U}}$.) Define the chance variables

$$M_p = V G_{\mathcal{V}},\tag{331a}$$

$$M_r = U G_{\mathcal{U}} \oplus W G_{\mathcal{W}} = \left(U W\right) \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{W}} \end{pmatrix}, \qquad (331b)$$

where M_p is computed in the field \mathbb{F}_{2^p} and M_r in \mathbb{F}_{2^r} . Note that $M_p \in \mathbb{F}_{2^p}^{\delta}$ and $M_r \in \mathbb{F}_{2^r}^{\delta}$. Since both (164) in the guessing version and (169) in the list version imply that s = p + r, Alice can choose the ℓ -th hint to comprise the ℓ -th components of M_p and M_r , so

$$M_{\ell} = ([M_p]_{\ell}, [M_r]_{\ell}), \quad \ell \in \{1, \dots, \delta\}.$$
(332)

For this choice of the hints Bob can recover (V, W, U) no matter which ν hints he observes, because

$$G_{\mathcal{V}}, \quad \begin{pmatrix} G_{\mathcal{U}} \\ G_{\mathcal{W}} \end{pmatrix}$$

are generator matrices of MDS codes. Hence, in the guessing version (165) follows from (328), and in the list version (170) follows from (329).

The proof of (166) and (171) is more involved. Recall that Eve observes a size- η set $\mathcal{E} \subset \{1, \ldots, \delta\}$ and the components $M_{\mathcal{E}}$ of M indexed by \mathcal{E} . Index the possible sets that \mathcal{E} could denote by the elements of some size- $\binom{\delta}{\eta}$ set \mathcal{K} , and denote by $\mathcal{E}(k)$ the set that is indexed by k. The proof of (166) and (171) builds on the following two intermediate claims, which we prove next:

1. Eve's ambiguity can be alternatively expressed as

$$\mathscr{A}_{\mathbb{E}}(P_{X,Y}) = \min_{K, G(\cdot|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)} \mathbb{E}\left[G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)^{\rho}\right],$$
(333)

where K is a chance variable of support \mathcal{K} , and where the minimization is over all conditional PMFs of K given (X, Y, \mathbf{M}) and all guessing functions $G(\cdot|Y, \mathbf{M}_{\mathcal{E}(K)}, K)$.

2. We can assume w.l.g. that Eve must guess not only X but the pair (X, U).

We first prove Claim 1, i.e., that

$$\min_{G_{\mathcal{E}}(\cdot|Y, M_{\mathcal{E}})} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, M_{\mathcal{E}})^{\rho}\right] \\
= \min_{K, G(\cdot|Y, M_{\mathcal{E}(K)}, K)} \mathbb{E}\left[G(X|Y, M_{\mathcal{E}(K)}, K)^{\rho}\right].$$
(334)

Note that

$$\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, \boldsymbol{M}_{\mathcal{E}}) = \min_{k} G_{\mathcal{E}(k)}(X|Y, \boldsymbol{M}_{\mathcal{E}(k)});$$
(335)

and for any given $G_{\mathcal{E}(k)}(\cdot|Y, M_{\mathcal{E}(k)}), k \in \mathcal{K}$, define

$$K = \arg\min_{k} G_{\mathcal{E}(k)}(X|Y, \boldsymbol{M}_{\mathcal{E}(k)}),$$
(336)

and introduce the guessing function $G(\cdot|Y, M_{\mathcal{E}(K)}, K)$ satisfying that, for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $\mathbf{m}_{\mathcal{E}(k)} \in \mathbb{F}_{2^s}^{\eta}$, and $k \in \mathcal{K}$,

$$G(x|y, \mathbf{m}_{\mathcal{E}(k)}, k) = G_{\mathcal{E}(k)}(x|y, \mathbf{m}_{\mathcal{E}(k)}).$$
(337)

We then obtain that

$$\mathbb{E}\left[G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)^{\rho}\right] = \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, \boldsymbol{M}_{\mathcal{E}})^{\rho}\right],\tag{338}$$

and consequently that

$$\min_{G_{\mathcal{E}}(\cdot|Y,\boldsymbol{M}_{\mathcal{E}})} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y,\boldsymbol{M}_{\mathcal{E}})^{\rho}\right] \\
\geq \min_{K, G(\cdot|Y,\boldsymbol{M}_{\mathcal{E}},K)} \mathbb{E}\left[G(X|Y,\boldsymbol{M}_{\mathcal{E}}(K),K)^{\rho}\right].$$
(339)

To see that equality holds, note that, irrespective of K and $G(\cdot|Y, M_{\mathcal{E}(K)}, K)$,

$$\mathbb{E}\left[G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)^{\rho}\right] \ge \mathbb{E}\left[\min_{k} G(X|Y, \boldsymbol{M}_{\mathcal{E}(k)}, k)^{\rho}\right].$$
(340)

For any given $G(\cdot|Y, \mathbf{M}_{\mathcal{E}(K)}, K)$ introduce the collection of guessing functions $G_{\mathcal{E}(k)}(\cdot|Y, \mathbf{M}_{\mathcal{E}(K)}), k \in \mathcal{K}$ that, for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and $\mathbf{m}_{\mathcal{E}(k)} \in \mathbb{F}_{2^s}^{\eta}$, satisfy

$$G_{\mathcal{E}(k)}(x|y,\mathbf{m}_{\mathcal{E}(k)}) = G(x|y,\mathbf{m}_{\mathcal{E}(k)},k).$$
(341)

We then obtain from (340) that

$$\mathbb{E}\left[G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)^{\rho}\right] \ge \mathbb{E}\left[\min_{k} G_{\mathcal{E}(k)}(X|Y, \boldsymbol{M}_{\mathcal{E}(k)})^{\rho}\right],\tag{342}$$

and consequently that

$$\min_{G_{\mathcal{E}}(\cdot|Y,\boldsymbol{M}_{\mathcal{E}})} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y,\boldsymbol{M}_{\mathcal{E}})^{\rho}\right] \\
\leq \min_{K, G(\cdot|Y,\boldsymbol{M}_{\mathcal{E}},K)} \mathbb{E}\left[G(X|Y,\boldsymbol{M}_{\mathcal{E}}(K),K)^{\rho}\right].$$
(343)

From (339) and (343) we conclude that (334) holds.

We next prove Claim 2. To this end we shall use Claim 1. Let K be any chance variable of finite support \mathcal{K} , and note that W is deterministic given (X, Y). By (331b)

$$[U G_{\mathcal{U}}]_{\mathcal{E}(K)} = [M_r]_{\mathcal{E}(K)} \ominus [W G_{\mathcal{V}}]_{\mathcal{E}(K)}, \qquad (344)$$

where the computation is in the field \mathbb{F}_{2^r} . Consequently, $[U G_{\mathcal{U}}]_{\mathcal{E}(K)}$ is deterministic given $(X, Y, \mathbf{M}_{\mathcal{E}(K)}, K)$. Because $G_{\mathcal{U}}$ is a generator matrix of an MDS code, and because $|\mathcal{E}(K)| = \eta$, it follows that U is deterministic given $(X, Y, \mathbf{M}_{\mathcal{E}(K)}, K)$, i.e., that there exists some mapping

$$g\colon \mathcal{X} \times \mathcal{Y} \times \mathbb{F}_{2^r}^{\eta} \times \mathcal{K} \to \mathcal{U}$$

for which

$$U = g(X, Y, \boldsymbol{M}_{\mathcal{E}(K)}, K).$$
(345)

Given any guessing function $G(\cdot|Y, M_{\mathcal{E}(K)}, K)$ for X, introduce some guessing function $G(\cdot, \cdot|Y, M_{\mathcal{E}(K)}, K)$ for (X, U) satisfying that

$$G(X, g(X, Y, \boldsymbol{M}_{\mathcal{E}(K)}, K) | Y, \boldsymbol{M}_{\mathcal{E}(K)}, K) = G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K),$$
(346)

and note that

$$G(X, U|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K) = G(X|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K).$$
(347)

This proves Claim 2.

Having established Claims 1 and 2, we are now ready to prove (166) and (171):

$$\min_{\substack{G_{\mathcal{E}}(\cdot|Y, M_{\mathcal{E}}) \\ \equiv}} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, M_{\mathcal{E}})^{\rho}\right] \\
\stackrel{(a)}{=} \min_{K, G(\cdot|Y, M_{\mathcal{E}}, K)} \mathbb{E}\left[G(X|Y, M_{\mathcal{E}(K)}, K)^{\rho}\right]$$
(348)

$$\stackrel{(b)}{=} \min_{K, G_{\mathcal{E}}(\cdot|Y, \boldsymbol{M}_{\mathcal{E}}, K)} \mathbb{E} \left[G(X, U|Y, \boldsymbol{M}_{\mathcal{E}(K)}, K)^{\rho} \right]$$
(349)

$$\stackrel{(c)}{\geq} 2^{\rho(H_{\tilde{\rho}}(X,U|Y) - \eta s - \log\binom{\delta}{\eta} - \log(1 + \ln|\mathcal{X}|))} \tag{350}$$

$$\stackrel{(d)}{\geq} 2^{\rho(H_{\tilde{\rho}}(X|Y) - \eta(s-r) - \eta \log \delta - \log(1 + \ln |\mathcal{X}|))},\tag{351}$$

where (a) holds by (334); (b) holds by (347); (c) follows from Corollary 7 and the fact that $(M_{\mathcal{E}(K)}, K)$ takes values in a set of size $2^{\eta s} {\delta \choose \eta}$; and (d) holds because ${\delta \choose \eta} \leq \delta^{\eta}$ and

$$H_{\tilde{\rho}}(X, U|Y)$$

$$\stackrel{(e)}{=} \frac{1}{\rho} \log \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} \sum_{u \in \mathbb{F}_{2r}^{\eta}} \left(P_{X,Y}(x, y) / 2^{\eta r} \right)^{\tilde{\rho}} \right)^{1+\rho}$$

$$= \frac{1}{\rho} \log \left(\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{X,Y}(x, y)^{\tilde{\rho}} \right)^{1+\rho} 2^{\rho \eta r} \right)$$

$$= H_{\tilde{\rho}}(X|Y) + \eta r, \qquad (352)$$

where (e) holds because U is independent of (X, Y) and uniform over the set $\mathbb{F}_{2^r}^{\eta}$ of size $2^{\eta r}$. This concludes the proof of the achievability results.

It remains to establish the converse results, i.e., (167)-(168) in the guessing version and (172)-(173) in the list version. To this end we first note that

$$\mathscr{A}_{\mathrm{B}}^{(g)}(P_{X,Y}) = \min_{G_{\mathcal{B}}(\cdot|Y,\boldsymbol{M}_{\mathcal{B}})} \mathbb{E}\left[\max_{\mathcal{B}} G_{\mathcal{B}}(X|Y,\boldsymbol{M}_{\mathcal{B}})^{\rho}\right]$$

$$\geq \min_{G_{\mathcal{B}}(\cdot|Y,\boldsymbol{M}_{\mathcal{B}})} \max_{\mathcal{B}} \mathbb{E}\left[G_{\mathcal{B}}(X|Y,\boldsymbol{M}_{\mathcal{B}})^{\rho}\right], \qquad (353a)$$

$$\mathscr{A}_{\mathrm{B}}^{(l)}(P_{X,Y}) = \mathbb{E}\left[\max_{\boldsymbol{P}} \left|\mathcal{L}_{\boldsymbol{M}_{\mathcal{B}}}^{Y}\right|^{\rho}\right]$$

$$(\Gamma_{X,Y}) = \mathbb{E}\left[\max_{\mathcal{B}} |\mathcal{L}_{M_{\mathcal{B}}}|\right]$$
$$\geq \max_{\mathcal{B}} \mathbb{E}\left[|\mathcal{L}_{M_{\mathcal{B}}}^{Y}|^{\rho}\right].$$
(353b)

Because $\mathcal{B} \subseteq \{1, \ldots, \delta\}$ is a size- ν set, in the guessing version (167) follows from (353a) and Corollary 7, and in the list version (172) follows from (353b) and Theorem 4. To prove (168) and (173), we first note that

$$A_{\mathrm{E}}(P_{X,Y}) = \min_{G_{\mathcal{E}}(\cdot|Y, M_{\mathcal{E}})} \mathbb{E}\left[\min_{\mathcal{E}} G_{\mathcal{E}}(X|Y, M_{\mathcal{E}})^{\rho}\right]$$
(354)

$$\leq \min_{\mathcal{E}, G_{\mathcal{E}}(\cdot|Y, M_{\mathcal{E}})} \mathbb{E} \left[G_{\mathcal{E}}(X|Y, M_{\mathcal{E}})^{\rho} \right].$$
(355)

Corollary 6 implies that, for every size- ν set $\mathcal{B} \subseteq \{1, \ldots, \delta\}$ and every size- η set $\mathcal{E} \subset \mathcal{B}$,

$$\min_{G_{\mathcal{B}}(\cdot|Y,\boldsymbol{M}_{\mathcal{B}})} \mathbb{E}\left[G_{\mathcal{B}}(X|Y,\boldsymbol{M}_{\mathcal{B}})^{\rho}\right] \ge 2^{-\rho(\nu-\eta)s} \min_{G_{\mathcal{E}}(\cdot|Y,\boldsymbol{M}_{\mathcal{E}})} \mathbb{E}\left[G_{\mathcal{E}}(X|Y,\boldsymbol{M}_{\mathcal{E}})^{\rho}\right];$$
(356)

and, because

$$\min_{G_{\mathcal{B}}(\cdot|Y, \boldsymbol{M}_{\mathcal{B}})} \mathbb{E} \left[G_{\mathcal{B}}(X|Y, \boldsymbol{M}_{\mathcal{B}})^{\rho} \right] \leq \mathbb{E} \left[\left| \mathcal{L}_{\boldsymbol{M}_{\mathcal{B}}}^{Y} \right| \right],$$

(355) and (356) imply that in both versions Eve's ambiguity exceeds Bob's by at most a factor of $2^{\rho(\nu-\eta)s}$, i.e., $\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \leq 2^{\rho(\nu-\eta)s} \mathscr{A}_{\mathrm{B}}^{(g)}(P_{X,Y})$ and $\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \leq 2^{\rho(\nu-\eta)s} \mathscr{A}_{\mathrm{B}}^{(l)}(P_{X,Y})$. Since Eve can ignore the hints that she observes and guess X based on Y alone, we obtain from Theorem 3 that, for every size- η set $\mathcal{E} \subset \{1, \ldots, \delta\}$,

$$\min_{G_{\mathcal{E}}(\cdot|Y,\boldsymbol{M}_{\mathcal{E}})} \mathbb{E}\left[G_{\mathcal{E}}(X|Y,\boldsymbol{M}_{\mathcal{E}})^{\rho}\right] \le 2^{\rho H_{\tilde{\rho}}(X|Y)};$$
(357)

and (355) and (357) imply that in both versions Eve's ambiguity cannot exceed $2^{\rho H_{\bar{\rho}}(X|Y)}$, i.e., $A_{\rm E}(P_{X,Y}) \leq 2^{\rho H_{\bar{\rho}}(X|Y)}$. This concludes the proof of (168) and (173) and consequently that of the converse results.

G A Proof of Corollary 26

Proof. For the guessing version, the results in (175)-(176) follow from Theorem 24 if we let

$$\tilde{r} = \frac{\nu s + \rho^{-1} \log(\mathscr{U}_{\rm B} - 1) - H_{\tilde{\rho}}(X|Y) - 1}{\eta}, \qquad (358)$$

$$r = \begin{cases} 0 & \lfloor \tilde{r} \rfloor \in (-\infty, \log \delta), \\ \lfloor \tilde{r} \rfloor & \lfloor \tilde{r} \rfloor \in [\log \delta, s - \log \delta), \\ s - \lceil \log \delta \rceil & \lfloor \tilde{r} \rfloor \in [s - \log \delta, s), \\ s & \lfloor \tilde{r} \rfloor \in [s, \infty), \end{cases}$$
(359)
$$p = s - r,$$
(360)

and note that

 $r \neq s \implies \tilde{r} - r < \log \delta + 1.$

To obtain the results in (178)–(179) for the list version, let

$$\tilde{r} = \frac{\nu s - \log\left(2^{H_{\tilde{\rho}}(X|Y) - \frac{1}{\rho}\log(\mathscr{U}_{\mathrm{B}} - 1) + 2} + \log|\mathcal{X}| + 2\right)}{\eta},\tag{361}$$

and choose r as in (359). Then, (170) implies that Bob's ambiguity satisfies (178). Since

$$r \neq s \implies \tilde{r} - r < \log \delta + 1,$$

we obtain from (171) that, if $r \neq s$, then

$$\mathscr{A}_{\rm E}(P_{X,Y}) > 2^{\rho(H_{\tilde{\rho}}(X|Y) + (\nu - \eta)s - 2\eta \log \delta - \eta - \log(1 + \ln |\mathcal{X}|))} \\ \times \left(2^{H_{\tilde{\rho}}(X|Y) - \frac{1}{\rho} \log(\mathscr{U}_{\rm B} - 1) + 2} + \log |\mathcal{X}| + 2 \right)^{-\rho}.$$
(362)

Because

$$\frac{1}{a+b} \ge \frac{1}{2a} \land \frac{1}{2b}, \quad a, b > 0,$$

the second factor satisfies the lower bound

$$\left(2^{H_{\tilde{\rho}}(X|Y) - \frac{1}{\rho} \log(\mathscr{U}_{\mathrm{B}} - 1) + 2} + \log |\mathcal{X}| + 2 \right)^{-\rho} \\ \geq 2^{-\rho(H_{\tilde{\rho}}(X|Y) - \frac{1}{\rho} \log(\mathscr{U}_{\mathrm{B}} - 1) + 3)} \wedge \left(2(\log |\mathcal{X}| + 2) \right)^{-\rho}.$$
 (363)

We are now ready to conclude the proof of (179): if $r \neq s$, then (179) follows from (362) and (363); and if r = s, then (171) implies that

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \ge 2^{\rho(H_{\tilde{\rho}}(X|Y) - \eta \log \delta - \log(1 + \ln |\mathcal{X}|))}$$
(364)

and consequently that (179) holds.

H A Proof of Theorem 27

Proof. If we choose $\mathcal{B} = \{1, \ldots, \nu\}$, then in the guessing version (180a) follows from (353a) and Corollary 7, and in the list version (180b) follows from (353b) and Theorem 4. For $\mathcal{B} = \{1, \ldots, \nu\}$ and $\mathcal{E} = \{\nu - \eta + 1, \ldots, \nu\}$, Corollary 6 implies that,

$$\min_{G_{\mathcal{B}}(\cdot|Y,\boldsymbol{M}_{\mathcal{B}})} \mathbb{E}\left[G_{\mathcal{B}}(X|Y,\boldsymbol{M}_{\mathcal{B}})^{\rho}\right] \ge 2^{-\rho \sum_{\ell=1}^{\eta-\nu} s_{\ell}} \min_{G_{\mathcal{E}}(\cdot|Y,\boldsymbol{M}_{\mathcal{E}})} \mathbb{E}\left[G_{\mathcal{E}}(X|Y,\boldsymbol{M}_{\mathcal{E}})^{\rho}\right].$$
(365)

Since

$$\min_{G_{\mathcal{B}}(\cdot|Y, \boldsymbol{M}_{\mathcal{B}})} \mathbb{E}\left[G_{\mathcal{B}}(X|Y, \boldsymbol{M}_{\mathcal{B}})^{\rho}\right] \leq \mathbb{E}\left[\left|\mathcal{L}_{\boldsymbol{M}_{\mathcal{B}}}^{Y}\right|\right],$$

(355) and (365) imply that in both versions Eve's ambiguity exceeds Bob's by at most a factor of $2^{\rho \sum_{\ell=1}^{n-\nu} s_{\ell}}$. That is,

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \le 2^{\rho \sum_{\ell=1}^{\eta-\nu} s_{\ell}} \mathscr{A}_{\mathrm{B}}^{(g)}(P_{X,Y})$$

and

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}) \leq 2^{\rho \sum_{\ell=1}^{\eta-\nu} s_{\ell}} \mathscr{A}_{\mathrm{B}}^{(l)}(P_{X,Y}).$$

Moreover, (355) and (357) imply that in both versions Eve's ambiguity cannot exceed $2^{\rho H_{\bar{\rho}}(X|Y)}$. That is,

$$A_{\mathrm{E}}(P_{X,Y}) \le 2^{\rho H_{\tilde{\rho}}(X|Y)}$$

which concludes the proof of (181).

I A Proof of Theorem 29

Proof. We first prove (182). If $\nu R_s < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$, then (167) in the guessing version and (172) in the list version imply that the privacy-exponent is negative infinity. We hence assume that $\nu R_s > H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$.

We start by showing that the privacy-exponent cannot exceed the RHS of (182). To this end, suppose that (57) holds and consequently

$$\limsup_{n \to \infty} \frac{\log \left(\mathscr{A}_{\mathcal{B}}(P_{X^n, Y^n}) \right)}{n} = 0.$$
(366)

Combining (168) with (366) in the guessing version and (173) in the list version implies that

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}))}{n} \le \rho(R_s(\nu - \eta) \wedge H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})).$$
(367)

Hence, the privacy-exponent cannot exceed the RHS of (182).

We next show that the privacy-exponent cannot be smaller than the RHS of (182). To this end fix $0 < \epsilon < \nu R_s - H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})$ and let

$$\mathscr{U}_{\mathcal{B}}(n) = 1 + 2^{-n\epsilon}.$$
(368)

Note that $\mathscr{U}_{\mathrm{B}}(n)$ converges to one as *n* tends to infinity. By Corollary 26 we can guarantee that Bob's ambiguity not exceed $\mathscr{U}_{\mathrm{B}}(n)$ whenever *n* is sufficiently large and that

$$\liminf_{n \to \infty} \frac{\log(A_{\rm E}(P_{X^n, Y^n}))}{n} \ge \rho\Big(\Big(R_s(\nu - \eta) - \epsilon\Big) \wedge H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y})\Big). \tag{369}$$

By letting ϵ tend to zero we thus find that the privacy-exponent cannot be smaller than the RHS of (182).

To prove (183), we first note that if $\nu R_s < H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}) - \rho^{-1}E_{\rm B}$, then (167) in the guessing version and (172) in the list version imply that the modest privacy-exponent is negative infinity. We hence assume that $\nu R_s \ge -\rho^{-1}E_{\rm B}$.

We start by showing that the modest privacy-exponent cannot exceed the RHS of (183). To this end, suppose that (59) holds. Due to (168) in the guessing version and (173) in the list version, it follows that

$$\limsup_{n \to \infty} \frac{\log \left(\mathscr{A}_{\mathrm{E}}(P_{X^n, Y^n}) \right)}{n} \le \left(\rho R_s(\nu - \eta) + E_{\mathrm{B}} \right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X} | \boldsymbol{Y}).$$
(370)

Hence, the privacy-exponent cannot exceed the RHS of (183).

We next show that the privacy-exponent cannot be smaller than the RHS of (183). To this end let

$$\mathscr{U}_{\mathrm{B}}(n) = 2^{\rho n E_{\mathrm{B}}}.\tag{371}$$

By Corollary 26 we can guarantee that Bob's ambiguity not exceed $\mathscr{U}_{\mathrm{B}}(n)$ whenever n is sufficiently large and that

$$\liminf_{n \to \infty} \frac{\log(A_{\rm E}(P_{X^n, Y^n}))}{n} \ge \left(\rho R_s(\nu - \eta) + E_{\rm B}\right) \wedge \rho H_{\tilde{\rho}}(\boldsymbol{X}|\boldsymbol{Y}).$$
(372)

This proves that the modest privacy-exponent cannot be smaller than the RHS of (183). \Box

J A Proof of Lemma 31

Proof. To prove (195), fix some optimal guessing function $\hat{G}^{\star}(\cdot|Y^n, Z)$ with corresponding success function $G^{\star}_{\Delta}(\cdot|Y^n, Z)$. The success function $G^{\star}_{\Delta}(\cdot|Y^n, Z)$ minimizes $\mathbb{E}\left[G^{\star}_{\Delta}(X|Y^n, Z)^{\rho}\right]$. Let $\psi(\cdot|Y^n, Z)$ be the corresponding reconstruction function, i.e., the unique mapping satisfying that

$$\psi(\mathbf{x}|\mathbf{y},z) = \hat{\mathbf{x}} \iff G^{\star}_{\Delta}(\mathbf{x}|\mathbf{y},z) = \hat{G}^{\star}(\hat{\mathbf{x}}|\mathbf{y},z), \ \forall (\mathbf{x},\hat{\mathbf{x}},\mathbf{y},z) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n \times \mathcal{Y}^n \times \mathcal{Z}.$$
(373)

For every $\mathbf{y} \in \mathcal{Y}^n$ consider a guessing order on $\hat{\mathcal{X}}^n$ where we first guess the elements of the set

$$\left\{ \hat{\mathbf{x}} \in \hat{\mathcal{X}}^n \colon \min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}} | \mathbf{y}, z) = 1 \right\}$$

in some arbitrary order followed by the elements of the set

$$\Big\{ \hat{\mathbf{x}} \in \hat{\mathcal{X}}^n \colon \min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}} | \mathbf{y}, z) = 2 \Big\},\$$

and where we continue until concluding by guessing the elements of $\hat{\mathcal{X}}^n$ for which $\min_{z \in \mathcal{Z}} \hat{G}^*(\hat{\mathbf{x}} | \mathbf{y}, z)$ is maximum. Let $\hat{G}(\cdot | Y^n)$ be the corresponding guessing function. For every $\hat{\mathbf{x}}, \hat{\mathbf{x}}' \in \hat{\mathcal{X}}^n$ and $\mathbf{y} \in \mathcal{Y}^n$ a necessary condition for $\hat{G}(\hat{\mathbf{x}}' | \mathbf{y}) \leq \hat{G}(\hat{\mathbf{x}} | \mathbf{y})$ is that

$$\min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}}' | \mathbf{y}, z) \le \min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}} | \mathbf{y}, z).$$

In addition, for every $z' \in \mathcal{Z}$ the mapping $\hat{G}^{\star}(\cdot | \mathbf{y}, z') \colon \hat{\mathcal{X}}^n \to [1 : |\hat{\mathcal{X}}|^n]$ is one-to-one, and consequently the number of $\hat{\mathbf{x}}' \in \hat{\mathcal{X}}^n$ satisfying

$$\hat{G}^{\star}(\hat{\mathbf{x}}' | \mathbf{y}, z') \le \min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}} | \mathbf{y}, z)$$

is $\min_{z \in \mathcal{Z}} \hat{G}^{\star}(\hat{\mathbf{x}} | \mathbf{y}, z)$. Consequently,

$$\hat{G}\left(\psi(X^{n}|Y^{n},Z)|Y^{n}\right) \leq |\mathcal{Z}|\min_{z\in\mathcal{Z}}\hat{G}^{\star}\left(\psi(X^{n}|Y^{n},Z)|Y^{n},z\right)$$
(374)

$$\leq |\mathcal{Z}| \hat{G}^{\star} \big(\psi(X^n | Y^n, Z) \big| Y^n, Z \big). \tag{375}$$

From (375) it follows that the success function $G_{\Delta}(\cdot|Y^n)$ corresponding to $\hat{G}(\cdot|Y^n)$ satisfies

$$G_{\Delta}(X^{n}|Y^{n}) \stackrel{(a)}{\leq} \hat{G}\left(\psi(X^{n}|Y^{n},Z)|Y^{n}\right)$$
(376)

$$\overset{\leq}{\leq} |\mathcal{Z}| \, \hat{G}^{\star} \left(\psi(X^n | Y^n, Z) | Y^n, Z \right) \tag{377}$$

$$\stackrel{(c)}{=} |\mathcal{Z}| G^{\star}_{\Delta}(X^n | Y^n, Z), \tag{378}$$

where (a) holds because $d^{(n)}(X^n, \psi(X^n|Y^n, Z)) \leq \Delta$; (b) holds by (375); and (c) holds because $\psi(\cdot|Y^n, Z)$ satisfies (373). Since $\hat{G}^*(\cdot|Y^n, Z)$ is an optimal guessing function, this concludes the proof of (195).

To prove (196), fix some optimal guessing function $\hat{G}^{\star}(\cdot|Y^n)$ with a corresponding success function $G^{\star}_{\Delta}(\cdot|Y^n)$. The success function $G^{\star}_{\Delta}(\cdot|Y^n)$ minimizes $\mathbb{E}\left[G^{\star}_{\Delta}(X|Y^n)^{\rho}\right]$. Let $\psi(\cdot|Y^n)$ be the corresponding reconstruction function for which (189) holds when we substitute $\hat{G}^{\star}(\hat{\mathbf{x}}|\mathbf{y})$ for $\hat{G}(\hat{\mathbf{x}}|\mathbf{y})$ and $G^{\star}_{\Delta}(\mathbf{x}|\mathbf{y})$ for $G_{\Delta}(\mathbf{x}|\mathbf{y})$ in (189). Let $f: \hat{\mathcal{X}}^n \times \mathcal{Y}^n \to \mathcal{Z}$ be some mapping for which $f(\hat{\mathbf{x}}, \mathbf{y}) = f(\hat{\mathbf{x}}', \mathbf{y})$ implies either $\left[\hat{G}^{\star}_{\Delta}(\hat{\mathbf{x}}|\mathbf{y})/|\mathcal{Z}|\right] \neq \left[\hat{G}^{\star}_{\Delta}(\hat{\mathbf{x}}'|\mathbf{y})/|\mathcal{Z}|\right]$ or $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$. The mapping f could be any mapping for which, for every $(\hat{\mathbf{x}}, \mathbf{y}) \in \hat{\mathcal{X}}^n \times \mathcal{Y}^n$, $f(\hat{\mathbf{x}}, \mathbf{y})$ is—up to relabeling the elements of \mathcal{Z} —the remainder of the Euclidean division of $\hat{G}^{\star}(\hat{\mathbf{x}}|\mathbf{y}) - 1$ by $|\mathcal{Z}|$. Define the chance variable $\hat{X}^n = \psi(X^n|Y^n)$, which takes values in $\hat{\mathcal{X}}^n$. Lemma 5 implies that for $Z = f(\hat{X}^n, Y^n)$ there exists some guessing function $\hat{G}(\cdot|Y^n, Z)$ for \hat{X}^n for which

$$\mathbb{E}\left[\hat{G}(\hat{X}^n|Y^n,Z)^{\rho}\right] = \mathbb{E}\left[\left[\hat{G}(\hat{X}^n|Y^n)/|\mathcal{Z}|\right]^{\rho}\right].$$
(379)

In fact, in the proof of Lemma 5 it is shown that there exists some guessing function $\hat{G}(\cdot|Y^n, Z)$ for \hat{X}^n for which

$$\hat{G}(\hat{X}^n|Y^n, Z) = \left[\hat{G}(\hat{X}^n|Y^n) / |\mathcal{Z}|\right].$$
(380)

Let $\hat{G}(\cdot|Y^n, Z)$ be a guessing function as in (380) with corresponding success function $G_{\Delta}(\cdot|Y^n, Z)$. Note that

$$G_{\Delta}(X^n|Y^n, Z) \stackrel{(a)}{\leq} \hat{G}(\psi(X^n|Y^n)|Y^n, Z)$$
(381)

$$\stackrel{(b)}{=} \left| \hat{G}^{\star} \left(\psi(X^n | Y^n) | Y^n \right) / |\mathcal{Z}| \right|$$
(382)

$$\stackrel{(c)}{=} \left[G^{\star}_{\Delta}(X^n | Y^n) / |\mathcal{Z}| \right], \tag{383}$$

where (a) holds because $d^{(n)}(X^n, \psi(X^n|Y^n)) \leq \Delta$; (b) holds because $\hat{X}^n = \psi(X^n|Y^n)$ and by (380); and (c) holds because $\psi(\cdot|Y^n)$ satisfies (189) when we substitute $\hat{G}^*(\hat{\mathbf{x}}|\mathbf{y})$ for $\hat{G}(\hat{\mathbf{x}}|\mathbf{y})$ and $G^*_{\Delta}(\mathbf{x}|\mathbf{y})$ for $G_{\Delta}(\mathbf{x}|\mathbf{y})$ in (189). Since $\hat{G}^*(\cdot|Y^n)$ is an optimal guessing function, this concludes the proof of (196).

K A Proof of Theorem 34

Proof. As to the first part, suppose we are given a stochastic task-encoder (193) and a decoder with lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ satisfying (194). For every $\mathbf{y} \in \mathcal{Y}^n$ order the lists $\{\mathcal{L}_z^{\mathbf{y}}\}_{z \in \mathcal{Z}}$ in increasing order of their cardinalities, and order the elements in each list in some arbitrary way. Now consider the guessing order where we first guess the elements of the first (and smallest) list in their respective order followed by those elements in the second list that have not yet been guessed (i.e., that are not contained in the first list). We continue until concluding by guessing those elements of the last (and longest) list that have not been previously guessed. Let $\hat{G}(\cdot|Y^n)$ be the corresponding guessing function, let $G_{\Delta}(\cdot|Y^n)$ be its success function, and let $\psi(\cdot|Y^n)$ be its reconstruction function (which satisfies (189)). Observe that

$$\mathbb{E}\left[G_{\Delta}(X^{n}|Y^{n})^{\rho}\right] \stackrel{(a)}{=} \mathbb{E}\left[\hat{G}\left(\psi(X^{n}|Y^{n})|Y^{n}\right)^{\rho}\right]$$
(384)

$$=\sum_{\mathbf{x},\mathbf{y}} P_{X,Y}^{n}(\mathbf{x},\mathbf{y}) \left| \left\{ \hat{\mathbf{x}} \colon \hat{G}(\hat{\mathbf{x}}|\mathbf{y}) \le \hat{G}(\psi(\mathbf{x}|\mathbf{y})|\mathbf{y}) \right\} \right|^{\rho}$$
(385)

$$\stackrel{(b)}{\leq} \sum_{\mathbf{x},\mathbf{y}} P_{X,Y}^{n}(\mathbf{x},\mathbf{y}) \left| \mathcal{Z} \right|^{\rho} \min_{z: \ \psi(\mathbf{x}|\mathbf{y}) \in \mathcal{L}_{z}^{\mathbf{y}}} \left| \mathcal{L}_{z}^{\mathbf{y}} \right|^{\rho}$$
(386)

$$\stackrel{(c)}{\leq} |\mathcal{Z}|^{\rho} \mathbb{E}\Big[\left| \mathcal{L}_{Z}^{Y^{n}} \right|^{\rho} \Big], \tag{387}$$

where (a) holds because $\psi(\cdot|Y^n)$ satisfies (189); (b) holds because for every $\mathbf{x} \in \mathcal{X}^n$, $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$, and $\mathbf{y} \in \mathcal{Y}^n$, a necessary condition for $\hat{G}(\hat{\mathbf{x}}|\mathbf{y}) \leq \hat{G}(\psi(\mathbf{x}|\mathbf{y})|\mathbf{y})$ is that $\hat{\mathbf{x}} \in \mathcal{L}_{z}^{\mathbf{y}}$ for some $\tilde{z} \in \mathcal{Z}$ satisfying $|\mathcal{L}_{\tilde{z}}^{\mathbf{y}}| \leq \min_{z: |\psi(\mathbf{x}|\mathbf{y}) \in \mathcal{L}_{z}^{\mathbf{y}}|} |\mathcal{L}_{z}^{\mathbf{y}}|$, and because the number of lists whose size does not exceed $\min_{z: |\psi(\mathbf{x}|\mathbf{y}) \in \mathcal{L}_{z}^{\mathbf{y}}|} |\mathcal{L}_{z}^{\mathbf{y}}|$ is at most $|\mathcal{Z}|$; and (c) is true because by (194) the list $\mathcal{L}_{Z}^{Y^{n}}$ contains a reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^{n}$ of X^{n} that satisfies the fidelity criterion (185), and because (189) implies that

$$\hat{G}(\psi(\mathbf{x}|\mathbf{y})|\mathbf{y}) \le \hat{G}(\hat{\mathbf{x}}|\mathbf{y}), \ \forall \, \hat{\mathbf{x}} \text{ s.t. } d^{(n)}(X^n, \hat{\mathbf{x}}) \le \Delta,$$
(388)

and consequently that

$$\min_{z \colon \psi(\mathbf{x}|\mathbf{y}) \in \mathcal{L}_{z}^{\mathbf{y}}} |\mathcal{L}_{z}^{\mathbf{y}}| \leq \min_{z \colon \hat{\mathbf{x}} \in \mathcal{L}_{z}^{\mathbf{y}}} |\mathcal{L}_{z}^{\mathbf{y}}|, \ \forall \, \hat{\mathbf{x}} \text{ s.t. } d^{(n)}(X^{n}, \hat{\mathbf{x}}) \leq \Delta.$$
(389)

This concludes the proof of (203).

2

As to the second part, suppose we are given a positive integer $\omega \leq |\hat{\mathcal{X}}|^n$ satisfying (204) and a guessing function $\hat{G}(\cdot|Y^n)$ with corresponding success function $G_{\Delta}(\cdot|Y^n)$ and reconstruction function $\psi(\cdot|Y^n)$ satisfying (189). Define the chance variable $\hat{\mathcal{X}}^n = \psi(\mathcal{X}^n|Y^n)$, which takes values in $\hat{\mathcal{X}}^n$. Theorem 8 implies that $\hat{G}(\cdot|Y^n)$ and ω induce a $\{0,1\}$ -valued conditional PMF

$$\mathbb{P}[Z=z|\hat{X}^n=\hat{\mathbf{x}},Y^n=\mathbf{y}],\ \forall\ (\hat{\mathbf{x}},\mathbf{y},z)\in\hat{\mathcal{X}}^n\times\mathcal{Y}^n\times\mathcal{Z},$$
(390)

whose associated decoding lists

$$\hat{\mathcal{L}}_{z}^{\mathbf{y}} = \left\{ \hat{\mathbf{x}} \in \hat{\mathcal{X}}^{n} \colon \mathbb{P}[\hat{X}^{n} = \hat{\mathbf{x}} | Y^{n} = \mathbf{y}, Z = z] > 0 \right\}, \ \forall (\mathbf{y}, z) \in \mathcal{Y}^{n} \times \mathcal{Z}$$
(391)

satisfy

$$\mathbb{E}\left[\left|\hat{\mathcal{L}}_{Z}^{Y^{n}}\right|^{\rho}\right] \leq \mathbb{E}\left[\left[\hat{G}(\hat{X}^{n}|Y^{n})/\omega\right]^{\rho}\right].$$
(392)

Define the $\{0, 1\}$ -valued conditional PMF

$$\mathbb{P}[Z = z | X^{n} = \mathbf{x}, Y^{n} = \mathbf{y}]$$

= $\mathbb{P}[Z = z | \hat{X}^{n} = \psi(\mathbf{x} | \mathbf{y}), Y^{n} = \mathbf{y}], \ \forall (\mathbf{x}, \mathbf{y}, z) \in \mathcal{X}^{n} \times \mathcal{Y}^{n} \times \mathcal{Z},$ (393)

and the lists

$$\mathcal{L}_{z}^{\mathbf{y}} = \hat{\mathcal{L}}_{z}^{\mathbf{y}}, \ \forall \left(\mathbf{y}, z\right) \in \mathcal{Y}^{n} \times \mathcal{Z}.$$
(394)

Because $\hat{X}^n = \psi(X^n | Y^n)$, (391), (393), and (394) imply that

$$\psi(X^n|Y^n) \in \mathcal{L}_Z^{Y^n}.$$
(395)

Since

$$d^{(n)}(\mathbf{x},\psi(\mathbf{x}|\mathbf{y})) \leq \Delta, \ \forall (\mathbf{x},\mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n,$$

this implies that the decoding lists $\{\mathcal{L}_z^{\mathbf{y}}\}$ satisfy (194). Hence, (393) is a deterministic taskencoder (whose conditional PMF (193) is $\{0, 1\}$ -valued) for which the decoder with lists (394) satisfies (194). We are now ready to conclude the proof of (205):

$$\mathbb{E}\left[\left|\mathcal{L}_{Z}^{Y^{n}}\right|^{\rho}\right] \stackrel{(a)}{=} \mathbb{E}\left[\left|\hat{\mathcal{L}}_{Z}^{Y^{n}}\right|^{\rho}\right]$$
(396)

$$\stackrel{(b)}{\leq} \mathbb{E}\left[\left[\hat{G}(\hat{X}^n|Y^n)/\omega\right]^{\rho}\right] \tag{397}$$

$$\stackrel{(c)}{=} \mathbb{E}\Big[\big[G_{\Delta}(X^n | Y^n) / \omega \big]^{\rho} \Big], \tag{398}$$

where (a) holds by (394); (b) holds by (392); and (c) holds because $\hat{X}^n = \psi(X^n | Y^n)$, where $\psi(\cdot | Y^n)$ satisfies (189).

L A Proof of Theorem 38

Proof. We first prove (217). If $R_1 + R_2 < E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$, then Corollary 33 in the guessing version and Corollary 36 in the list version imply that the privacy-exponent is negative

infinity. We hence assume that $R_1 + R_2 > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$. In this case Corollary 33 in the guessing version and Corollary 36 in the list version imply that the constraint

$$\lim_{n \to \infty} \mathscr{A}_{\mathcal{B}}(P^n_{X,Y}, \Delta) = 1$$
(399)

can be met.

We first show that the privacy-exponent cannot exceed the RHS of (217). To this end we note that it holds for every $n \in \mathbb{N}$ that

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}) = \min_{\hat{G}^{(1)}(\cdot|Y^{n},M_{1}),\,\hat{G}^{(2)}(\cdot|Y^{n},M_{2})} \mathbb{E}\Big[G_{\Delta}^{(1)}(X^{n}|Y^{n},M_{1})^{\rho} \wedge G_{\Delta}^{(2)}(X^{n}|Y^{n},M_{2})^{\rho}\Big] \quad (400)$$

$$\leq \min_{k \in \{1,2\}} \left(\min_{\hat{G}^{(k)}(\cdot|Y^n, M_k)} \mathbb{E}\left[G^{(k)}_{\Delta}(X^n|Y^n, M_k)^{\rho} \right] \right).$$

$$\tag{401}$$

By Corollary 32 it holds for every $k \in \{1, 2\}$ and $l \in \{1, 2\} \setminus \{k\}$ that

$$\min_{\hat{G}(\cdot|Y^{n},M_{1},M_{2})} \mathbb{E}\left[G_{\Delta}(X^{n}|Y^{n},M_{1},M_{2})^{\rho}\right] \\
\geq |\mathcal{M}_{l}|^{-\rho} \min_{\hat{G}^{(k)}(\cdot|Y^{n},M_{k})} \mathbb{E}\left[G_{\Delta}^{(k)}(X^{n}|Y^{n},M_{k})^{\rho}\right].$$
(402)

Because

$$\min_{\hat{G}(\cdot|Y^n,M_1,M_2)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n,M_1,M_2)^{\rho}\right] \leq \mathbb{E}\left[\left|\mathcal{L}_{M_1,M_2}^{Y^n}\right|^{\rho}\right],$$

(401) and (402) imply that in both versions Eve's ambiguity exceeds Bob's by at most a factor of $|\mathcal{M}_1|^{\rho} \wedge |\mathcal{M}_2|^{\rho}$. That is,

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n},\Delta) \leq \left(|\mathcal{M}_{1}| \wedge |\mathcal{M}_{2}|\right)^{\rho} \mathscr{A}_{\mathrm{B}}(P_{X,Y}^{n},\Delta).$$

$$\tag{403}$$

Suppose that (399) holds and consequently

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathcal{B}}(P_{X,Y}^n, \Delta))}{n} = 0.$$
(404)

From (403) and (404) it follows that

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^n, \Delta))}{n} \le \rho(R_1 \wedge R_2).$$
(405)

Eve can ignore the hint that she observes and guess a reconstruction for X^n based on Y^n alone. Hence, we obtain from Theorem 30 that

$$\limsup_{n \to \infty} \frac{\log\left(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}, \Delta)\right)}{n} \le \rho E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta).$$
(406)

From (405) and (406) we conclude that the privacy-exponent cannot exceed the RHS of (217):

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}, \Delta))}{n} \le \rho\Big(R_{1} \wedge R_{2} \wedge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)\Big).$$
(407)

We next show that the privacy-exponent cannot be smaller than the RHS of (217). By possibly relabeling the hints, we can assume w.l.g. that $R_2 = R_1 \wedge R_2$. Fix some $\epsilon > 0$ satisfying

$$\epsilon \le R_1 + R_2 - E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta).$$
 (408)

Choose a nonnegative rate-triple $(R_s, \tilde{R}_1, \tilde{R}_2)$ as follows:

1. If $R_2 \leq E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)/2$, then choose

$$R_{\rm s} = 0, \quad \tilde{R}_1 = E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - R_2 + \epsilon, \quad \tilde{R}_2 = R_2.$$
(409)

2. Else if $E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)/2 < R_2 \leq E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$, then choose

$$R_{\rm s} = 2R_2 - E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) - \epsilon, \quad \tilde{R}_1 = \tilde{R}_2 = E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) - R_2 + \epsilon.$$
(410)

(To guarantee that $R_{\rm s} \ge 0$, we assume in this case that $\epsilon > 0$ is sufficiently small so that, in addition to (408), also

$$\epsilon < 2R_2 - E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) \tag{411}$$

holds.) 3. Else if $E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) < R_2$, then choose

$$R_{\rm s} = R_2, \quad \tilde{R}_1 = \tilde{R}_2 = 0.$$
 (412)

Having chosen $(R_s, \tilde{R}_1, \tilde{R}_2)$, choose the triple $(c_s, c_1, c_2) \in \mathbb{N}^3$ to be

$$(c_{\rm s}, c_1, c_2) = (2^{nR_{\rm s}}, 2^{n\tilde{R}_1}, 2^{n\tilde{R}_2}).$$
(413)

For each $\nu \in \{s, 1, 2\}$, let V_{ν} be a chance variable taking values in the set $\mathcal{V}_{\nu} = \{0, \dots, c_{\nu} - 1\}$. Because our choice of $(R_s, \tilde{R}_1, \tilde{R}_2)$ satisfies

$$R_{\rm s} + \tilde{R}_1 + \tilde{R}_2 > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta), \tag{414}$$

Corollary 36 implies that there exist $\{0, 1\}$ -valued conditional PMFs

$$\mathbb{P}\big[(V_{\mathrm{s}}, V_1, V_2) = (v_{\mathrm{s}}, v_1, v_2) \big| X^n = \mathbf{x}, Y^n = \mathbf{y}\big]$$

and decoders, whose lists

$$\left\{\mathcal{L}_{v_{\mathrm{s}},v_{1},v_{2}}^{\mathbf{y}}\right\}_{\left(\mathbf{y},v_{\mathrm{s}},v_{1},v_{2}\right)\in\mathcal{Y}^{n}\times\mathcal{V}_{\mathrm{s}}\times\mathcal{V}_{1}\times\mathcal{V}_{2}}$$

satsify

$$\exists \hat{\mathbf{x}} \in \mathcal{L}_{V_{\mathrm{s}},V_{1},V_{2}}^{Y^{n}} \text{ s.t. } d^{(n)}(X^{n},\hat{\mathbf{x}}) \leq \Delta,$$

$$(415)$$

for which

$$\lim_{n \to \infty} \mathbb{E} \left[\left| \mathcal{L}_{V_{\mathrm{s}}, V_{1}, V_{2}}^{Y^{n}} \right|^{\rho} \right] = 1.$$
(416)

Because

$$\min_{\hat{G}(\cdot|Y^n, V_{\mathrm{s}}, V_1, V_2)} \mathbb{E}\left[G_{\Delta}(X^n|Y^n, V_{\mathrm{s}}, V_1, V_2)^{\rho}\right] \le \mathbb{E}\left[\left|\mathcal{L}_{V_{\mathrm{s}}, V_1, V_2}^{Y^n}\right|^{\rho}\right],$$

(416) implies that

$$\lim_{n \to \infty} \min_{\hat{G}(\cdot|Y^n, V_{\rm s}, V_1, V_2)} \mathbb{E} \Big[G_{\Delta}(X^n \big| Y^n, V_{\rm s}, V_1, V_2)^{\rho} \Big] = 1.$$
(417)

Our choice of $(R_s, \tilde{R}_1, \tilde{R}_2)$ satisfies

$$R_1 \ge R_{\rm s} + \tilde{R}_1, \quad R_2 \ge R_{\rm s} + \tilde{R}_2,$$
(418)

and hence we can for every blocklength n choose some conditional PMF (212) that assigns positive probability only to $c_{\rm s}c_1$ elements of \mathcal{M}_1 and $c_{\rm s}c_2$ elements of \mathcal{M}_2 . Therefore, we can assume w.l.g. that $\mathcal{M}_1 = \mathcal{V}_{\rm s} \times \mathcal{V}_1$ and $\mathcal{M}_2 = \mathcal{V}_{\rm s} \times \mathcal{V}_2$ and choose $\mathcal{M}_1 = (V_{\rm s} \oplus_{c_{\rm s}} U, V_1)$ and $\mathcal{M}_2 = (U, V_2)$, where $(V_{\rm s}, V_1, V_2)$ is drawn according to the above conditional PMF, and where U is independent of $(X^n, Y^n, V_{\rm s}, V_1, V_2)$ and uniform over $\mathcal{V}_{\rm s}$. For this choice (399) follows from (416) in the list version and from (417) in the guessing version.

It remains to show that for the above choice of the conditional PMFs (212)

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^n, \Delta))}{n} \ge \rho \Big(R_1 \wedge R_2 \wedge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) \Big).$$
(419)

Define the triple of chance variables

$$(I, \hat{U}, \hat{V}) \triangleq \begin{cases} (1, V_s \oplus_{c_s} U, V_1) & \text{if } G_{\Delta}^{(1)}(X^n | Y^n, M_1) \le G_{\Delta}^{(2)}(X^n | Y^n, M_2), \\ (2, U, V_2) & \text{otherwise} \end{cases}$$
(420)

with alphabet $\mathcal{I} \times \mathcal{V}_{s} \times \hat{\mathcal{V}}$, where $\mathcal{I} = \{1, 2\}$ and $\hat{\mathcal{V}} = \{0, \ldots, c_{1} \vee c_{2} - 1\}$. From (Y^{n}, I, U, \hat{V}) Eve can guess a reconstruction $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^{n}$ of X^{n} using either $\hat{G}^{(1)}(\cdot|Y^{n}, M_{1})$ or $\hat{G}^{(2)}(\cdot|Y^{n}, M_{2})$ depending on the value of I. That is, Eve can use some guessing function $\hat{G}(\cdot|Y^{n}, I, \hat{U}, \hat{V})$ satisfying that, for every $\mathbf{y} \in \mathcal{Y}^{n}$, $i \in \mathcal{I}$, $\hat{u} \in \mathcal{V}_{s}$, and $\hat{v} \in \{0, \ldots, c_{i} - 1\}$,

$$\hat{G}(\hat{\mathbf{x}}|\mathbf{y}, i, \hat{u}, \hat{v}) = \hat{G}^{(i)}(\hat{\mathbf{x}}|\mathbf{y}, (\hat{u}, \hat{v})), \qquad (421)$$

where by (420) the success function $G_{\Delta}(\cdot|Y^n, I, \hat{U}, \hat{V})$ corresponding to $\hat{G}(\cdot|Y^n, I, \hat{U}, \hat{V})$ satisfies

$$G_{\Delta}(X^n|Y^n, I, \hat{U}, \hat{V})$$

= $G_{\Delta}^{(I)}(X^n|Y^n, (\hat{U}, \hat{V}))$ (422)

$$=G_{\Delta}^{(I)}(X^n|Y^n, M_I) \tag{423}$$

$$= G_{\Delta}^{(1)}(X^n | Y^n, M_1)^{\rho} \wedge G_{\Delta}^{(2)}(X^n | Y^n, M_2).$$
(424)

Let $\psi(\cdot|Y^n, I, \hat{U}, \hat{V})$ be the reconstruction function corresponding to $\hat{G}(\cdot|Y^n, I, \hat{U}, \hat{V})$, i.e., the unique mapping satisfying that

$$\begin{pmatrix} \psi(\mathbf{x}|\mathbf{y}, i, \hat{u}, \hat{v}) = \hat{\mathbf{x}} \iff G_{\Delta}(\mathbf{x}|\mathbf{y}, i, \hat{u}, \hat{v}) = \hat{G}(\hat{\mathbf{x}}|\mathbf{y}, i, \hat{u}, \hat{v}) \end{pmatrix}, \\ \forall (\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}, i, \hat{u}, \hat{v}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n \times \mathcal{Y}^n \times \mathcal{I} \times \mathcal{V}_{\mathrm{s}} \times \hat{\mathcal{V}},$$
(425)

and define the chance variable $\hat{X}^n=\psi(X^n|Y^n,I,\hat{U},\hat{V}).$ Note that

$$\mathbb{E}\left[\hat{G}(\hat{X}^n|Y^n, I, \hat{U}, \hat{V})^{\rho}\right] \ge \min_{G(\cdot, \cdot, \cdot|Y^n, I, \hat{U}, \hat{V})} \mathbb{E}\left[G(\hat{X}^n, I, \hat{U}|Y^n, I, \hat{U}, \hat{V})^{\rho}\right].$$
(426)

This implies that

$$\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}, \Delta) \geq \min_{G(\cdot, \cdot, \cdot|Y^{n}, I, \hat{U}, \hat{V})} \mathbb{E}\left[G(\hat{X}^{n}, I, \hat{U}|Y^{n}, I, \hat{U}, \hat{V})^{\rho}\right]$$
(427)

$$\stackrel{(a)}{\geq} \left(\left| \mathcal{I} \right| \left| \mathcal{V}_{s} \right| \left| \hat{\mathcal{V}} \right| \right)^{-\rho} \min_{G(\cdot, \cdot, \cdot, |Y^{n})} \mathbb{E} \left[G(\hat{X}^{n}, I, \hat{U} | Y^{n})^{\rho} \right]$$
(428)

$$\geq 2^{-\rho - n\rho(R_{\mathrm{s}} + \tilde{R}_1 \vee \tilde{R}_2)} \min_{G(\cdot, \cdot, \cdot | Y^n)} \mathbb{E} \left[G(\hat{X}^n, I, \hat{U} | Y^n)^{\rho} \right], \tag{429}$$

where (a) follows from Corollary 6 and the fact that (I, \hat{U}, \hat{V}) takes values in the set

$$\left\{(1,\hat{u},\hat{v})\colon (\hat{u},\hat{v})\in \mathcal{V}_{s}\times\mathcal{V}_{1}\right\}\cup\left\{(2,\hat{u},\hat{v})\colon (\hat{u},\hat{v})\in \mathcal{V}_{s}\times\mathcal{V}_{2}\right\},\$$

which is of size

$$|\mathcal{V}_{s} \times \mathcal{V}_{1}| + |\mathcal{V}_{s} \times \mathcal{V}_{2}| = c_{s}(c_{1} + c_{2}).$$

From (429) it follows that

$$\liminf_{n \to \infty} \frac{\log\left(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}, \Delta)\right)}{n} \\
\geq \liminf_{n \to \infty} \min_{G(\cdot, \cdot, \cdot |Y^{n})} \frac{\log\left(\mathbb{E}\left[G(\hat{X}^{n}, I, \hat{U}|Y^{n})^{\rho}\right]\right)}{n} - \rho(R_{\mathrm{s}} + \tilde{R}_{1} \lor \tilde{R}_{2}). \tag{430}$$

Therefore, if we can show that

$$\liminf_{n \to \infty} \min_{G(\cdot, \cdot, \cdot|Y^n)} \frac{\log\left(\mathbb{E}\left[G(\hat{X}^n, I, \hat{U}|Y^n)^{\rho}\right]\right)}{n} \ge \rho\left(E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) + R_s\right),$$
(431)

then we can let ϵ tend to zero to conclude from (430) that (419) holds:

$$\liminf_{n \to \infty} \frac{\log(\mathscr{A}_{\mathrm{E}}(P_{X,Y}^{n}, \Delta))}{n} \ge \rho \Big(R_{2} \wedge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) \Big)$$
(432)

$$\geq \rho \Big(R_1 \wedge R_2 \wedge E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) \Big). \tag{433}$$

We next conclude the proof of (419) by establishing (431). By Theorem 3

$$\liminf_{n \to \infty} \min_{G(\cdot, \cdot, \cdot|Y^n)} \frac{\log\left(\mathbb{E}\left[G(\hat{X}^n, I, \hat{U}|Y^n)^{\rho}\right]\right)}{n} \ge \rho H_{\tilde{\rho}}(\hat{X}^n, I, \hat{U}|Y^n).$$
(434)

In [4, Appendix B] it is shown that for every pair of chance variables (A, B) taking values in some finite set $\mathcal{A} \times \mathcal{B}$ according to som PMF $P_{A,B}$

$$H_{\tilde{\rho}}(A|B) = \max_{\substack{Q \in \mathscr{P}(\mathcal{B}), \\ V \in \mathscr{P}(\mathcal{A}|\mathcal{B})}} H(V|Q) - \rho^{-1} D(Q \times V||P_{A,B}),$$
(435)

where $\mathscr{P}(\mathcal{B})$ denotes the set of PMFs on \mathcal{B} , and $\mathscr{P}(\mathcal{A}|\mathcal{B})$ denotes the set of transition laws from \mathcal{B} to \mathcal{A} . We shall use (435) to lower-bound the RHS of (434), where we will substitute (\hat{X}^n, I, \hat{U}) for A and Y^n for B in (435). To that end denote by V_n the conditional PMF of (\hat{X}^n, I, \hat{U}) given (X^n, Y^n, U) , and denote by \tilde{V}_n the conditional PMF of $(Y^n, \hat{X}^n, I, \hat{U})$ given (X^n, Y^n, U) . Note that V_n and \tilde{V}_n are both $\{0, 1\}$ -valued. Fix any PMF $Q_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$, let P_U denote the uniform distribution on \mathcal{V}_s , and define the PMF on $\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{U} \times \hat{\mathcal{X}}^n \times \mathcal{I} \times \mathcal{U}$

$$Q_{X^n,Y^n,U,\hat{X}^n,I,\hat{U}} = \left(Q_{X,Y}^n \times P_U\right) \times V_n$$

As to $D(Q \times V || P_{A,B})$, we then find that

$$D\left(Q_Y^n \times \left(\left(Q_{X|Y}^n \times P_U\right) V_n\right) \middle| \middle| P_Y^n \times \left(\left(P_{X|Y}^n \times P_U\right) V_n\right)\right)$$
(436)

$$= D((Q_{X,Y}^n \times P_U) V_n || (P_{X,Y}^n \times P_U) V_n)$$
(437)

$$\stackrel{(a)}{\leq} D(Q_{X,Y}^n \times P_U || P_{X,Y}^n \times P_U) \tag{438}$$

$$= D(Q_{X,Y}^n || P_{X,Y}^n)$$
(439)

$$= nD(Q_{X,Y}||P_{X,Y}), (440)$$

where (a) follows from the Data-Processing inequality [23, Lemma 3.11]. As to H(V|Q), we find that

$$H((Q_{X|Y}^n \times P_U) V_n | Q_Y^n) \tag{441}$$

$$\stackrel{(a)}{\geq} I\left(Q_{X|Y}^n \times P_U, V_n \middle| Q_Y^n\right) \tag{442}$$

$$\overset{(b)}{=} I(Q_{X|Y}^{n}, P_{U}V_{n}|Q_{Y}^{n}) + I(P_{U}, V_{n}|Q_{X,Y}^{n})$$
(443)

$$\stackrel{(c)}{=} I(Q_{X|Y}^n, P_U V_n | Q_Y^n) + \log |\mathcal{V}_{\rm s}| \tag{444}$$

$$\stackrel{(d)}{\geq} I(Q_{X|Y}^{n}, Q_{\hat{X}^{n}|X^{n},Y^{n}}|Q_{Y}^{n}) + \log|\mathcal{V}_{s}|$$
(445)

$$\stackrel{\scriptscriptstyle(c)}{\geq} nR_{X|Y}(Q_{X,Y},\Delta) + \log |\mathcal{V}_{\rm s}|,\tag{446}$$

where (a) holds because entropy is nonnegative; (b) follows from chain rule; (c) holds because U is independent of (X^n, Y^n) and uniform over its support \mathcal{V}_s , and because U is deterministic given $(X^n, Y^n, \hat{X}^n, I, \hat{U})$ (which holds by (420) and because V_s is deterministic given (X^n, Y^n) ; (d) holds for the conditional PMF

$$Q_{\hat{X}^n|X^n,Y^n}(\hat{\mathbf{x}}|\mathbf{x},\mathbf{y}) = \sum_{u,i,\hat{u}} P_U(u) V_n(\hat{\mathbf{x}},i,\hat{u}|\mathbf{x},\mathbf{y},u), \,\forall \, (\mathbf{x},\hat{\mathbf{x}},\mathbf{y}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n \times \mathcal{Y}^n, \quad (447)$$

because conditioning cannot increase entropy; and (e) follows from the conditional Rate-Distortion theorem [25] and

$$\mathbb{P}_{Q_{X,Y}^n \times Q_{\hat{X}^n \mid X^n, Y^n}} \left[d^{(n)} \left(X^n, \hat{X}^n \right) \le \Delta \right] = 1,$$

$$(448)$$

which holds by (447) and because

$$\left(V_n(\hat{\mathbf{x}}, i, \hat{u} | \mathbf{x}, \mathbf{y}, u) > 0 \implies \exists \, \hat{v} \in \hat{\mathcal{V}} \colon \hat{\mathbf{x}} = \psi(\mathbf{x} | \mathbf{y}, i, \hat{u}, \hat{v}) \right), \forall (\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}, i, \hat{u}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n \times \mathcal{Y}^n \times \mathcal{I} \times \mathcal{U}.$$
(449)
More precisely, (e) can be established as follows. Draw $(\underline{X^n}, \underline{Y^n}, \underline{\hat{X}^n})$ from $\mathcal{X}^n \times \mathcal{Y}^n \times \hat{\mathcal{X}}^n$ according to the PMF $Q_{X,Y}^n \times Q_{\hat{X}^n|X^n,Y^n}$. By (448)

$$\mathbb{E}\left[d^{(n)}(\underline{X^n},\underline{\hat{X}^n})\right] = \mathbb{E}_{Q^n_{X,Y} \times Q_{\hat{X}^n | X^n, Y^n}}\left[d^{(n)}(X^n, \hat{X}^n)\right] \le \Delta.$$
(450)

Consequently, we find that

$$I(Q_{X|Y}^{n}, Q_{\hat{X}^{n}|X^{n},Y^{n}}|Q_{Y}^{n}) = I(\underline{X^{n}}; \underline{\hat{X}^{n}}|\underline{Y^{n}})$$
(451)

$$\stackrel{(f)}{=} \sum_{i=1}^{n} I\left(\underline{X_i}; \underline{\hat{X}^n} | \underline{Y^n}, \underline{X^{i-1}}\right) \tag{452}$$

$$\stackrel{(g)}{\geq} \sum_{i=1}^{n} I(\underline{X_i}; \underline{\hat{X}_i} | \underline{Y_i}) \tag{453}$$

$$\stackrel{(h)}{=} n \left(\frac{1}{n} \sum_{i=1}^{n} I(Q_{X|Y}, Q_{\hat{X}_i|X_i, Y_i} | Q_Y) \right)$$
(454)

$$\stackrel{(i)}{\geq} nI\left(Q_{X|Y}, \frac{1}{n}\sum_{i=1}^{n}Q_{\hat{X}_{i}|X_{i},Y_{i}}\middle|Q_{Y}\right)$$
(455)

$$\stackrel{(j)}{\geq} n \min_{\substack{Q_{\hat{X}|X,Y}:\\\mathbb{E}[d(X,\hat{X}) \leq \Delta]}} I(Q_{X|Y}, Q_{\hat{X}|X,Y}|Q_Y) \tag{456}$$

$$\stackrel{(k)}{=} R_{X|Y}(Q_{X,Y}, \Delta), \tag{457}$$

where (f) follows from the chain rule; (g) holds because $\underline{X_i}$ and $(\underline{Y^{i-1}}, \underline{Y_{i+1}^n}, \underline{X^{i-1}})$ are independent, and because conditioning cannot increase entropy; (h) holds for the conditional PMFs $Q_{\hat{X}_i|X_i,Y_i}$, $i \in [1:n]$ that satisfy

$$\begin{aligned} Q_{\hat{X}_{i}|X_{i},Y_{i}}(\hat{x}_{i}|x_{i},y_{i}) &= \sum_{\substack{x^{i-1}, \hat{x}^{i-1}, y^{i-1}, \\ x^{n}_{i+1}, \hat{x}^{n}_{i+1}, y^{n}_{i+1}, \\ x^{n}_{i+1}, \hat{x}^{n}_{i+1}, y^{n}_{i+1}, y^{n}_{i+1}, \\ \forall \left(x_{i}, \hat{x}_{i}, y_{i}\right) \in \mathcal{X} \times \hat{\mathcal{X}} \times \mathcal{Y}; \end{aligned}$$

(i) holds because mutual information is convex in the transition law (here $Q_{\hat{X}_i|X_i,Y_i}$); (j) holds because (450) implies that

$$\Delta \ge \mathbb{E}\left[d^{(n)}(\underline{X^n}, \underline{\hat{X}^n})\right] \tag{458}$$

$$= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}d(\underline{X}_{i},\underline{\hat{X}}_{i})\right]$$
(459)

$$= \mathbb{E}_{Q_{X,Y} \times \left(\frac{1}{n} \sum_{i=1}^{n} Q_{\hat{X}_{i}|X_{i},Y_{i}}\right)} \left[d(X,\hat{X})\right];$$
(460)

and (k) holds by the definition of the rate-distortion function under the PMF $Q_{X,Y}$ (187). This concludes the proof of (446). Having established (446), we are now ready to conclude the proof of (431). By substituting (\hat{X}^n, I, \hat{U}) for A and Y^n for B in (435), we obtain from (435), (440), and (446) that

$$H_{\tilde{\rho}}(\hat{X}^{n}, I, \hat{U} | Y^{n})$$

$$\geq H((Q_{X|Y}^{n} \times P_{U}) V_{n} | Q_{Y}^{n})$$

$$-\rho^{-1} D(Q_{Y}^{n} \times ((Q_{X|Y}^{n} \times P_{U}) V_{n}) | | P_{Y}^{n} \times ((P_{X|Y}^{n} \times P_{U}) V_{n}))$$

$$(461)$$

$$\geq n \Big(R_{X|Y}(Q_{X,Y}, \Delta) - \rho^{-1} D(Q_{X,Y}||P_{X,Y}) \Big) + \log |\mathcal{V}_{s}|.$$
(462)

Because this holds for every PMF $Q_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$, and by the definition of $E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta)$ (188),

$$H_{\tilde{\rho}}(\hat{X}^{n}, I, \hat{U}|Y^{n}) \\ \geq n \sup_{Q_{X,Y}} \left(R_{X|Y}(Q_{X,Y}, \Delta) - \rho^{-1} D(Q_{X,Y}||P_{X,Y}) \right) + \log |\mathcal{V}_{s}|$$
(463)

$$= n E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) + \log |\mathcal{V}_{\rm s}|.$$

$$\tag{464}$$

This, $|\mathcal{V}_{\rm s}| = 2^{nR_{\rm s}}$, and (434) imply (431). This concludes the proof of (217).

We next prove (218). If $R_1 + R_2 < E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_B$, then Corollary 33 in the guessing version and Corollary 36 in the list version imply that the modest privacy-exponent is negative infinity. We hence assume that $R_1 + R_2 > E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_B$. We can now use the same line of argument as in the proof of (217) but with (404) replaced by

$$\limsup_{n \to \infty} \frac{\log(\mathscr{A}_{\mathcal{B}}(P_{X,Y}^n, \Delta))}{n} \le E_{\mathcal{B}}$$
(465)

to show that the modest privacy-exponent cannot exceed the RHS of (218). To show that the modest privacy-exponent is lower-bounded by the RHS of (218), we argue as for the privacy-exponent, except that here we choose the nonnegative triple $(R_s, \tilde{R}_1, \tilde{R}_2)$ as follows:

1. If $R_2 \leq (E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_{\rm B})/2$, then choose

$$R_{\rm s} = 0, \quad \tilde{R}_1 = E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_{\rm B} - R_2, \quad \tilde{R}_2 = R_2.$$
 (466)

2. Else if $\left(E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) - \rho^{-1}E_{\mathrm{B}}\right)/2 < R_2 \leq E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) - \rho^{-1}E_{\mathrm{B}}$, then choose

$$R_{\rm s} = 2R_2 - E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) + \rho^{-1}E_{\rm B},$$

$$\tilde{R}_1 = \tilde{R}_2 = E_{X|Y}^{(\rho)}(P_{X,Y}, \Delta) - \rho^{-1}E_{\rm B} - R_2.$$
(467)

3. Else if $E_{X|Y}^{(\rho)}(P_{X,Y},\Delta) - \rho^{-1}E_{\mathrm{B}} < R_2$, then choose

$$R_{\rm s} = R_2, \quad \dot{R}_1 = \ddot{R}_2 = 0.$$
 (468)

References

- N. Merhav and E. Arikan, "The Shannon cipher system with a guessing wiretapper," IEEE Trans. Inf. Theory, vol. 45, no. 6, pp. 1860–1866, Sep. 1999.
- [2] J. L. Massey, "Guessing and entropy," in Proc. of IEEE Int. Symp. on Inf. Theory (ISIT), Jun. 1994, p. 204.
- [3] E. Arikan, "An inequality on guessing and its application to sequential decoding," *IEEE Trans. Inf. Theory*, vol. 42, no. 1, pp. 99–105, Jan. 1996.
- [4] C. Bunte and A. Lapidoth, "Encoding tasks and Rényi entropy," *IEEE Trans. Inf. Theory*, vol. 60, no. 9, pp. 5065–5076, Sep. 2014.
- [5] —, "On the listsize capacity with feedback," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 6733–6748, Nov. 2014.
- [6] A. Lapidoth and C. Pfister, "A method for the construction of optimal task encoders," in Proc. of IEEE Int. Symp. on Inf. Theory (ISIT), Jun. 2015, pp. 2540–2544.
- [7] E. Arikan and N. Merhav, "Guessing subject to distortion," *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 1041–1056, May 1998.
- [8] A. Subramanian and S. W. McLaughlin, "MDS codes on the erasure-erasure wiretap channel," arXiv:0902.3286 [cs.IT], 2009.
- [9] N. Cai and R. W. Yeung, "Secure network coding on a wiretap network," *IEEE Trans.* Inf. Theory, vol. 57, no. 1, pp. 424–435, Jan. 2011.
- [10] E. S. S. El Rouayheb and A. Sprintson, "Secure network coding for wiretap networks of type ii," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1361–1371, Mar. 2012.
- [11] G. R. Blakley, "Safeguarding cryptographic keys," in Proc. of AFIPS Nat. Comput. Conf. (NCC), Jun. 1979, pp. 313–317.
- [12] —, "How to share a secret," Commun. ACM, vol. 22, no. 11, pp. 612–613, Nov. 1979.
- [13] G. R. Blakley and C. Meadows, "Security of ramp schemes," in Advances in Cryptology, ser. Lecture Notes in Computer Science, 1985, vol. 196, pp. 242–268.
- [14] C.-E. Pfister and W. G. Sullivan, "Rényi entropy, guesswork moments, and large deviations," *IEEE Trans. Inf. Theory*, vol. 50, no. 11, pp. 2794–2800, Nov. 2004.
- [15] R. Sundaresan, "Guessing under source uncertainty," *IEEE Trans. Inf. Theory*, vol. 53, no. 1, pp. 269–287, Jan. 2007.
- [16] M. K. Hanawal and R. Sundaresan, "Guessing revisited: A large deviations approach," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 70–78, Jan. 2011.

- [17] Y. Hayashi and H. Yamamoto, "Coding theorems for the Shannon cipher system with a guessing wiretapper and correlated source outputs," *IEEE Trans. Inf. Theory*, vol. 54, no. 6, pp. 2808–2817, Jun. 2008.
- [18] M. K. Hanawal and R. Sundaresan, "The Shannon cipher system with a guessing wiretapper: General sources," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2503–2516, Apr. 2011.
- [19] S. Arimoto, "Information measures and capacity of order α for discrete memoryless channels," *Topics in Inf. Theory*, vol. 17, no. 6, pp. 41–52, 1977.
- [20] S. Fehr and S. Berens, "On the conditional rényi entropy," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 6801–6810, Nov 2014.
- [21] C. E. Shannon, "The zero error capacity of a noisy channel," *IRE Trans. Inf. Theory*, vol. 2, no. 3, pp. 8–19, Sep. 1956.
- [22] —, "Communication theory of secrecy systems," Bell Systems Technical Journal, vol. 28, no. 3, pp. 565–715, Oct. 1949.
- [23] I. Csiszár and J. Körner, Information Theory, 2nd ed. Cambridge University Press, 2011.
- [24] R. M. Roth, Introduction to Coding Theory. Cambridge University Press, 2006.
- [25] B. M. Leiner and R. M. Gray, "Rate-distortion for ergodic sources with side information," *IEEE Trans. Inf. Theory*, vol. 12, no. 5, pp. 672–675, Sep. 1974.