# New Bounds for Permutation Codes in Ulam Metric 

Faruk Göloğlu, Jüri Lember, Ago-Erik Riet, and Vitaly Skachek<br>Faculty of Mathematics and Computer Science<br>University of Tartu, Tartu 50409, Estonia


#### Abstract

New bounds on the cardinality of permutation codes equipped with the Ulam distance are presented. First, an integer-programming upper bound is derived, which improves on the Singleton-type upper bound in the literature for some lengths. Second, several probabilistic lower bounds are developed, which improve on the known lower bounds for large minimum distances. The results of a computer search for permutation codes are also presented.


Index Terms-Permutation codes, rank modulation, Singleton bound, sphere-packing bound, Ulam distance.

## I. Introduction

A permutation code is a subset of the symmetric group $\mathbb{S}_{n}$, equipped with a distance metric. Permutation codes are of potential use in various applications, such as communications over Gaussian channels [12], [20], power-line communications [3], [7], and coding for flash memories used with rank modulation [4]. Permutation codes were extensively studied in the literature over the last decades. In most of these studies, permutation codes are equipped with the Hamming and the Kendall $\tau$ metric [5], [8], [13].

Permutation codes were recently proposed for storing information in non-volatile (flash) memories under rank modulation [4], [10], [11], [23]. The main idea of the rank modulation scheme is that the information is stored in the form of rankings of the cell charges, rather than in terms of the absolute values of the charges. Rank-modulation codes represent a family of codes capable of handling errors of the form of adjacent transpositions [2], [18] or translocations [9]. Such error patterns are typical for memory systems, where leakage of electric charge occurs over time.

There are two types of errors where a permutation code equipped with the Ulam or Kendall $\tau$ metric could be of use. One such type is overshoot errors, in which a cell receives more charge than it is supposed to. The second type is the errors, in which a defective cell loses charge more quickly than normal. Both of these types of errors constitute one error in the Ulam metric or a number of errors in the Kendall $\tau$ metric. Thus, codes in the Ulam or Kendall $\tau$ metrics seem to be appropriate for error detection and correction in the paradigm of rank modulation.

The problem of estimating the maximum size of a code in the Ulam metric for given parameters is very difficult. Different mathematical tools could be applied to this problem. In this work, we demonstrate that novel bounds on the maximum size of a code can be obtained by an integer-programming method and by probability estimation techniques. These two approaches deal with different regimes: the probability bounds
are useful for large $n$ (some results, like Proposition IV.3, are of the asymptotic nature only), whilst the integer-programming approach works well with (relatively) small $n$.

## II. Notation

Denote by $\mathbb{Z}_{0}^{+}$the set of non-negative integers. We also use the notation $[n] \triangleq\{1,2, \cdots, n\}$.

A permutation $\sigma:[n] \rightarrow[n]$ is a bijection. Let $\mathbb{S}_{n}$ denote the set of all permutations of the set [ $n$ ], i.e., the symmetric group of order $n!$. For any $\sigma \in \mathbb{S}_{n}$, we write $\sigma=[\sigma(1), \sigma(2), \cdots, \sigma(n)]$, where $\sigma(i)$ is the image of $i \in[n]$ under the permutation $\sigma$. This is called the one-line notation of permutation $\sigma$. The identity permutation $[1,2, \cdots, n]$ is denoted by $e$, while $\sigma^{-1}$ stands for the inverse of the permutation $\sigma$.

Let d : $\mathbb{S}_{n} \times \mathbb{S}_{n} \rightarrow \mathbb{Z}_{0}^{+}$be a metric defined for pairs of permutations. A permutation code of length $n$ and minimum distance $d$ in a metric $d$ is a subset $\mathcal{C}$ of $\mathbb{S}_{n}$, such that for all $\tau, \sigma \in \mathcal{C}, \tau \neq \sigma$, we have $\mathrm{d}(\tau, \sigma) \geq d$. Such a code will be also called an $(n, d)$ code in a metric d .

Definition II.1. Assume that $1 \leq i<j \leq n$. A permutation $\tau \in \mathbb{S}_{n}$ is a right translocation if

$$
\tau=[1, \cdots, i-1, i+1, i+2, \cdots, j, i, j+1, \cdots, n]
$$

A permutation $\tau \in \mathbb{S}_{n}$ is a left translocation if

$$
\tau=[1, \cdots, j-1, i, j, j+1, \cdots, i-1, i+1, \cdots, n]
$$

Next, we define the composition of two permutations.
Definition II.2. Let $\tau$ and $\sigma$ be two permutations in $\mathbb{S}_{n}$. Then, their composition $\tau \sigma$ is a permutation in $\mathbb{S}_{n}$ defined as

$$
\forall i \in[n]:(\tau \sigma)(i)=\tau(\sigma(i))
$$

Under composition of permutations, $\mathbb{S}_{n}$ forms a group, called the symmetric group of order $n$.

Definition II.3. The Ulam distance $\mathrm{d}_{U}(\sigma, \rho)$ is the smallest integer $m$ such that there exists a sequence of (right and left) translocations $\tau_{1}, \tau_{2}, \ldots, \tau_{m}$, such that $\rho=\sigma \tau_{1} \tau_{2} \cdots \tau_{m}$.

Definition II.4. A subsequence of length $m$ of $\sigma=$ $[\sigma(1), \ldots, \sigma(n)]$ is a sequence of the form $\left[\sigma\left(i_{1}\right), \ldots, \sigma\left(i_{m}\right)\right]$, where $i_{1}<i_{2}<\ldots i_{m}$. Let $\tau, \sigma \in \mathbb{S}_{n}$. The longest common subsequence of $\tau$ and $\sigma$ is a subsequence of both $\tau$ and $\sigma$ of the longest possible length.

We denote the length of a longest common subsequence of $\tau$ and $\sigma$ by $\mathrm{L}(\tau, \sigma)$. Similarly, $\mathrm{L}(\sigma) \triangleq \mathrm{L}(\sigma, e)$, the length of
a longest increasing subsequence of $\tau$. It is well known (9] that for any $\sigma \in \mathbb{S}_{n}$,

$$
\begin{equation*}
\mathrm{d}_{U}(\tau, \sigma)=n-\mathrm{L}(\tau, \sigma) \tag{1}
\end{equation*}
$$

## III. Integer-programming bound for the Ulam METRIC

## A. Known bounds

Denote by $\mathcal{A}(n, d)$ the maximum size of a code over $\mathbb{S}_{n}$ equipped with the Ulam metric. The following theorem provides bounds on $\mathcal{A}(n, d)$ [9].

Proposition III.1. For all $n, d \in \mathbb{Z}_{0}^{+}$with $n \geq d \geq 1$,

$$
\begin{equation*}
\frac{(n-d+1)!}{\binom{n}{d-1}} \leq \mathcal{A}(n, d) \leq(n-d+1)! \tag{2}
\end{equation*}
$$

The right-hand side of (2) will be referred to as the Singleton bound in the sequel.

## B. Integer-programming bound

In this section, we derive an integer-programming upper bound on $\mathcal{A}(n, d)$.

Let $\mathcal{C} \subseteq \mathbb{S}_{n}$ be a permutation code of Ulam distance $d$. It follows from (1), that any subsequence of length $n-d+1$ appears at most once in any codeword of $\mathcal{C}$ (in other words, any two codewords in $\mathcal{C}$ cannot have the same subsequence of length $n-d+1$ or more). We use this fact in order to define integer variables $X_{b, a}$ for all $1 \leq a \leq n, 1 \leq b \leq n$. More specifically,

$$
X_{b, a}=|\{\sigma \in \mathcal{C}: \sigma(b)=a\}|
$$

In other words, $X_{b, a}$ counts a number of codewords with $a$ in position $b$.

Assume that $\sigma \in \mathcal{C}$, such that $\sigma(b)=a$. Then, the number of different subsequences of $\sigma$ of length $n-d+1$ of the form $(\underbrace{\bullet, \cdots, \bullet}_{\ell}, a, \underbrace{\bullet, \cdots, \bullet})$, where $\sigma(b)=a$ and $1 \leq \ell \leq n$, is given by

$$
\binom{b-1}{\ell} \cdot\binom{n-b}{n-d-\ell}
$$

On the other hand, there are $\frac{(n-1)!}{(d-1)!}$ different sequences of the form $(\underbrace{\bullet, \cdots, \bullet}_{\ell}, a, \underbrace{\bullet, \cdots, \bullet}_{n-d-\ell})$.

By a simple counting argument, we obtain that for all $a \in$ $[n]$,

$$
\begin{equation*}
\sum_{b=1}^{n}\binom{b-1}{\ell} \cdot\binom{n-b}{n-d-\ell} \cdot X_{b, a} \leq \frac{(n-1)!}{(d-1)!} \tag{3}
\end{equation*}
$$

The total number of the codewords can be obtained, for example, by $\sum_{a=1}^{n} X_{b, a}$, for any $b \in[n]$. Therefore, we add constraints

$$
\forall b \in[n-1]: \sum_{a=1}^{n} X_{b, a}=\sum_{a=1}^{n} X_{b+1, a}
$$

and an objective function

$$
\max \sum_{a=1}^{n} X_{1, a}
$$

By combining this, we obtain the following linear program in Figure 1 where its maximum provides an upper bound on $\mathcal{A}(n, d)$.

$$
\begin{array}{cl}
\max & \sum_{a=1}^{n} X_{1, a} \\
\text { s.t. } & \forall a \in[n], \forall \ell \in[n-d+1]: \\
\sum_{b=1}^{n}\binom{b-1}{\ell} \cdot\binom{n-b}{n-d-\ell} \cdot X_{b, a} \leq \frac{(n-1)!}{(d-1)!} \\
& \forall b \in[n-1]: \quad \sum_{a=1}^{n} X_{b, a}=\sum_{a=1}^{n} X_{b+1, a} \\
& \forall a, b \in[n]: \quad X_{b, a} \geq 0
\end{array}
$$

Fig. 1. General integer-programming bound.
Next, observe that $X_{b, a}$ should be an integer. Therefore, we are interested in an integral solution to this linear-programming problem. This provides a tighter upper bound than the fractional solution to the same LP problem.

Example III.1. Take $n=5$ and $d=3$. The corresponding integer linear-programming problem is shown in Figure 2

$$
\begin{aligned}
& \max \sum_{a=1}^{5} X_{1, a} \\
& \text { s.t. } \forall a \in[5]: \\
& 1 \cdot\binom{4}{2} \cdot X_{1, a}+1 \cdot\binom{3}{2} \cdot X_{2, a}+1 \cdot\binom{2}{2} \cdot X_{3, a} \leq 12 \\
& 1 \cdot 3 \cdot X_{2, a}+2 \cdot 2 \cdot X_{3, a}+3 \cdot 1 \cdot X_{4, a} \leq 12 \\
&\binom{2}{2} \cdot 1 \cdot X_{3, a}+\binom{3}{2} \cdot 1 \cdot X_{4, a}+\binom{4}{2} \cdot 1 \cdot X_{5, a} \leq 12 \\
& \forall b \in[4]: \quad \sum_{a=1}^{5} X_{b, a}=\sum_{a=1}^{5} X_{b+1, a} \\
& \forall a, b \in[5]: \quad X_{b, a} \geq 0
\end{aligned}
$$

Fig. 2. Integer program for $n=5$ and $d=3$.
After simplification, this integer-programming problem becomes as in Figure 3

$$
\begin{array}{cll}
\max & \sum_{a=1}^{5} X_{1, a} & \\
\text { s.t. } & \forall a \in[n]: & 6 X_{1, a}+3 X_{2, a}+X_{3, a} \leq 12 \\
& & 3 X_{2, a}+4 X_{3, a}+3 X_{4, a} \leq 12 \\
& X_{3, a}+3 X_{4, a}+6 X_{5, a} \leq 12 \\
& \forall b \in[4]: & \sum_{a=1}^{5} X_{b, a}=\sum_{a=1}^{5} X_{b+1, a} \\
& \forall a, b \in[5]: & X_{b, a} \geq 0
\end{array}
$$

Fig. 3. Simplified integer program.
By solving the integer-programming problem in Figure 3, we obtain that the maximum of the objective is obtained, for
example, for $X_{b, a}=1$ for all $a, b \in[n]$. This corresponds to the upper bound $\mathcal{A}(n, d) \leq 5$, which improves on the value 6 obtained by using the Singleton bound. The actual value of $\mathcal{A}(n, d)$ in this case is 4 .

We remark, that the proposed integer LP problem can be further tightened by using additional constraints. For example, one can define additional variables $X_{\left(b_{1}, a_{1}\right),\left(b_{2}, a_{2}\right), \cdots,\left(b_{t}, a_{t}\right)}$, where all $a_{i}, b_{i}, t \in[n]$. Such a variable will count the number of permutations $\sigma$, such that $\sigma\left(b_{i}\right)=a_{i}$ for all $i \in[t]$. Additional constraints can be defined in a manner similar to (3), with respect to variables $X_{\left(b_{1}, a_{1}\right),\left(b_{2}, a_{2}\right), \cdots,\left(b_{t}, a_{t}\right)}$.

## IV. Probabilistic bounds

## A. Asymptotic version of the lower bound

In what follows, we consider an $(n, d)$ Ulam code. Denote $\Delta \triangleq d-1$. Recall the bounds in Proposition III.1. By using

$$
\begin{aligned}
& m!\geq\left(\frac{m}{e}\right)^{m}=\exp [m(\ln m-1)] \\
& \text { and } \quad \frac{1}{m+1} \exp \left[m h_{e}(\alpha)\right] \leq\binom{ m}{\alpha m} \leq \exp \left[m h_{e}(\alpha)\right]
\end{aligned}
$$

we obtain
$\frac{(n-\Delta)!}{\binom{n}{\Delta}} \geq \exp \left[(n-\Delta)(\ln (n-\Delta)-1)-n h_{e}\left(\frac{\Delta}{n}\right)\right]$.
Here $h_{e}(p)$, where $p \in[0,1]$, is the binary entropy function with base $e$, i.e. $h_{e}(p) \triangleq-p \ln (p)-(1-p) \ln (1-p)$. Hence, (4) is an asymptotic lower bound on $\mathcal{A}(n, d)$. Consider a special case of it, when $\Delta=n-c \sqrt{n}, c$ is a constant. Then $1-\frac{\Delta}{n}=$ $c / \sqrt{n}$, and (4) becomes

$$
\begin{align*}
& \exp \left[\sqrt{n} c\left(\frac{1}{2} \ln n+\ln c-1\right)-n h_{e}\left(1-\frac{c}{\sqrt{n}}\right)\right] \\
&= \exp [\sqrt{n} c(2 \ln c-1) \\
&\left.+n\left(1-\frac{c}{\sqrt{n}}\right) \ln \left(1-\frac{c}{\sqrt{n}}\right)\right] \\
& \geq \exp [\sqrt{n} c(2 \ln c-1)-c \sqrt{n}] \\
&= \exp [2 \sqrt{n} c \cdot(\ln c-1)] . \tag{5}
\end{align*}
$$

Hence, with $\Delta_{n}=n-c \sqrt{n}$, we have

$$
\begin{equation*}
\lim \inf _{n} \frac{1}{\sqrt{n}} \ln \left(\frac{(c \sqrt{n})!}{\binom{n}{c \sqrt{n}}}\right) \geq 2 c \cdot(\ln c-1) \tag{6}
\end{equation*}
$$

Let us now show that $2 c \cdot(\ln c-1)$ is actually the limit. Indeed, it holds $m!=(1+o(1)) \sqrt{2 \pi m}\left(\frac{m}{e}\right)^{m}$ (for large values of $m$ ). Then, provided that $(n-\Delta) \rightarrow \infty$,

$$
\begin{aligned}
\ln \left(\frac{(n-\Delta)!}{\binom{n}{\Delta}}\right) \leq(n-\Delta) & (\ln (n-\Delta)-1) \\
+\frac{1}{2} \ln (2 \pi(n-\Delta)) & +\ln (1+o(1)) \\
& -n h_{e}\left(\frac{\Delta}{n}\right)+\ln (n+1)
\end{aligned}
$$

Hence, since $\Delta=n-c \sqrt{n}$,

$$
\begin{aligned}
& \ln \left(\frac{(c \sqrt{n})!}{\binom{n}{c \sqrt{n})} \leq} \begin{array}{l}
c \sqrt{n}\left(2 \ln c-1+\frac{\sqrt{n}}{c}\left(1-\frac{c}{\sqrt{n}}\right) \ln \left(1-\frac{c}{\sqrt{n}}\right)\right) \\
\quad+\frac{1}{2} \ln (2 \pi c \sqrt{n})+\ln (1+o(1))+\ln (n+1)
\end{array} .\right.
\end{aligned}
$$

Since

$$
\frac{\sqrt{n}}{c} \ln \left(1-\frac{c}{\sqrt{n}}\right) \rightarrow-1
$$

we obtain that

$$
\begin{equation*}
\limsup _{n} \frac{1}{\sqrt{n}} \ln \left(\frac{(c \sqrt{n})!}{\binom{n}{c \sqrt{n}}}\right) \leq 2 c(\ln c-1) \tag{7}
\end{equation*}
$$

By combining (6) with (7), we have the following result.
Proposition IV.1. Let $\Delta=n-c \sqrt{n}$, where $c$ is a constant. Then,

$$
\begin{equation*}
\lim _{n} \frac{1}{\sqrt{n}} \ln \left(\frac{\left(n-\Delta_{n}\right)!}{\binom{n}{\Delta_{n}}}\right)=2 c \cdot(\ln c-1) \tag{8}
\end{equation*}
$$

## B. Bounds using longest increasing subsequence

The following bounds hold for the Ulam metric (for the Kendall $\tau$ metric see, e.g., [2]):

$$
\begin{equation*}
\frac{n!}{|\mathcal{B}(\Delta)|} \leq \mathcal{A}(n, d) \leq \frac{n!}{\left|\mathcal{B}\left(\frac{\Delta}{2}\right)\right|} \tag{9}
\end{equation*}
$$

where $|\mathcal{B}(r)| \triangleq\left|\left\{\sigma: \mathrm{d}_{U}(e, \sigma) \leq r\right\}\right|$ is the number of permutations in the ball centered at the identity $e$ and having radius $r$. The number of permutations in a ball $\mathcal{B}(r)$ is difficult to estimate. However, under the uniform distribution over all permutations in $\mathbb{S}_{n}$, the ratio $|\mathcal{B}(r)| / n$ ! is just the probability that a randomly chosen permutation is at distance at most $r$ from $e$. In terms of the longest increasing subsequences, thus,

$$
\begin{equation*}
\frac{|\mathcal{B}(r)|}{n!}=P\left(n-\mathrm{L}_{n} \leq r\right)=P\left(\mathrm{~L}_{n} \geq n-r\right) \tag{10}
\end{equation*}
$$

where $L_{n}$ is the length of a longest increasing subsequence of a random permutation under the uniform distribution. In terms of $L_{n}$, the inequalities (9) can be rewritten as

$$
\frac{1}{P\left(\mathrm{~L}_{n} \geq n-\Delta\right)} \leq \mathcal{A}(n, \Delta+1) \leq \frac{1}{P\left(\mathrm{~L}_{n} \geq n-\Delta / 2\right)}
$$

By combining this with (2), when $\Delta$ is even, we obtain the following probability estimates

$$
\begin{align*}
& P\left(\mathrm{~L}_{n} \geq n-\Delta / 2\right) \leq \frac{\binom{n}{\Delta}}{(n-\Delta)!} \\
& \text { and } \quad P\left(\mathrm{~L}_{n} \geq n-\Delta\right) \geq \frac{1}{(n-\Delta)!} \tag{11}
\end{align*}
$$

The study of the properties of the random variable $L_{n}$ has a long history, starting with the pioneering paper of Ulam [22], where the question of asymptotic behavior of $\mathrm{E}\left[\mathrm{L}_{n}\right]$ was stated. This so-called Ulam's problem deserved attention of many
researchers over several decades. In a sense, the problem was solved by in the celebrated paper [1], where the limit law of (properly centered and scaled) $L_{n}$ was found. In particular, they showed that for every $t \in \mathbb{R}$ (as $n$ increases),

$$
\begin{equation*}
P\left(\frac{\mathrm{~L}_{n}-2 \sqrt{n}}{n^{\frac{1}{6}}} \leq t\right) \rightarrow F(t) \tag{12}
\end{equation*}
$$

where $F(t)$ is the distribution function of the Tracy-Widom law. For a historical overview of Ulam's problem, the proof of (12), as well as the state of the art, we refer the reader to the book [19]. Since the random variable $L_{n}$ has been studied for a relatively long time, one hopes that a proper upper estimate on the probability $P\left(\mathrm{~L}_{n} \geq n-\Delta\right)$ (or, alternatively, a lower estimate on the probability $\left.P\left(\mathrm{~L}_{n} \geq n-\Delta / 2\right)\right)$ gives also a good lower (upper) bound on $\mathcal{A}(n, d)$.

In what follows, we aim at bounding the probability $P\left(\mathrm{~L}_{n} \geq\right.$ $n-\Delta)$ from above. The following simple estimate can be found in [19, page 9]:

$$
\begin{equation*}
P\left(\mathrm{~L}_{n} \geq n-\Delta\right) \leq \frac{\binom{n}{\Delta}}{(n-\Delta)!} \tag{13}
\end{equation*}
$$

That estimate gives another proof of the lower bound (2). In order to improve it, the probability estimate has to be superior to (13). When $n-2 \geq \Delta \geq 1$, then the inequality in (13) is strict, and that follows from the use of the Markov inequality in the proof. Hence, the lower bound in (9) is always tighter than the bound (2). We have the following result.

Proposition IV.2. The inequality

$$
\frac{(n-\Delta)!}{\binom{n}{\Delta}} \leq \frac{n!}{|\mathcal{B}(\Delta)|}
$$

holds and for $0<\Delta<n-1$, the inequality is strict.
Proof: Apply (13) and (10).
Bounds for $d=n-c \sqrt{n}$ : One of the first probability estimates on $P\left(\mathrm{~L}_{n}>n-d\right)$ was established by Kim [14]. Thus, for any $t \in\left(0, n^{\frac{1}{3}} / 20\right]$, it holds that

$$
\begin{equation*}
P\left(\mathrm{~L}_{n}-2 \sqrt{n} \geq t n^{\frac{1}{6}}\right) \leq \exp \left[-\frac{4}{3} t^{\frac{3}{2}}+\phi(t)\right] \tag{14}
\end{equation*}
$$

where

$$
\phi(t)=\left(\frac{t}{27 n^{\frac{1}{3}}}+\frac{5 \ln n}{t^{\frac{1}{2}} n^{\frac{1}{3}}}\right) t^{\frac{3}{2}}
$$

That estimate leads to the lower bound on $\mathcal{A}(d, n)$ for $n-$ $c \sqrt{n}$, where $c \in(2,2+1 / 20]$, which is approximately

$$
\exp \left[(c-2)^{\frac{3}{2}}\left(\frac{38-c}{27}\right) \sqrt{n}\right]
$$

This is the same order as the bound $\exp [2 \sqrt{n} c(\ln c-1)]$, but the constant in the expression is smaller. This bound holds only for $c$ very close to 2 and above 2 .

The best code rate estimate for large $d$ is given by the following large deviation principle [1]. For every $c>2$,

$$
\begin{equation*}
\lim _{n} \frac{1}{\sqrt{n}} \ln P\left(\mathrm{~L}_{n}>c \sqrt{n}\right)=-I(c) \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
-I(c) & =-2 c \cosh ^{-1}\left(\frac{c}{2}\right)+2 \sqrt{c^{2}-4} \\
& =-2 c \ln \left(\frac{c}{2}+\sqrt{\frac{c^{2}}{4}-1}\right)+2 \sqrt{c^{2}-4} \tag{16}
\end{align*}
$$

In terms of the lower bound, 15) can be stated as follows.
Proposition IV.3. For every constant $c>2$, the following convergence holds:

$$
\begin{aligned}
\lim _{n} \frac{1}{\sqrt{n}} \ln \left(\frac{n!}{\left|\mathcal{B}\left(\Delta_{n}\right)\right|}\right) & =I(c)>2 c(\ln c-1) \\
& =\lim _{n} \frac{1}{\sqrt{n}} \ln \left(\frac{\left(n-\Delta_{n}\right)!}{\binom{n}{\Delta_{n}}}\right)
\end{aligned}
$$

where $\Delta_{n}=n-c \sqrt{n}-1$ and $-I(c)$ is given in (16).
Proof: Use (10) together with (15) and (8). Note that (8) is formally proven for $\Delta_{n}=n-c \sqrt{n}$, but it also holds for $\Delta_{n}=n-c \sqrt{n}-1$.

This proposition yields an asymptotic improvement on the lower bound in (2).

We note that any probability estimate that is better than the very simple estimate in (13), gives a better lower bound on $\mathcal{A}(n, d)$ in comparison with the existing lower bound (2). Except for large $d$, there are no better estimates known. On the other hand, any good upper bound on $|\mathcal{B}(n-\Delta)|$ entails also a good estimate on the probability $P\left(\mathrm{~L}_{n} \geq \Delta\right)$. Since the probabilities $P\left(\mathrm{~L}_{n} \geq \Delta\right)$ are closely related to the Tracy-Widom distribution, such a link between coding and probability theory might be valuable.

## V. Computational results

In this section, we present computational results related to the optimal codes. It turns out that there exist non-trivial Ulam-metric codes, which attain the Singleton bound with equality. We call such codes Singleton-optimal. Singletonoptimal Ulam $(n, d)$ codes are also known as perfect deletioncorrecting codes on $n$ distinct symbols, capable of correcting $d-1$ deletions and as directed Steiner systems [15]. They exist for every $n$ and $d=2$ [15], and also for $n=6$ and $d=3$ [17]. It was also found by the exhaustive search in [17] that $\mathcal{A}(7,4)=12$. We have complemented these results for other pairs $(n, d)$.

Table $\square$ summarizes what is known about $\mathcal{A}(n, d)$. These results are obtained by using computer search, and they improve on the theoretical bounds in many cases. Table II summarizes the experimental results on the existence of the Singleton-optimal $(n, d)$ Ulam codes.

In order to obtain these results, we construct the graph on the vertex set $\mathbb{S}_{n}$ with an edge if and only if the corresponding vertices are at least a distance $d$ away. Our goal is to find a clique of the maximum size.

We assign colors to the vertices of this graph, such that color class of a permutation corresponds to the relative ordering of symbols $1,2, \ldots, n-d+1$ in the one-line notation of

|  | $d=2$ | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n=4$ | 6 | 2 | - | - | - | - |
| $n=5$ | 24 | 4 | 2 | - | - | - |
| $n=6$ | 120 | 24 | 4 | 2 | - | - |
| $n=7$ | 720 | $\geq 59$ and $<120$ | 12 | 4 | 2 | - |
| $n=8$ | 5040 | $?$ | $<120$ | $\leq 12$ | 4 | 2 |
| $n=9$ | 40320 | $?$ | $?$ | $<120$ | $\leq 12$ | 2 |

TABLE I
Known maximum sizes of codes in the Ulam metric.
the permutation. The existence of a Singleton-optimal code becomes equivalent to the property that the clique number of the graph is equal to its chromatic number. Thus, in order to obtain a Singleton-optimal code, we need to pick exactly one vertex from each color class, such that the induced graph forms a clique.

To obtain a maximum-size code when a Singleton-optimal code does not exist, we need to pick at most one vertex from each color class. This makes the respective exhaustive search computationally much harder.

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|  | $d=2$ | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n=4$ | yes | yes | - | - | - | - |
| $n=5$ | yes | no | yes | - | - | - |
| $n=6$ | yes | yes | no | yes | - | - |
| $n=7$ | yes | no | no | no | yes | - |
| $n=8$ | yes | $?$ | no | no | no | yes |
| $n=9$ | yes | $?$ | $?$ | no | no | no |

TABLE II
The existence of Singleton-optimal codes in the Ulam metric.
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