

First- and Second-Order Coding Theorems for Mixed Memoryless Channels with General Mixture

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Abstract

This paper investigates the first- and second-order maximum achievable rates of codes with/without cost constraints for mixed channels whose channel law is characterized by a general mixture of (at most) uncountably many stationary and memoryless discrete channels. These channels are referred to as mixed memoryless channels with general mixture and include the class of mixed memoryless channels of finitely or countably memoryless channels as a special case. For mixed memoryless channels with general mixture, the first-order coding theorem which gives a formula for the ε -capacity is established, and then a direct part of the second-order coding theorem is provided. A subclass of mixed memoryless channels whose component channels can be ordered according to their capacity is introduced, and the first- and second-order coding theorems are established. It is shown that the established formulas reduce to several known formulas for restricted scenarios.

I. INTRODUCTION

Investigation of the maximum achievable rate of codes whose probability of decoding error does not exceed $\varepsilon \in [0, 1)$ for various coding systems has been one of major research topics in information theory. The first-order optimum rate for channel codes with such a property is referred to as the ε -capacity. Inspired by the recent results of second-order coding theorems given, for example, by Hayashi [6] and Polyanskiy, Poor, and Verdú [11] for stationary memoryless channels, this research topic has become of greater importance from both theoretical and practical viewpoints.

It is well-known that stationary memoryless channels with finite input and/or output alphabets have the so-called *strong converse property*, and the ε -capacity coincides with the channel capacity (ε -capacity with $\varepsilon = 0$) [19]. On the other hand, allowing a decoding error probability up to ε , the maximum achievable rate may be improved for non-stationary and/or non-ergodic channels. The simplest example is a class of *mixed channels* [5], also referred to as averaged channels [1], [8] or decomposable channels [18], whose probability distribution is characterized by a mixture of multiple stationary memoryless channels. This channel is stationary but non-ergodic and is of theoretical importance when extensions of coding theorems for ergodic channels are addressed.

For general channels including mixed channels, a general formula for the ε -capacity has been given by Verdú and Han [14]. This formula, however, involves limit operations with respect to code length n , and thus is infeasible to compute in general. On the other hand, for mixed channels of uncountably many stationary and memoryless discrete channels, which will be called general *mixed memoryless channels*, a single-letter characterization of the channel capacity has been given by Ahlswede [1] for the case without cost constraints and by Han [5] for the case with cost constraints. These characterizations are of importance because the channel capacity may be computed with complexity independent of n . Recently, Yagi and Nomura [20] has provided a single-letter characterization of the ε -capacity with/without cost constraints for mixed channels of at most countably many stationary memoryless channels. Regarding the ε -capacity for mixed memoryless channels with general mixture, however, no characterizations have been given in the literature. The regular decomposable channel which consists of memoryless channels [18], is one of a few examples for which a single-letter characterization of the ε -capacity is known. In addition, the

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second-order optimum rate has been characterized only for a few classes of mixed memoryless channels such as the mixed channel of two memoryless additive channels [12], the mixed channel of finitely many stationary and memoryless discrete channels which can be ordered according to their capacities [21], and block fading channels characterized as the mixed channel consisting of additive Gaussian noise channels [22].

This paper first gives a single-letter characterization of the ε -capacity with/without cost constraints for mixed memoryless channels with general mixture (Theorem 1). The established formula reduces to the one for the channel capacity given by [1] and [5] when ε is zero. The achievability and converse proofs of Theorem 1 proceed in a parallel manner: (i) the upper or lower bound on the error probability is characterized by the type (empirical distribution) of codewords and (ii) the convergence of a subsequence of types to a certain probability distribution is discussed. Next, a direct coding theorem (achievability) is given for the second-order optimum rate (Theorem 2). In the proof of Theorem 2, an upper bound on the error probability is derived based on the random coding argument of a fixed type, and it is a key to specify the type of codewords so that the speed of the convergence of the mutual information computed by this type to the target first-order coding rate is fast enough (cf. Equation (98)). For a fixed code, on the other hand, we cannot guarantee that the speed of the convergence of such mutual information to the target first-order coding rate is fast enough, and this fact has prevented us from establishing the converse part of the second-order coding theorem. In order to circumvent this problem, we will introduce a subclass of mixed memoryless channels with general mixture, called *well-ordered mixed memoryless channels*, whose component channels can be ordered as discussed in [21]. For this channel class, the first- and second-order coding theorems are established. It is shown that the established formulas reduce to several known formulas for restricted scenarios. All coding theorems are proved based on the *information spectrum methods* (c.f. [5], [17]). In particular, we use a proof technique for the converse part such that the proof proceeds based on an arbitrarily chosen converging subsequence of types of codewords, which may simplify even the proof of the second-order coding theorem for stationary memoryless channels such as in [6].

This paper is organized as follows: The problem addressed in this paper is stated in Sect. II. We next establish the first-order coding theorem in Sect. III-A and a direct part of the second-order coding theorem in Sect. III-B for mixed memoryless channels with general mixture. These theorems are proved in Sect. IV; several lemmas used to prove the theorems are first provided in Sect. IV-A, and then proofs of the coding theorems are given in Sect. IV-B and IV-C, respectively. Section V discusses well-ordered mixed memoryless channels, introduced in Sect. V-A, and the first- and second-order coding theorems are stated in Sect. V-B along with the proofs in Sect. V-C and V-D. Some concluding remarks are given in Sect. VI.

II. PROBLEM FORMULATION

A. Mixed Memoryless Channel under General Mixture

Consider a channel $W^n : \mathcal{X}^n \rightarrow \mathcal{Y}^n$, without any assumption on the memory structure, which stochastically maps an input sequence $\mathbf{x} \in \mathcal{X}^n$ of length n into an output sequence $\mathbf{y} \in \mathcal{Y}^n$ of length n . Here, \mathcal{X} and \mathcal{Y} denote *finite* input and output alphabets, respectively. A sequence $\mathbf{W} := \{W^n\}_{n=1}^{\infty}$ of channels W^n is referred to as a *general channel* [5].

We consider a *mixed channel*¹ with a general probability measure [5, Sect 3.3]. Let Θ be an arbitrary probability space and assign a general channel $\mathbf{W}_\theta = \{W_\theta^n\}_{n=1}^{\infty}$ to each $\theta \in \Theta$, which are called *component channels* or simply *components*. Here, we assume that each \mathbf{W}_θ has the same input alphabet \mathcal{X} and output alphabet \mathcal{Y} . With an arbitrary probability measure w on Θ , we define a *mixed channel* $\mathbf{W} = \{W^n\}_{n=1}^{\infty}$

¹Mixed channels are also referred to as *averaged channels* [8] or *decomposable channels* [18].

with the conditional probability distribution given by

$$W^n(\mathbf{y}|\mathbf{x}) = \int_{\Theta} W_{\theta}^n(\mathbf{y}|\mathbf{x}) dw(\theta) \quad (\forall n = 1, 2, \dots; \forall \mathbf{x} \in \mathcal{X}^n, \forall \mathbf{y} \in \mathcal{Y}^n). \quad (1)$$

In this paper, we focus on the case where the component channels are stationary memoryless discrete channels. Then, a component channel can be denoted simply by $\mathbf{W}_{\theta} = \{W_{\theta}: \mathcal{X} \rightarrow \mathcal{Y}\}$. A mixed channel given by (1) with stationary memoryless discrete channels $\mathbf{W}_{\theta} = \{W_{\theta}\}$ is referred to as a general *mixed memoryless channel* for simplicity.

Let \mathcal{C}_n be a code of length n and the number of codewords $|\mathcal{C}_n| = M_n$. We denote the codeword corresponding to message $i \in \{1, 2, \dots, M_n\}$ by \mathbf{u}_i , i.e., $\mathcal{C}_n = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{M_n}\}$. We assume that the decoding region D_i of \mathbf{u}_i satisfies

$$\bigcup_{i=1}^{M_n} D_i = \mathcal{Y}^n \quad \text{and} \quad D_i \cap D_j = \emptyset \quad (i \neq j). \quad (2)$$

The *average* probability of decoding error over \mathbf{W} is defined as

$$\varepsilon_n := \frac{1}{M_n} \sum_{i=1}^{M_n} W^n(D_i^c | \mathbf{u}_i), \quad (3)$$

where D_i^c denotes the complement set of D_i in \mathcal{Y}^n . Such a code \mathcal{C}_n is referred to as an (n, M_n, ε_n) code.

We consider a cost function $c_n(\cdot)$ for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{X}^n$, defined as

$$c_n(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^n c(x_i), \quad (4)$$

where $c: \mathcal{X} \rightarrow [0, \infty)$. A sequence \mathbf{x} is said to satisfy cost constraint Γ if

$$c_n(\mathbf{x}) \leq \Gamma, \quad (5)$$

and an (n, M_n, ε_n) code \mathcal{C}_n is said to satisfy cost constraint Γ if every codeword $\mathbf{u}_i \in \mathcal{C}_n$ satisfies cost constraint Γ .

Remark 1: If $\Gamma \geq \max_{x \in \mathcal{X}} c(x)$, then (5) holds for any $\mathbf{x} \in \mathcal{X}^n$. This case corresponds to the coding system without cost constraints, which is indicated simply by $\Gamma = +\infty$. \square

B. Optimum Coding Rates

Definition 1: A first-order coding rate $R \geq 0$ is said to be $(\varepsilon|\Gamma)$ -*achievable* if there exists a sequence of (n, M_n, ε_n) codes satisfying cost constraint Γ such that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \leq \varepsilon \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \geq R. \quad (6)$$

The supremum of all $(\varepsilon|\Gamma)$ -achievable rates is called the first-order $(\varepsilon|\Gamma)$ -*capacity* and is denoted by $C_{\varepsilon}(\Gamma)$. We also write as $C_{\varepsilon} = C_{\varepsilon}(+\infty)$ for simplicity. \square

Set $\Gamma_0 := \min_{x \in \mathcal{X}} c(x)$. If $\Gamma < \Gamma_0$, then obviously $C_{\varepsilon}(\Gamma) = 0$ because no sequences $\mathbf{x} \in \mathcal{X}^n$ satisfy cost constraint Γ , and hence no $R > 0$ is $(\varepsilon|\Gamma)$ -achievable.

Let $M_{n,\varepsilon}^*$ denote the maximum size of codes of length n and error probability less than or equal to ε satisfying cost constraint Γ . The first-order $(\varepsilon|\Gamma)$ -capacity indicates that $M_{n,\varepsilon}^*$ behaves as

$$\log M_{n,\varepsilon}^* = nC_{\varepsilon}(\Gamma) + o(n)$$

for sufficiently large n . For coding systems whose first-order capacity had been characterized, our next target may be to characterize the second-order term of $\log M_{n,\varepsilon}^*$. This motivates us to introduce the *second-order coding rates*, and its maximum value denoted by $D_\varepsilon(R|\Gamma)$ with respect to the first-order coding rate $R = C_\varepsilon(\Gamma)$ roughly satisfies the relation

$$\log M_{n,\varepsilon}^* \simeq nC_\varepsilon(\Gamma) + \sqrt{n}D_\varepsilon(R|\Gamma) + o(\sqrt{n})$$

for sufficiently large n . Second-order achievable rates and their optimum value are now formally defined as follows.

Definition 2: A second-order coding rate S is said to be $(\varepsilon, R|\Gamma)$ -achievable if there exists a sequence of (n, M_n, ε_n) codes satisfying cost constraint Γ such that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \leq \varepsilon \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \frac{M_n}{e^{nR}} \geq S. \quad (7)$$

The supremum of all $(\varepsilon, R|\Gamma)$ -achievable rates is called the second-order $(\varepsilon, R|\Gamma)$ -capacity and is denoted by $D_\varepsilon(R|\Gamma)$. We also write as $D_\varepsilon(R) = D_\varepsilon(R|\Gamma) + \infty$ for simplicity. \square

Remark 2: It is easily verified that if $R < C_\varepsilon(\Gamma)$ then $D_\varepsilon(R|\Gamma) = +\infty$ for all $\varepsilon \in [0, 1)$ from the definition of capacities. Also, if $R > C_\varepsilon(\Gamma)$ then $D_\varepsilon(R|\Gamma) = -\infty$ for all $\varepsilon \in [0, 1)$. Therefore, only the case $R = C_\varepsilon(\Gamma)$ is of our main interest. \square

III. CODING THEOREMS FOR GENERAL MIXED MEMORYLESS CHANNEL

A. First-Order Coding Theorems

The following theorem gives a single-letter characterization for the first-order $(\varepsilon|\Gamma)$ -capacity of mixed memoryless channels with general mixture.

Theorem 1: Let \mathbf{W} be a general mixed memoryless channel with measure w . For any fixed $\varepsilon \in [0, 1)$ and $\Gamma \geq \Gamma_0$, the first-order $(\varepsilon|\Gamma)$ -capacity is given by

$$C_\varepsilon(\Gamma) = \sup_{P: \mathbb{E}c(X_P) \leq \Gamma} \sup \left\{ R \mid \int_{\{\theta \mid I(P, W_\theta) < R\}} dw(\theta) \leq \varepsilon \right\}, \quad (8)$$

where X_P indicates the input random variable subject to distribution P on \mathcal{X} , and $I(P, W_\theta)$ denotes the mutual information with input P and channel $W_\theta : \mathcal{X} \rightarrow \mathcal{Y}$ (cf. Csiszár and Körner [3]). \square

The proof of this theorem is given in Sect. IV.

Remark 3: If Θ is a singleton, Theorem 1 reduces to the well-known formula

$$C_\varepsilon(\Gamma) = \sup_{P: \mathbb{E}c(X_P) \leq \Gamma} I(P, W) \quad (0 \leq \forall \varepsilon < 1), \quad (9)$$

which means that the strong converse holds in this case (cf. [3], [19]), unlike in the general case $|\Theta| > 1$. For Θ which is a finite or countable infinite set, formula (8) of the first-order capacity $C_\varepsilon(\Gamma)$ reduces to the formula given by Yagi and Nomura [20]. For mixed memoryless channels with general mixture, on the other hand, in the special case of $\varepsilon = 0$, formula (8) reduces to

$$C_0(\Gamma) = \sup_{P: \mathbb{E}c(X_P) \leq \Gamma} w\text{-ess.inf} I(P, W_\theta), \quad (10)$$

which coincides with the formula given by Han [5, Theorem 3.6.5], where $w\text{-ess.inf}$ denotes the essential infimum of $I(P, W_\theta)$ with respect to the probability measure w . \square

When Θ is a singleton, it is known that the $C_\varepsilon(\Gamma)$ is concave in Γ and is strictly increasing over the range $\Gamma_0 \leq \Gamma \leq \Gamma^*$, where Γ^* denotes the smallest Γ at which $C_\varepsilon(\Gamma)$ coincides with C_ε (without cost constraints) (cf. Blahut [2]). For the case of $|\Theta| > 1$, $C_\varepsilon(\Gamma)$ is indeed non-decreasing, but there are examples of mixed memoryless channels for which $C_\varepsilon(\Gamma)$ is not strictly increasing in $\Gamma_0 \leq \Gamma \leq \Gamma^*$. This also indicates that $C_\varepsilon(\Gamma)$ need not be concave in Γ .

In the case without cost constraints, Theorem 1 reduces to the following corollary.

Corollary 1: Let \mathbf{W} be a general mixed memoryless channel with measure w . For any fixed $\varepsilon \in [0, 1)$, the first-order ε -capacity is given by

$$C_\varepsilon = \sup_P \sup \left\{ R \mid \int_{\{\theta \mid I(P, W_\theta) < R\}} dw(\theta) \leq \varepsilon \right\}, \quad (11)$$

where \sup_P denotes the supremum over the set $\mathcal{P}(\mathcal{X})$ of all probability distributions on \mathcal{X} . \square

Remark 4: The direct part of formula (11) was first demonstrated by Han [5, Lemma 3.3.3]. In the special case of $\varepsilon = 0$, we have an alternative formula of C_0 as in (10) (by replacing the supremum over $\{P \mid \mathbb{E}c(X_P) \leq \Gamma\}$ with the supremum over $\mathcal{P}(\mathcal{X})$), which coincides with the formula given by Ahlswede [1]. See also [5, Remark 3.3.3] for the equivalence between these characterizations. \square

B. Second-Order Coding Theorems

We now turn to analyzing second-order coding rates. Let $\Psi_{\theta, P}$ denote the Gaussian cumulative distribution function with zero mean and variance

$$V_{\theta, P} := \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(x) W_\theta(y|x) \left(\log \frac{W_\theta(y|x)}{P W_\theta(y)} - D(W_\theta(\cdot|x) \| P W_\theta) \right)^2, \quad (12)$$

that is,

$$\Psi_{\theta, P}(z) := G\left(\frac{z}{\sqrt{V_{\theta, P}}}\right), \quad G(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt, \quad (13)$$

where

$$P W_\theta(y) := \sum_x P(x) W_\theta(y|x) \quad (14)$$

denotes the output distribution on \mathcal{Y} due to the input distribution P on \mathcal{X} via channel W_θ , and $D(W_\theta(\cdot|x) \| P W_\theta)$ denotes the divergence between $W_\theta(\cdot|x)$ and $P W_\theta$. It is known that there are stationary memoryless channels W_θ for which $V_{\theta, P} = 0$ for some $P \in \mathcal{P}(\mathcal{X})$ (cf. [11], [14]). In such a case, with an abuse of notation, we interpret $\Psi_{\theta, P}(z) = G(z/\sqrt{V_{\theta, P}})$ as the step function which is defined to take zero for $z < 0$ and one otherwise.

For the second-order coding rate, we have the following direct theorem (achievability).

Theorem 2 (Direct Part): Let \mathbf{W} be a general mixed memoryless channel with measure w . For $\varepsilon \in [0, 1)$, $\Gamma \geq \Gamma_0$, and $R \geq 0$, it holds that

$$D_\varepsilon(R|\Gamma) \geq \sup_{P: \mathbb{E}c(X_P) \leq \Gamma} \sup \left\{ S \mid G_w(R, S|P) \leq \varepsilon \right\} =: \overline{D}_\varepsilon(R|\Gamma), \quad (15)$$

where

$$G_w(R, S|P) := \int_{\{\theta \mid I(P, W_\theta) < R\}} dw(\theta) + \int_{\{\theta \mid I(P, W_\theta) = R\}} \Psi_{\theta, P}(S) dw(\theta). \quad (16)$$

\square

The proof of this theorem is given in Sect. IV.

Remark 5: The two terms on the right-hand side of (16) can be summarized into the following single term:

$$\int_{\Theta} dw(\theta) \lim_{n \rightarrow \infty} \Psi_{\theta, P}(\sqrt{n}(R - I(P, W_\theta)) + S), \quad (17)$$

which is called the *canonical representation* (cf. Nomura and Han [9], [10]). Let us here focus on the crucial case of $R = C_\varepsilon(\Gamma)$. In view of formula (8) for the ε -capacity $C_\varepsilon(\Gamma)$ it is not difficult to check that, for any P such that $\text{Ec}(X_P) \leq \Gamma$,

$$\int_{\{\theta | I(P, W_\theta) < C_\varepsilon(\Gamma)\}} dw(\theta) \leq \varepsilon, \quad (18)$$

$$\int_{\{\theta | I(P, W_\theta) \leq C_\varepsilon(\Gamma)\}} dw(\theta) \geq \varepsilon \quad (19)$$

hold. Thus, we may consider the following canonical equation for S :

$$\int_{\Theta} dw(\theta) \lim_{n \rightarrow \infty} \Psi_{\theta, P}(\sqrt{n}(C_\varepsilon(\Gamma) - I(P, W_\theta)) + S) = \varepsilon. \quad (20)$$

Notice here, in view of (18) and (19), that equation (20) always has a solution. Let $S_P(\varepsilon)$ denote the solution of this equation, where $S_P(\varepsilon) = +\infty$ if the solution is not unique (notice that this case occurs if $\int_{\{\theta | I(P, W_\theta) = C_\varepsilon(\Gamma)\}} dw(\theta) = 0$, which equivalently means that the second term on the right-hand side in (16) is zero). Then, the $\overline{D}_\varepsilon(C_\varepsilon(\Gamma)|\Gamma)$ (i.e., $R = C_\varepsilon(\Gamma)$) in (15) can be rewritten in a simpler form as

$$\overline{D}_\varepsilon(C_\varepsilon(\Gamma)|\Gamma) = \sup_{P: \text{Ec}(X_P) \leq \Gamma} S_P(\varepsilon). \quad (21)$$

We sometimes prefer this simple expression rather than in (15). \square

Remark 6: Denote the right-hand side of (15) again by $\overline{D}_\varepsilon(R|\Gamma)$. If Θ is a singleton, it can be easily verified that

$$\overline{D}_\varepsilon(R|\Gamma) = \begin{cases} -\infty & \text{if } R > C_\varepsilon(\Gamma) \\ \sup_{\substack{P: I(P, W) = R \\ \text{Ec}(X_P) \leq \Gamma}} \sup \{S \mid \Psi_P(S) \leq \varepsilon\} & \text{if } R = C_\varepsilon(\Gamma) \\ +\infty & \text{if } R < C_\varepsilon(\Gamma), \end{cases} \quad (22)$$

where setting the singleton set Θ as $\Theta = \{\theta_0\}$ we use Ψ_P instead of Ψ_{P, θ_0} . In particular, if

$$R = C_\varepsilon(\Gamma) = \sup_{\substack{P: I(P, W) = R \\ \text{Ec}(X_P) \leq \Gamma}} I(P, W), \quad (23)$$

then it follows from Theorem 4 with $|\Theta| = 1$ later in Sect. V that

$$D_\varepsilon(C_\varepsilon(\Gamma)|\Gamma) = \overline{D}_\varepsilon(C_\varepsilon(\Gamma)|\Gamma) = \begin{cases} \sqrt{V_{\max}} G^{-1}(\varepsilon) & \text{if } \varepsilon \geq \frac{1}{2} \\ \sqrt{V_{\min}} G^{-1}(\varepsilon) & \text{if } \varepsilon < \frac{1}{2}, \end{cases} \quad (24)$$

where

$$V_{\max} := \max_{\substack{P: I(P, W) = C_\varepsilon(\Gamma) \\ \text{Ec}(X_P) \leq \Gamma}} V_P, \quad (25)$$

$$V_{\min} := \min_{\substack{P: I(P, W) = C_\varepsilon(\Gamma) \\ \text{Ec}(X_P) \leq \Gamma}} V_P \quad (26)$$

by using V_P instead of V_{P, θ_0} . Formula (24) is due to Hayashi [6] (with cost constraint), Polyanskiy, Poor, and Verdú [11] (without cost constraints), and Strassen [14] (without cost constraints under the maximum error probability criterion). \square

Similarly to the first-order coding theorem, Theorem 2 reduces to the following corollary in the case where there are no cost constraints.

Corollary 2: Let \mathcal{W} be a general mixed memoryless channel with measure w . For $\varepsilon \in [0, 1)$ and $R \geq 0$, it holds that

$$D_\varepsilon(R) \geq \sup_P \sup \left\{ S \mid G_w(R, S|P) \leq \varepsilon \right\}. \quad (27)$$

\square

IV. PROOFS OF THEOREMS 1 AND 2

A. Lemmas

We state several lemmas which are used to prove Theorems 1 and 2. We first provide error bounds for codes of fixed length, which hold for any general channel.

Lemma 1 (Feinstein's Upper Bound [4]): For any input variable X^n with values in \mathcal{X}^n , there exists an (n, M_n, ε_n) code such that

$$\varepsilon_n \leq \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n|X^n)}{P_{Y^n}(Y^n)} \leq \frac{1}{n} \log M_n + \eta \right\} + e^{-n\eta}, \quad (28)$$

where² Y^n is the output variable due to X^n via channel W^n and $\eta > 0$ is an arbitrary positive number. \square

The following lemma was first established in [7, Lemma 4] in the context of quantum channel coding. The proof for the classical version is stated in [6, Sect. IX-B]³.

Lemma 2 (Hayashi-Nagaoka's Lower Bound [7]): Let Q^n be an arbitrary probability distribution on \mathcal{Y}^n . Every (n, M_n, ε_n) code \mathcal{C}_n satisfies

$$\varepsilon_n \geq \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n|X^n)}{Q^n(Y^n)} \leq \frac{1}{n} \log M_n - \eta \right\} - e^{-n\eta}, \quad (29)$$

where X^n denotes the random variable subject to the uniform distribution on \mathcal{C}_n , Y^n denotes the output variable due to X^n via channel W^n , and $\eta > 0$ is an arbitrary positive number. \square

We next state lemmas for mixed channels. We first arrange a so-called *expurgated parameter space* which possesses a useful property and is still asymptotically dominant over the whole parameter space. Given a set of arbitrary i.i.d. product probability distributions $Q_\theta^n = Q_\theta^{\times n}$ on \mathcal{Y}^n , let Q^n be given as

$$Q^n(\mathbf{y}) := \int_{\Theta} Q_\theta^n(\mathbf{y}) d\omega(\theta) \quad (\forall \mathbf{y} \in \mathcal{Y}^n), \quad (30)$$

and define

$$\Theta(\mathbf{y}) := \left\{ \theta \in \Theta \mid Q_\theta^n(\mathbf{y}) \leq e^{\sqrt[n]{n}} Q^n(\mathbf{y}) \right\} \quad (\forall \mathbf{y} \in \mathcal{Y}^n) \quad (31)$$

and

$$\tilde{\Theta}(\mathbf{x}, \mathbf{y}) := \left\{ \theta \in \Theta \mid W_\theta^n(\mathbf{y}|\mathbf{x}) \leq e^{\sqrt[n]{n}} W^n(\mathbf{y}|\mathbf{x}) \right\} \quad (\forall (\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n). \quad (32)$$

Let $S_k, k = 1, 2, \dots, N_n$, denote a type (empirical distribution) on \mathcal{Y}^n , where N_n is the number of all distinct types. Let $\tilde{S}_k, k = 1, 2, \dots, \tilde{N}_n$, denote a joint type on $\mathcal{X}^n \times \mathcal{Y}^n$, where \tilde{N}_n is the number of all distinct joint types. Since Q_θ^n is an i.i.d. product probability distribution, the subset $\Theta(\mathbf{y})$ depends only on the type S_k of \mathbf{y} , and therefore it can be denoted as $\Theta(S_k)$ instead of $\Theta(\mathbf{y})$. Likewise, since $W_\theta^n(\mathbf{y}|\mathbf{x})$ is stationary and memoryless, the subset $\tilde{\Theta}(\mathbf{x}, \mathbf{y})$ depends only on the joint type \tilde{S}_k of (\mathbf{x}, \mathbf{y}) , and therefore it can be denoted as $\tilde{\Theta}(\tilde{S}_k)$ instead of $\tilde{\Theta}(\mathbf{x}, \mathbf{y})$. Using

$$\Theta_n := \bigcap_{k=1}^{N_n} \Theta(S_k) \quad \text{and} \quad \tilde{\Theta}_n := \bigcap_{k=1}^{\tilde{N}_n} \tilde{\Theta}(\tilde{S}_k), \quad (33)$$

we define another set

$$\Theta_n^* := \Theta_n \cap \tilde{\Theta}_n. \quad (34)$$

²For random variables U and V , we let P_U denote the probability distribution of U and $P_{U|V}$ denote the conditional probability distribution of U given V .

³Later, we shall generalize this lemma to the mixed channel consisting of general component channels in Lemma 7, whose proof is given in Appendix D.

Lemma 3: Let \mathbf{W} be a general mixed memoryless channel with measure w . Given a set of arbitrary i.i.d. product probability distributions Q_θ^n on \mathcal{Y}^n , let Q^n be defined by (30). Then, it holds that

$$\int_{\Theta_n^*} dw(\theta) \geq 1 - 2(n+1)^{|\mathcal{X}| \cdot |\mathcal{Y}|} e^{-\sqrt[4]{n}}. \quad (35)$$

(Proof) See Appendix A. \square

The following lemmas play a key role in proving the coding theorems for mixed channels.

Lemma 4 (Upper Decomposition Lemma): Let \mathbf{W} be a general mixed memoryless channel with measure w . Then, it holds that

$$\Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n | X^n)}{P_{Y^n}(Y_\theta^n)} \leq z_n \right\} \leq \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq z_n + \frac{\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\} + e^{-\sqrt{n}\gamma} \quad (\forall \theta \in \Theta_n^*), \quad (36)$$

where $\gamma > 0$ and $z_n > 0$ are arbitrary numbers, and Y_θ^n indicates the output variable due to the input X^n via channel W_θ^n .

(Proof) See Appendix B. \square

Lemma 5 (Lower Decomposition Lemma): Let \mathbf{W} be a general mixed memoryless channel with measure w . Given a set of arbitrary i.i.d. product probability distributions Q_θ^n on \mathcal{Y}^n , let Q^n be defined by (30). Then, it holds that

$$\Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n | X^n)}{Q^n(Y_\theta^n)} \leq z_n \right\} \geq \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{Q_\theta^n(Y_\theta^n)} \leq z_n - \frac{\gamma}{\sqrt{n}} - \frac{1}{\sqrt[4]{n^3}} \right\} - e^{-\sqrt{n}\gamma} \quad (\forall \theta \in \Theta_n^*), \quad (37)$$

where $\gamma > 0$ and $z_n > 0$ are arbitrary numbers, and Y_θ^n indicates the output variable due to the input X^n via channel W_θ^n .

(Proof) See Appendix C. \square

Remark 7: As we shall show in the proof of Theorem 1 in the next subsection, there exists an interesting duality between the achievability proof and the converse proof based on Lemmas 4 and 5. Using Upper/Lower Decomposition Lemma has been the standard technique in the analysis of the optimum coding rate in various problems in information theory such as source coding [5, Sect. 1.4], [10], random number generation [9], and hypothesis testing [5, Sect. 4.2] for mixed sources. The proof of Theorem 1 demonstrates that we may also use this standard technique for mixed memoryless channels. Later, we shall also demonstrate in Sect. V-D that Lemma 7 can be used as a powerful alternative to Lemmas 2 and 5, and it saves several steps of the converse proof. \square

B. Proof of Theorem 1

(Proof of Direct Part)

Define

$$\overline{C}_\varepsilon(\Gamma) := \sup_{P: \text{Ec}(X_P) \leq \Gamma} \sup \left\{ R \mid \int_{\{\theta | I(P, W_\theta) < R\}} dw(\theta) \leq \varepsilon \right\}, \quad (38)$$

and then for any small $\delta > 0$ there exists an input distribution $P_0 \in \mathcal{P}(\mathcal{X})$ such that $\text{Ec}(X_{P_0}) \leq \Gamma$ and

$$\sup \left\{ R \mid \int_{\{\theta | I(P_0, W_\theta) < R\}} dw(\theta) \leq \varepsilon \right\} \geq \overline{C}_\varepsilon(\Gamma) - \delta. \quad (39)$$

We fix such a P_0 and show that

$$R = \overline{C}_\varepsilon(\Gamma) - 4\delta. \quad (40)$$

is $(\varepsilon|\Gamma)$ -achievable.

Without loss of generality, we assume that the elements in $\mathcal{X} = \{1, 2, \dots, |\mathcal{X}|\}$ are indexed so that $c(1) \geq c(2) \geq \dots \geq c(|\mathcal{X}|)$. We define the type P_n on \mathcal{X}^n so that

$$P_n(x) = \frac{\lfloor nP_0(x) \rfloor}{n} \quad (x = 1, 2, \dots, |\mathcal{X}| - 1), \quad (41)$$

$$P_n(|\mathcal{X}|) = 1 - \sum_{x=1}^{|\mathcal{X}|-1} P_n(x). \quad (42)$$

Then, it is readily shown that

$$\sum_{x \in \mathcal{X}} P_n(x) c(x) \leq \Gamma, \quad (43)$$

$$|P_n(x) - P_0(x)| \leq \frac{|\mathcal{X}|}{n} \quad (\forall x \in \mathcal{X}), \quad (44)$$

and

$$\lim_{n \rightarrow \infty} P_n(x) = P_0(x) \quad (\forall x \in \mathcal{X}), \quad (45)$$

where (43) follows because P_0 satisfies $\sum_{x \in \mathcal{X}} P_0(x) c(x) \leq \Gamma$.

Let T_n be the set of all sequences $\mathbf{x} \in \mathcal{X}^n$ of type P_n , and consider the input random variable X^n uniformly distributed on T_n . Using Lemma 1 with $\frac{1}{n} \log M_n = R$ and $\eta = \frac{\gamma}{\sqrt{n}}$, where $\gamma > 0$ is an arbitrary positive number, we obtain the following chain of expansions

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varepsilon_n &\leq \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n|X^n)}{P_{Y^n}(Y^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \\ &= \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \\ &= \limsup_{n \rightarrow \infty} \left[\int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \right. \\ &\quad \left. + \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \right] \\ &\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \\ &\quad + \limsup_{n \rightarrow \infty} \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} \\ &\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\} + \limsup_{n \rightarrow \infty} \int_{\Theta - \Theta_n^*} dw(\theta) \\ &= \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{\gamma}{\sqrt{n}} \right\}. \end{aligned} \quad (46)$$

Here, we have used

$$\int_{\Theta - \Theta_n^*} dw(\theta) \leq 2(n+1)^{|\mathcal{X}| \cdot |\mathcal{Y}|} e^{-\sqrt[4]{n}} \quad (47)$$

(cf. Lemma 3) to obtain (46). We apply Lemma 4 with $z_n = R + \frac{\gamma}{\sqrt{n}}$ to (46) to obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varepsilon_n &\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq R + \frac{2\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\} \\ &\leq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq R + \frac{2\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\} \\ &\leq \int_{\Theta} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq R + \frac{2\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\}, \end{aligned} \quad (48)$$

where the inequality in (48) is due to Fatou's lemma. Now notice that

$$\begin{aligned} P_{Y_\theta^n}(\mathbf{y}) &= \frac{1}{|T_n|} \sum_{\mathbf{x} \in T_n} W_\theta^n(\mathbf{y} | \mathbf{x}) \\ &\leq (n+1)^{|\mathcal{X}|} \sum_{\mathbf{x} \in T_n} e^{-nH(P_n)} W_\theta^n(\mathbf{y} | \mathbf{x}) \\ &= (n+1)^{|\mathcal{X}|} \sum_{\mathbf{x} \in T_n} \prod_{i=1}^n P_n(x_i) W_\theta(y_i | x_i) \\ &= (n+1)^{|\mathcal{X}|} (P_n W_\theta)^{\times n}(\mathbf{y}) \quad (\forall \mathbf{y} \in \mathcal{Y}^n), \end{aligned} \quad (49)$$

where $(P_n W_\theta)^{\times n}$ denotes the n product distribution of

$$P_n W_\theta(y) := \sum_{x \in \mathcal{X}} P_n(x) W_\theta(y | x) \quad (\forall y \in \mathcal{Y}). \quad (50)$$

Plugging inequality (49) into (48), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varepsilon_n &\leq \int_{\Theta} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{2\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} + \frac{|\mathcal{X}|}{n} \log(n+1) \right\} \\ &\leq \int_{\Theta} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \delta \right\}. \end{aligned} \quad (51)$$

Inequality (51) implies that there exists $\mathbf{x}_n \in \mathcal{X}^n$ of type P_n such that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \leq \int_{\Theta} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \delta \mid X^n = \mathbf{x}_n \right\} \quad (52)$$

Now, we can write as

$$\frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} = \frac{1}{n} \sum_{i=1}^n \log \frac{W_\theta(Y_{\theta,i} | x_i)}{P_n W_\theta(Y_{\theta,i})}, \quad (53)$$

where

$$\begin{aligned} \mathbf{x}_n &= (x_1, x_2, \dots, x_n), \\ Y_\theta^n &= (Y_{\theta,1}, Y_{\theta,2}, \dots, Y_{\theta,n}). \end{aligned}$$

Notice here that $Y_{\theta,1}, Y_{\theta,2}, \dots, Y_{\theta,n}$ are conditionally independent random variables given $X^n = \mathbf{x}_n$ (under the conditional distribution $W_\theta^n(\cdot | \mathbf{x}_n)$), and therefore the right-hand side of (53) is a sum of conditionally independent random variables given $X^n = \mathbf{x}_n$ with conditional mean

$$\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n \log \frac{W_\theta(Y_{\theta,i} | x_i)}{P_n W_\theta(Y_{\theta,i})} \mid X^n = \mathbf{x}_n \right\} = I(P_n, W_\theta) \quad (54)$$

and conditional variance

$$\begin{aligned}
& \mathbb{V} \left\{ \frac{1}{n} \sum_{i=1}^n \log \frac{W_\theta(Y_{\theta,i}|x_i)}{P_n W_\theta(Y_{\theta,i})} \middle| X^n = \mathbf{x}_n \right\} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_n(x) W_\theta(y|x) \left(\log \frac{W_\theta(y|x)}{P_n W_\theta(y)} - D(W_\theta(\cdot|x) || P_n W_\theta) \right)^2 \\
&= V_{\theta, P_n}.
\end{aligned} \tag{55}$$

Then, we can invoke the weak law of large numbers to the probability term $\Pr\{\cdot\}$ in (52). To do so, we split the parameter space Θ as follows:

$$\Theta_1 := \{\theta \in \Theta | I(P_0, W_\theta) < R + \delta\}, \tag{56}$$

$$\Theta_2 := \{\theta \in \Theta | I(P_0, W_\theta) = R + \delta\}, \tag{57}$$

$$\Theta_3 := \{\theta \in \Theta | I(P_0, W_\theta) > R + \delta\}. \tag{58}$$

It is easily verified that

$$\limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \delta \middle| X^n = \mathbf{x}_n \right\} = \begin{cases} 1, & \text{if } \theta \in \Theta_1 \\ 0, & \text{if } \theta \in \Theta_3 \end{cases} \tag{59}$$

by virtue of the weak law of large numbers and (45), where we should notice that the inequality

$$\max_P V_{\theta, P} < +\infty \quad (\forall \theta \in \Theta) \tag{60}$$

holds due to Han [5, Remark 3.1.1] and Polyanskiy et al. [11, Lemma 62]. Then, (52) is rewritten as

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \varepsilon_n &\leq \int_{\Theta_1 \cup \Theta_2} dw(\theta) = \int_{\{\theta | I(P_0, W_\theta) \leq \overline{C}_\varepsilon(\Gamma) - 3\delta\}} dw(\theta) \\
&\leq \int_{\{\theta | I(P_0, W_\theta) < \overline{C}_\varepsilon(\Gamma) - 2\delta\}} dw(\theta) \leq \varepsilon,
\end{aligned} \tag{61}$$

where the last inequality follows from (39). Hence, $R = \overline{C}_\varepsilon(\Gamma) - 4\delta$ is $(\varepsilon|\Gamma)$ -achievable. \square

(Proof of Converse Part)

Assume that R is $(\varepsilon|\Gamma)$ -achievable. By the definition of $(\varepsilon|\Gamma)$ -achievable rates, there exists an (n, M_n, ε_n) code \mathcal{C}_n with cost constraint Γ such that, for an arbitrary $\delta > 0$,

$$\frac{1}{n} \log M_n \geq R - \delta \quad (\forall n > n_0). \tag{62}$$

By Lemma 2 with $\eta = \frac{\gamma}{\sqrt{n}}$, any (n, M_n, ε_n) code \mathcal{C}_n satisfies

$$\varepsilon_n \geq \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n | X^n)}{Q^n(Y^n)} \leq \frac{1}{n} \log M_n - \frac{\gamma}{\sqrt{n}} \right\} - e^{-\sqrt{n}\gamma}, \tag{63}$$

where X^n denotes the random variable subject to the uniform distribution on the code \mathcal{C}_n and $\gamma > 0$ is an arbitrary number. The output distribution Q^n in (63) is set as follows: Letting Q_θ^n be an output distribution on \mathcal{Y}^n indexed by $\theta \in \Theta$ such as

$$Q_\theta^n(\mathbf{y}) := \frac{1}{N_n} \sum_{P_n \in \mathcal{T}_n} (P_n W_\theta)^{\times n}(\mathbf{y}) \quad (\forall \theta \in \Theta, \forall \mathbf{y} \in \mathcal{Y}^n) \tag{64}$$

where \mathcal{T}_n denotes the set of all types on \mathcal{X}^n of size $N_n := |\mathcal{T}_n|$, and $(P_n W_\theta)^{\times n}$ denotes the n product distribution of $P_n W_\theta$. Using this $\{Q_\theta^n\}_{\theta \in \Theta}$, we define Q^n as

$$Q^n(\mathbf{y}) := \int_{\Theta} Q_\theta^n(\mathbf{y}) dw(\theta) \quad (\forall \mathbf{y} \in \mathcal{Y}^n), \quad (65)$$

where we notice that $Q_\theta^n(\mathbf{y})$ depends only on the type of \mathbf{y} , and so does $Q^n(\mathbf{y})$.

Since R is $(\varepsilon|\Gamma)$ -achievable, the following expansion follows from (62) and (63):

$$\begin{aligned} \varepsilon &\geq \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n|X^n)}{Q^n(Y^n)} \leq R - \delta - \frac{\gamma}{\sqrt{n}} \right\} \\ &\geq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 2\delta \right\} \\ &\geq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 2\delta \right\}. \end{aligned} \quad (66)$$

Applying Lemma 5 with $z_n = R - 2\delta$ to (66) yields

$$\begin{aligned} \varepsilon &\geq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 2\delta - \frac{\gamma}{\sqrt{n}} - \frac{1}{\sqrt[4]{n^3}} \right\} \\ &\geq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \\ &= \limsup_{n \rightarrow \infty} \left[\int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \right. \\ &\quad \left. - \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \right] \\ &\geq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \\ &\quad - \limsup_{n \rightarrow \infty} \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \\ &\geq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} - \limsup_{n \rightarrow \infty} \int_{\Theta - \Theta_n^*} dw(\theta) \\ &= \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\}. \end{aligned} \quad (67)$$

Here, we have used (47) to obtain (67). Notice that

$$\begin{aligned} &\int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \\ &= \sum_{\mathbf{x} \in \mathcal{C}_n} \frac{1}{M_n} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x})}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \mid X^n = \mathbf{x} \right\}, \end{aligned} \quad (68)$$

and therefore there exists a codeword $\mathbf{x}_n \in \mathcal{C}_n$ such that

$$\begin{aligned} &\int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \right\} \\ &\geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x}_n)}{Q_\theta^n(Y_\theta^n)} \leq R - 3\delta \mid X^n = \mathbf{x}_n \right\} \quad (\forall n > n_0). \end{aligned} \quad (69)$$

Let P_n denote the type of such \mathbf{x}_n . By (64), the right-hand side of (69) can be lower bounded as

$$\begin{aligned} & \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{Q_{\theta}^n(Y_{\theta}^n)} \leq R - 3\delta \middle| X^n = \mathbf{x}_n \right\} \\ & \geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)} \leq R - 3\delta - \frac{\log N_n}{n} \middle| X^n = \mathbf{x}_n \right\} \\ & \geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)} \leq R - 4\delta \middle| X^n = \mathbf{x}_n \right\} \quad (\forall n \geq \tilde{n}_0), \end{aligned} \quad (70)$$

where we have used the inequality $N_n \leq (n+1)^{|\mathcal{X}|}$ to obtain (70). Combining (67), (69), and (70) yields

$$\varepsilon \geq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)} \leq R - 4\delta \middle| X^n = \mathbf{x}_n \right\}. \quad (71)$$

Since $\{P_n\}_{n > \tilde{n}_0}$ is a sequence in $\mathcal{P}(\mathcal{X})$ (compact set), it always contains a converging subsequence $\{P_{n_1}, P_{n_2}, \dots\}$, where $n_1 < n_2 < \dots \rightarrow \infty$. We denote the convergent point by P_0 ;

$$\lim_{i \rightarrow \infty} P_{n_i} = P_0, \quad (72)$$

where it should be noticed that P_0 satisfies cost constraint: $\mathbb{E}c(X_{P_0}) \leq \Gamma$ because P_n satisfies the same cost constraint Γ . For notational simplicity, we relabel n_k as $m = n_1, n_2, \dots$. Then, in view of

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)} \leq R - 4\delta \middle| X^n = \mathbf{x}_n \right\} \\ & \geq \limsup_{m \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} \leq R - 4\delta \middle| X^m = \mathbf{x}_m \right\}, \end{aligned} \quad (73)$$

(71) becomes

$$\begin{aligned} \varepsilon & \geq \limsup_{m \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} \leq R - 4\delta \middle| X^m = \mathbf{x}_m \right\} \\ & \geq \int_{\Theta} dw(\theta) \liminf_{m \rightarrow \infty} \Pr \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} \leq R - 4\delta \middle| X^m = \mathbf{x}_m \right\} \end{aligned} \quad (74)$$

$$\geq \int_{\Theta_1} dw(\theta) \liminf_{m \rightarrow \infty} \Pr \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} \leq R - 4\delta \middle| X^m = \mathbf{x}_m \right\}, \quad (75)$$

where the inequality in (74) is due to Fatou's lemma, and Θ_1 is defined as

$$\Theta_1 := \{\theta \in \Theta \mid I(P_0, W_{\theta}) < R - 4\delta\}. \quad (76)$$

Set $\mathbf{x}_m = (x_1, x_2, \dots, x_m)$, and then

$$\frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} = \frac{1}{m} \sum_{i=1}^m \log \frac{W_{\theta}(Y_{\theta,i} | x_i)}{(P_m W_{\theta})(Y_{\theta,i})} \quad (77)$$

is a sum of conditionally independent random variables given $X^m = \mathbf{x}_m$, and its expectation and variance under $W_{\theta}^m(\cdot | \mathbf{x}_m)$ are given by

$$\mathbb{E} \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} \middle| X^m = \mathbf{x}_m \right\} = I(P_m, W_{\theta}) \quad (78)$$

and

$$\begin{aligned}
& \mathbb{V} \left\{ \frac{1}{m} \log \frac{W_\theta^m(Y_\theta^m | \mathbf{x}_m)}{(P_m W_\theta)^{\times m}(Y_\theta^m)} \middle| X^m = \mathbf{x}_m \right\} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_m(x) W_\theta(y|x) \left(\log \frac{W_\theta(y|x)}{(P_m W_\theta)(y)} - D(W_\theta(\cdot|x) || P_m W_\theta) \right)^2 \\
&= V_{\theta, P_m},
\end{aligned} \tag{79}$$

respectively. Hence, the weak law of large numbers guarantees

$$\liminf_{m \rightarrow \infty} \Pr \left\{ \frac{1}{m} \log \frac{W_\theta^m(Y_\theta^m | \mathbf{x}_m)}{(P_m W_\theta)^{\times m}(Y_\theta^m)} \leq R - 4\delta \middle| X^m = \mathbf{x}_m \right\} = 1 \quad (\forall \theta \in \Theta_1). \tag{80}$$

Thus, (75) is rewritten as

$$\varepsilon \geq \int_{\Theta_1} dw(\theta) = \int_{\{\theta | I(P_0, W_\theta) < R - 4\delta\}} dw(\theta). \tag{81}$$

Therefore, from the definition of $\overline{C}_\varepsilon(\Gamma)$ (cf. (38)), we have

$$R - 4\delta \leq \overline{C}_\varepsilon(\Gamma). \tag{82}$$

On the other hand, since $\delta > 0$ is arbitrary, we conclude that $R \leq \overline{C}_\varepsilon(\Gamma)$. \square

C. Proof of Theorem 2

We first define

$$\overline{D}_\varepsilon(R|\Gamma) := \sup_{P: \text{Ec}(X_P) \leq \Gamma} \sup \{S | G_w(R, S|P) \leq \varepsilon\}, \tag{83}$$

where see (16) as for the definition of $G_w(R, S|P)$. Then, for any $\delta > 0$ there exists an input distribution $P_0 \in \mathcal{P}(\mathcal{X})$ such that $\text{Ec}(X_{P_0}) \leq \Gamma$, where X_{P_0} denotes the random variable subject to P_0 , and

$$\sup \{S | G_w(R, S|P_0) \leq \varepsilon\} \geq \overline{D}_\varepsilon(R|\Gamma) - \delta. \tag{84}$$

We shall show that $S = \overline{D}_\varepsilon(R|\Gamma) - 4\delta$ is $(\varepsilon, R|\Gamma)$ -achievable.

Fix a P_0 satisfying (84) and a constant $\gamma > 0$ such that $\delta > 2\gamma$. By Lemma 1 with

$$\frac{1}{n} \log M_n = R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta) \tag{85}$$

and $\eta = \frac{\gamma}{\sqrt{n}}$, we have

$$\varepsilon_n \leq \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n | X^n)}{P_{Y^n}(Y^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} + e^{-\sqrt{n}\gamma}. \tag{86}$$

We choose a type P_n on \mathcal{X}^n so as to be specified by (43)–(45). Let X^n be the uniformly distributed input random variable on T_n , defined to be the set of all sequences $\mathbf{x} \in \mathcal{X}^n$ of type P_n . Then, we have

$$P_{Y_\theta^n}(\mathbf{y}) \leq (n+1)^{|\mathcal{X}|} (P_n W_\theta)^{\times n}(\mathbf{y}) \quad (\forall \mathbf{y} \in \mathcal{Y}^n) \tag{87}$$

by (49). Then, by (86), we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \varepsilon_n &\leq \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y^n|X^n)}{P_{Y^n}(Y^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \\
&\leq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \\
&= \limsup_{n \rightarrow \infty} \left[\int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \right. \\
&\quad \left. + \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \right] \\
&\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \\
&\quad + \limsup_{n \rightarrow \infty} \int_{\Theta - \Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\} \\
&= \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n|X^n)}{P_{Y^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma) \right\}, \quad (88)
\end{aligned}$$

where the last equality is due to (47).

Now by (87) and Lemma 4 with $z_n = R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + \gamma)$,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \varepsilon_n &\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + 2\gamma) + \frac{1}{\sqrt[4]{n^3}} \right\} \\
&\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 4\delta + 2\gamma) \right. \\
&\quad \left. + \frac{1}{\sqrt[4]{n^3}} + \frac{|\mathcal{X}| \log(n+1)}{n} \right\} \\
&\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \right\}. \quad (89)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|X^n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \right\} \\
&= \sum_{\mathbf{x} \in T_n} \Pr\{X^n = \mathbf{x}\} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x})}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x} \right\}, \quad (90)
\end{aligned}$$

there exists an $\mathbf{x}_n \in T_n$ such that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \varepsilon_n &\leq \limsup_{n \rightarrow \infty} \int_{\Theta_n^*} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\} \\
&\leq \limsup_{n \rightarrow \infty} \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\} \\
&\leq \int_{\Theta} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n|\mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\}, \quad (91)
\end{aligned}$$

where the last inequality is due to Fatou's lemma.

Now, again since

$$\frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)}$$

is a sum of conditionally independent random variables given $X^n = \mathbf{x}_n$, by virtue of (45), (53)–(55) and the weak law of large numbers, we have

$$\limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\} = \begin{cases} 1, & \text{if } \theta \in \Theta_1 \\ 0, & \text{if } \theta \in \Theta_3 \end{cases}, \quad (92)$$

where Θ_i ($i = 1, 2, 3$) is defined as

$$\Theta_1 := \{\theta \in \Theta \mid I(P_0, W_\theta) < R\}, \quad (93)$$

$$\Theta_2 := \{\theta \in \Theta \mid I(P_0, W_\theta) = R\}, \quad (94)$$

$$\Theta_3 := \{\theta \in \Theta \mid I(P_0, W_\theta) > R\}. \quad (95)$$

Thus,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varepsilon_n &\leq \int_{\Theta_1} dw(\theta) \\ &+ \int_{\Theta_2} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq R + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\}. \end{aligned} \quad (96)$$

Denoting the second term on the right-hand side by B , we have

$$\begin{aligned} B &= \int_{\{\theta \mid I(P_0, W_\theta) = R\}} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} \leq I(P_0, W_\theta) + \frac{1}{\sqrt{n}} (\overline{D}_\varepsilon(R|\Gamma) - 3\delta) \mid X^n = \mathbf{x}_n \right\}, \\ &= \int_{\{\theta \mid I(P_0, W_\theta) = R\}} dw(\theta) \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\sqrt{n}} \left(\log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} - nI(P_n, W_\theta) \right) \right. \\ &\quad \left. \leq \overline{D}_\varepsilon(R|\Gamma) - 3\delta + \sqrt{n}(I(P_0, W_\theta) - I(P_n, W_\theta)) \mid X^n = \mathbf{x}_n \right\}. \end{aligned} \quad (97)$$

Now, we notice that, owing to (44),

$$\lim_{n \rightarrow \infty} \sqrt{n}(I(P_0, W_\theta) - I(P_n, W_\theta)) = 0 \quad (98)$$

and

$$\lim_{n \rightarrow \infty} V_{\theta, P_n} = V_{\theta, P_0}, \quad (99)$$

and therefore, for $\theta \in \Theta_2$ with $V_{\theta, P_0} > 0$ the central limit theorem assures that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\sqrt{n}} \left(\log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} - nI(P_n, W_\theta) \right) \right. \\ &\quad \left. \leq \overline{D}_\varepsilon(R|\Gamma) - 3\delta + \sqrt{n}(I(P_0, W_\theta) - I(P_n, W_\theta)) \mid X^n = \mathbf{x}_n \right\} \\ &= \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\sqrt{n}} \left(\log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_n W_\theta)^{\times n}(Y_\theta^n)} - nI(P_n, W_\theta) \right) \leq \overline{D}_\varepsilon(R|\Gamma) - 3\delta \mid X^n = \mathbf{x}_n \right\} \\ &\leq \Psi_{\theta, P_0}(\overline{D}_\varepsilon(R|\Gamma) - 2\delta). \end{aligned} \quad (100)$$

For $\theta \in \Theta_2$ with $V_{\theta, P_0} = 0$, we interpret $\Psi_{\theta, P_0}(z)$ as the step function which takes zero for $z < 0$ and one otherwise. It is easily verified that (100) also holds for such $\theta \in \Theta_2$, and hence

$$B \leq \int_{\{\theta | I(P_0, W_\theta) = R\}} \Psi_{\theta, P_0}(\overline{D}_\varepsilon(R|\Gamma) - 2\delta) dw(\theta). \quad (101)$$

Thus, by (96),

$$\limsup_{n \rightarrow \infty} \varepsilon_n \leq \int_{\{\theta | I(P_0, W_\theta) < R\}} dw(\theta) + \int_{\{\theta | I(P_0, W_\theta) = R\}} \Psi_{\theta, P_0}(\overline{D}_\varepsilon(R|\Gamma) - 2\delta) dw(\theta) \leq \varepsilon, \quad (102)$$

where the last inequality follows from (84), implying that $\overline{D}_\varepsilon(R|\Gamma) - 4\delta$ is $(\varepsilon, R|\Gamma)$ -achievable. \square

V. CODING THEOREMS FOR WELL-ORDERED MIXED MEMORYLESS CHANNEL

A. Well-Ordered Mixed Memoryless Channel

So far, in Sect. III-B, we have established Theorem 2 on the second-order capacity for the mixed memoryless channel with general mixture; however, unfortunately, this theorem lacks the converse part. Thus, in this section, we are led to introduce a subclass of general mixed memoryless channels for which the second-order coding theorem is established, including both of the direct and converse parts.

Definition 3: Let $W_\Theta = \{W_\theta : \mathcal{X} \rightarrow \mathcal{Y}\}_{\theta \in \Theta}$ be a family of stationary memoryless channels. Let $c_{\theta, \Gamma}$ denote the capacity of component channel W_θ with cost constraint Γ ($\geq \Gamma_0$), that is,

$$c_{\theta, \Gamma} = \max_{P: \mathbb{E}c(X_P) \leq \Gamma} I(P, W_\theta), \quad (103)$$

and let $\Pi_{\theta, \Gamma}$ denote the set of input probability distributions P on \mathcal{X} that achieve $c_{\theta, \Gamma}$. It should be noted that $\Pi_{\theta, \Gamma}$ is a bounded closed set. If W_Θ is closed and, for any $\theta \in \Theta$ and any $P \in \Pi_{\theta, \Gamma}$, it holds that

$$\begin{aligned} c_{\theta, \Gamma} &= I(P, W_{\theta'}) \quad \text{for } \theta' \in \Theta \text{ s.t. } c_{\theta, \Gamma} = c_{\theta', \Gamma} \text{ and} \\ c_{\theta, \Gamma} &< I(P, W_{\theta'}) \quad \text{for } \theta' \in \Theta \text{ s.t. } c_{\theta, \Gamma} < c_{\theta', \Gamma}, \end{aligned} \quad (104)$$

then W_Θ is said to be *well-ordered with cost constraint Γ* , or simply Γ -*well-ordered*. A mixed memoryless channel \mathbf{W} with Γ -well-ordered W_Θ is referred to as Γ -*well-ordered* mixed memoryless channel. \square

Remark 8: For a Γ -well-ordered mixed memoryless channel, it is not difficult to check that

$$\Pi_{\theta, \Gamma} = \Pi_{\theta', \Gamma} \quad \text{if } c_{\theta, \Gamma} = c_{\theta', \Gamma} \text{ for } \theta, \theta' \in \Theta, \quad (105)$$

that is, two component channels with equal capacity have the same set of capacity-achieving input distributions. \square

Remark 9: The assumption that W_Θ is closed is made just due to a technical reason. Even in the case where W_Θ is not closed, if its closure denoted by $W_{\overline{\Theta}}$ (with extended parameter space $\overline{\Theta}$) is Γ -well-ordered, all coding theorems we shall establish also hold for the mixed channel \mathbf{W} with the original W_Θ . \square

Example 1: For two channels W_θ and $W_{\theta'}$, channel $W_{\theta'}$ is said to be *more capable* than W_θ if $I(P, W_\theta) \leq I(P, W_{\theta'})$ for all $P \in \mathcal{P}(\mathcal{X})$ [3]. If $W_{\theta'}$ is more capable than W_θ for all $\theta, \theta' \in \Theta$ such that $c_\theta \leq c_{\theta'}$, then W_Θ is Γ -well-ordered for all $\Gamma \geq \Gamma_0$, where c_θ denotes the capacity of W_θ with no cost constraints. The followings are examples of such W_Θ :

- A family of binary symmetric channels which forms a closed set.
- More generally, a closed set of additive noise channels for which additive noise $Z \sim W_\theta(\cdot|\cdot)$ is a degraded version of additive noise $Z' \sim W_{\theta'}(\cdot|\cdot)$ for all $\theta, \theta' \in \Theta$ such that $c_\theta \leq c_{\theta'}$. \square

Example 2: In the special case of $\Gamma = +\infty$ (that is, without cost constraints), we may find much more examples of Γ -well-ordered W_Θ . A family of output-symmetric channels which forms a closed set is

Γ -well-ordered since the capacity-achieving input distribution is uniform on \mathcal{X} and unique (cf. Shannon [13]). \square

Set $E_{\Theta, \Gamma} := \{c_{\theta, \Gamma} \mid \theta \in \Theta\}$. We show an important property of Γ -well-ordered mixed memoryless channels.

Lemma 6: If W_{Θ} is closed, then $E_{\Theta, \Gamma}$ is bounded and closed for all $\Gamma \geq \Gamma_0$.

(Proof) Boundedness of $E_{\Theta, \Gamma}$ is obvious, so we shall show its closedness. Let a function $f : \mathcal{P}(\mathcal{X} \rightarrow \mathcal{Y}) \rightarrow [0, +\infty)$ be defined as

$$f(W) := \max_{P \in \mathcal{P}_C} I(P, W) \quad (106)$$

for a given closed convex set $\mathcal{P}_C \subseteq \mathcal{P}(\mathcal{X})$, where $\mathcal{P}(\mathcal{X} \rightarrow \mathcal{Y})$ denotes the set of all channel matrices $W : \mathcal{X} \rightarrow \mathcal{Y}$. Since $I(P, W)$ is continuous with respect to (P, W) , the $f(W)$ is a continuous function of W . The image of a closed set by a continuous function is also closed. Hence, since $\mathcal{P}_C := \{P \in \mathcal{P}(\mathcal{X}) \mid \text{Ec}(X_P) \leq \Gamma\}$ is closed and convex, we can conclude that $E_{\Theta, \Gamma} = f(W_{\Theta})$ is closed. \square

B. Coding Theorems

We first provide a characterization of the first-order capacity $C_{\varepsilon}(\Gamma)$, which is different from the one in Theorem 1, for Γ -well-ordered mixed memoryless channels. This alternative characterization is of simpler form and is of great use to analyze the second-order capacity later.

Theorem 3: Let \mathbf{W} be a Γ -well-ordered mixed memoryless channel with general measure w . For any fixed $\varepsilon \in [0, 1)$ and $\Gamma \geq \Gamma_0$, the first-order $(\varepsilon|\Gamma)$ -capacity is given by

$$C_{\varepsilon}(\Gamma) = \sup \left\{ R \mid \int_{\{\theta \mid c_{\theta, \Gamma} < R\}} dw(\theta) \leq \varepsilon \right\}. \quad (107)$$

\square

Remark 10: Due to the closedness of $E_{\Theta, \Gamma}$, for every $\varepsilon \in [0, 1)$ there exists some $\bar{\theta} \in \Theta$ such that $C_{\varepsilon}(\Gamma) = c_{\bar{\theta}, \Gamma}$. This fact is shown in the the proof of the converse part of Theorem 3 in Sect. V-C. \square

Remark 11: The characterization (107) with $\Gamma = +\infty$ is a generalization of the one given by Winkelbauer [18] in the sense that the class of Γ -well-ordered mixed channels with $\Gamma = +\infty$ is wider than the class of *regular decomposable* channels with stationary memoryless components. On the other hand, the regular decomposability allows component channels to be stationary and ergodic, which means that the characterization (107) with $\Gamma = +\infty$ is a particularization of the one given in [18]. \square

Now, we turn to discussing the second-order capacity of Γ -well-ordered mixed memoryless channels. In contrast to mixed memoryless channels with general mixture, for which only the direct part of the second-order coding theorem (Theorem 2) has been given, Γ -well-orderedness allows us to establish the converse theorem as well.

Theorem 4: Let \mathbf{W} be a Γ -well-ordered mixed memoryless channel with general measure w . Then, for $\varepsilon \in [0, 1)$, $\Gamma \geq \Gamma_0$, and $R \geq 0$, it holds that

$$\begin{aligned} D_{\varepsilon}(R|\Gamma) &= \sup_{P: \text{Ec}(X_P) \leq \Gamma} \sup \left\{ S \mid \int_{\{\theta \mid I(P, W_{\theta}) < R\}} dw(\theta) + \int_{\{\theta \mid I(P, W_{\theta}) = R\}} \Psi_{\theta, P}(S) dw(\theta) \leq \varepsilon \right\} \\ &= \sup_{P \in \Pi_{\bar{\theta}, \Gamma}} \sup \left\{ S \mid \int_{\{\theta \mid I(P, W_{\theta}) < R\}} dw(\theta) + \int_{\{\theta \mid I(P, W_{\theta}) = R\}} \Psi_{\theta, P}(S) dw(\theta) \leq \varepsilon \right\} \\ &= \sup_{P \in \Pi_{\bar{\theta}, \Gamma}} \sup \left\{ S \mid \int_{\{\theta \mid c_{\theta, \Gamma} < R\}} dw(\theta) + \int_{\{\theta \mid c_{\theta, \Gamma} = R\}} \Psi_{\theta, P}(S) dw(\theta) \leq \varepsilon \right\}, \end{aligned} \quad (108)$$

where $\bar{\theta} \in \Theta$ gives the $(\varepsilon|\Gamma)$ -capacity, that is $C_{\varepsilon}(\Gamma) = c_{\bar{\theta}, \Gamma}$. \square

Remark 12: Formula (108) has been established for the case of $|\Theta| < +\infty$ by Yagi and Nomura [21]. When the component channels are output-symmetric and $\Gamma = +\infty$, the first supremum (with respect to P) on the right-hand side of (108) is attained by only the *uniform* inputs, which may facilitate the proof of the coding theorem. \square

Remark 13: It is not difficult to check from formula (107) that

$$\int_{\{\theta | c_{\theta, \Gamma} < C_\varepsilon(\Gamma)\}} dw(\theta) \leq \varepsilon, \quad (109)$$

$$\int_{\{\theta | c_{\theta, \Gamma} \leq C_\varepsilon(\Gamma)\}} dw(\theta) \geq \varepsilon \quad (110)$$

hold, and like in Remark 5 here also we may consider the following canonical equation for S :

$$\int_{\Theta} dw(\theta) \lim_{n \rightarrow \infty} \Psi_{\theta, P}(\sqrt{n}(C_\varepsilon(\Gamma) - c_{\theta, \Gamma}) + S) = \varepsilon. \quad (111)$$

Notice here, in view of (109) and (110), that equation (111) always has a solution. Let $S_P(\varepsilon)$ denote the solution of this equation, where $S_P(\varepsilon) = +\infty$ if the solution is not unique. Then, the $D_\varepsilon(C_\varepsilon(\Gamma)|\Gamma)$ (i.e., $R = C_\varepsilon(\Gamma)$) can be rewritten in a simpler form as

$$D_\varepsilon(C_\varepsilon(\Gamma)|\Gamma) = \sup_{P \in \Pi_{\bar{\theta}, \Gamma}} S_P(\varepsilon), \quad (112)$$

which is again sometimes preferable to the expression in (108). \square

C. Proof of Theorem 3

(Proof of Converse Part)

By definition, it holds that $I(P, W_\theta) \leq c_{\theta, \Gamma}$ for all $\theta \in \Theta$ if P satisfies $\text{Ec}(X_P) \leq \Gamma$. Therefore, by (8) in Theorem 1, we have

$$\begin{aligned} C_\varepsilon(\Gamma) &\leq \sup_{P: \text{Ec}(X_P) \leq \Gamma} \sup \left\{ R \mid \int_{\{\theta | c_{\theta, \Gamma} < R\}} dw(\theta) \leq \varepsilon \right\} \\ &= \sup \left\{ R \mid \int_{\{\theta | c_{\theta, \Gamma} < R\}} dw(\theta) \leq \varepsilon \right\}. \end{aligned} \quad (113)$$

\square

(Proof of Direct Part)

Set

$$\bar{R} = \sup \left\{ R \mid \int_{\{\theta | c_{\theta, \Gamma} < R\}} dw(\theta) \leq \varepsilon \right\} \quad (114)$$

for notational simplicity. Consider an increasing sequence $R_1 \leq R_2 \leq \dots \rightarrow \bar{R}$ such that

$$\int_{\{\theta | c_{\theta, \Gamma} < R_i\}} dw(\theta) \leq \varepsilon \quad (\forall i = 1, 2, \dots). \quad (115)$$

Then, we have

$$\int_{\{\theta | c_{\theta, \Gamma} < \bar{R}\}} dw(\theta) \leq \varepsilon \quad (116)$$

by the continuity of probability measures. Now suppose that \bar{R} is not an accumulation point of $E_{\Theta, \Gamma}$ to show a contradiction. Then, there exists some $\nu > 0$ such that

$$(\bar{R} - \nu, \bar{R} + \nu) \cap E_{\Theta, \Gamma} = \emptyset. \quad (117)$$

This implies that $\{\theta \mid \bar{R} \leq c_{\theta, \Gamma} < \bar{R} + \nu\} = \emptyset$, and hence, we have

$$\int_{\{\theta \mid c_{\theta, \Gamma} < \bar{R} + \nu\}} dw(\theta) = \int_{\{\theta \mid c_{\theta, \Gamma} < \bar{R}\}} dw(\theta) \leq \varepsilon, \quad (118)$$

which contradicts the definition of \bar{R} . Therefore, \bar{R} is an accumulation point of $E_{\Theta, \Gamma}$. Since $E_{\Theta, \Gamma}$ is a closed set by Lemma 6, it holds that $\bar{R} \in E_{\Theta, \Gamma}$, and there exists some $\bar{\theta} \in \Theta$ such that $\bar{R} = c_{\bar{\theta}, \Gamma}$.

Fixing $P \in \Pi_{\bar{\theta}, \Gamma}$ arbitrarily, we have

$$\begin{aligned} & \int_{\{\theta \mid I(P, W_\theta) < \bar{R}\}} dw(\theta) \\ &= \int_{\{\theta \mid I(P, W_\theta) < \bar{R}, c_{\theta, \Gamma} < \bar{R}\}} dw(\theta) + \int_{\{\theta \mid I(P, W_\theta) < \bar{R}, c_{\theta, \Gamma} \geq \bar{R}\}} dw(\theta) \\ &= \int_{\{\theta \mid I(P, W_\theta) < \bar{R}, c_{\theta, \Gamma} < \bar{R}\}} dw(\theta), \end{aligned} \quad (119)$$

where the last equality follows from the fact that there are no $\theta \in \Theta$ such that $c_{\theta, \Gamma} \geq \bar{R} = c_{\bar{\theta}, \Gamma}$ and $I(P, W_\theta) < \bar{R}$ for $P \in \Pi_{\bar{\theta}, \Gamma}$ by the definition of Γ -well-orderedness. Noticing that $\{\theta \mid I(P, W_\theta) < \bar{R}, c_{\theta, \Gamma} < \bar{R}\} = \{\theta \mid c_{\theta, \Gamma} < \bar{R}\}$ for $P \in \Pi_{\bar{\theta}, \Gamma}$ in (119), we have

$$\int_{\{\theta \mid I(P, W_\theta) < \bar{R}\}} dw(\theta) = \int_{\{\theta \mid c_{\theta, \Gamma} < \bar{R}\}} dw(\theta) \leq \varepsilon, \quad (120)$$

and formula (8) in Theorem 1 indicates that $\bar{R} \leq C_\varepsilon(\Gamma)$. \square

D. Proof of Theorem 4

(Proof of Direct Part)

It apparently holds, with $G_w(R, S|P)$ as in (16), that

$$\sup_{P: \text{Ec}(X_P) \leq \Gamma} \sup \left\{ S \mid G_w(R, S|P) \leq \varepsilon \right\} \geq \sup_{P \in \Pi_{\bar{\theta}, \Gamma}} \sup \left\{ S \mid G_w(R, S|P) \leq \varepsilon \right\} \quad (121)$$

since any $P \in \Pi_{\bar{\theta}, \Gamma}$ satisfies cost constraint: $\text{Ec}(X_P) \leq \Gamma$. Therefore, by Theorem 2, any S such that

$$S < \sup_{P \in \Pi_{\bar{\theta}, \Gamma}} \sup \left\{ S \mid G_w(R, S|P) \leq \varepsilon \right\} \quad (122)$$

is $(\varepsilon, R|\Gamma)$ -achievable. \square

(Proof of Converse Part)

Although the converse part can be established on the basis of Lemmas 2 and 5 in a manner similar to the converse proof of Theorem 1, here instead of these lemmas, we use the following simple but powerful lower bound on the probability of decoding error, which is of independent interest and facilitates the proof of this converse part.

Lemma 7: Let $\{Q_\theta^n\}_{\theta \in \Theta}$ be a family of arbitrarily fixed output distributions on \mathcal{Y}^n . Every (n, M_n, ε_n) code \mathcal{C}_n for the mixed channel W^n given in (1) satisfies

$$\varepsilon_n \geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{Q_\theta^n(Y_\theta^n)} \leq \frac{1}{n} \log M_n - \eta \right\} - e^{-n\eta} \quad (123)$$

with an arbitrary number $\eta > 0$, where X^n is uniformly distributed on \mathcal{C}_n .

(Proof) See Appendix D. \square

Remark 14: It should be noted that Lemma 7 holds for arbitrary alphabets \mathcal{X}, \mathcal{Y} (not necessarily finite). \square

Since formula (108) trivially holds in the cases $R < C_\varepsilon(\Gamma)$ (with $D_\varepsilon(R|\Gamma) = +\infty$) and $R > C_\varepsilon(\Gamma)$ (with $D_\varepsilon(R|\Gamma) = -\infty$), hereafter we shall prove only for the case $R = C_\varepsilon(\Gamma)$, which is of our main interest. Assume that S is $(\varepsilon, R|\Gamma)$ -achievable. Then, by definition, for any given $\gamma > 0$ there exists an (n, M_n, ε_n) code with cost constraint Γ such that

$$\frac{1}{n} \log M_n \geq R + \frac{S - \gamma}{\sqrt{n}} \quad (\forall n \geq n_0). \quad (124)$$

Following a technique developed by Hayashi [6], let Q_θ^n be the output distribution on \mathcal{Y}^n indexed by $\theta \in \Theta$ such that

$$Q_\theta^n(\mathbf{y}) = \sum_{P_n \in \mathcal{T}_n} \frac{(P_n W_\theta)^{\times n}(\mathbf{y})}{N_n + 1} + \frac{(P_\theta W_\theta)^{\times n}(\mathbf{y})}{N_n + 1} \quad (\forall \theta \in \Theta, \forall \mathbf{y} \in \mathcal{Y}^n), \quad (125)$$

where \mathcal{T}_n with $N_n = |\mathcal{T}_n|$ denotes the set of all types on \mathcal{X}^n and P_θ is an arbitrary input distribution in $\Pi_{\theta, \Gamma}$. It should be noted that the capacity-achieving output distribution $P_\theta W_\theta$ for W_θ is the same for all $P_\theta \in \Pi_{\theta, \Gamma}$, and this enables us to choose a particular $P_\theta \in \Pi_{\theta, \Gamma}$ later. Using this $\{Q_\theta^n\}_{\theta \in \Theta}$, we define Q^n as in (65). Lemma 7 by replacing η with $\frac{\gamma}{\sqrt{n}}$ assures that the sequence of (n, M_n, ε_n) codes \mathcal{C}_n (satisfying cost constraint Γ) such that

$$\begin{aligned} \varepsilon_n &\geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{Q_\theta^n(Y_\theta^n)} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \right\} - e^{-\sqrt{n}\gamma} \\ &= \sum_{\mathbf{x}_n \in \mathcal{X}^n} P_{X^n}(\mathbf{x}_n) \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{Q_\theta^n(Y_\theta^n)} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \middle| X^n = \mathbf{x}_n \right\} - e^{-\sqrt{n}\gamma}, \end{aligned} \quad (126)$$

where P_{X^n} is the uniform distribution on \mathcal{C}_n . This implies that there exists a codeword \mathbf{x}_n such that

$$\varepsilon_n \geq \int_{\Theta} dw(\theta) \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{Q_\theta^n(Y_\theta^n)} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \middle| X^n = \mathbf{x}_n \right\} - e^{-\sqrt{n}\gamma}. \quad (127)$$

Now, we partition the parameter space Θ as follows:

$$\Theta_1 := \{\theta \in \Theta \mid c_{\theta, \Gamma} < R\}, \quad (128)$$

$$\Theta_2 := \{\theta \in \Theta \mid c_{\theta, \Gamma} = R\}, \quad (129)$$

$$\Theta_3 := \{\theta \in \Theta \mid c_{\theta, \Gamma} > R\}. \quad (130)$$

Using these partitioned spaces, we further bound (127) as

$$\varepsilon_n \geq \int_{\Theta_1} dw(\theta) B_{\theta, n} + \int_{\Theta_2} dw(\theta) B_{\theta, n} - e^{-\sqrt{n}\gamma}, \quad (131)$$

where we have set

$$B_{\theta, n} := \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{Q_\theta^n(Y_\theta^n)} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \middle| X^n = \mathbf{x}_n \right\}. \quad (132)$$

Let $P_n \in \mathcal{T}_n$ denote the type of \mathbf{x}_n (obviously, this P_n satisfies $\text{Ec}(X_{P_n}) \leq \Gamma$, where X_{P_n} denotes the random variable subject to P_n). By (125), the probability term $B_{\theta, n}$ is lower bounded in two ways as

$$B_{\theta, n} \geq \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | \mathbf{x}_n)}{(P_\theta W_\theta)^{\times n}(Y_\theta^n)} + \frac{\log(N_n + 1)}{n} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \middle| X^n = \mathbf{x}_n \right\} =: \alpha_{\theta, n}. \quad (133)$$

and

$$B_{\theta,n} \geq \Pr \left\{ \frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)} + \frac{\log(N_n + 1)}{n} \leq R + \frac{S - 2\gamma}{\sqrt{n}} \middle| X^n = \mathbf{x}_n \right\} =: \beta_{\theta,n} \quad (134)$$

It should be noted that both $B_{\theta,n}$ and $\alpha_{\theta,n}$ do not depend on the choice of $P_{\theta} \in \Pi_{\theta,\Gamma}$ in (125) since $P_{\theta}W_{\theta}$ is unique. Notice that $\frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_n W_{\theta})^{\times n}(Y_{\theta}^n)}$ in (134) (cf. (53)) is a sum of conditionally independent random variables given $X^n = \mathbf{x}_n$ (under $W_{\theta}^n(\cdot | \mathbf{x}_n)$) with mean $I(P_n, W_{\theta})$ and variance V_{θ,P_n} , which is given as in (55). Moreover,

$$\frac{1}{n} \log \frac{W_{\theta}^n(Y_{\theta}^n | \mathbf{x}_n)}{(P_{\theta} W_{\theta})^{\times n}(Y_{\theta}^n)} = \frac{1}{n} \sum_{i=1}^n \log \frac{W_{\theta}(Y_{\theta,i} | x_i)}{P_{\theta} W_{\theta}(Y_{\theta,i})} \quad (135)$$

is a sum of conditionally independent random variables given $X^n = \mathbf{x}_n = (x_1, x_2, \dots, x_n)$ (under $W_{\theta}^n(\cdot | \mathbf{x}_n)$) with mean

$$\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n \log \frac{W_{\theta}(Y_{\theta,i} | x_i)}{P_{\theta} W_{\theta}(Y_{\theta,i})} \middle| X^n = \mathbf{x}_n \right\} = \sum_{x \in \mathcal{X}} P_n(x) D(W_{\theta}(\cdot | x) \| P_{\theta} W_{\theta}) \quad (136)$$

and variance

$$\begin{aligned} \mathbb{V} \left\{ \frac{1}{n} \sum_{i=1}^n \log \frac{W_{\theta}(Y_{\theta,i} | x_i)}{P_{\theta} W_{\theta}(Y_{\theta,i})} \middle| X^n = \mathbf{x}_n \right\} &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_n(x) W_{\theta}(y | x) \left(\log \frac{W_{\theta}(y | x)}{P_{\theta} W_{\theta}(y)} - D(W_{\theta}(\cdot | x) \| P_{\theta} W_{\theta}) \right)^2 \\ &=: V_{\theta,P_{\theta}}(\mathbf{x}_n). \end{aligned} \quad (137)$$

Since $\{P_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{P}(\mathcal{X})$, which is compact, it always contains a converging subsequence $\{P_{n_1}, P_{n_2}, \dots\}$, where $n_1 < n_2 < \dots \rightarrow \infty$. We denote the convergent point by P_0 ;

$$\lim_{i \rightarrow \infty} P_{n_i} = P_0, \quad (138)$$

where it should be noticed that P_0 also satisfies cost constraint: $\mathbb{E}c(X_{P_0}) \leq \Gamma$. For notational simplicity, we relabel n_k as $m = n_1, n_2, \dots$. For this subsequence, we shall evaluate

$$A_m^{(1)} := \int_{\Theta_1} dw(\theta) B_{\theta,m} \quad \text{and} \quad A_m^{(2)} := \int_{\Theta_2} dw(\theta) B_{\theta,m} \quad (m = n_1, n_2, \dots) \quad (139)$$

where (131) is now expressed as $\varepsilon_m \geq A_m^{(1)} + A_m^{(2)} - e^{-\sqrt{m}\gamma}$.

We first evaluate $A_m^{(1)}$. Fix $\theta \in \Theta_1$ arbitrarily. In this case, $I(P_m, W_{\theta}) \leq c_{\theta,\Gamma} < R$, so $\beta_{\theta,m}$ on the right-hand side of (134) becomes

$$\begin{aligned} \beta_{\theta,m} &= \Pr \left\{ \frac{1}{m} \log \frac{W_{\theta}^m(Y_{\theta}^m | \mathbf{x}_m)}{(P_m W_{\theta})^{\times m}(Y_{\theta}^m)} - I(P_m, W_{\theta}) \leq R - I(P_m, W_{\theta}) + \frac{S - 2\gamma}{\sqrt{m}} - \frac{\log(N_m + 1)}{m} \middle| X^m = \mathbf{x}_m \right\} \\ &\rightarrow 1 \quad (m \rightarrow \infty), \end{aligned} \quad (140)$$

where the convergence is due to the weak law of large numbers. By (134), (140), and Fatou's lemma, we obtain

$$\begin{aligned} \liminf_{m \rightarrow \infty} A_m^{(1)} &\geq \liminf_{m \rightarrow \infty} \int_{\Theta_1} dw(\theta) \beta_{\theta,m} \\ &\geq \int_{\Theta_1} dw(\theta) \liminf_{m \rightarrow \infty} \beta_{\theta,m} \\ &= \int_{\Theta_1} dw(\theta) = \int_{\{\theta \mid c_{\theta,\Gamma} < R\}} dw(\theta). \end{aligned} \quad (141)$$

Next, we turn to evaluating $A_m^{(2)}$. We consider two cases according to whether the convergent point $P_0 = \lim_{m \rightarrow \infty} P_m$ is in $\Pi_{\bar{\theta}, \Gamma}$ or not, where $c_{\bar{\theta}, \Gamma} = R = C_\varepsilon(\Gamma)$. More precisely, we will bound $A_m^{(2)}$ from below in two ways as

$$A_m^{(2)} = \int_{\Theta_2} dw(\theta) B_{\theta, m} \geq \begin{cases} \int_{\Theta_2} dw(\theta) \alpha_{\theta, m} & \text{if } P_0 \in \Pi_{\bar{\theta}, \Gamma} \\ \int_{\Theta_2} dw(\theta) \beta_{\theta, m} & \text{if } P_0 \notin \Pi_{\bar{\theta}, \Gamma} \end{cases}. \quad (142)$$

(i) Consider the case of $P_0 \notin \Pi_{\bar{\theta}, \Gamma}$. We define

$$\mathcal{V}_\tau := \{P \mid I(P, W_\theta) > c_{\bar{\theta}, \Gamma} - \tau\} \quad (\forall \tau > 0), \quad (143)$$

where $c_{\theta, \Gamma} = c_{\bar{\theta}, \Gamma} = R = C_\varepsilon(\Gamma)$ as we are now considering the case of $\theta \in \Theta_2$. Then, for each $\theta \in \Theta_2$ there exists some $\tau_\theta > 0$ such that $P_0 \notin \mathcal{V}_{2\tau_\theta}$. This implies that $P_m \notin \mathcal{V}_{\tau_\theta}$ for all $m > m_0$ with some positive number $m_0 > 0$. Then, by Chebyshev's inequality, it holds that

$$\beta_{\theta, m} \geq 1 - \frac{\max_P V_{\theta, P}}{\left(S - 2\gamma + \sqrt{m}\tau_\theta - \frac{\log(N_m+1)}{\sqrt{m}}\right)^2} \quad (\forall m > m_0), \quad (144)$$

where (60) holds, indicating that $\beta_{\theta, m} \rightarrow 1$ ($m \rightarrow \infty$). By Fatou's lemma and (134), we obtain

$$\begin{aligned} \liminf_{m \rightarrow \infty} A_m^{(2)} &\geq \int_{\Theta_2} dw(\theta) \liminf_{m \rightarrow \infty} B_{\theta, m} \\ &\geq \int_{\Theta_2} dw(\theta) \liminf_{m \rightarrow \infty} \beta_{\theta, m} \\ &\geq \int_{\Theta_2} dw(\theta) \liminf_{m \rightarrow \infty} \left(1 - \frac{\max_P V_{\theta, P}}{\left(S - 2\gamma + \sqrt{m}\tau_\theta - \frac{\log(N_m+1)}{\sqrt{m}}\right)^2}\right) \\ &= \int_{\Theta_2} dw(\theta). \end{aligned} \quad (145)$$

(ii) Next, consider the case of $P_0 \in \Pi_{\bar{\theta}, \Gamma}$. Since $c_{\theta, \Gamma} = c_{\bar{\theta}, \Gamma}$ for $\theta \in \Theta_2$ and hence $\Pi_{\theta, \Gamma} = \Pi_{\bar{\theta}, \Gamma}$ (cf. Remark 8), in (125) we can choose $P_\theta \in \Pi_{\theta, \Gamma}$ for each $\theta \in \Theta_2$ so that

$$\lim_{m \rightarrow \infty} P_m = P_0 = P_\theta, \quad (146)$$

where we notice that $B_{\theta, n}$ and $\alpha_{\theta, n}$ do not depend on the choice of $P_\theta \in \Pi_{\theta, \Gamma} = \Pi_{\bar{\theta}, \Gamma}$. Since again $P_\theta \in \Pi_{\theta, \Gamma}$ and $c_{\theta, \Gamma} = R = C_\varepsilon(\Gamma)$ for $\theta \in \Theta_2$, we have

$$\sum_{x \in \mathcal{X}} P_m(x) D(W_\theta(\cdot|x) \| P_\theta W_\theta) \leq c_{\theta, \Gamma} = R \quad (147)$$

by the Kuhn-Tucker theorem. Indeed, the Kuhn-Tucker theorem asserts that for finite \mathcal{X} and \mathcal{Y} , it holds for all $x \in \mathcal{X}$ that

$$D(W_\theta(\cdot|x) \| P_\theta W_\theta) \leq c_{\theta, \Gamma} + \lambda_0(c(x) - \Gamma) \quad (148)$$

with some $\lambda_0 \geq 0$ (cf. [5, Lemma 3.7.1]). By taking the average with P_m for both sides of (148), we obtain

$$\sum_{x \in \mathcal{X}} P_m(x) D(W_\theta(\cdot|x) \| P_\theta W_\theta) \leq c_{\theta, \Gamma} + \lambda_0 \left(\sum_{x \in \mathcal{X}} P_m(x) c(x) - \Gamma \right), \quad (149)$$

which implies the inequality in (147) since P_m satisfies cost constraint: $\text{Ec}(X_{P_m}) \leq \Gamma$. By (147), we have

$$\begin{aligned} \alpha_{\theta,m} &= \Pr \left\{ \frac{1}{\sqrt{m}} \left(\log \frac{W_\theta^m(Y_\theta^m | \mathbf{x}_m)}{(P_\theta W_\theta)^{\times m}(Y_\theta^m)} - mR \right) \leq S - 2\gamma - \frac{\log(N_m + 1)}{\sqrt{m}} \middle| X^m = \mathbf{x}_m \right\} \\ &\geq \Pr \left\{ \frac{1}{\sqrt{m}} \left(\log \frac{W_\theta^m(Y_\theta^m | \mathbf{x}_m)}{(P_\theta W_\theta)^{\times m}(Y_\theta^m)} - m \sum_{x \in \mathcal{X}} P_m(x) D(W_\theta(\cdot | x) \| P_\theta W_\theta) \right) \right. \\ &\quad \left. \leq S - 2\gamma - \frac{\log(N_m + 1)}{\sqrt{m}} \middle| X^m = \mathbf{x}_m \right\} =: f_{\theta,m}. \end{aligned} \quad (150)$$

Since $V_{\theta,P_\theta}(\mathbf{x}_m) < +\infty$ and the third moment of $\frac{1}{n} \log \frac{W_\theta^m(Y_\theta^m | \mathbf{x}_m)}{(P_\theta W_\theta)^{\times m}(Y_\theta^m)}$ is also bounded (cf. [5, Remark 3.1.1], [11, Lemma 62], [15, Lemma 7]), by the Berry-Esséen theorem and the relations in (136) and (137), we have

$$\left| f_{\theta,m} - G \left(\frac{S - 2\gamma - \frac{\log(N_m + 1)}{\sqrt{m}}}{\sqrt{V_{\theta,P_\theta}(\mathbf{x}_m)}} \right) \right| \leq \frac{\nu_0}{\sqrt{m}} \quad (\forall m = n_1, n_2, \dots), \quad (151)$$

where $G(\cdot)$ is defined as in (13) and $\nu_0 > 0$ is a positive constant. Notice here that $V_{\theta,P_\theta}(\mathbf{x}_m) \rightarrow V_{\theta,P_\theta}$ as $m \rightarrow \infty$ owing to (146). For $\theta \in \Theta_2$ with $V_{\theta,P_\theta} > 0$, we have $V_{\theta,P_\theta}(\mathbf{x}_m) > 0$ for all $m \geq m_1$ with some $m_1 > 0$. Since $\log(N_m + 1) \leq |\mathcal{X}| \log(m + 1)$ and $G(\cdot)$ is continuous, by letting $m \rightarrow \infty$ we obtain

$$\begin{aligned} \liminf_{m \rightarrow \infty} f_{\theta,m} &\geq G \left(\frac{S - 3\gamma}{\sqrt{V_{\theta,P_\theta}}} \right) \\ &= \Psi_{\theta,P_\theta}(S - 3\gamma), \end{aligned} \quad (152)$$

where we have used the relation in (13) for the equality. For $\theta \in \Theta_2$ with $V_{\theta,P_\theta} = 0$, $G(z/\sqrt{V_{\theta,P_\theta}})$ is the step function which takes zero for $z < 0$ and one otherwise. Then, we have (152) for such $\theta \in \Theta_2$, too. Putting (133), (139), (150), and (152) together, we obtain

$$\begin{aligned} \liminf_{m \rightarrow \infty} A_m^{(2)} &\geq \liminf_{m \rightarrow \infty} \int_{\Theta_2} dw(\theta) \alpha_{\theta,m} \\ &\geq \liminf_{m \rightarrow \infty} \int_{\Theta_2} dw(\theta) f_{\theta,m} \\ &\geq \int_{\Theta_2} dw(\theta) \liminf_{m \rightarrow \infty} f_{\theta,m} \\ &\geq \int_{\Theta_2} \Psi_{\theta,P_\theta}(S - 3\gamma) dw(\theta) \\ &\geq \inf_{P \in \Pi_{\bar{\theta},\Gamma}} \int_{\Theta_2} \Psi_{\theta,P}(S - 3\gamma) dw(\theta), \end{aligned} \quad (153)$$

where we have used Fatou's lemma in the third inequality and the relation $P_\theta \in \Pi_{\theta,\Gamma} = \Pi_{\bar{\theta},\Gamma}$ in the last inequality.

To finalize the evaluation of $A_m^{(2)}$ for both the two cases, combining (145) and (153) leads to

$$\begin{aligned} \liminf_{m \rightarrow \infty} A_m^{(2)} &\geq \inf_{P \in \Pi_{\bar{\theta},\Gamma}} \int_{\{\theta | c_{\theta,\Gamma} = R\}} \Psi_{\theta,P}(S - 3\gamma) dw(\theta) \\ &= \inf_{P \in \Pi_{\bar{\theta},\Gamma}} \int_{\{\theta | I(P, W_\theta) = R\}} \Psi_{\theta,P}(S - 3\gamma) dw(\theta) \end{aligned} \quad (154)$$

because $\Theta_2 = \{\theta \mid c_{\theta,\Gamma} = R\} = \{\theta \mid I(P, W_\theta) = R\}$ for any $P \in \Pi_{\bar{\theta},\Gamma}$ with $R = c_{\bar{\theta},\Gamma}$.

Now, we are in a position to synthesize all evaluations. By the definition of achievability, it follows from (131), which means $\varepsilon_m \geq A_m^{(1)} + A_m^{(2)} - e^{-\sqrt{m}\gamma}$, (141), and (154) that

$$\begin{aligned}
\varepsilon &\geq \limsup_{n \rightarrow \infty} \varepsilon_n \\
&\geq \limsup_{m \rightarrow \infty} \varepsilon_m \\
&\geq \limsup_{m \rightarrow \infty} (A_m^{(1)} + A_m^{(2)}) \\
&\geq \liminf_{m \rightarrow \infty} A_m^{(1)} + \liminf_{m \rightarrow \infty} A_m^{(2)} \\
&\geq \int_{\{\theta \mid c_{\theta,\Gamma} < R\}} dw(\theta) + \inf_{P \in \Pi_{\bar{\theta},\Gamma}} \int_{\{\theta \mid I(P, W_\theta) = R\}} \Psi_{\theta,P}(S - 3\gamma) dw(\theta) \\
&= \inf_{P \in \Pi_{\bar{\theta},\Gamma}} \left\{ \int_{\{\theta \mid c_{\theta,\Gamma} < R\}} dw(\theta) + \int_{\{\theta \mid I(P, W_\theta) = R\}} \Psi_{\theta,P}(S - 3\gamma) dw(\theta) \right\}. \tag{155}
\end{aligned}$$

We note that $\Theta_1 = \{\theta \mid c_{\theta,\Gamma} < R\} = \{\theta \mid I(P, W_\theta) < R\}$ for any $P \in \Pi_{\bar{\theta},\Gamma}$ with $R = c_{\bar{\theta},\Gamma}$ due to the definition of Γ -well-orderedness, so it follows from (155) that

$$S - 3\gamma \leq \sup_{P \in \Pi_{\bar{\theta},\Gamma}} \sup \left\{ S \mid \int_{\{\theta \mid I(P, W_\theta) < R\}} dw(\theta) + \int_{\{\theta \mid I(P, W_\theta) = R\}} \Psi_{\theta,P}(S) dw(\theta) \leq \varepsilon \right\}. \tag{156}$$

Since $\gamma > 0$ is arbitrary, we completed the proof of the converse part. \square

VI. CONCLUDING REMARKS

In this paper, we have established the coding theorem for the $(\varepsilon|\Gamma)$ -capacity of mixed memoryless channels with general mixture. For mixed memoryless channels with general mixture, a direct part of the second-order coding theorem has also been provided. The class of Γ -well-ordered mixed memoryless channels, whose component channels are ordered according to their capacity with cost constraint Γ , has been introduced to further analyze the second-order $(\varepsilon, R|\Gamma)$ -capacity. The Γ -well-orderedness allows us to establish a second-order converse theorem, which coincides with the direct theorem for mixed memoryless channels with general mixture. The obtained results include several known results as special cases such as capacity characterizations for mixed memoryless channels with general mixture [1], [5] and for regular decomposable channels with stationary memoryless components [18], an ε -capacity characterization for mixed memoryless channels with countable mixture [20], and second-order (ε, R) -capacity characterizations for additive-noise channels with finite mixture [12] and for well-ordered memoryless channels with finite mixture [21].

Tomamichel and Tan [16] have recently discussed mixed memoryless channels with finite Θ by treating them as memoryless channels with finite states. In other words, channel state $\theta \in \Theta$ is selected with probability $w(\theta)$ before the transmission of a codeword of length n . In the scenario where the encoder and decoder can observe channel state θ , characterizations for the $(\varepsilon|\Gamma)$ -capacity and $(\varepsilon, R|\Gamma)$ -capacity have been discussed. Indeed, when Θ is finite and the encoder and decoder can access to the channel state information, the $(\varepsilon|\Gamma)$ -capacity and $(\varepsilon, R|\Gamma)$ -capacity are characterized as the natural counterparts of those in (107) and (108), respectively, even for mixed memoryless channels whose component channels are not necessarily Γ -well-ordered. We can easily extend this result to mixed memoryless channels with general mixture (general states).

As noted in Sect. V-D, Lemma 7 holds for mixed channels with general input and output alphabets (\mathcal{X} and \mathcal{Y}), and we can also establish the converse part of the first-order coding theorem which corresponds to Theorem 1 in the case with finite \mathcal{X} and general \mathcal{Y} . However, the proof of a direct part in this case

may be trickier because we rely on the upper-decomposition technique of Lemma 4 (that is, the method of types). Extensions of the established formulas for mixed channels with general input and/or output alphabets are interesting and practically important research subjects.

APPENDIX A PROOF OF LEMMA 3

Given an arbitrary i.i.d. product probability distribution Q_θ^n on \mathcal{Y}^n , let Q^n be given as in (30). Since $Q^n(\mathbf{y})$ is the expectation of $Q_\theta^n(\mathbf{y})$ with respect to $w(\theta)$, Markov's inequality implies that

$$\Pr\{\theta \in \Theta(\mathbf{y})\} \geq 1 - e^{-\sqrt[4]{n}} \quad (\forall \mathbf{y} \in \mathcal{Y}^n). \quad (157)$$

We also have

$$\begin{aligned} \Pr\{\theta \in \Theta_n^c\} &= \Pr\{\theta \in \cup_k \Theta(S_k)^c\} \\ &\leq \sum_k \Pr\{\theta \in \Theta(S_k)^c\} \leq (n+1)^{|\mathcal{Y}|} e^{-\sqrt[4]{n}}. \end{aligned} \quad (158)$$

Here, A^c denotes the complement of a set A . Therefore,

$$\Pr\{\theta \in \Theta_n\} \geq 1 - (n+1)^{|\mathcal{Y}|} e^{-\sqrt[4]{n}}. \quad (159)$$

In a similar way, we also have

$$\Pr\{\theta \in \tilde{\Theta}_n\} \geq 1 - (n+1)^{|\mathcal{X}| \cdot |\mathcal{Y}|} e^{-\sqrt[4]{n}}. \quad (160)$$

Then, it holds for $\Theta_n^* = \Theta_n \cap \tilde{\Theta}_n$ that

$$\Pr\{\theta \in \Theta_n^*\} \geq 1 - 2(n+1)^{|\mathcal{X}| \cdot |\mathcal{Y}|} e^{-\sqrt[4]{n}}, \quad (161)$$

thus, yielding (35). \square

APPENDIX B PROOF OF LEMMA 4

The proof is implicitly contained in Han [5]. We summarize it here for the reader's convenience.

For a given $\gamma > 0$, we define a set

$$D_n = \left\{ \mathbf{y} \in \mathcal{Y}^n \left| \frac{1}{n} \log P_{Y_\theta^n}(\mathbf{y}) - \frac{1}{n} \log P_{Y^n}(\mathbf{y}) \leq -\frac{\gamma}{\sqrt{n}} \right. \right\} \quad (162)$$

for $\theta \in \Theta$. Then, it holds that

$$\begin{aligned} \Pr\{Y_\theta^n \in D_n\} &= \sum_{\mathbf{y} \in D_n} P_{Y_\theta^n}(\mathbf{y}) \\ &\leq \sum_{\mathbf{y} \in D_n} P_{Y^n}(\mathbf{y}) e^{-\sqrt{n}\gamma} \\ &\leq e^{-\sqrt{n}\gamma}. \end{aligned} \quad (163)$$

Hence, for any real number z_n we have

$$\begin{aligned} \Pr\left\{-\frac{1}{n} \log P_{Y^n}(Y_\theta^n) \leq z_n\right\} &\leq \Pr\left\{-\frac{1}{n} \log P_{Y^n}(Y_\theta^n) \leq z_n, Y_\theta^n \notin D_n\right\} + \Pr\{Y_\theta^n \in D_n\} \\ &\leq \Pr\left\{-\frac{1}{n} \log P_{Y_\theta^n}(Y_\theta^n) \leq z_n + \frac{\gamma}{\sqrt{n}}\right\} + e^{-\sqrt{n}\gamma} \end{aligned} \quad (164)$$

for all $\theta \in \Theta$. By using the above inequality we have

$$\begin{aligned}
\Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n | X^n)}{P_{Y^n}(Y_\theta^n)} \leq z_n \right\} &= \Pr \left\{ \frac{1}{n} \log W^n(Y_\theta^n | X^n) - \frac{1}{n} \log P_{Y^n}(Y_\theta^n) \leq z_n \right\} \\
&\leq \Pr \left\{ \frac{1}{n} \log W^n(Y_\theta^n | X^n) - \frac{1}{n} \log P_{Y_\theta^n}(Y_\theta^n) \leq z_n + \frac{\gamma}{\sqrt{n}} \right\} + e^{-\sqrt{n}\gamma} \\
&\leq \Pr \left\{ \frac{1}{n} \log W_\theta^n(Y_\theta^n | X^n) - \frac{1}{n} \log P_{Y_\theta^n}(Y_\theta^n) \leq z_n + \frac{\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\} + e^{-\sqrt{n}\gamma} \\
&= \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{P_{Y_\theta^n}(Y_\theta^n)} \leq z_n + \frac{\gamma}{\sqrt{n}} + \frac{1}{\sqrt[4]{n^3}} \right\} + e^{-\sqrt{n}\gamma} \tag{165}
\end{aligned}$$

for $\theta \in \Theta_n^*$, where the last inequality is due to the inequality $W_\theta^n(\mathbf{y}|\mathbf{x}) \leq e^{\frac{\gamma}{\sqrt{n}}} W^n(\mathbf{y}|\mathbf{x})$ for $\theta \in \Theta_n^*$. This completes the proof. \square

APPENDIX C PROOF OF LEMMA 5

This proof is also implicitly contained in Han [5] in the case of $Q^n = P_{Y^n}$, where P_{Y^n} denotes the output distribution on \mathcal{Y}^n due to input X^n via channel W^n . Similarly to (164), we obtain

$$\Pr \left\{ \frac{1}{n} \log W^n(Y_\theta^n | X^n) \leq z_n \right\} \geq \Pr \left\{ \frac{1}{n} \log W_\theta^n(Y_\theta^n | X^n) \leq z_n - \frac{\gamma}{\sqrt{n}} \right\} - e^{-\sqrt{n}\gamma} \tag{166}$$

for $\theta \in \Theta$. Using this inequality, we have

$$\begin{aligned}
\Pr \left\{ \frac{1}{n} \log \frac{W^n(Y_\theta^n | X^n)}{Q^n(Y_\theta^n)} \leq z_n \right\} &= \Pr \left\{ \frac{1}{n} \log W^n(Y_\theta^n | X^n) - \frac{1}{n} \log Q^n(Y_\theta^n) \leq z_n \right\} \\
&\geq \Pr \left\{ \frac{1}{n} \log W_\theta^n(Y_\theta^n | X^n) - \frac{1}{n} \log Q^n(Y_\theta^n) \leq z_n - \frac{\gamma}{\sqrt{n}} \right\} - e^{-\sqrt{n}\gamma} \\
&\geq \Pr \left\{ \frac{1}{n} \log W_\theta^n(Y_\theta^n | X^n) - \frac{1}{n} \log Q_\theta^n(Y_\theta^n) \leq z_n - \frac{\gamma}{\sqrt{n}} - \frac{1}{\sqrt[4]{n^3}} \right\} - e^{-\sqrt{n}\gamma} \\
&= \Pr \left\{ \frac{1}{n} \log \frac{W_\theta^n(Y_\theta^n | X^n)}{Q_\theta^n(Y_\theta^n)} \leq z_n - \frac{\gamma}{\sqrt{n}} - \frac{1}{\sqrt[4]{n^3}} \right\} - e^{-\sqrt{n}\gamma} \tag{167}
\end{aligned}$$

for $\theta \in \Theta_n^*$, where the last inequality is due to the inequality $Q_\theta^n(\mathbf{y}) \leq e^{\frac{\gamma}{\sqrt{n}}} Q^n(\mathbf{y})$ for $\theta \in \Theta_n^*$. Thus, we complete the proof. \square

APPENDIX D PROOF OF LEMMA 7

For any given (n, M_n, ε_n) code $\mathcal{C}_n = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{M_n}\}$, it follows from (1) and (3) that

$$\begin{aligned}
\varepsilon_n &= \frac{1}{M_n} \sum_{i=1}^{M_n} W^n(D_i^c | \mathbf{u}_i) \\
&= \frac{1}{M_n} \sum_{i=1}^{M_n} \int_{\Theta} dw(\theta) W_\theta^n(D_i^c | \mathbf{u}_i) \\
&= \int_{\Theta} dw(\theta) \left\{ \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i^c | \mathbf{u}_i) \right\}, \tag{168}
\end{aligned}$$

where the equality in (168) is obtained by exchanging the integral and the sum of finitely many terms. Here, the term inside the brace $\{\cdot\}$ in (168) corresponds to the average error probability with the decoding region $\{D_i\}_{i=1}^{M_n}$ over W_θ^n . Then, a simple but key observation is that each of such terms indexed by $\theta \in \Theta$, which is characterized by the common decoding region $\{D_i\}_{i=1}^{M_n}$, may be lower bounded separately using another set depending on $\theta \in \Theta$.

Define the set

$$B_{\theta,i} := \left\{ \mathbf{y} \in \mathcal{Y}^n \mid \frac{1}{n} \log \frac{W_\theta^n(\mathbf{y}|\mathbf{u}_i)}{Q_\theta^n(\mathbf{y})} \leq \frac{1}{n} \log M_n - \eta \right\}. \quad (169)$$

Then the term inside the brace $\{\cdot\}$ in (168) can be bounded as

$$\begin{aligned} \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i^c|\mathbf{u}_i) &\geq \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i^c \cap B_{\theta,i}|\mathbf{u}_i) \\ &= \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(B_{\theta,i}|\mathbf{u}_i) - \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i \cap B_{\theta,i}|\mathbf{u}_i), \end{aligned} \quad (170)$$

where the equality in (170) follows from the relation

$$D_i^c \cap B_{\theta,i} = B_{\theta,i} \setminus (D_i \cap B_{\theta,i}). \quad (171)$$

We focus on the second term in (170). By definition, every $\mathbf{y} \in B_{\theta,i}$ satisfies

$$\frac{1}{M_n} W_\theta^n(\mathbf{y}|\mathbf{u}_i) \leq Q_\theta^n(\mathbf{y}) e^{-n\eta}. \quad (172)$$

Then the second term in (170) is bounded as

$$\begin{aligned} \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i \cap B_{\theta,i}|\mathbf{u}_i) &= \frac{1}{M_n} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in D_i \cap B_{\theta,i}} W_\theta^n(\mathbf{y}|\mathbf{u}_i) \\ &\leq e^{-n\eta} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in D_i \cap B_{\theta,i}} Q_\theta^n(\mathbf{y}), \end{aligned} \quad (173)$$

$$\leq e^{-n\eta} \sum_{i=1}^{M_n} Q_\theta^n(D_i) = e^{-n\eta}, \quad (174)$$

where (172) is used to obtain (173), and (2) is used to obtain the equality in (174).

Plugging (174) into (170) yields⁴

$$\frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(D_i^c|\mathbf{u}_i) \geq \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(B_{\theta,i}|\mathbf{u}_i) - e^{-n\eta}. \quad (175)$$

Thus, the left-hand side of (168) is lower bounded as

$$\varepsilon_n \geq \int_{\Theta} dw(\theta) \left\{ \frac{1}{M_n} \sum_{i=1}^{M_n} W_\theta^n(B_{\theta,i}|\mathbf{u}_i) \right\} - e^{-n\eta}, \quad (176)$$

which is equivalent to (123). \square

⁴Inequality (175) is Hayashi-Nagaoka's lower bound on the probability of decoding error, which has been originally established for the quantum channel setting [7], for the component channel W_θ^n . The derivation is essentially the same but slightly more direct than the original derivation (cf. [6, Sect. IX-B]).

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