On Caching with More Users than Files

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Abstract—Caching appears to be an efficient way to reduce peak hour network traffic congestion by storing some content at the user's cache without knowledge of later demands. Recently, Maddah-Ali and Niesen proposed a two-phase, placement and delivery phase, coded caching strategy for centralized systems (where coordination among users is possible in the placement phase), and for decentralized systems. This paper investigates the same setup under the further assumption that the number of users is larger than the number of files. By using the same uncoded placement strategy of Maddah-Ali and Niesen, a novel coded delivery strategy is proposed to profit from the multicasting opportunities that arise because a file may be demanded by multiple users. The proposed delivery method is proved to be optimal under the constraint of uncoded placement for centralized systems with two files; moreover it is shown to outperform known caching strategies for both centralized and decentralized systems.

I. INTRODUCTION

Caching is a popular method to smooth out network traffic in broadcasting systems, where some content is cached into the user's memory during off peak hours in the hope that the prestored content will be required by the user during peak hours and thus, reducing the number of broadcast transmissions from the server to the users.

System model: In this paper, we study a system with N files available at a server that is connected to K users; each user has a cache of size M to store files; users are connected to the server via a shared error-free link. The caching procedure assumes two phases. (1) Placement phase: where users store (coded or uncoded) pieces of the files within their cache without knowledge of later demands. When the file pieces are not network coded we say that the placement phase is uncoded, otherwise that it is coded. (2) Delivery phase: where each user demands a specific file and, based on the users' demands and cache content, the server broadcasts packets so that each user can recover the demanded file. The objective of the system designer is to provide a two-phase scheme so that the number of transmitted packets, or load, in the delivery phase for the worst-case demands is minimized.

Coordinated cache placement: Maddah-Ali and Niesen proposed [1] a coded caching scheme that utilizes an uncoded combinatorial cache construction in the placement phase and a linear network code in the delivery phase, where users store contents in a coordinated manner. The worst-case load of the Maddah-Ali and Niesen scheme (refer to as MNS) was shown to be no larger than $K\left(1-\frac{M}{N}\right)\min\left\{\frac{1}{1+K\frac{M}{N}},\frac{N}{K}\right\}$, which has the additional global caching gain $\frac{1}{1+KM/N}$ compared to the conventional uncoded caching scheme. MNS was shown to be optimal [2] under the constraint of uncoded cache placement and $N \ge K$, and order optimal [1] to within a factor of 12 of the cut-set outer bound. The authors in [3] showed that a scheme based on coded cache placement, originally proposed in [1] for N = 2, is optimal when N < K and MK < 1while providing a load of N(1-M) which coincides with the cut-set outer bound. Recently, reference [4] studied the case N=2 and $M \leq \frac{K-1}{K}$, and proposed a scheme with coded cache placement yielding a lower load than MNS.

Un-coordinated cache placement: The previously mentioned works assumed that the K connected users are the same during both phases. However, this may not always be the case in practice (e.g. due to user mobility) where a user may be connected to one server during his placement phase but to a different one during his delivery phase. In this decentralized scenario, each server must carry out independently the two phases of caching and thus, the coordination (among users) during the placement phase is not possible. In [5], Maddah-Ali and Niesen proposed that each user fills its cache randomly and independently of the others. During the delivery phase, the bits of N files are organized into sub-files depending on which users know, each of which is delivered by using the delivery strategy in [1] for centralized systems. The corresponding load was shown to be $K\left(1-\frac{M}{N}\right)\min\left\{\frac{N}{KM}\left(1-\left(1-\frac{M}{N}\right)K\right),\frac{N}{K}\right\}$, where the factor $\frac{N}{KM}\left[1-\left(1-\frac{M}{N}\right)K\right]$ represents an additional of the last of the second se tional global caching gain compared to the conventional uncoded caching.

A delivery phase with load equal to the fractional local chromatic number (described in [6]) of the directed graph formed by the users' demands and caches was shown in [7], [8] for centralized and decentralized scenarios, respectively. Since the computation of the fractional local chromatic number is NP-hard, the authors in [9], [10] proposed approximate algorithms to simplify computations.

Our contribution: In [2], we showed that for $N \ge K$ and under the constraint of uncoded cache placement, MNS is optimal. In this work, motivated by practical considerations (e.g., a server has several popular music or video files that are widely demanded by different users), we study the case N < Kwhere same sub-files may be demanded by multiple users.

It is worthing noting that MNS cannot be used to multicast files since it considers each sub-file demanded by each user as a district sub-file. With the goal of multicasting messages, we design a delivery phase for the case of N < K that is applicative to both centralized and decentralized scenarios. The proposed delivery method is shown to achieve the optimal load under the constraint of uncoded placement for centralized systems with two files and to outperform known caching strategies for both centralized and decentralized scenarios.

Paper Outline: The rest of the paper is organized as follows. Section II presents the system model. Section III introduces the main results. Section IV compares by numerical results the proposed scheme to existing ones. Finally, Section V presents summary and discussion while some technical proofs are relegated to the Appendix.

Notations: Calligraphic symbols denotes sets; $|\cdot|$ is used to represent the cardinality of a set or the length of a file; we denote $[1:K] := \{1, 2, ..., K\}$ and $\mathcal{A} \setminus \mathcal{B} := \{x \in \mathcal{A} | x \notin \mathcal{B}\};$ \oplus represents the bit-wise XOR operation, and $\binom{K}{t}$ is the binomial coefficient.

II. SYSTEM MODEL AND PROBLEM STATEMENT

Consider a broadcasting caching system that consists of a center server with N files, denoted by (F_1, F_2, \ldots, F_N) , and K users connected to it through an error-free link. Each file has $F \gg 1$ bits. Here we assume N < K and that each file is requested by each user with identical probability.

During the placement phase, user $i \in [1:K]$ stores content from N files in his cache of size MF bits without knowledge of later demands, where $M \in [0, N]$. We denote the content in the cache of user i by Z_i ; we also let $\mathbf{Z} := (Z_1, \ldots, Z_K)$. Centralized systems allow for coordination among users in the placement phase, while decentralized systems do not. In the delivery phase, each user demands one file and the demand vector $\mathbf{d} := (d_1, d_2, \ldots, d_K)$ is revealed to the server, where $d_i \in [1:N]$ is the file demanded by user $i \in [1:K]$.

Given (\mathbf{Z}, \mathbf{d}) , the server broadcasts a message $X_{\mathbf{d},\mathbf{Z}}$ with normalized length (by the file size F) $R(\mathbf{d}, M)$. It is required that user $i \in [1:K]$ recovers his desired file F_{d_i} from $X_{\mathbf{d},\mathbf{Z}}$ and Z_i with high probability. The objective is to minimize the worst-case network load: $R(M) = \min \max R(\mathbf{d}, M)$.

III. MAIN RESULTS

We propose a caching scheme that attains the following memory-load tradeoffs for centralized systems.

Theorem 1 (Centralized). For centralized systems, the lower convex envelope of $R_p(M)$ is achievable with $t \in [0:K]$ and

$$R_{\rm c}(M) = \begin{cases} N(1-M), & M = \frac{1}{K}, \\ R_{\rm co}(M), & M = t\frac{N}{K}, 0 \le M < M_{th}, \\ \frac{K(1-\frac{M}{N})}{1+K\frac{M}{N}}, & M = t\frac{N}{K}, M_{th} \le M \le N, \end{cases}$$
(1)

$$R_{\rm co}(M) = N - M - \frac{M(N-1)K(N-M)}{N^2(K-1)},$$
(2)

$$M_{\rm th} := N \frac{NK - 2N + 1 - \sqrt{f(N,K)}}{2K(N-1)},\tag{3}$$

$$f(N,K) := (NK - 2N + 1)^2 - 4(N - 1)(K - N)(K - 1).$$
(4)

The same idea applied to decentralized systems attains the following memory-load tradeoff.

Theorem 2 (Decentralized). For decentralized systems, the lower convex envelope of $R_d(M)$ is achievable with $M \in [0, N]$, q := M/N and

$$R_{\rm d}(M) = N(1-q)C(t_{\rm th}, K-1, q) - (N-1)q(q-1)$$

$$C(t_{\rm th} - 1, K-2, q) + \frac{1-q}{q}(1 - C(t_{\rm th} + 1, K, q)), \quad (5)$$

$$C(x, y, q) := \sum_{i=0}^{x} {\binom{y}{i}} q^{i} (1-q)^{y-i}.$$
 (6)

We next derive an outer bound under the constraint of uncoded cache placement and K > N and prove the optimality of the proposed achievable scheme for N = 2.

Theorem 3 (Optimality for N = 2). The minimal load under the constraint of uncoded cache placement and K > N = 2for the aforementioned centralized systems, is $R_{co}(M)$ in (2) and is achieved by the proposed scheme.

The rest of the Section is firstly devoted to the proof of Theorem 1 and Theorem 2. The main idea is to consider the multicasting opportunities that arise for the case of N < K. Due to space limitation, the proof of Theorem 3 is only outlined.

A. Proof of Theorem 1

We start by describing our scheme and computing the load for $M = t\frac{N}{K}$, where $t \in [0 : K]$. The complete memoryload tradeoff is obtained as the lower convex envelope of the derived points, which can be achieved by memory sharing.

Placement Phase: The cache placement phase is as in the MNS. Each file is split into $\binom{K}{t}$ non-overlapping sub-files of identical size given by $\frac{F}{\binom{K}{t}}$, where $t = K\frac{M}{N} \in [0:K]$. Each sub-file of F_i is denoted by $F_{i,\mathcal{W}}$ where $\mathcal{W} \subseteq [1:K]$ such that $|\mathcal{W}| = t$. User $j \in [1:K]$ stores $F_{i,\mathcal{W}}$ for all $i \in [1:N]$ in his cache if and only if $j \in \mathcal{W}$.

Delivery Phase: The delivery phase is divided into two steps. We consider the worst case demand where each file is demanded by at least one user. Let \mathcal{G}_i be the set of users who demand file F_i , for $i \in [1:N]$.

Step 1: We divide the sub-files of F_i into several groups indicated as $\mathcal{O}_{i,J} := \{F_{i,\mathcal{W}} : \mathcal{W} \setminus \mathcal{G}_i = \mathcal{J}\}$, where $\mathcal{J} \subseteq [1 : K] \setminus \mathcal{G}_i$ and $\max\{0, t - \mathcal{G}_i\} \leq |\mathcal{J}| \leq t$. There are $\binom{|\mathcal{G}_i|}{t - |\mathcal{J}|}$ sub-files in $\mathcal{O}_{i,J}$. Each user in \mathcal{G}_i wants to recover all the subfiles in $\mathcal{O}_{i,J}$ and knows $\binom{|\mathcal{G}_i|-1}{t - |\mathcal{J}|-1}$ of them. Note that when $|\mathcal{J}| = t$, we assume $\binom{|\mathcal{G}_i|-1}{t - |\mathcal{J}|-1} = 0$. The authors in [5] showed that this kind of problem can be solved by using m - d random linear combinations of all the m bits, where m and d are number of bits to encode and minimum number of bits known at each decoder. Since m and d tend to infinite, the m - drandom linear combinations are linearly independent with high probability, thus each decoder can recover all the m bits with high probability. Hence, in order to delivery all the sub-files in $\mathcal{O}_{i,J}$ to the users in \mathcal{G}_i , we can use $\left[\binom{|\mathcal{G}_i|}{t-|\mathcal{J}|} - \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|-1}\right] \frac{F}{\binom{K}{t}}$ random linear combinations of all the $\binom{|\mathcal{G}_i|}{t-|\mathcal{J}|} \frac{F}{\binom{K}{t}}$ bits in $\mathcal{O}_{i,J}$. We define $\mathcal{C}_{i,J}$ as the code for $\mathcal{O}_{i,J}$. With the *Pascal's triangle*

$$\binom{|\mathcal{G}_i|}{t-|\mathcal{J}|} - \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|-1} = \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|}, \quad (7)$$

it can be seen easily that $C_{i,J}$ has $\binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|} \frac{F}{\binom{K}{t}}$ bits. Note that when $|\mathcal{J}| = t - \mathcal{G}_i$, the right side of (7) is 0. Let $v_{i,t} := \max\{0, t - \mathcal{G}_i + 1\}$. As a consequence, for each \mathcal{J} where $\mathcal{J} \subseteq [1:K] \setminus \mathcal{G}_i$ and $v_{i,t} \leq |\mathcal{J}| \leq t$, we use random linear combinations as described above to encode $\mathcal{O}_{i,J}$. We define \mathcal{C}_i as the set of $\mathcal{C}_{i,J}$ for all $\mathcal{J} \subseteq [1:K] \setminus \mathcal{G}_i$ and $v_{i,t} \leq |\mathcal{J}| \leq t$. The number of bits in \mathcal{C}_i is equal to (see Appendix)

$$\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_i:v_{i,t}\leq|\mathcal{J}|\leq t} \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|} \frac{F}{\binom{K}{t}} = \binom{K-1}{t} \frac{F}{\binom{K}{t}}.$$
 (8)

Let C_{step1} denote set of bits in C_i for all $i \in [1:N]$, where C_{step1} has $N\binom{K-1}{t} \frac{F}{\binom{K}{t}}$ bits. If the server transmits C_{step1} , each user would be able to recover his desired file with very high probability. However, by doing so we have some redundancy left, which motivates the next step.

Step 2: For file F_i and user $j \notin G_i$, user j knows some bits in C_i . More precisely, user j knows $C_{i,J}$ if $j \in \mathcal{J}$ and hence, the number of bits in C_i known by j is (see Appendix)

$$\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_i:v_{i,t}\leq|\mathcal{J}|\leq t,j\in\mathcal{J}} \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|} \frac{F}{\binom{K}{t}} = \binom{K-2}{t-1} \frac{F}{\binom{K}{t}}.$$
(9)

Considering C_i for all i such that user $j \notin G_i$, the total number of bits known by j is $(N-1)\binom{K-2}{t-1}\frac{F}{\binom{K}{t}}$. We can use $\left[N\binom{K-1}{t} - (N-1)\binom{K-2}{t-1}\right]\frac{F}{\binom{K}{t}}$ random linear combinations to encode C_{step1} . As a result by letting $t = \frac{KM}{N} \in [0:K]$, the load of our scheme is $R_{\text{co}}(M)$ in (2).

Compared to the delivery method in the MNS, whose load is $K(1 - \frac{M}{N})\min\left\{\frac{1}{1+K\frac{M}{N}}, \frac{N}{K}\right\}$, it can be shown that for $0 \le M \le N$,

$$R_{\rm co}(M) \le K(1 - \frac{M}{N})\frac{N}{K} = N - M.$$

We can also find that if $0 \le M < M_{\text{th}}$,

$$R_{\rm co}(M) < K(1 - \frac{M}{N}) \frac{1}{1 + K\frac{M}{N}}$$

and that if $M_{\text{th}} \leq M \leq N$,

$$R_{\rm co}(M) \ge K(1 - \frac{M}{N})\frac{1}{1 + K\frac{M}{N}},$$

where the threshold $M_{\rm th}$ was given in (3).

Finally, by memory sharing the caching scheme in [3] (which is optimal for M = 1/K) together with the above proposed scheme, we have the load is no larger than the lower convex envelope of $R_p(M)$ described in (1).



Fig. 1: Groups of sub-files for the example N = 2, K = 5, M = 4/5 and $\mathbf{d} = (1, 1, 1, 2, 2)$.

Example: In order to clarify the steps of the proposed scheme, we analyse here in detail the case N = 2, K = 5, M = 4/5, and $F_1 = A$, $F_2 = B$. With these parameters we have $t = \frac{KM}{N} = 2$ and we therefore split each of the two files A and B into $\binom{K}{t} = 10$ non-overlapping sub-files of size equal to $\frac{F}{10}$. For simplicity in the following we omit the braces when we indicate sets, i.e., A_{12} represents $A_{\{12\}}$.

In the placement phase we set $Z_j = \{F_{i,\mathcal{W}} : j \in \mathcal{W}, |\mathcal{W}| = t\}$ for $i = \{1,2\}$, e.g., $Z_1 = \{A_{12}, A_{13}, A_{14}, A_{15}, B_{12}, B_{13}, B_{14}, B_{15}\}$. In the delivery phase, since $M_{\text{th}} = 1.2 > M = 4/5 = 0.8$, the novel proposed two-step method is used. We consider the worse-case demand vector $\mathcal{G}_1 = \{1, 2, 3\}$ and $\mathcal{G}_2 = \{4, 5\}$.

In step 1, we divide the sub-files of A into several groups, $\mathcal{O}_{1,J} = \{F_{1,\mathcal{W}} : \mathcal{W} \setminus \mathcal{G}_1 = \mathcal{J}\}$, where $\mathcal{J} \subseteq \{4,5\}$ and $|\mathcal{J}| \leq 2$. Similarly, $\mathcal{O}_{2,J} = \{F_{2,\mathcal{W}} : \mathcal{W} \setminus \mathcal{G}_2 = \mathcal{J}\}$, where $\mathcal{J} \subseteq \{1,2,3\}$ and $|\mathcal{J}| \leq 2$. The groups can be seen in Fig. 1, identified by different colors. For each group $\mathcal{O}_{i,J}$, each user in \mathcal{G}_i wants to recover all the $\binom{|\mathcal{G}_i|}{t-|\mathcal{J}|}$ sub-files in this group and knows $\binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|-1}$ of them. For instance, for $\mathcal{O}_{1,4} = \{A_{14}, A_{24}, A_{34}\}$, each of the users in $\mathcal{G}_1 = \{1, 2, 3\}$ wants to recover $\mathcal{O}_{1,4}$ whose length is $\frac{3F}{10}$, while user 1 knows A_{14} , user 2 knows A_{24} , and user 3 knows A_{34} . We can use $\frac{3F}{10} - \frac{F}{10}$ random linear combinations of the bits in $\mathcal{O}_{1,4}$. So $\mathcal{C}_{1,4}$ has $\frac{F}{5}$ bits. By using the same method to encode all the groups, the numbers of bits in $\mathcal{C}_{1,0}, \mathcal{C}_{1,4}, \mathcal{C}_{1,5}, \mathcal{C}_{1,45}, \mathcal{C}_{2,0}, \mathcal{C}_{2,1},$ $\mathcal{C}_{2,2}, \mathcal{C}_{2,3}, \mathcal{C}_{2,12}, \mathcal{C}_{2,13}, \mathcal{C}_{2,23}$ are $\frac{F}{10}, \frac{F}{5}, \frac{F}{5}, \frac{F}{10}, 0, \frac{F}{10}, \frac{F$

In step 2, it is easy to check that among all the codes C_{step1} , user 1 knows $C_{2,1}$, $C_{2,12}$ and $C_{2,13}$, i.e., $\frac{3F}{10}$ bits. Similarly, in C_{step1} each user knows $\frac{3F}{10}$ bits. Hence we can use $\frac{6F}{5} - \frac{3F}{10} = \frac{9F}{10}$ random linear combinations of the bits in C_{step1} . As a result each user can recover each sub-files of his desired file and the load is 0.9 while the MNS in [1] requires 1. This represents 10% saving over the MNS scheme.

B. Proof of Theorem 2

Following similar steps to [5], we can extend our proposed delivery method to decentralized systems as well. Note that since in decentralized systems no coordination among users is possible, we can not utilize the caching scheme in [1]. Placement Phase: The cache placement phase is the same as in [5]. For each $M \in [0, N]$, user k independently caches a subset of $\frac{MF}{N}$ bits of each file, chosen uniformly at random. Given the cache content of all the users, we can group the bits of the files into sets $F_{i,W}$, where $F_{i,W}$ is the set of bits of file i which are only known by the users in $W \subseteq [1:K]$. By Law of Large Numbers we have

$$\frac{|F_{i,\mathcal{W}}|}{F} \approx \left(\frac{M}{N}\right)^{|\mathcal{W}|} \left(1 - \frac{M}{N}\right)^{K-|\mathcal{W}|}, \text{ for } F \gg 1.$$

Delivery Phase: We divide the sub-files into groups, $DG_i = \{F_{i,\mathcal{W}} : |\mathcal{W}| = i\}$ where $i \in [0 : K - 1]$. The delivery phase described for centralized systems can be used for the sub-files of DG_i for each i.

If we transmit all the coded bits of the groups, each user can recover his desired file. The load of the proposed method for decentralized systems is thus

$$\begin{aligned} R_{\mathrm{d}}(M) &= \sum_{i=0}^{\lfloor t_{\mathrm{th}} \rfloor} \left(N \binom{K-1}{i} - (N-1) \binom{K-2}{i-1} \right) \\ q^{i} \left(1-q\right)^{K-i} &+ \sum_{i=\lfloor t_{\mathrm{th}} \rfloor+1}^{K-1} \binom{K}{i+1} q^{i} \left(1-q\right)^{K-i}, \end{aligned}$$

where $t_{\text{th}} := KM_{\text{th}}/N$ and $q := \frac{M}{N}$. After some simple algebraic manipulations, it is easy to check $R_d(M)$ can be expressed as in (5). Finally, the memory-load trade-off of the proposed scheme is the lower convex envelope of $R_d(M)$.

C. Sketch of the Proof of Theorem 3

Assume each file is demanded by at least one user. We denote the worst-case load under the constraint of uncoded placement by $R_{\rm u}(M)$. We choose N users with different demands in the user set [1 : K]. The chosen user set is denoted by $C = \{c_1, c_2, ..., c_N\}$ where $c_1 < c_2 < ... < c_N$ and $c_i \in [1 : K]$. We assume user c_i demands d_{c_i} , where $d_{c_i}, i \in [1 : N]$ and $d_{c_i} \neq d_{c_j}$ if $i \neq j$. By considering uncoded placement and that other users do not require any file, the delivery phase is an index coding problem where each message is demanded by only one user. We denote the worst-case load of the above case by n(M). It is obvious that $R_{\rm u}(M) > n(M)$. Hence we can use the same method as [2] based on the index coding graph where each node represents a sub-file demanded by one user as argued in [2]. The only difference is that $\mathbf{u} = (u_1, u_2, ..., u_N)$ is a permutation of \mathcal{C} . So by following [2], it is not difficult to generate the following outer bound for n(M),

$$n(M) \ge \sum_{i=0}^{K} \frac{\binom{K-1}{i} + \binom{K-2}{i} + \dots + \binom{K-N}{i}}{N\binom{K}{i}} x_i, \qquad (10)$$

$$x_0 + x_1 + \dots + x_K = N, \qquad (11)$$

$$x_1 + 2x_2 + \dots + ix_i + \dots + Kx_K = KM, \qquad (12)$$

where x_t is the total length of the sub-files that are known by t users, $t \in [0 : K]$. For N = 2, we eliminate x_t for $t \in [0 : K]$ in the system of inequalities (10)-(12) and get



Fig. 2: The memory-load trade-off for a centralized system with N = 2 and K = 10.

an outer bound for the load n(M). In [2] we proposed an elimination method to this kind of problem. Please find the details of the elimination in Appendix. Finally, we can see that the above outer bound for n(M) coincides with the lower convex envelope $R_{\rm co}(M)$ in (2) for N = 2. Next we give an example to understand the elimination method.

Example: In the Section III-A where N = 2, K = 5 and M = 0.8, it was shown that the proposed delivery scheme leads to a load equal to 0.9. Now we prove its optimality.

From expressions (10)-(12), we have that

$$n(M) \ge \sum_{i=0}^{4} \frac{(5-i)(8-i)}{40} x_i,$$
(13)

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 2, (14)$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5M.$$
⁽¹⁵⁾

Then we sum $(14) \times \frac{19}{20}$ and $(15) \times \frac{-1}{4}$, to find

$$-\frac{7}{10}x_1 - \frac{9}{20}x_2 - \frac{1}{5}x_3 + \frac{1}{20}x_4 + \frac{3}{10}x_5 + \frac{19}{10} - \frac{5}{4}M = 0.$$
(16)

At last we take (16) into (13), and we can have

$$n(M) \ge \frac{19}{10} - \frac{5}{4}M + \frac{1}{20}x_0 + \frac{1}{20}x_3 + \frac{3}{20}x_4 + \frac{3}{10}x_5$$
$$\ge \frac{19}{10} - \frac{5}{4}M.$$

When M = 4/5, $R_u(M) \ge n(M) \ge 0.9$ which is equal to the load of the proposed scheme. By using the same method we can know that for any K > N = 2 and $M \in [0, N]$, the proposed scheme is optimal under the constraint of uncoded placement.

IV. NUMERICAL RESULTS

A. Centralized Systems

We compare the achievable load with our proposed scheme in (1) with that of the schemes in [1], [4], [7]. Since the scheme in [3] is optimal when $0 \le M \le \frac{1}{K}$, we memoryshare each considered scheme with the one in [3]. Note that [7]



Fig. 3: The memory-load trade-off for a decentralized system with N = 4 and K = 8.

uses the local chromatic number, whose computation is NPhard; here in order to simplify the computations we use the approximate algorithms GCC, HgLC and GRASP proposed in [9], [10]. Numerically we find that for centralized system GCC performs better than the other simplification methods. Therefore, in order to have a less cluttered figure, we only plot GCC. We also do the numerical evaluations for the MNS and the scheme in [4]. Fig. 2 shows the memory-load trade-offs for a centralized system with N = 2 and K = 10. We can see the scheme in [7] with GCC has the same performance as the MNS and the improvement from the scheme in [4] to the MNS is negligible. Furthermore, the proposed scheme improves on the MNS. For instance, when M = 1, the proposed scheme (with load 0.722) reduces 9.1% of the load of the MNS (0.794).

B. Decentralized Systems

In decentralized scenarios, the caching schemes in [3] and [4] with coordinated cache placement can not be used. The scheme in [8] is similar to the one in [7], where the main difference relies on the use of random placement for the first one. In order to compute the local chromatic number, the approximate GCC, HgLC and GRASP algorithms are used. The authors in [9], [10] claimed that for decentralized system with uniform demands and infinite file size, GCC performs better than other simplification methods. Hence, we compare the scheme in [7] with GCC, $R_d(M)$ in (5) and the decentralized MNS in [5] in the numerical evaluations.

Fig. 3 shows the memory-load trade-offs for a decentralized system with N = 4 and K = 8. The decentralized MNS and the scheme in [7] with GCC have the same performance, while the proposed scheme performs better than the other ones. For instance, when M = 1.2, the proposed scheme (with load 1.894) reduces 9.9% of the load of the decentralized MNS (2.102).

V. CONCLUSION AND FURTHER WORK

We investigated the caching problem for centralized and decentralized systems with more users than files, which implies a file may be demanded by several users. We proposed a novel delivery method leveraging multicasting opportunities with the cache placements of the MNS. We showed that under the constraint of uncoded placement and K > N = 2, the proposed scheme is optimal for centralized systems. Furthermore, numerical results showed that our proposed scheme outperforms previous schemes for both centralized and decentralized systems.

Further work includes studying coded cache placement and coded delivery schemes while establishing outer bounds and optimality results beyond those derived in this paper.

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APPENDIX

Firstly we recall the Vandermonde's identity:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

From (8),

$$\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_{i}:\max\{0,t-\mathcal{G}_{i}+1\}\leq|\mathcal{J}|\leq t} \binom{|\mathcal{G}_{i}|-1}{t-|\mathcal{J}|}$$
$$=\sum_{k=\max\{0,t-\mathcal{G}_{i}+1\}}\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_{i}:|\mathcal{J}|=k}\binom{|\mathcal{G}_{i}|-1}{t-k}$$
$$=\sum_{k=\max\{0,t-\mathcal{G}_{i}+1\}}\binom{|\mathcal{G}_{i}|-1}{t-k}\binom{|\mathcal{G}_{i}|-1}{t-k}\binom{K-|\mathcal{G}_{i}|}{k} = \binom{K-1}{t}.$$

Similarly from (9),

$$\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_i:\max\{0,t-\mathcal{G}_i+1\}\leq|\mathcal{J}|\leq t,j\in\mathcal{J}} \binom{|\mathcal{G}_i|-1}{t-|\mathcal{J}|}$$
$$=\sum_{k=\max\{0,t-\mathcal{G}_i+1\}}\sum_{\mathcal{J}\subseteq[1:K]\setminus\mathcal{G}_i:|\mathcal{J}|\leq k,j\in\mathcal{J}} \binom{|\mathcal{G}_i|-1}{t-k}$$
$$=\sum_{k=\max\{1,t-\mathcal{G}_i+1\}}\binom{|\mathcal{G}_i|-1}{t-k}\binom{|\mathcal{G}_i|-1}{t-k}\binom{K-|\mathcal{G}_i|-1}{k-1} = \binom{K-2}{t-1}$$

Finally we will show the elimination of x_t for $t \in [0:K]$ in the system of inequalities (10)-(12).

If N = 2, (10)-(12) becomes

$$n(M) \ge \sum_{i=0}^{K} \frac{(K-i)(2K-i-2)}{2K(K-1)} x_i, \qquad (17)$$

$$x_0 + x_1 + \dots + x_K = 2, \qquad (18)$$

$$x_1 + 2x_2 + \dots + ix_i + \dots + Kx_K = KM.$$
(19)

For a $q \in [1:K]$ we want to eliminate x_q and x_{q-1} in (17) by the help of (18) and (19).

From (18), we have

$$\frac{2K^2 - 2K - q^2 + q}{2K(K-1)}(x_{q-1} + x_q)$$

$$=\frac{2K^2-2K-q^2+q}{2K(K-1)}(2-\sum_{i\in[0:K]:i\neq q-1,q}x_i).$$
 (20)

From (19), we have

$$\frac{2q - 3K + 1}{2K(K - 1)}(q - 1)x_{q-1} + \frac{2q - 3K + 1}{2K(K - 1)}qx_q$$

= $\frac{2q - 3K + 1}{2K(K - 1)}KM - \frac{2q - 3K + 1}{2K(K - 1)}\sum_{i \in [0:K]: i \neq q-1, q} ix_i.$
(21)

Then we sum (20) and (21),

$$\frac{(K-q)(2K-q-2)}{2K(K-1)}x_{q-1} + \frac{(K-q+1)(2K-q-1)}{2K(K-1)}x_q$$

$$= \frac{2K^2 - 2K - q^2 + q}{2K(K-1)}(2 - \sum_{i \in [0:K]: i \neq q-1, q} x_i) + \frac{2q - 3K + 1}{2(K-1)}M + \frac{2q - 3K + 1}{2K(K-1)}\sum_{i \in [0:K]: i \neq q-1, q} ix_i$$

$$= \frac{2q - 3K + 1}{2(K-1)}M + \frac{2K^2 - 2K - q^2 + q}{K(K-1)} + \sum_{i \in [0:K]: i \neq q-1, q} \frac{2K^2 + 2K + 3Ki + q^2 - q - 2qi - i}{2K(K-1)}x_i.$$
(22)

Take (22) into (17),

$$n(M) \ge \sum_{i=0}^{K} \frac{(K-i)(2K-i-2)}{2K(K-1)} x_i$$

$$\ge \frac{2K^2 - 2K - q^2 + q}{K(K-1)} + \frac{2q - 3K + 1}{2(K-1)} M$$

$$+ \sum_{i=0}^{K} \frac{(q-i)(q-i-1)}{2K(K-1)} x_i$$

$$\ge \frac{2K^2 - 2K - q^2 + q}{K(K-1)} + \frac{2q - 3K + 1}{2(K-1)} M. \quad (23)$$

When M = Nq/K, (23) becomes

$$n(M) \ge \frac{2K^2 - 2K - q^2 + q}{K(K-1)} + \frac{2q - 3K + 1}{2(K-1)}\frac{2q}{K}$$
$$= \frac{2(K-q)}{K} - \frac{q(K-q)}{K(K-1)}.$$

When M = N(q-1)/K, (23) becomes

$$n(M) \ge \frac{2K^2 - 2K - q^2 + q}{K(K - 1)} + \frac{2q - 3K + 1}{2(K - 1)} \frac{2(q - 1)}{K}$$
$$= \frac{2(K - q + 1)}{K} - \frac{(q - 1)(K - q + 1)}{K(K - 1)}.$$

Hence for $\frac{N(q-1)}{K} \leq M \leq \frac{Nq}{K}$, we can see the linear outer bound of n(M) in (23), as well as $R_{\rm u}(M)$, coincides with the lower convex envelop of the load of our proposed load in (2).

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