## Almost Cover-Free Codes and Designs

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**Abstract.** An s-subset of codewords of a binary code X is said to be  $(s,\ell)$ -bad in X if the code X contains a subset of other  $\ell$  codewords such that the conjunction of the  $\ell$  codewords is covered by the disjunctive sum of the s codewords. Otherwise, the s-subset of codewords of X is said to be  $(s,\ell)$ -good in X. A binary code X is said to be a cover-free (CF)  $(s,\ell)$ -code if the code X does not contain  $(s,\ell)$ -bad subsets. In this paper, we introduce a natural probabilistic generalization of CF  $(s,\ell)$ -codes, namely: a binary code is said to be an almost CF  $(s,\ell)$ -code if almost all s-subsets of its codewords are  $(s,\ell)$ -good. We discuss the concept of almost cover-free  $(s,\ell)$ -codes arising in combinatorial group testing problems connected with the nonadaptive search of defective supersets (complexes). We develop a random coding method based on the ensemble of binary constant weight codes to obtain lower bounds on the capacity of such codes. From the main result it is seen that the lower asymptotic bound on the rate for almost CF  $(s,\ell)$ -codes is essentially greater than the rate for ordinary CF  $(s,\ell)$ -codes

*Index terms*. Nonadaptive group testing, search model of defective supersets, cover-free codes and designs, almost cover-free codes, capacity, error probability exponent, random coding bounds.

### 1 Statement of Problem and Results

### 1.1 Notations and Definitions

In what follows, the symbol  $\triangleq$  denotes definitional equalities. For any positive integer n put  $[n] \triangleq \{1, 2, \ldots, n\}$ . Let N and t be positive integers, |A| – the size of set A. The standard symbol  $\lfloor a \rfloor$  ( $\lceil a \rceil$ ) will be used to denote the largest (least) integer  $\leq a$  ( $\geq a$ ). Introduce a binary  $N \times t$  matrix  $X = \|x_i(j)\|$  having N rows  $\boldsymbol{x}_i \triangleq (x_i(1), x_i(2), \ldots, x_i(t))$ ,  $i \in [N]$ , and t columns  $\boldsymbol{x}(j) \triangleq (x_1(j), x_2(j), \ldots, x_N(j))$ ,  $j \in [t]$ . Any such matrix X is called a binary code of length N and size  $t = \lfloor 2^{RN} \rfloor$  (briefly, (N, R)-code), where a fixed parameter R > 0 is called the rate of code X [1]. A column  $\boldsymbol{x}(j) \in \{0,1\}^N$  is called a j-th codeword. The number of 1's in column x(j), i.e.,  $|\boldsymbol{x}(j)| \triangleq \sum_{i=1}^{N} x_i(j)$ , is called the weight of x(j),  $j \in [t]$ . A code X is called a constant weight binary code of weight w, w if for any w is for any w if w is the weight w if w is called w in the weight w is called a constant weight binary code of weight w, w if for any w is for any w if w is the weight w in the weight w is called a constant weight binary code of weight w, w if for any w is for any w if w is the weight w is called w in the set w in the same w is called w in the set w in the set w in the set w in the set w is called w in the set w in the set

For binary vectors  $\mathbf{u} \triangleq (u_1, \dots, u_N) \in \{0, 1\}^N$  and  $\mathbf{v} \triangleq (v_1, \dots, v_N) \in \{0, 1\}^N$ , we introduce the component-wise disjunction (or disjunctive (Boolean) sum)  $\mathbf{u} \bigvee \mathbf{v}$  and conjunction  $\mathbf{u} \bigwedge \mathbf{v}$ :

$$\mathbf{u} \bigvee \mathbf{v} \triangleq (u_1 \vee v_1, \dots, u_N \vee v_N), \quad \mathbf{u} \bigwedge \mathbf{v} \triangleq (u_1 \wedge v_1, \dots, u_N \wedge v_N),$$

where  $0 \lor 0 = 0$ ,  $0 \lor 1 = 1 \lor 0 = 1 \lor 1 = 1$ ,  $0 \land 0 = 0 \land 1 = 1 \land 0 = 0$ , and  $1 \land 1 = 1$ . We say that **u** is *covered* by **v** (**v**  $\succeq$  **u**) if **u**  $\bigvee$  **v** = **v**.

#### 1.2 Almost Cover-Free Codes

Let s and  $\ell$  be positive integers such that  $s + \ell \leq t$  and  $\mathcal{P}_s(t) \triangleq \{\mathcal{S} : \mathcal{S} \subset [t], |\mathcal{S}| = s\}$  is the collection of all s-subsets of the set [t]. Note that the size of the collection  $\mathcal{P}_s(t)$  is  $|\mathcal{P}_s(t)| = {t \choose s}$ .

**Definition 1.** Let  $X = (\boldsymbol{x}(1), \boldsymbol{x}(2), \dots, \boldsymbol{x}(t))$  be an arbitrary binary code of length N and size t. A set  $S \in \mathcal{P}_s(t)$  is said to be  $(s, \ell)$ -bad for the code X if there exists a set  $\mathcal{L}$ ,  $\mathcal{L} \subset [t] \setminus S$  of size  $|\mathcal{L}| = \ell$  such that

$$\bigvee_{j \in \mathcal{S}} \boldsymbol{x}(j) \succeq \bigwedge_{j \in \mathcal{L}} \boldsymbol{x}(j). \tag{1}$$

Otherwise, the set  $S \in \mathcal{P}_s(t)$  is called an  $(s,\ell)$ -good set for the code X. In other words, a set S,  $S \in \mathcal{P}_s(t)$ , is  $(s,\ell)$ -good for the code X if for any set  $\mathcal{L}$ ,  $\mathcal{L} \subset [t] \setminus S$  of size  $|\mathcal{L}| = \ell$ , the conjunction  $\bigwedge_{j \in \mathcal{L}} \boldsymbol{x}(j)$  is not covered by the disjunction  $\bigvee_{k \in S} \boldsymbol{x}(k)$ .

Let the symbol  $\mathbf{B}(s,\ell,X)$  ( $\mathbf{G}(s,\ell,X)$ ) denote the collection of all  $(s,\ell)$ -bad ( $(s,\ell)$ -good) sets  $\mathcal{S}, \mathcal{S} \in \mathcal{P}_s(t)$  for the code X and  $|\mathbf{B}(s,\ell,X)|$  ( $|\mathbf{G}(s,\ell,X)|$ ) is the size of the corresponding collection. Obviously,

$$0 \le |\mathbf{B}(s,\ell,X)| \le {t \choose s}, \quad 0 \le |\mathbf{G}(s,\ell,X)| \le {t \choose s}, \quad |\mathbf{B}(s,\ell,X)| + |\mathbf{G}(s,\ell,X)| = {t \choose s}.$$

Note an evident statement.

**Proposition 1.** For  $s \ge 2$  and  $\ell \ge 1$ , any  $(s, \ell+1)$ -good  $((s, \ell)$ -bad) set for a code X is  $(s, \ell)$ -good  $((s, \ell+1)$ -bad) set for the code X, i.e., the injections are true:  $\mathbf{B}(s, \ell, X) \subset \mathbf{B}(s, \ell+1, X)$  and  $\mathbf{G}(s, \ell+1, X) \subset \mathbf{G}(s, \ell, X)$ .

**Definition 2.** Let  $\epsilon$ ,  $0 \le \epsilon \le 1$ , be a fixed parameter. A code X is said to be an *almost* cover-free  $(s, \ell)$ -code of error probability  $\epsilon$  or, briefly,  $CF(s, \ell, \epsilon)$ -code if

$$\frac{|\mathbf{B}(s,\ell,X)|}{\binom{t}{s}} \le \epsilon \iff |\mathbf{G}(s,\ell,X)| \ge (1-\epsilon)\binom{t}{s}. \tag{2}$$

**Example 1.** Consider  $5 \times 5$  code X:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$
 (3)

Then  $\mathbf{G}(2,2,X) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}\} \text{ and } X \text{ is an CF } (2,2,\frac{1}{2})\text{-code.} \}$ 

From Definition 2 and Proposition 1, it follows

**Proposition 2.** Any CF  $(s, \ell + 1, \epsilon)$ -code is an CF  $(s, \ell, \epsilon)$ -code.

Actually, we have the similar property of monotonicity for CF  $(s, \ell, \epsilon)$ -codes for the case when the parameter  $\ell$  is fixed.

**Proposition 3.** Let  $s \geq 2$  and  $\ell \geq 1$ . If X is an arbitrary  $CF(s,\ell,\epsilon)$ -code of size t and length N, then there exists an  $CF(s-1,\ell,\epsilon)$ -code X' of size t-1 and length N.

**Proof of Proposition 3.** Let  $\mathbf{B}(s,\ell,X,i) \triangleq \{S: i \in S \in \mathbf{B}(s,\ell,X)\}$  denote the collection of all  $(s,\ell)$ -bad sets S for the code X, containing the element  $i \in [t]$ . Note that the cardinalities  $|\mathbf{B}(s,\ell,X,i)|$ ,  $0 \leq |\mathbf{B}(s,\ell,X,i)| \leq {t-1 \choose s-1}$ ,  $i \in [t]$ , satisfy the equality:

$$\sum_{i=1}^{t} |\mathbf{B}(s,\ell,X,i)| = s \cdot |\mathbf{B}(s,\ell,X)| \le s {t \choose s} \epsilon,$$

where the last inequality follows from (2). This means that there exists  $j \in [t]$ , such that

$$|\mathbf{B}(s,\ell,X,j)| \le \frac{1}{t} s {t \choose s} \epsilon = {t-1 \choose s-1} \epsilon.$$

Then one can check that the code X' obtained from X by deleting the column x(j) is an CF  $(s-1,\ell,\epsilon)$ -code of size t-1 and length N.

For the particular case  $\epsilon = 0$ , the concept of CF  $(s, \ell, \epsilon)$ -code can be considered as a natural probabilistic generalization of the combinatorial concept of cover-free  $(s, \ell)$ -code that is defined in [3]-[4] as the incidence matrix of a family of finite sets in which no intersection of  $\ell$  sets is covered by the union of s others. For the case  $\ell = 1$ , cover-free codes and their applications were introduced in [6]. For  $\ell \geq 2$ , cover-free  $(s, \ell)$ -codes along with their applications to key distribution patterns were firstly suggested in [5].

Let  $t(N, s, \ell)$  be the maximal size of cover-free  $(s, \ell)$ -codes of length N and let  $N(t, s, \ell)$  be the minimal length of cover-free  $(s, \ell)$ -codes of size t. Then the number

$$R(s,\ell) \triangleq \overline{\lim}_{N \to \infty} \frac{\log_2 t(N,s,\ell)}{N} = \overline{\lim}_{t \to \infty} \frac{\log_2 t}{N(t,s,\ell)}$$
(4)

is called [4] the rate of cover-free  $(s, \ell)$ -codes. In the recent papers [8, 9], one can find a detailed survey of the best known lower and upper bounds on the rate  $R(s, \ell)$ .

Using the conventional information-theoretic terminology accepted in the probabilistic coding theory [1]-[2], introduce

**Definition 3.** Let R, R > 0, be a fixed parameter. Taking into account inequality (2) define the *error* for CF  $(s, \ell, \epsilon)$ -codes:

$$\epsilon(s, \ell, R, N) \triangleq \min_{X: t = \lceil 2^{RN} \rceil} \left\{ \frac{|\mathbf{B}(s, \ell, X)|}{\binom{t}{s}} \right\}, \quad R > 0,$$
 (5)

where the minimum is taken over all (N, R)-codes X. The function

$$\mathbf{E}(s,\ell,R) \triangleq \overline{\lim}_{N \to \infty} \frac{-\log_2 \epsilon(s,\ell,R,N)}{N}, \quad R > 0, \tag{6}$$

is said to be the error exponent for CF  $(s, \ell, \epsilon)$ -codes, the number

$$C(s,\ell) \triangleq \sup\{R : \mathbf{E}(s,\ell,R) > 0\}$$
(7)

is said to be the *capacity* for almost CF  $(s, \ell)$ -codes and rate  $R(s, \ell)$  defined by (4) is said to be *zero-error* capacity for almost CF  $(s, \ell)$ -codes.

For the particular case  $\ell = 1$ , Definitions 1-3 were suggested in our paper [11], in which we introduce the concept of almost disjunctive list-decoding codes. The best presently known

constructions of such codes were proposed in [7]. Bounds on the rate for these constructions were computed in the recent paper [13].

Definitions 1-3 and Proposition 1-3 lead to

**Theorem 1.** (Monotonicity properties.) The following inequalities hold true

$$R(s+1,\ell) \le R(s,\ell) \le R(s,\ell-1), \quad C(s+1,\ell) \le C(s,\ell) \le C(s,\ell-1),$$
  
 $\mathbf{E}(s+1,\ell,R) \le \mathbf{E}(s,\ell,R) \le \mathbf{E}(s,\ell-1,R) \quad s \ge 1, \quad \ell \ge 2, \quad R > 0.$  (8)

### 1.3 Almost Cover-Free Designs

By  $\hat{\mathcal{P}}_s(\ell, t)$  denote the collection of supersets  $p, p \triangleq (P_1, P_2, \dots, P_s), P_i \subset \mathcal{P}_\ell(t), i \in [s]$ , where each p consists of s disjoint sets  $P \subset [t]$  of size  $|P| = \ell$ , i.e.:

$$\hat{\mathcal{P}}_s(\ell,t) \triangleq \left\{ \mathsf{p} = (P_1, P_2, \dots, P_s), \begin{array}{c} P_i \subset [t], \ |P_i| = \ell, \\ P_i \cap P_j = \varnothing \text{ for } i \neq j, i, j \in [s], \end{array} \right\}.$$
 (9)

Obviously, the collection  $\hat{\mathcal{P}}_s(\ell,t)$  has the cardinality

$$|\hat{\mathcal{P}}_s(\ell, t)| = \frac{1}{s!} {t \choose s\ell} {s\ell \choose (s-1)\ell} \cdots {2\ell \choose \ell}.$$
(10)

For a superset  $p \in \hat{\mathcal{P}}_s(\ell, t)$  and a code X, introduce the binary vector  $\mathbf{r}(p, X) \in \{0, 1\}^N$  as follows:

$$\mathbf{r}(\mathsf{p},X) \triangleq \bigvee_{P \in \mathsf{p}} \bigwedge_{j \in P} \mathbf{x}(j), \quad \mathbf{r}(\mathsf{p},X) \triangleq (r_1, r_2, \dots, r_N). \tag{11}$$

One can see that the *i*-th component of  $\mathbf{r}(\mathbf{p}, X)$  can be written in the form:

$$r_i = \begin{cases} 1, & \text{if there exists } P \in \mathsf{p} \text{ such that } x_i(j) = 1 \text{ for all } j \in P, \\ 0, & \text{otherwise.} \end{cases}$$
 (12)

**Definition 4.** Let  $X = (\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(t))$  be an arbitrary binary code of length N and size t. A superset  $\mathsf{p}, \mathsf{p} \in \hat{P}_s(\ell, t)$ , is said to be an  $(s, \ell)$ -bad superset for the code X, if there exists another superset  $\mathsf{p}' \in \hat{P}_s(\ell, t), \mathsf{p} \neq \mathsf{p}'$ , such that  $\mathbf{r}(\mathsf{p}, X) = \mathbf{r}(\mathsf{p}', X)$ . Otherwise, the superset  $\mathsf{p}$  is said to be  $(s, \ell)$ -good superset for the code X.

Let the symbol  $\hat{\mathbf{B}}(s,\ell,X)$  ( $\hat{\mathbf{G}}(s,\ell,X)$ ) denote the collection of all  $(s,\ell)$ -bad  $((s,\ell)$ -good) supersets  $\mathbf{p}, \mathbf{p} \in \hat{P}_s(\ell,t)$ , for the code X and  $|\hat{\mathbf{B}}(s,\ell,X)|$  ( $|\hat{\mathbf{G}}(s,\ell,X)|$ ) is the size of the corresponding collection. Obviously,

$$0 \le |\hat{\mathbf{B}}(s,\ell,X)| \le |\hat{\mathcal{P}}_s(\ell,t)|, \ 0 \le |\mathbf{G}(s,\ell,X)| \le |\hat{\mathcal{P}}_s(\ell,t)|, \ |\hat{\mathbf{B}}(s,\ell,X)| + |\hat{\mathbf{G}}(s,\ell,X)| = |\hat{\mathcal{P}}_s(\ell,t)|.$$

**Definition 5.** Let  $\epsilon$ ,  $0 \le \epsilon \le 1$ , be a fixed parameter. A code X is said to be an *almost* cover-free  $(s, \ell)$ -design of error probability  $\epsilon$  or, briefly, CF  $(s, \ell, \epsilon)$ -design if

$$\frac{|\hat{\mathbf{B}}(s,\ell,X)|}{|\hat{\mathcal{P}}_s(\ell,t)|} \le \epsilon \iff |\hat{\mathbf{G}}(s,\ell,X)| \ge (1-\epsilon)|\hat{\mathcal{P}}_s(\ell,t)|. \tag{13}$$

**Example 2.** For the code X described in (3), the collection of (2,2)-bad supersets

$$\hat{\mathbf{B}}(s,\ell,X) = \{(\{1;2\},\{4;5\}), (\{1;3\},\{4;5\}), (\{1;4\},\{2;3\}), (\{1;5\},\{2;3\})\}.$$

It follows that X is an CF  $(2, 2, \frac{4}{15})$ -design.

**Definition 6.** Let R, R > 0, be a fixed parameter. Taking into account inequality (13) define the *error* for CF  $(s, \ell, \epsilon)$ -designs:

$$\hat{\epsilon}(s,\ell,R,N) \triangleq \min_{X:t=\lceil 2^{RN}\rceil} \left\{ \frac{|\hat{\mathbf{B}}(s,\ell,X)|}{|\hat{\mathcal{P}}_s(\ell,t)|} \right\}, \quad R > 0, \tag{14}$$

where the minimum is taken over all (N, R)-codes X. The function

$$\hat{\mathbf{E}}(s,\ell,R) \triangleq \overline{\lim}_{N \to \infty} \frac{-\log_2 \hat{\epsilon}(s,\ell,R,N)}{N}, \quad R > 0, \tag{15}$$

is said to be the error exponent for CF  $(s, \ell, \epsilon)$ -designs, the number

$$\hat{C}(s,\ell) \triangleq \sup\{R : \hat{\mathbf{E}}(s,\ell,R) > 0\}$$

is said to be the *capacity* for almost CF  $(s, \ell)$ -designs.

For the particular case  $\ell=1$ , Definitions 4-6 were already introduced in [12] to describe the model called *planning screening experiments*. In [12], it was proved that the capacity of almost CF (s,1)-designs  $\hat{C}(s,1)=1/s$ . One can see that Definitions 4-6 represent a natural generalization of almost CF (s,1)-designs. We conjecture that for any  $s \geq 2$  and  $\ell \geq 2$ , the capacity  $\hat{C}(s,\ell)=1/(s\,\ell)$ .

In Section 2, we establish

**Theorem 2.** (Upper Bounds on Capacities  $C(s,\ell)$  and  $\hat{C}(s,\ell)$ ) The following inequalities hold true

$$C(s,\ell) \le \hat{C}(s,\ell) \le 1/(s\,\ell), \quad \mathbf{E}(s,\ell,R) \le \hat{\mathbf{E}}(s,\ell,R) \quad s \ge 1, \quad \ell \ge 1, \quad R > 0. \tag{16}$$

However, in spite of the greater capacity, using of CF  $(s, \ell, \epsilon)$ -designs for the superset identification problem  $p \in \hat{P}_s(\ell, t)$  is practically unacceptable, since it requires much greater complexity, which is evidently equal to the complexity of exhaustive search  $|\hat{\mathcal{P}}_s(\ell, t)| \sim t^{s\ell}$ . It will be shown in Section 1.5 that CF  $(s, \ell, \epsilon)$ -codes are efficient CF  $(s, \ell, \epsilon)$ -designs and for such codes the algorithm of identification supersets  $\mathbf{p} \in \hat{\mathcal{P}}_s(\ell, t)$ , is essentially faster than the trivial one, and its complexity is proportional to  $t^{\ell}$ .

# **1.4** Lower Bounds on $R(s, \ell)$ , $C(s, \ell)$

The best presently known upper and lower bounds on the rate  $R(s, \ell)$  of cover-free  $(s, \ell)$ -codes were presented in [8, 9]. If  $\ell \geq 1$  is fixed and  $s \to \infty$ , then these bounds have the following asymptotic form:

$$R(s,\ell) \le \frac{(\ell+1)^{\ell+1}}{2e^{\ell-1}} \frac{\log_2 s}{s^{\ell+1}} (1+o(1)), \tag{17}$$

$$R(s,\ell) \ge \frac{(\ell+1)^{\ell+1}}{e^{\ell+1}} \frac{\log_2 s}{s^{\ell+1}} (1 + o(1)). \tag{18}$$

In the present paper, we suggest a modification of the random coding method developed in [8] and [11], which permits us to obtain a lower bound on the capacity  $C(s, \ell)$ . Let

$$[x]^+ \triangleq \begin{cases} x & \text{if } x \ge 0, \\ 0 & \text{if } x < 0, \end{cases}$$
 and  $h(a) \triangleq -a \log_2 a - (1-a) \log_2 (1-a), \ 0 < a < 1,$ 

denote the positive part function and the binary entropy function. In Section 3, we prove

**Theorem 3.** (Random coding lower bound  $\underline{C}(s,\ell)$ ). The following two claims hold true. Claim 1. For  $\ell \geq 2$  the capacity  $C(s,\ell)$  for almost cover-free codes satisfies inequality

$$C(s,\ell) \ge \underline{C}(s,\ell) \triangleq \frac{1}{\ell} \max_{0 \le Q \le 1} \mathcal{D}(\ell,Q,\hat{q}),$$
 (19)

where the function  $\mathcal{D}(\ell, Q, \hat{q})$  is defined in the parametric form

$$\mathcal{D}(\ell, Q, \hat{q}) \triangleq (1 - Q)\ell \log_2 z - (1 - \hat{q}) \log_2 [1 - (1 - z)^{\ell}] + \tag{20}$$

$$+\ell \left(\frac{(1-Q)}{z}(1-z) - \left(\frac{(1-Q)}{z} - \hat{q}\right)(1-z)^{\ell}\right) \log_2[1-z] + \ell h(Q),$$

and parameters z and  $\hat{q}$  are uniquely determined by the following equations

$$Q = \frac{(1-z)(1-(1-z)^{\ell}) - (1-\hat{q})z(1-z)^{\ell}}{1-(1-z)^{\ell}}, \qquad \hat{q} = 1 - (1-Q)^{s}.$$
 (21)

**Claim 2.** For a fixed parameter  $\ell \geq 2$  and  $s \to \infty$ , the lower asymptotic bound on  $C(s,\ell)$  is:

$$C(s,\ell) \ge \frac{\log_2 e}{s^{\ell}} \cdot \frac{\ell^{\ell-1}}{e^{\ell}} (1 + o(1)). \tag{22}$$

#### 1.5 Boolean Model for Nonadaptive Search of Supersets

Denote by  $\mathcal{P}_s(\ell,t)$  the following collection of supersets p,  $p \triangleq (P_1, P_2, \dots, P_k)$ ,  $P_i \subset [t]$ ,  $i \in [k]$ ,  $k \leq s$ , where each superset p is composed of not more than s subsets  $P \subset [t]$  of size  $|P| \leq \ell$ , i.e.:

$$\mathcal{P}_s(\ell,t) \triangleq \left\{ \mathsf{p} = (P_1, P_2, \dots, P_k) : k \le s, \begin{array}{c} P_i \subset [t], |P_i| \le \ell, \\ P_i \not\subseteq P_j \text{ for } i \ne j, i, j \in [k], \end{array} \right\}.$$
 (23)

For a superset  $p \in \mathcal{P}_s(\ell, t)$  and a code X, the vector  $\mathbf{r}(p, X)$  is defined in the same way as in (11).

**Definition 7.** [4] A binary  $N \times t$  matrix X is called a *cover-free*  $(s, \ell)$ -design or, briefly,  $CF(s, \ell)$ -design if for any  $p', p'' \in \mathcal{P}_s(\ell, t), p' \neq p''$ , the vector  $\mathbf{r}(p', X) \neq \mathbf{r}(p'', X)$ .

Let us first remind the well-known application of CF (s,1)-designs which is called the boolean search model for sets [6]. Suppose a set of t samples is given. We identify it with the set [t]. Assume we know that some of them are positive. The number of positive samples is bounded above by the given integer s. Our aim is to detect the whole set of positive samples which is referred to as positive set  $P \subset [t]$ . We use group tests, i.e., take a subset (group)  $G \subset [t]$  and check whether G contains at least one positive sample (i.e.,  $G \cap P \neq \emptyset$ ) or not.

In the present paper we consider a generalization of this model which is called the boolean search model for supersets [4]. Assume that a positive superset  $p \in \mathcal{P}_s(\ell, t)$  is fixed instead

of positive set. Our aim is to detect it using a number of group tests, where each test checks whether a testing group G contains at least one set  $P \in p$  or not. One can see that for  $\ell = 1$  each set  $P \in p$  is composed of exactly one sample and the model coincides with the boolean search model for sets. Now assume that we use N tests. They can be encoded by a code  $X = ||x_i(j)|$ . A column (codeword) x(j) corresponds to the j-th sample; a row  $x_i$  corresponds to the i-th test. We put  $x_i(j) \triangleq 1$  iff the j-th sample is included into the i-th testing group; otherwise we put  $x_i(j) \triangleq 0$ . Then it is easy to see that the outcomes (12) of all N tests form the binary vector  $\mathbf{r}(p, X)$  (11), where  $p \in \mathcal{P}_s(\ell, t)$  is the (unknown) positive superset. Thus, the code X should be designed in such a way that we should be able to detect a superset p given the vector  $\mathbf{r}(p, X)$ . Obviously, it is possible if and only if X is an CF  $(s, \ell)$ -design (see Definition 7). Note that we deal with the nonadaptive search model in which we are not allowed to use the outcomes of the previous tests to form the future ones. The given boolean search model for supersets (also called the search model for complexes) when all tests are performed simultaneously arises from the needs of molecular biology. It was firstly suggested in [10].

In addition, one can easily understand the necessity of the additional condition in (23): no set  $P \subset [t]$  which is an element of a superset  $p \in \mathcal{P}_s(\ell, t)$ , can be included into another set  $P' \in p$ . Indeed, if this holds, then we can consider another superset  $\hat{p} \in \mathcal{P}_s(\ell, t)$  having the form  $\hat{p} = p \setminus \{P'\}$ . Evidently, for any binary  $N \times t$  matrix X, the outcomes  $\mathbf{r}(p, X)$  and  $\mathbf{r}(\hat{p}, X)$  are identical. Thus, we cannot distinguish these supersets. In [4], we established

**Proposition 4.** [4] 1) Any cover-free  $(s, \ell)$ -code is an cover-free  $(s, \ell)$ -design. 2) Any cover-free  $(s, \ell)$ -design is an cover-free  $(s, \ell)$ -code and an cover-free  $(s, \ell - 1)$ -code.

Let X be an arbitrary binary  $N \times t$  matrix and  $\mathbf{p}^{(\mathrm{un})} \in \mathcal{P}_s(\ell, t)$  be an unknown superset. Any fixed set  $P' \subset [t]$ ,  $|P'| \leq \ell$ , is called acceptable for the known vector  $\mathbf{r}^{(\mathrm{kn})} \triangleq \mathbf{r}(\mathbf{p}^{(\mathrm{un})}, X)$  if the conjunction  $\bigwedge_{j \in P'} \boldsymbol{x}(j)$  is covered by  $\mathbf{r}^{(\mathrm{kn})}$ , i.e.,

$$\mathbf{r}^{(\mathrm{kn})} \, = \, \mathbf{r}(\mathsf{p}^{(\mathrm{un})}, X) \, = \, \bigvee_{P \in \mathsf{p}^{(\mathrm{un})}} \bigwedge_{j \in P} \boldsymbol{x}(j) \, \succeq \, \bigwedge_{j \in P'} \boldsymbol{x}(j).$$

An acceptable set P' is called a *minimal* acceptable set if no subset  $P'' \subseteq P'$  is acceptable. In the boolean search model for supersets, an effective decoding algorithm is based on the following evident

**Proposition 5.** [4] If X is an cover-free  $(s,\ell)$ -code, then any superset  $p \in \mathcal{P}_s(\ell,t)$  is composed of all minimal acceptable sets for the vector  $\mathbf{r}(\mathbf{p},X)$ . This means that one can uniquely decode  $\mathbf{p}^{(\mathrm{un})}$  on the base of known vector  $\mathbf{r}^{(\mathrm{kn})} = \mathbf{r}(\mathbf{p}^{(\mathrm{un})},X)$ , and the decoding complexity is proportional to  $\binom{t}{1} + \cdots + \binom{t}{\ell}$ , which does not depend on s. When  $t \to \infty$  and  $\ell$  is fixed, then this complexity  $\sim t^{\ell}/\ell!$ .

Note that in the general case of cover-free  $(s, \ell)$ -design and the trivial decoding algorithm, we need to check all possible supersets  $p \in \mathcal{P}_s(\ell, t)$ , i.e., calculate the vector  $\mathbf{r} = \mathbf{r}(p, X)$  for all possible supersets p and compare this vector with the known result  $\mathbf{r}^{(kn)}$ . If s and  $\ell$  are fixed and  $t \to \infty$ , then the number of such comparisons (decoding complexity) is proportional to

$$|\mathcal{P}_s(\ell,t)| \ge {t \choose \ell \choose s} \sim \frac{t^{s\ell}}{s!(\ell!)^s}.$$
 (24)

Thus, CF  $(s, \ell)$ -codes form a class of CF  $(s, \ell)$ -designs for which the decoding algorithm based on Proposition 5 is strongly better than the trivial one.

Let  $\ell \geq 1$  be fixed and  $s \to \infty$ . Taking into account (18) we conclude that for sufficiently large t the use of  $CF(s,\ell)$ -codes gives the bounds:

$$\frac{\log_2 s}{s^{\ell+1}} \cdot \frac{(\ell+1)^{\ell+1}}{2e^{\ell-1}} \ (1+o(1)) \ge \log_2 t/N \ \ge \ \frac{\log_2 s}{s^{\ell+1}} \cdot \frac{(\ell+1)^{\ell+1}}{e^{\ell+1}} \ (1+o(1)).$$

In virtue of Theorem 3, the capacity for CF  $(s, \ell, \epsilon)$ -codes  $C(s, \ell)$  can be interpreted as the theoretical tightest upper bound on the information rate  $\log_2 t/N$  with error probability  $\epsilon \to 0$ . Therefore, the bound (22) means that for  $\ell \geq 2$ ,  $s \to \infty$  and sufficiently large t, using of CF  $(s, \ell, \epsilon)$ -codes guarantees the inequality:

$$\log_2 t/N \, \geq \, \frac{\log_2 e}{s^\ell} \cdot \frac{\ell^{\ell-1}}{e^\ell} \, (1+o(1)).$$

# 2 Proof of Theorem 2.

For any superset  $p \in \hat{P}_s(\ell, t)$ ,  $p = \{P_1, P_2, \dots, P_s\}$ , define a set T(p) of its projections

$$T(\mathsf{p}) \triangleq \{ \mathcal{S} \in \mathcal{P}_s(t) : S = \{a_1, a_2, \dots, a_s\}, a_i \in P_i, P_i \in \mathsf{p}, i \in [s] \}.$$

One can see  $|T(p)| = \ell^s$ . Observe that if all sets  $S \in T(p)$  are  $(s, \ell)$ -good for the code X, then the superset p is also a  $(s, \ell)$ -good superset for the code X.

Assume that a code X is an CF  $(s,\ell,\epsilon)$ -code. It means that the number (2) of bad  $(s,\ell)$ -sets doesn't exceed  $\epsilon \cdot \binom{t}{s}$ . Given a bad  $(s,\ell)$ -set  $B \in \mathcal{P}_s(t)$  for the code X, one can check that the number of  $\mathsf{p} \in \hat{P}_s(\ell,t)$  such that  $B \in T(\mathsf{p})$  is at most  $\binom{t-s}{s(\ell-1)}\binom{s(\ell-1)}{(s-1)(\ell-1)}\cdots\binom{2(\ell-1)}{\ell-1}$ . This implies that the number of bad  $(s,\ell)$ -supersets is at most  $\epsilon \cdot \binom{t}{s}\binom{t-s}{s(\ell-1)}\binom{s(\ell-1)}{(s-1)(\ell-1)}\cdots\binom{2(\ell-1)}{\ell-1}$  or  $\epsilon \cdot \ell^s \cdot |\hat{\mathcal{P}}_s(\ell,t)|$ , where  $|\hat{\mathcal{P}}_s(\ell,t)|$  is computed in (10). Therefore, the code X is also an CF  $(s,\ell,\epsilon\cdot\ell^s)$ -design. In other words, we proved the relations  $C(s,\ell) \leq \hat{C}(s,\ell)$  and  $\mathbf{E}(s,\ell,R) \leq \hat{\mathbf{E}}(s,\ell,R)$ .

Now, fix R > 0 and  $\epsilon > 0$  and suppose that the code X is an CF  $(s, \ell, \epsilon)$ -design of length N and size  $t \triangleq \lfloor 2^{RN} \rfloor$ . Observe that for any two various good (see Def. 4) supersets  $\mathsf{p},\,\mathsf{p}' \in \hat{\mathbf{G}}(s,\ell,X),\,\mathsf{p} \neq \mathsf{p}'$ , two binary vectors  $\mathbf{r}(\mathsf{p},X)$  and  $\mathbf{r}(\mathsf{p}',X)$  defined by (11) are distinct, i.e.,  $\mathbf{r}(\mathsf{p},X) \neq \mathbf{r}(\mathsf{p},X)$ . Thus, from the definition (13) of CF  $(s,\ell,\epsilon)$ -design, it follows

$$(1 - \epsilon) \cdot |\hat{\mathcal{P}}_s(\ell, t)| = (1 - \epsilon) \cdot \frac{1}{s!} {t \choose s\ell} {s\ell \choose (s-1)\ell} \cdots {2\ell \choose \ell} \le 2^N, \quad t = \lfloor 2^{RN} \rfloor. \tag{25}$$

Comparing left and right-hand sides of inequality (25) leads to the lower asymptotic bound

$$\hat{\epsilon}(s,\ell,R,N) \ge 1 - 2^N \cdot \left(\frac{1}{s!} {t \choose s\ell} {s\ell \choose (s-1)\ell} \cdots {2\ell \choose \ell} \right)^{-1} = 1 - 2^{-N[(s\ell \cdot R - 1) + o(1)]}, \quad N \to \infty.$$

This inequality means that the condition  $R < 1/(s\ell)$  is necessary for  $\hat{\mathbf{E}}(s,\ell,R) > 0$ . It follows  $\hat{C}(s,\ell) \leq \frac{1}{s\ell}$ .

### 3 Proof of Theorem 3

**Proof of Claim 1.** For an arbitrary code X, the number  $|\mathbf{B}(s,\ell,X)|$  of  $(s,\ell)$ -bad sets in the code X can be represented in the form:

$$|\mathbf{B}(s,\ell,X)| \triangleq \sum_{S \in \mathcal{P}_s(t)} \psi(X,S), \qquad \psi(X,S) \triangleq \begin{cases} 1 & \text{if the set } S \in \mathbf{B}(s,\ell,X), \\ 0 & \text{otherwise.} \end{cases}$$
 (26)

Let Q, 0 < Q < 1, and R, 0 < R < 1, be fixed parameters. Define the ensemble  $\{N, t, Q\}$  of binary  $(N \times t)$ -matrices  $X = (\boldsymbol{x}(1), \boldsymbol{x}(2), \dots \boldsymbol{x}(t))$ , where columns  $\boldsymbol{x}(i)$ ,  $i \in [t]$ ,  $t \triangleq \lfloor 2^{RN} \rfloor$ , are chosen independently and equiprobably from the set consisting of  $\binom{N}{\lfloor QN \rfloor}$  columns of the fixed weight  $\lfloor QN \rfloor$ . Fix two subsets  $\mathcal{S}, \mathcal{L} \subset [t]$  such that  $|\mathcal{S}| = s$ ,  $|\mathcal{L}| = \ell$  and  $\mathcal{S} \cap \mathcal{L} = \emptyset$ . From (26) it follows that for  $\{N, t, Q\}$ , the expectation  $|\mathbf{B}(s, \ell, X)|$  of the number  $|\mathbf{B}(s, \ell, X)|$  is

$$\overline{|\mathbf{B}(s,\ell,X)|} = |\mathcal{P}_s(t)| \operatorname{Pr} \{ \mathcal{S} \in \mathbf{B}(s,\ell,X) \}.$$

Therefore, the expectation of the error probability for almost cover-free  $(s, \ell)$ -codes is

$$\mathcal{E}^{(N)}(s,\ell,R,Q) \triangleq |\mathcal{P}_s(t)|^{-1} |\overline{\mathbf{B}(s,\ell,X)}| = \Pr\{\mathcal{S} \in \mathbf{B}(s,\ell,X)\},$$
 (27)

where the code size  $t = \lfloor 2^{RN} \rfloor$ . The evident random coding upper bound on the error probability (5) for cover-free  $(s, \ell)$ -codes is formulated as the following inequality:

$$\epsilon(s, \ell, R, N) \triangleq \min_{X: t = \lfloor 2^{RN} \rfloor} \left\{ \frac{|\mathbf{B}(s, \ell, X)|}{|\mathcal{P}_s(t)|} \right\} \leq \mathcal{E}^{(N)}(s, \ell, R, Q) \quad \text{for any } 0 < Q < 1.$$
 (28)

The expectation  $\mathcal{E}^{(N)}(s,\ell,R,Q)$  defined by (27) can be represented as follows

$$\mathcal{E}^{(N)}(s,\ell,R,Q) = \sum_{k=\lfloor QN \rfloor}^{\min\{N,s\lfloor QN \rfloor\}} \Pr \left\{ \mathcal{S} \in \mathbf{B}(s,\ell,X) \middle/ \left| \bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i) \right| = k \right\} \cdot \mathcal{P}_2^{(N)}(s,Q,k) \le 1$$

$$\leq \sum_{k=\lfloor QN\rfloor}^{\min\{N,s\lfloor QN\rfloor\}} \mathcal{P}_2^{(N)}(s,Q,k) \cdot \min\left\{1; \binom{t-s}{\ell} \mathcal{P}_1^{(N)}(\ell,Q,k)\right\}, \tag{29}$$

where we apply the total probability formula and the standard union bound for the conditional probability

$$\Pr\left\{\bigcup_{i} C_i / C\right\} \le \min\left\{1; \sum_{i} \Pr\{C_i / C\}\right\},$$

and introduce the notations

$$\mathcal{P}_{1}^{(N)}(\ell, Q, k) \triangleq \Pr \left\{ \bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i) \succeq \bigwedge_{j \in \mathcal{L}} \boldsymbol{x}(j) \middle/ \left| \bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i) \right| = k \right\}$$
(30)

and

$$\mathcal{P}_{2}^{(N)}(s,Q,k) \triangleq \Pr\left\{ \left| \bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i) \right| = k \right\}, \quad \lfloor QN \rfloor \le k \le \min\{N, s \lfloor QN \rfloor\}.$$
 (31)

Let  $k \triangleq |qN|$  and the functions

$$\mathcal{D}(\ell, Q, q) \triangleq \lim_{N \to \infty} \frac{-\log_2 \left[ \mathcal{P}_1^{(N)}(\ell, Q, k) \right]}{N} \tag{32}$$

and

$$\mathcal{A}(s, Q, q) \triangleq \lim_{N \to \infty} \frac{-\log_2 \left[ \mathcal{P}_2^{(N)}(s, Q, k) \right]}{N}$$
 (33)

denote the exponents of the logarithmic asymptotic behavior for the probability of events (30) and (31) for the ensemble  $\{N, t, Q\}$  respectively. Define  $\hat{q} \triangleq 1 - (1 - Q)^s$ .

In Appendix we will prove

**Lemma 1.** The function A(s,Q,q) of the parameter  $q, Q < q < \min\{1,sQ\}$ , defined by (33) can be represented in the parametric form

$$\mathcal{A}(s,Q,q) \triangleq (1-q)\log_2(1-q) + q\log_2\left[\frac{Qy^s}{1-y}\right] + sQ\log_2\frac{1-y}{y} + sh(Q), \tag{34}$$

$$q = Q \frac{1 - y^s}{1 - y}, \qquad 0 < y < 1. \tag{35}$$

In addition, the function  $\mathcal{A}(s,Q,q)$  is  $\cup$ -convex, monotonically decreases in the interval  $(Q,1-(1-Q)^s)$ , monotonically increases in the interval  $(1-(1-Q)^s,\min\{1,sQ\})$  and its unique minimal value which is equal to 0 is attained at  $q=\hat{q}\triangleq 1-(1-Q)^s$ , i.e.,

$$\min_{Q < q < \min\{1, sQ\}} \mathcal{A}(s, Q, q) = \mathcal{A}(s, Q, \hat{q}) = 0, \quad 0 < Q < 1.$$

**Lemma 2.** For  $\ell \geq 2$ , the value of the function  $\mathcal{D}(\ell, Q, q)$  defined by (32) at point  $q = \hat{q}$  is equal to

$$\mathcal{D}(\ell, Q, \hat{q}) = (1 - Q) \ell \log_2 z - (1 - \hat{q}) \log_2 [1 - (1 - z)^{\ell}] + \ell \left( \frac{(1 - Q)}{z} (1 - z) - \left( \frac{(1 - Q)}{z} - \hat{q} \right) (1 - z)^{\ell} \right) \log_2 [1 - z] + \ell h(Q),$$

where z is uniquely determined by the following equation

$$Q = \frac{(1-z)(1-(1-z)^{\ell}) - (1-\hat{q})z(1-z)^{\ell}}{1-(1-z)^{\ell}}.$$

The inequality (29) and the random coding bound (28) imply that the error probability exponent (15) satisfies the inequality

$$\mathbf{E}(s,\ell,R) \ge \underline{\mathbf{E}}(s,\ell,R) \triangleq \max_{0 \le Q \le 1} E(s,\ell,R,Q), \tag{36}$$

$$E(s,\ell,R,Q) \triangleq \min_{Q < q < \min\{1,sQ\}} \left\{ \mathcal{A}(s,Q,q) + \left[ \mathcal{D}(\ell,Q,q) - \ell R \right]^+ \right\}. \tag{37}$$

Lemma 1 states that  $\mathcal{A}(s,Q,q) > 0$  if  $q \neq \hat{q}$ . In particular, the condition  $q \neq \hat{q}$  implies  $E(s,\ell,R,Q) > 0$ . Therefore, if  $\ell R < \mathcal{D}(\ell,Q,\hat{q})$  then  $E(s,\ell,R,Q) > 0$ , what, in turn, means (see (7) and (36)) that

$$C(s,\ell) \ge \underline{C}(s,\ell) \triangleq \frac{1}{\ell} \max_{0 \le Q \le 1} \mathcal{D}(\ell,Q,\hat{q}), \text{ where } \hat{q} = 1 - (1-Q)^s.$$

Thus, the lower bound (19) is established.  $\square$ 

**Proof of Claim 2.** Let  $\ell \geq 2$  be fixed and  $s \to \infty$ . Substituting  $z = s/(s+\ell)$  in (19)-(21) yields

$$Q = \frac{(1-z)(1-(1-z)^\ell) - (1-\hat{q})z(1-z)^\ell}{1-(1-z)^\ell} = \frac{\ell}{s+\ell} - \frac{\ell^\ell e^{-\ell}}{s^\ell} + O\left(\frac{1}{s^{\ell+1}}\right),$$

$$\hat{q} = 1 - (1 - Q)^s = 1 - e^{-\frac{s\ell}{s+\ell} + O(\frac{1}{s})} = 1 - e^{-\ell} + O(\frac{1}{s})$$

and

$$C(s,\ell) \geq \frac{1}{\ell} \max_{0 \leq Q \leq 1} \mathcal{D}(\ell,Q,\hat{q}) = \frac{1}{\ell} \max_{0 \leq z \leq 1} \mathcal{D}(\ell,Q(z),\hat{q}(z)) \geq \frac{1}{\ell} \mathcal{D}(\ell,Q(s/(s+\ell)),\hat{q}(s/(s+\ell))),$$

where

$$\mathcal{D}(\ell, Q, \hat{q}) \triangleq (1 - Q) \, \ell \, \log_2 z - (1 - \hat{q}) \log_2 [1 - (1 - z)^{\ell}] + \\ + \ell \left( \frac{(1 - Q)}{z} (1 - z) - \left( \frac{(1 - Q)}{z} - \hat{q} \right) (1 - z)^{\ell} \right) \log_2 [1 - z] + \ell h(Q).$$

Therefore, one can write

$$\begin{split} C(s,\ell) &\geq \left(\frac{s}{s+\ell} + \frac{\ell^{\ell}e^{-\ell}}{s^{\ell}} + O\left(\frac{1}{s^{\ell+1}}\right)\right) \log_2\left[\frac{s}{s+\ell}\right] - \left(\frac{e^{-\ell}}{\ell} + O\left(\frac{1}{s}\right)\right) \log_2\left[1 - \left(\frac{\ell}{s+\ell}\right)^{\ell}\right] + \\ &+ \left(1 + O\left(\frac{1}{s^{\ell}}\right)\right) \frac{\ell}{s+\ell} \log_2\left[\frac{\ell}{s+\ell}\right] - \left(e^{-\ell} + O\left(\frac{1}{s}\right)\right) \left(\frac{\ell}{s+\ell}\right)^{\ell} \log_2\left[\frac{\ell}{s+\ell}\right] - \\ &- \left(\frac{\ell}{s+\ell} - \frac{\ell^{\ell}e^{-\ell}}{s^{\ell}} + O\left(\frac{1}{s^{\ell+1}}\right)\right) \log_2\left[\frac{\ell}{s+\ell} - \frac{\ell^{\ell}e^{-\ell}}{s^{\ell}} + O\left(\frac{1}{s^{\ell+1}}\right)\right] - \\ &- \left(\frac{s}{s+\ell} + \frac{\ell^{\ell}e^{-\ell}}{s^{\ell}} + O\left(\frac{1}{s^{\ell+1}}\right)\right) \log_2\left[\frac{s}{s+\ell} + \frac{\ell^{\ell}e^{-\ell}}{s^{\ell}} + O\left(\frac{1}{s^{\ell+1}}\right)\right] = \\ &= \frac{\ell^{\ell-1} \log_2 e}{e^{\ell} \, s^{\ell}} + O\left(\frac{\log_2 s}{s^{\ell+1}}\right). \end{split}$$

This completes the proof of Claim 2.  $\Box$ 

## 4 Appendix

**Proof of Lemma 1.** Let  $s \ge 2$ , 0 < Q < 1,  $Q < q < \min\{1, sQ\}$  be fixed parameters. Assume also  $k \triangleq |qN|$  and  $N \to \infty$ . With the help of the *type* (see [2], [8]) terminology:

$$\{n(\mathbf{a})\}, \quad \mathbf{a} \triangleq (a_1, a_2, \dots, a_s) \in \{0, 1\}^s, \quad 0 \le n(\mathbf{a}) \le N, \quad \sum_{\mathbf{a}} n(\mathbf{a}) = N,$$

the probability of event (31) in the ensemble  $\{N, t, Q\}$  can be written as follows:

$$\mathcal{P}_{2}^{(N)}(s,Q,k) = \binom{N}{\lfloor QN \rfloor}^{-s} \cdot \sum_{(39)} \frac{N!}{\prod_{\mathbf{a}} n(\mathbf{a})!}, \quad \lfloor QN \rfloor \le k \le \min\{N, s \lfloor QN \rfloor\},$$
(38)

and in the right-hand side of (38), the sum is taken over all types  $\{n(\mathbf{a})\}$  provided that

$$n(\mathbf{0}) = N - k, \qquad \sum_{\mathbf{a}: a_i = 1} n(\mathbf{a}) = \lfloor QN \rfloor \quad \text{for any } i \in [s].$$
 (39)

For every type  $\{n(\mathbf{a})\}$  we will consider the corresponding distribution  $\tau : \tau(\mathbf{a}) = \frac{n(\mathbf{a})}{N}, \quad \forall \ \mathbf{a} \in \{0,1\}^s$ . Applying the Stirling approximation, we obtain the following logarithmic asymptotic behavior of a term in the sum (38):

$$-\log_2 \frac{N!}{\prod_{\mathbf{a}} n(\mathbf{a})!} {N \choose \lfloor QN \rfloor}^{-s} = NF(\tau, Q, q)(1 + o(1)), \text{ where}$$

$$F(\tau, Q, q) = \sum_{\mathbf{a}} \tau(\mathbf{a}) \log_2 \tau(\mathbf{a}) + sH(Q). \tag{40}$$

Thus, one can reduce the calculation of  $\mathcal{A}(s,Q,q)$  defined by (33) to the search of the minimum:

$$\mathcal{A}(s,Q,q) = \min_{\tau \in (55):(56)} F(\tau,Q,q) \triangleq F(\hat{\tau},Q,q), \tag{41}$$

$$\{\tau: \ \forall \ \mathbf{a} \quad 0 < \tau(\mathbf{a}) < 1\},\tag{42}$$

$$\sum_{\mathbf{a}} \tau(\mathbf{a}) = 1, \qquad \tau(\mathbf{0}) = 1 - q, \qquad \sum_{\mathbf{a}: a_i = 1} \tau(\mathbf{a}) = Q \quad \forall \ i \in [s], \tag{43}$$

where the restrictions (56) are induced by the definition of type and the properties (39).

To find the minimum (54) and the extremal distribution  $\{\hat{\tau}\}$  we use the method of Lagrange multipliers. The Lagrangian is

$$\Lambda \triangleq \sum_{\tau(\mathbf{a})} \tau(\mathbf{a}) \log_2 \tau(\mathbf{a}) + sh(Q) + \lambda_0 \left(\tau(\mathbf{0}) + q - 1\right) + \\
+ \sum_{i=1}^{s} \lambda_i \left(\sum_{\mathbf{a}: \mathbf{a}=1} \tau(\mathbf{a}) - Q\right) + \lambda_{s+1} \left(\sum_{\mathbf{a}} \tau(\mathbf{a}) - 1\right).$$

Therefore, the necessary conditions for the extremal distribution  $\{\hat{\tau}\}$  are

$$\begin{cases} \frac{\partial \Lambda}{\partial \tau(\boldsymbol{\theta})} = \log_2 \hat{\tau}(\boldsymbol{\theta}) + \log_2 e + \lambda_0 + \lambda_{s+1} = 0, \\ \frac{\partial \Lambda}{\partial \tau(\mathbf{a})} = \log_2 \hat{\tau}(\mathbf{a}) + \log_2 e + \lambda_{s+1} + \sum_{i=1}^s a_i \lambda_i = 0 & \text{for any } \mathbf{a} \neq \boldsymbol{\theta}. \end{cases}$$
(44)

It turns out that the matrix of second derivatives of the Lagrangian is diagonal and positive definite in the region (55), and the function  $F(\tau, Q)$  defined by (40) is strictly  $\cup$ -convex in the region (55). The Karush-Kuhn-Tacker theorem states that each solution  $\tau \in (55)$  satisfying system (44) and constraints (56) gives a local minimum of  $F(\tau, Q)$ . Thus, if there exists a solution of the system (44) and (56) in the region (55), then it is unique and gives a minimum in the minimization problem (54) - (56).

Note that the symmetry of problem yields the equality  $v \triangleq \lambda_1 = \lambda_2 = \cdots = \lambda_s$ . Let  $u \triangleq \log_2 e + \lambda_{s+1}$  and  $w \triangleq \lambda_0$ . One can rewrite (56) and (44) as follows:

$$\begin{cases}
1) & \log_2 \hat{\tau}(\mathbf{a}) + u + v \sum_{i=1}^s a_i = 0 & \text{for any } \mathbf{a} \neq \mathbf{0}, \\
2) & \log_2 \hat{\tau}(\mathbf{0}) + u + w = 0, \\
3) & \hat{\tau}(\mathbf{0}) = 1 - q, \\
4) & \sum_{\mathbf{a}} \hat{\tau}(\mathbf{a}) = 1, \\
5) & \sum_{\mathbf{a}: a_i = 1} \hat{\tau}(\mathbf{a}) = Q & \text{for any } i \in [s].
\end{cases} \tag{45}$$

Let  $y \triangleq \frac{1}{1+2^{-v}}$ . The first equation of the system (45) means that

$$\hat{\tau}(\mathbf{a}) = \frac{1}{2^u y^s} (1 - y)^{\sum a_j} y^{s - \sum a_j} \quad \text{for any } \mathbf{a} \neq \mathbf{0}.$$
 (46)

Substituting (46) into the equation 5) allows us to obtain

$$\sum_{\mathbf{a}: a_i = 1} \frac{1}{2^u y^s} (1 - y)^{\sum a_j} y^{s - \sum a_j} = \frac{1 - y}{2^u y^s},$$

and therefore the solution u is determined by the equality

$$u = \log_2 \left[ \frac{1 - y}{Qy^s} \right]. \tag{47}$$

Substituting (46), (47) and the third equation of (45) into the equation 4) of the system (45) we have

$$q = \sum_{\mathbf{a} \neq 0} \hat{\tau}(\mathbf{a}) = \frac{Q(1 - y^s)}{1 - y},$$

i.e. the equation (35). Thus, the conditions (56) and (44) have the unique solution  $\tau$  in the region (55):

$$\hat{\tau}(\boldsymbol{\theta}) = 1 - q, \qquad \hat{\tau}(\mathbf{a}) = \frac{Q}{1 - y} (1 - y)^{\sum a_j} y^{s - \sum a_j} \quad \text{for any } \mathbf{a} \neq \boldsymbol{\theta},$$
 (48)

where the parameters q and y are related by the equation (35). To get the exact formula (34), the substitution of (48) into (40) is sufficient.

Let us prove the properties of the function (34). Note that the function  $q(y) = Q \frac{1-y^s}{1-y}$  (35) monotonically increases in the interval  $y \in (0,1)$  and correspondingly takes the values Q and sQ at the ends of the interval. That is why one can consider the function (34) as the function  $\mathcal{F}(s,Q,y) \triangleq \mathcal{A}(s,Q,q(y))$  of the parameter y in the interval  $y \in (0,y_1)$ , where  $q(y_1) = \min\{1,sQ\}$ . The derivative of the function  $\mathcal{F}(s,Q,y)$  equals

$$\mathcal{F}'(s,Q,y) = q'(y)\log_2\left[\frac{Qy^s}{1 - Q - y + Qy^s}\right]. \tag{49}$$

Thus,  $\mathcal{F}(s,Q,y)$  decreases in the interval  $y \in (0,1-Q)$ , increases in the interval  $y \in (1-Q,y_1)$ , is  $\cup$ -convex, attains the minimal value 0 at  $y_0 = 1 - Q$  and  $q(y_0) = 1 - (1-Q)^s$ .

**Proof of Lemma 2.** Now, compute the conditional probability

$$\mathcal{P}_1^{(N)}(\ell,Q,k) \triangleq \Pr \left\{ \bigvee_{i \in \mathcal{S}} oldsymbol{x}(i) \succeq \bigwedge_{j \in \mathcal{L}} oldsymbol{x}(j) \middle/ \left| \bigvee_{i \in \mathcal{S}} oldsymbol{x}(i) \right| = k 
ight. 
ight\}$$

Let  $q, Q \leq q \leq \min\{1, sQ\}$ , be fixed and  $k \triangleq \lfloor qN \rfloor$ ,  $\lfloor QN \rfloor \leq k \leq s \lfloor QN \rfloor$ . In terms of types (see [2], [8]):

$$\{n(\mathbf{a})\}, \quad \mathbf{a} \triangleq (a_1, a_2, \dots, a_s) \in \{0, 1\}^{\ell}, \quad 0 \le n(\mathbf{a}) \le N, \quad \sum_{\mathbf{a} \in \{0, 1\}^{\ell}} n(\mathbf{a}) = N, \quad (50)$$

one can rewrite the probability in the following form

$$\mathcal{P}_{1}^{(N)}(\ell,Q,k) = \sum_{(52)} \frac{N!}{\prod_{\mathbf{a} \in \{0,1\}^{\ell}} n(\mathbf{a})!} \frac{\binom{k}{n(1)}}{\binom{N}{n(1)}} \binom{N}{\lfloor QN \rfloor}^{-\ell}, \tag{51}$$

where the summation is taken over all choices of types  $\{n(\mathbf{a})\}$  provided that

$$\sum_{\mathbf{a}: a:=1} n(\mathbf{a}) = \lfloor QN \rfloor \quad \text{for any } i \in [\ell].$$
 (52)

Applying the Stirling formula calculate the logarithmic behaviour of a term in (51)

$$\log_2\left[\frac{N!}{\prod\limits_{\mathbf{a}\in\{0,1\}^\ell}n(\mathbf{a})!}\frac{\binom{k}{n(\mathbf{1})}}{\binom{N}{n(\mathbf{1})}}\binom{N}{\lfloor QN\rfloor}^{-\ell}\right] = 2^{-NF(\tau,Q,q)(1+o(1))},$$

where

$$F(\tau, Q, q) \triangleq \sum_{\mathbf{a} \in \{0,1\}^{\ell}} \tau(\mathbf{a}) \log_2 \tau(\mathbf{a}) - q \cdot h\left(\frac{\tau(\mathbf{1})}{q}\right) + h(\tau(\mathbf{1})) + \ell \cdot h(Q).$$
 (53)

Here the probability distribution  $\{\tau(\mathbf{a})\}\$  is determined as

$$\tau(\mathbf{a}) \triangleq \frac{n(\mathbf{a})}{N}$$
 for any  $\mathbf{a} \in \{0, 1\}^{\ell}$ .

Since we are interested in

$$\mathcal{D}(\ell,Q,q) = \lim_{N \to \infty} -\frac{\log_2 \left[P_1^{(N)}(\ell,Q,k)\right]}{N},$$

we might estimate the following minimum

$$\mathcal{D}(\ell, Q, q) = \min_{\tau \in (55):(56)} F(\tau, Q, q) \triangleq F(\hat{\tau}, Q, q), \tag{54}$$

$$\left\{ \tau : \ \forall \ \mathbf{a} = (a_1, \dots, a_\ell) \in \{0, 1\}^\ell \quad 0 < \tau(\mathbf{a}) < 1 \right\},$$
 (55)

$$\sum_{\mathbf{a}} \tau(\mathbf{a}) = 1, \qquad \sum_{\mathbf{a}: a_i = 1} \tau(\mathbf{a}) = Q \quad \text{for any } i \in [\ell],$$
 (56)

where the restrictions (56) are induced by properties (50) and (50).

To find the minimum we apply the standard Lagrange method, i.e., consider the Lagrangian

$$\Lambda \triangleq \sum_{\mathbf{a} \in \{0,1\}^{\ell}} \tau(\mathbf{a}) \log_2 \tau(\mathbf{a}) - q \cdot h\left(\frac{\tau(\mathbf{1})}{q}\right) + h(\tau(\mathbf{1})) + \ell \cdot h(Q) + \\ + \mu_0 \cdot \left(\sum_{\mathbf{a}} \tau(\mathbf{a}) - 1\right) + \sum_{i=1}^{\ell} \mu_i \cdot \left(\sum_{\mathbf{a}: a_i = 1} \tau(\mathbf{a}) - Q\right). \tag{57}$$

Therefore, the necessary conditions for the extremal distribution  $\{\hat{\tau}\}$  are

$$\begin{cases} \frac{\partial \Lambda}{\partial \tau(\mathbf{a})} = \log_2 \hat{\tau}(\mathbf{a}) + \log_2 e + \mu_0 + \sum_{i: a_i = 1} \mu_i = 0 & \text{for any } \mathbf{a} \neq \mathbf{1}, \\ \frac{\partial \Lambda}{\partial \tau(\mathbf{1})} = \log_2 \hat{\tau}(\mathbf{1}) + \log_2 e + \sum_{i=0}^{\ell} \mu_i + \log_2 \left[ \frac{1 - \hat{\tau}(\mathbf{1})}{q - \hat{\tau}(\mathbf{1})} \right] = 0. \end{cases}$$
(58)

The matrix of second derivatives of the Lagrangian is obvious to be diagonal. Thus, this matrix is positive definite in the region (55) and the function  $F(\tau, Q, q)$  defined by (53) is strictly  $\cup$ -convex in the region (53). The Karush-Kuhn-Tacker theorem states that each solution  $\{\hat{\tau}\}$  satisfying system (58) and constraints (56) gives a local minimum of  $F(\tau, Q, q)$ . Thus, if there exists a solution of the system (58) and (56) in the region (55), then it is unique and gives a minimum in the minimization problem (54)-(56).

Note that the symmetry of problem yields the equality  $\mu \triangleq \mu_1 = \mu_2 = \cdots = \mu_\ell$ . Let  $\hat{\mu} \triangleq \log_2 e + \mu_0$ . One can rewrite (58) as

$$\begin{cases} \hat{\mu} + \mu \sum_{i=1}^{\ell} a_i + \log_2[\hat{\tau}(\mathbf{a})] = 0 & \text{for } \mathbf{a} \neq \mathbf{1}; \\ \hat{\mu} + \mu \ell + \log_2[\hat{\tau}(\mathbf{1})] + \log_2\left[\frac{1 - \hat{\tau}(\mathbf{1})}{q - \hat{\tau}(\mathbf{1})}\right] = 0; \end{cases}$$
(59)

The first equations of (59) lead to

$$\hat{\tau}(\mathbf{a}) = \frac{2^{-\hat{\mu}}}{z^{\ell}} \prod P(a_i) \text{ for } \mathbf{a} \neq \mathbf{1},$$

where we introduce the Bernoulli distribution

$$P(a) \triangleq \begin{cases} z \triangleq \frac{1}{1+2^{-\mu}} & \text{for } a = 0; \\ 1 - z \triangleq \frac{2^{-\mu}}{1+2^{-\mu}} & \text{for } a = 1; \end{cases}$$

In particular, it follows

$$\mu = \log_2 \left[ \frac{z}{1 - z} \right]. \tag{60}$$

Since (56) the sum of all probabilities equals 1 we get

$$\hat{\tau}(\mathbf{1}) = 1 - \sum_{k=0}^{\ell-1} {\ell \choose k} \frac{2^{-\hat{\mu}}}{z^{\ell}} z^{\ell-k} (1-z)^k = 1 - \frac{2^{-\hat{\mu}}}{z^{\ell}} \left( 1 - (1-z)^{\ell} \right).$$
 (61)

The relation (56) of constant weight leads to

$$Q = \frac{2^{-\hat{\mu}}}{z^{\ell}} \sum_{k=0}^{\ell-2} \binom{\ell-1}{k} z^{\ell-k-1} (1-z)^{k+1} + 1 - \frac{2^{-\hat{\mu}}}{z^{\ell}} \left( 1 - (1-z)^{\ell} \right) = 1 - \frac{2^{-\hat{\mu}}}{z^{\ell-1}}.$$

This gives the connection between  $\hat{\mu}$  and parameters Q and z

$$\hat{\mu} = -\log_2\left[(1-Q)z^{\ell-1}\right].$$
 (62)

Finally, substituting (60)-(62) to the second equation of (59) yields

$$\begin{split} -\log_2\left[(1-Q)z^{\ell-1}\right] + \ell\log_2\left[\frac{z}{1-z}\right] + \log_2\left[1 - \frac{(1-Q)}{z}\left(1 - (1-z)^{\ell}\right)\right] + \\ + \log_2\left[\frac{(1-Q)}{z}\left(1 - (1-z)^{\ell}\right)\right] - \log\left[q + \frac{(1-Q)}{z}\left(1 - (1-z)^{\ell}\right) - 1\right] = 0 \end{split}$$

It can be written in the equivalent form

$$\log_2 \left\lceil \frac{\left(1 - (1 - z)^{\ell}\right)}{(1 - z)^{\ell}} \right\rceil + \log_2 \left\lceil \frac{z - (1 - Q)\left(1 - (1 - z)^{\ell}\right)}{(q - 1)z + (1 - Q)\left(1 - (1 - z)^{\ell}\right)} \right\rceil = 0$$

This equation determines Q as a function of parameters z, q, s and  $\ell$ 

$$Q = \frac{(1-z)(1-(1-z)^{\ell}) - (1-q)z(1-z)^{\ell}}{1-(1-z)^{\ell}}.$$
(63)

Notice that for fixed parameters q, s and  $\ell$  there is a bijection between  $Q \in [0,1]$  and  $z \in [0,1]$ . From (62) and (63) it follows that

$$\frac{2^{-\hat{\mu}}}{z^{\ell}} = \frac{1-Q}{z} = \frac{1-q(1-z)^{\ell}}{1-(1-z)^{\ell}}.$$
(64)

Let us substitute  $q = \hat{q} = 1 - (1 - Q)^s$ . Thus

$$\hat{\tau}(\mathbf{1}) = \hat{q}(1-z)^{\ell}. \tag{65}$$

Remind (54) that

$$F(\hat{\tau}, Q, \hat{q}) = \sum_{\mathbf{a} \in \{0,1\}^{\ell}} \hat{\tau}(\mathbf{a}) \log_2 \hat{\tau}(\mathbf{a}) - \hat{q} \cdot h\left(\frac{\hat{\tau}(\mathbf{1})}{\hat{q}}\right) + h(\hat{\tau}(\mathbf{1})) + \ell \cdot h(Q).$$
 (66)

Let us rewrite the first sum of (66) applying (64):

$$\sum_{\mathbf{a} \in \{0,1\}^{\ell}} \hat{\tau}(\mathbf{a}) \log_2 \hat{\tau}(\mathbf{a}) = \sum_{i=0}^{\ell-1} {\ell \choose i} \frac{2^{-\hat{\mu}}}{z^{\ell}} (1-z)^i z^{\ell-i} \log_2 \left[ \frac{2^{-\hat{\mu}}}{z^{\ell}} (1-z)^i z^{\ell-i} \right] + \hat{\tau}(\mathbf{1}) \log_2 \hat{\tau}(\mathbf{1}) =$$

$$= \sum_{i=0}^{\ell-1} {\ell \choose i} \frac{2^{-\hat{\mu}}}{z^{\ell}} (1-z)^i z^{\ell-i} \log_2 \left[ \frac{2^{-\hat{\mu}}}{z^{\ell}} \right] + \sum_{i=0}^{\ell-1} {\ell \choose i} \frac{2^{-\hat{\mu}}}{z^{\ell}} (1-z)^i z^{\ell-i} \log_2 \left[ z^{\ell-i} \right] +$$

$$+ \sum_{i=0}^{\ell-1} {\ell \choose i} \frac{2^{-\hat{\mu}}}{z^{\ell}} (1-z)^{i} z^{\ell-i} \log_{2} \left[ (1-z)^{i} \right] + \hat{\tau}(\mathbf{1}) \log_{2} \hat{\tau}(\mathbf{1}) =$$

$$= \left( 1 - \hat{q}(1-z)^{\ell} \right) \log_{2} \left[ \frac{1 - \hat{q}(1-z)^{\ell}}{1 - (1-z)^{\ell}} \right] + (1-Q) \ell \log_{2} z +$$

$$+ \frac{(1-Q)}{z} \ell \left( (1-z) - (1-z)^{\ell} \right) \log_{2} [1-z] + \hat{\tau}(\mathbf{1}) \log_{2} \hat{\tau}(\mathbf{1}).$$

Taking into account (65) the second term of (66) is

$$-\hat{q}h\left(\frac{\hat{\tau}(\mathbf{1})}{\hat{q}}\right) = \tau(\mathbf{1})\log_2\left[\frac{\hat{\tau}(\mathbf{1})}{q}\right] + (q - \hat{\tau}(\mathbf{1}))\log_2\left[\frac{q - \hat{\tau}(\mathbf{1})}{q}\right] =$$

$$= \ell\hat{q}(1-z)^{\ell}\log_2[1-z] + \hat{q}(1-(1-z)^{\ell})\log_2[1-(1-z)^{\ell}].$$

The third term of (66) is

$$h(\hat{\tau}(\mathbf{1})) = -\hat{\tau}(\mathbf{1})\log_2\hat{\tau}(\mathbf{1}) - (1 - \hat{\tau}(\mathbf{1}))\log_2[1 - \hat{\tau}(\mathbf{1})].$$

Finally, the last term of (66) is  $\ell h(Q)$ . Therefore, the value  $\mathcal{D}(\ell,Q,\hat{q})=F(\hat{\tau},Q,\hat{q})$  can be written as

$$\mathcal{D}(\ell, Q, \hat{q}) = (1 - Q) \ell \log_2 z + \ell \left( \frac{(1 - Q)}{z} (1 - z) - \left( \frac{(1 - Q)}{z} - \hat{q} \right) (1 - z)^{\ell} \right) \log_2 [1 - z] - (1 - \hat{q}) \log_2 [1 - (1 - z)^{\ell}] + \ell h(Q).$$

Thus, we complete the proof of Lemma 2.  $\square$ 

### References

- [1] Gallager R.G., "Information Theory and Reliable Communication", *J. Wiley*, New York, 1968.
- [2] Csiszar I., Korner J. "Information Theory. Coding Theorems for Discrete Memoryless Systems", *Akademiai Kiado*, Budapest, 1981.
- [3] Erdos P., Frankl P., Furedi Z. Families of Finite Sets in Which No Set Is Covered by the Union of 2 Others // J. Combin. Theory. Ser. A. 1982. V. 33. P. 158-166.
- [4] D'yachkov A., Vilenkin P., Macula A., Torney D., "Families of Finite Sets in Which No Intersection of ℓ Sets Is Covered by the Union of s Others", Journal of Combinatorial Theory, Series A, vol. 99. pp. 195-218, 2002.
- [5] Mitchell C.J, Piper F.C., "Key storage in Secure Networks", *Discrete Applied Mathematics*, vol. 21, pp. 215-228, 1988.
- [6] Kautz W.H., Singleton R.C., "Nonrandom Binary Superimposed Codes", *IEEE Trans. Inform. Theory*, vol. 10, no. 4, pp. 363-377, 1964.
- [7] D'yachkov A.G., Macula A.J., Rykov V.V. New Applications and Results of Superimposed Code Theory Arising from the Potentialities of Molecular Biology // In the book "Numbers, Information and Complexity". P. 265-282, Kluwer Academic Publishers, 2000.
- [8] D'yachkov A.G., Vorobyev I.V., Polyanskii N.A., Shchukin V.Yu., "Bounds on the Rate of Disjunctive Codes", *Problems of Information Transmission*, vol. 50, no. 1, pp. 27-56, 2014.
- [9] D'yachkov A.G., Vorobyev I.V., Polyanskii N.A., Shchukin V.Yu., "Bounds on the Rate of Superimposed Codes", 2014 IEEE International Symposium on Information Theory, pp. 2341-2345, Honolulu, HI USA, Jun.29-Jul.4, 2014.
- [10] Torney D.C., "Sets Pooling Designs", Annals of Combinatorics, vol. 3, pp. 95-101, 1999.
- [11] D'yachkov A.G., Vorobyev I.V., Polyanskii N.A., Shchukin V.Yu., "Almost Disjunctive List-Decoding Codes", *Proc. of International Conference on Algebraic and Combinatorial Coding Theory (ACCT)*, Russia, pp. 115-126, Sep. 2014.
- [12] Malyutov M.B., "The Separating Property of Random Matrices", *Mathematical Notes*, vol.23, no. 1, pp. 84-91, 1978.
- [13] Bassalygo L.A., Rykov V.V. Multiple-access hyperchannel // Problems of Information Transmission, 2013. vol. 49. no. 4, pp. 299-307.