On the Smooth Rényi Entropy and Variable-Length Source Coding Allowing Errors

Shigeaki Kuzuoka Member, IEEE

Abstract

In this paper, we consider the problem of variable-length source coding allowing errors. The exponential moment of the codeword length is analyzed in the non-asymptotic regime and in the asymptotic regime. Our results show that the smooth Rényi entropy characterizes the optimal exponential moment of the codeword length.

Index Terms

 ε source coding, exponential moment, the smooth Rényi entropy, variable-length source coding

I. INTRODUCTION

Renato Renner and Stefan Wolf [1], [2] introduced a new information measure called the *smooth Rényi entropy*, which is a generalization of the Rényi entropy [3]. They showed that two special cases of the smooth Rényi entropy have clear operational meaning in the fixed-length source coding problem and the intrinsic randomness problem: (i) the smooth max Rényi entropy H_0^{ε} characterizes the minimum number of bits needed for with decoding error probability at most ε , and (ii) the smooth min Rényi entropy H_{∞}^{ε} characterizes the amount of uniform randomness that can be extracted from a random variable.

As the notations indicate, the smooth max/min Rényi entropies H_0^{ε} and H_{∞}^{ε} are defined as limits of the smooth Rényi entropy H_{α}^{ε} of order α ; see Section II for details. Hence it is natural to ask

Does the smooth Rényi entropy H^{ε}_{α} of order α have operational meaning?

In this study, we answer this question by demonstrating that the smooth Rényi entropy characterizes the optimal exponential moment of the codeword length of variable-length source code allowing errors. Our contributions in this paper are summarized as follows.

1

The work of S. Kuzuoka was supported in part by JSPS KAKENHI Grant Number 26820145.

S. Kuzuoka is with the Faculty of Systems Engineering, Wakayama University, 930 Sakaedani, Wakayama, 640-8510 Japan, e-mail:kuzuoka@ieee.org.

A. Contributions

We consider ε -variable-length source coding problem, that is, a variable-length source coding problem where decoding error is allowed as long as it is smaller than or equal to the given value $\varepsilon \ge 0$. Usually, in this setting, the average codeword length $\mathbb{E}[\ell(X)]$ is investigated; see, e.g., [4]. In this study, however, we adopt the criterion of minimizing the exponential moment of the codeword length $\ell(X)$, i.e., $\mathbb{E}[\exp{\lambda\ell(X)}]$ for a given parameter $\lambda > 0$.

Our first contribution is to give non-asymptotic upper and lower bounds on the exponential moment $\mathbb{E}[\exp{\lambda \ell(X)}]$ of the codeword length of ε -source codes. Our one-shot coding theorems (Theorems 1 and 2) demonstrate that the optimal exponential moment of the codeword length is characterized by the smooth Rényi entropy.

Our second contribution is a *general formula* (in the sense of Verdú-Han [5], [6]) for the asymptotic exponential rate of the exponential moment of the codeword length (Theorem 3). Moreover, to apply our general formula to the mixture of i.i.d. sources, we analyze the asymptotic behavior of the smooth Rényi entropy of the mixture of i.i.d. sources (Theorem 4).

B. Related Work

The smooth Rényi entropy was first introduced by Renner and Wolf [1], [2]. In our analysis, we use the result of Koga [7], where the smooth Rényi entropy is investigated by using majorization theory. As mentioned above, the smooth max and min Rényi entropies have clear operational meaning respectively in the fixed-length source coding [1], [2], [8] and the intrinsic randomness problem [1], [2], [9]. Recently it was shown that the smooth max Rényi entropy has an application also in variable-length lossless source coding [10], where it is shown that the smooth max Rényi entropy characterizes the threshold of codeword length under the condition that the overflow probability is at most ε . Similarly, the *smooth Rényi divergence* also finds applications in several coding problems; see, e.g., [11]–[13].

On the other hand, conventional Rényi entropy [3] also plays an important role in analyses of variable-length source coding [14], [15] and fixed-length coding [16]. In particular, Campbell [14] proposed the exponential moment of the codeword length as an alternative to the average codeword length as a criterion for variable-length lossless source coding, and gave upper and lower bounds on the exponential moment in terms of conventional Rényi entropy. Our one-shot coding theorems (Theorems 1 and 2) can be considered as a generalization of Campbell's result to the case where the decoding error is allowed. It should be mentioned here that a general problem for the optimization of the exponential moment of a given cost function was investigated by Merhav [17], [18].

Although we consider variable-length codes subject to prefix constraints in this paper, studies on variable-length codes *without* prefix constraints are also important [19], [20]. In particular, Courtade and Verdú [20] gave non-asymptotic upper and lower bounds on the distribution of codeword length by bounding the cumulant generating function of the optimum codeword lengths. It should be noted that in [19] and [20] codes are required to be injective so that the decoder can losslessly recover the source output from the codeword. The problem of variable-length

3

source coding *allowing errors* was investigated under the criterion of the average codeword length by Koga and Yamamoto [4] and Kostina *et al.* [21], [22].

C. Paper Organization

The rest of the paper is organized as follows. At first, we review the definition of the smooth Rényi entropy in Section II. Then, in Section III, non-asymptotic coding theorems for ε -variable-length source coding is given. The general formula for the optimal exponential moment of the codeword length achievable by ε -variable-length source codes is given in Section IV. Section V concludes the paper. To ensure that the main ideas are seamlessly communicated in the main text, we relegate all proofs to the appendices.

II. SMOOTH RÉNYI ENTROPY

Renner and Wolf [2] defined the smooth Rényi entropy as follows. Fix $\varepsilon \in [0, 1)$. Given a distribution P on a finite or countably infinite set \mathcal{X} , let $\mathcal{B}^{\varepsilon}(P)$ be the set of non-negative functions Q with domain \mathcal{X} such that $Q(x) \leq P(x)$, for all $x \in \mathcal{X}$, and $\sum_{x \in \mathcal{X}} Q(x) \geq 1 - \varepsilon$. Then, for $\alpha \in (0, 1) \cup (1, \infty)$, the ε -smooth Rényi entropy of order α is defined as¹

$$H^{\varepsilon}_{\alpha}(P) \triangleq \frac{1}{1-\alpha} \log r^{\varepsilon}_{\alpha}(P) \tag{1}$$

where

$$r_{\alpha}^{\varepsilon}(P) \triangleq \inf_{Q \in \mathcal{B}^{\varepsilon}(P)} \sum_{x \in \mathcal{X}} [Q(x)]^{\alpha}.$$
(2)

For basic properties of $H^{\varepsilon}_{\alpha}(P)$, see [2] and [7].

Remark 1. The definition of $H^{\varepsilon}_{\alpha}(P)$ above is slightly different from the original definition given in [1]. However, in [2], it is pointed out that this version is more appropriate for generalization to conditional smooth Rényi entropy. Our result in this paper demonstrates that this version is appropriate also for describing the variable-length source coding theorem allowing errors.

Remark 2. The max and min smooth Rényi entropies are defined respectively as

$$H_0^{\varepsilon}(P) \triangleq \lim_{\alpha \downarrow 0} H_{\alpha}^{\varepsilon}(P), \tag{3}$$

$$H^{\varepsilon}_{\infty}(P) \triangleq \lim_{\alpha \to \infty} H^{\varepsilon}_{\alpha}(P).$$
(4)

As shown in [1], $H^{\varepsilon}_{\alpha}(P)$ for $\alpha \in (0,1)$ is, up to an additive constant, equal to $H^{\varepsilon}_{0}(P)$. This fact may be one of the reasons that $H^{\varepsilon}_{\alpha}(P)$ has received less attentions. However, as shown in Theorems 1 and 2 below, $H^{\varepsilon}_{\alpha}(P)$ itself plays an important role in the evaluation of the exponential moment of the length function.

¹Throughout this paper, log denotes the natural logarithm.

III. ONE-SHOT CODING THEOREM

Let \mathcal{X} be a finite or countably infinite set and X be a random variable on \mathcal{X} with the distribution P. Without loss of generality, we assume P(X) > 0 for all $x \in \mathcal{X}$.

A variable-length source code $\Phi = (\varphi, \psi, C)$ is determined a triplet of a set $C \subset \{0, 1\}^*$ of finite-length binary strings, an encoder mapping $\varphi \colon \mathcal{X} \to C$, and a decoder mapping $\psi \colon C \to \mathcal{X}$. Without loss of generality, we assume that $C = \{\varphi(x) : x \in \mathcal{X}\}$. Further, we assume that C satisfies the prefix condition. The error probability of the code Φ is defined as

$$P_{e}(\Phi) \triangleq \Pr\left\{X \neq \psi(\varphi(X))\right\}.$$
(5)

The length of the codeword $\varphi(x)$ of x (in bits) is denoted by $\|\varphi(x)\|$. Let ℓ be the length function (in nats):

$$\ell(x) \triangleq \|\varphi(x)\| \log 2. \tag{6}$$

In this study, we focus on the exponential moment of the length function. For a given $\lambda > 0$, let us consider the problem of minimizing

$$\mathbb{E}_P\left[\exp\{\lambda\ell(X)\}\right] \tag{7}$$

subject to $P_e(\Phi) \leq \varepsilon$, where \mathbb{E}_P denotes the expectation with respect to the distribution P.

Remark 3. In Theorems 1 and 2 below, we allow the encoder mapping φ to be stochastic. Let $W_{\varphi}(c|x)$ be the probability that $x \in \mathcal{X}$ is encoded in $c \in \mathcal{C}$. Then, $P_{e}(\Phi)$ and $\mathbb{E}_{P}[\exp{\{\lambda \ell(X)\}}]$ are precisely written as

$$P_{e}(\Phi) = \sum_{x \in \mathcal{X}} P(x) \sum_{c: x \neq \psi(c)} W_{\varphi}(c|x)$$
(8)

and

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right] = \sum_{x\in\mathcal{X}} P(x) \sum_{c\in\mathcal{C}} W_{\varphi}(c|x) \exp\{\lambda \|c\| \log 2\}$$
(9)

where ||c|| is the length (in bits) of $c \in C$. Note that, without loss of optimality we can assume that the decoder mapping ψ is deterministic. Indeed, for a given W_{φ} , we can choose ψ so that

$$\psi(c) = \arg\max W_{\varphi}(c|x)P(x). \tag{10}$$

The following theorems demonstrate that the exponential moment $\mathbb{E}_P\left[\exp\{\lambda\ell(X)\}\right]$ is characterized by the smooth Rényi entropy $H^{\varepsilon}_{1/(1+\lambda)}(P)$.

Theorem 1. For any $\lambda > 0$ and $\varepsilon \in [0, 1)$, there exists a code Φ (with a stochastic encoder) such that $P_e(\Phi) \leq \varepsilon$ and

$$\mathbb{E}_P\left[\exp\{\lambda\ell(X)\}\right] \le 2^{2\lambda} \exp\left\{\lambda H^{\varepsilon}_{1/(1+\lambda)}(P)\right\} + \varepsilon 2^{\lambda}.$$
(11)

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right] \ge \exp\left\{\lambda H_{1/(1+\lambda)}^{\varepsilon}(P)\right\}.$$
(12)

Theorem 1 and Theorem 2 will be proved in respectively Appendix A and Appendix B.

In Theorem 1, we allow the encoder mapping φ to be stochastic. However, it is not hard to modify the theorem for the case where only deterministic encoder mappings are allowed. To see this, let $\mathcal{X} = \{1, 2, 3, ...\}$ and assume that $P(1) \ge P(2) \ge \cdots$. Then, let $k^* = k^*(\varepsilon)$ be the minimum integer such that $\sum_{i=1}^{k^*} P(i) \ge 1 - \varepsilon$ and let

$$Q^{*}(i) \triangleq \begin{cases} P(i), & i = 1, 2, \dots, k^{*} - 1, \\ 1 - \varepsilon - \sum_{i=1}^{k^{*} - 1} P(i), & i = k^{*}, \\ 0, & i > k^{*}. \end{cases}$$
(13)

Since $0 < 1/(1 + \lambda) < 1$ for all $\lambda > 0$, we can use (A) of Theorem 1 of [7] and obtain

$$\lambda H_{1/(1+\lambda)}^{\varepsilon}(P) = (1+\lambda) \log\left(\sum_{i\in\mathcal{X}} [Q^*(i)]^{1/(1+\lambda)}\right).$$
(14)

Based on this fact, we can modify the proof of Theorem 1 and obtain the following result (See Appendix C for details).

Proposition 1. For any $\lambda > 0$ and $\varepsilon \in [0, 1)$, there exists a code Φ with a deterministic encoder mapping φ such that $P_e(\Phi) \leq \varepsilon + \gamma_{\varepsilon}$ and

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right] \leq 2^{2\lambda}\exp\left\{\lambda H_{1/(1+\lambda)}^{\varepsilon+\gamma_{\varepsilon}}(P)\right\} + (\varepsilon+\gamma_{\varepsilon})2^{\lambda}$$
(15)

where $\gamma_{\varepsilon} \triangleq 1 - \varepsilon - \sum_{i=1}^{k^*(\varepsilon) - 1} P(i)$.

IV. GENERAL FORMULA

In this section, we consider coding problem for general sources. A general source

$$\boldsymbol{X} = \{X^n = (X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)})\}_{n=1}^{\infty}$$
(16)

is defined as a sequence of random variables X^n on the *n*-th Cartesian product \mathcal{X}^n of \mathcal{X} [6]. The distribution of X^n is denoted by P_{X^n} , which is not required to satisfy the consistency condition.

We consider a sequence of coding problems indexed by the blocklength n. A code of block length n is denoted by $\Phi_n = (\varphi_n, \psi_n, C_n)$. The length function of Φ_n is denoted by ℓ_n , i.e., $\ell_n(x^n) \triangleq \|\varphi_n(x^n)\| \log 2$ for all $x^n \in \mathcal{X}^n$. We are interested in the asymptotic behavior of $(1/n) \log \mathbb{E}_{P_{X^n}} [\exp\{\lambda \ell_n(X^n)\}]$.

A value E is said to be ε -achievable if there exists a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of codes satisfying

$$\limsup_{n \to \infty} \mathcal{P}_{\mathbf{e}}(\Phi_n) \le \varepsilon \tag{17}$$

and

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{P_{X^n}} \left[\exp\{\lambda \ell_n(X^n)\} \right] \le E.$$
(18)

The infimum of ε -achievable values is denoted by $E_{\lambda}^{\varepsilon}(\boldsymbol{X})$.

To characterize $E^{\varepsilon}_{\lambda}(X)$, we introduce the following notation.

$$H_{\alpha}^{\varepsilon}(\boldsymbol{X}) \triangleq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{1}{n} H_{\alpha}^{\varepsilon + \delta}(P_{X^{n}}).$$
⁽¹⁹⁾

It is worth to note that $H^{\varepsilon}_{\alpha}(\mathbf{X})$ is non-negative for all $\alpha \in (0,1)$ and $\varepsilon \in [0,1)$. Indeed, we can prove the stronger fact that

$$\liminf_{n \to \infty} \frac{1}{n} H^{\varepsilon}_{\alpha}(P_{X^n}) \ge 0, \quad \alpha \in (0, 1), \varepsilon \in [0, 1).$$
⁽²⁰⁾

We will prove (20) in Appendix D.

Now, we state our general formula, which will be proved in Appendix E.

Theorem 3. For any $\lambda > 0$ and $\varepsilon \in [0, 1)$,

$$E_{\lambda}^{\varepsilon}(\boldsymbol{X}) = \lambda H_{1/(1+\lambda)}^{\varepsilon}(\boldsymbol{X}).$$
⁽²¹⁾

In the following, we consider a mixture of i.i.d. sources. Let us consider m distributions P_{X_i} (i = 1, 2, ..., m)on \mathcal{X} . A general source \mathbf{X} is said to be a mixture of $P_{X_1}, P_{X_2}, ..., P_{X_m}$ if there exists $(\alpha_1, \alpha_2 ..., \alpha_m)$ satisfying $\sum_i \alpha_i = 1, \alpha_i > 0$ (i = 1, ..., m), and for all n = 1, 2, ... and all $x^n = (x_1, x_2, ..., x_n) \in \mathcal{X}^n$

$$P_{X^{n}}(x^{n}) = \sum_{i=1}^{m} \alpha_{i} P_{X^{n}_{i}}(x^{n})$$
(22)

$$=\sum_{i=1}^{m} \alpha_{i} \prod_{t=1}^{n} P_{X_{i}}(x_{t}).$$
(23)

For the later use, let $A_i \triangleq \sum_{j=1}^{i-1} \alpha_i$ (i = 1, 2, ..., m) and $A_{m+1} \triangleq 1$. Further, to simplify the analysis, we assume that

$$H(X_1) > H(X_2) > \dots > H(X_m) \tag{24}$$

where $H(X_i)$ is the entropy determined by P_{X_i} :

$$H(X_i) \triangleq \sum_{x \in \mathcal{X}} P_{X_i}(x) \log \frac{1}{P_{X_i}(x)}.$$
(25)

Then, $H^{\varepsilon}_{\alpha}(X)$ of the mixture X is characterized as in the following theorem.

Theorem 4. Let X be a mixture of i.i.d. sources satisfying (24). Fix $\alpha \in (0, 1)$, i, and $\varepsilon \in [A_i, A_{i+1})$. Then, we have

$$H_{\alpha}^{\varepsilon}(\boldsymbol{X}) = H(X_i). \tag{26}$$

Theorem 4 will be proved in Appendix F.

Remark 4. Letting m = 1 and $\varepsilon \downarrow 0$, Theorem (4) derives Lemma I.2 of [1].

Remark 5. Although Theorem 4 assumes that components are i.i.d., this assumption is not crucial. Indeed, the property of i.i.d. sources used in the proof of the theorem is only that the AEP [23] holds, i.e.,

$$\lim_{n \to \infty} \Pr\left\{ \left| \frac{1}{n} \log \frac{1}{P_{X_i^n}(X_i^n)} - H(X_i) \right| > \gamma \right\} = 0$$
(27)

for all i = 1, 2, ..., m and any $\gamma > 0$. Hence, it is straightforward to extend the theorem so that it can be applied for the mixture of stationary and ergodic sources. Moreover, since we use only the AEP, it can be seen that the assumption (24) is also not crucial. Assume that there exists some components $j_1 \neq j_2$ such as $H(X_{j_1}) = H(X_{j_2})$. Then, let us consider the modified mixture such that " j_2 th component is substituted by j_1 th component": i.e.,

$$P_{X^{n}}(x^{n}) = \sum_{i \neq j_{2}} \alpha'_{i} P_{X^{n}_{i}}(x^{n})$$
(28)

where $\alpha'_i = \alpha_i$ for $i \neq j_1$ and $\alpha'_{j_1} = \alpha_{j_1} + \alpha_{j_2}$. Then $H^{\varepsilon}_{\alpha}(\mathbf{X})$ the modified mixture is identical with the original one.

Combining Theorems 3 and 4, we have the coding theorem for the mixture of i.i.d. sources.

Corollary 1. Let X be a mixture of i.i.d. sources satisfying (24). Then, for any $\lambda > 0$ and $\varepsilon \in [0, 1)$,

$$E_{\lambda}^{\varepsilon}(\boldsymbol{X}) = \lambda H(X_i) \tag{29}$$

where *i* is determined so that $\varepsilon \in [A_i, A_{i+1})$.

V. CONCLUDING REMARKS

In this paper, we investigated the the exponential moment of the codeword length of variable-length source coding allowing decoding errors. Roughly speaking, our results demonstrate that the logarithm $\log \mathbb{E}_P \left[\exp\{\lambda \ell(X)\} \right]$ of the optimal exponential moment $\mathbb{E}_P \left[\exp\{\lambda \ell(X)\} \right]$ is characterized by the smooth Rényi entropy $H_{1/(1+\lambda)}^{\varepsilon}$.

Now, let us consider to take $\lambda \to \infty$. When λ is sufficiently large, the value $\log \mathbb{E}_P [\exp\{\lambda \ell(X)\}]$ is dominated by the longest codeword length $\max_{x \in \mathcal{X}} \ell(x)$. In other words, to minimize $\log \mathbb{E}_P [\exp\{\lambda \ell(X)\}]$, we need to minimize the longest codeword length $\max_{x \in \mathcal{X}} \ell(x)$. Therefore, roughly speaking, the difference between variablelength coding and fixed-length coding becomes smaller as λ is increased. On the other hand, we know that $H_0^{\varepsilon} = \lim_{\lambda \to \infty} H_{1/(1+\lambda)}^{\varepsilon}$ characterizes the optimal coding rate of fixed-length codes [1], [8]. The above argument implies that we can unify our result and results of [1], [8] in the limit of $\lambda \to \infty$ or equivalently $\alpha \to 0$.

On the other hand, since $\lambda > 0$ and thus $0 < 1/(1 + \lambda) < 1$, only the smooth Rényi entropy H_{α}^{ε} of the order $\alpha \in (0, 1)$ plays an important role in our coding theorems. It remains as a future work to investigate the operational meaning of the smooth Rényi entropy H_{α}^{ε} of the order $\alpha > 1$.

APPENDIX A

PROOF OF THEOREM 1

Fix $\delta > 0$ arbitrarily and choose $Q \in \mathcal{B}^{\varepsilon}(P)$ so that

$$\log \sum_{x \in \mathcal{X}} [Q(x)]^{1/(1+\lambda)} \le \frac{\lambda}{1+\lambda} H^{\varepsilon}_{1/(1+\lambda)}(P) + \delta.$$
(30)

Let $\mathcal{A} \triangleq \{x \in \mathcal{X} : Q(x) > 0\}$ and

$$\tilde{Q}^{(\lambda)}(x) = \frac{[Q(x)]^{1/(1+\lambda)}}{\sum_{x' \in \mathcal{A}} [Q(x')]^{1/(1+\lambda)}}.$$
(31)

Since

$$\sum_{x \in \mathcal{A}} 2^{-\{-\log_2 \tilde{Q}^{(\lambda)}(x)\}} \le 1$$
(32)

holds, we can construct $(\hat{\varphi}, \hat{\psi}, \hat{C})$ such that (i) $\hat{C} \triangleq \{\hat{\varphi}(x) : x \in A\}$ is prefix free, (ii) $\hat{\varphi} \colon A \to \hat{C}$ satisfies

$$\|\hat{\varphi}(x)\| = \left[-\log_2 \tilde{Q}^{(\lambda)}(x)\right],\tag{33}$$

and, (iii) $\hat{\varphi}$ and $\hat{\psi} \colon \hat{\mathcal{C}} \to \mathcal{A}$ satisfy $x = \psi(\varphi(x))$ for all $x \in \mathcal{A}$.

For each $x \in \mathcal{X}$, let $\gamma(x) = Q(x)/P(x)$. Note that $0 \leq \gamma(x) \leq 1$ and $\gamma(x) = 0$ for all $x \notin \mathcal{A}$. Since $Q \in \mathcal{B}^{\varepsilon}(P)$, we have

$$\sum_{x \in \mathcal{X}} P(x)\gamma(x) \ge 1 - \varepsilon.$$
(34)

Now, we construct a stochastic encoder as follows:

$$\varphi(x) = \begin{cases} 0 \circ \hat{\varphi}(x) & \text{with probability } \gamma(x) \\ 1 & \text{with probability } 1 - \gamma(x) \end{cases}$$
(35)

where \circ denotes the concatenation. That is, x is encoded to "0" following $\hat{\varphi}(x)$ with probability $\gamma(x)$, and "1" with probability $1 - \gamma(x)$. We can construct the corresponding decoder ψ so that $x = \psi(\varphi(x))$ for all $x \in \mathcal{X}$. The length function $\ell(x) = \|\varphi(x)\| \log 2$ satisfies that, if x is encoded to "0" following $\hat{\varphi}(x)$,

$$\ell(x) \le -\log \tilde{Q}^{(\lambda)}(x) + 2\log 2 \tag{36}$$

and otherwise $\ell(x) = \log 2$. Hence, we have

$$\mathbb{E}_{P} \left[\exp\{\lambda \ell(X)\} \right]$$

$$\leq \sum_{x \in \mathcal{X}} P(x)\gamma(x) \exp\{\lambda \left[-\log \tilde{Q}^{(\lambda)}(x) + 2\log 2 \right] \right\}$$

$$+ \sum_{x \in \mathcal{X}} P(x)(1 - \gamma(x)) \exp\{\lambda \log 2\}$$
(37)

$$\stackrel{\text{(a)}}{\leq} 2^{2\lambda} \sum_{x \in \mathcal{A}} Q(x) \exp\left\{-\lambda \log \tilde{Q}^{(\lambda)}(x)\right\} + \varepsilon 2^{\lambda}$$
(38)

$$=2^{2\lambda} \left\{ \sum_{x \in \mathcal{A}} [Q(x)]^{1/(1+\lambda)} \right\}^{(1+\lambda)} + \varepsilon 2^{\lambda}$$
(39)

$$\stackrel{\text{(b)}}{\leq} 2^{2\lambda} \exp\left\{\lambda H^{\varepsilon}_{1/(1+\lambda)}(P) + (1+\lambda)\delta\right\} + \varepsilon 2^{\lambda}$$
(40)

where the inequality (a) follows from (34) and (b) follows from (30). Since we can choose $\delta > 0$ arbitrarily small, we have (11).

APPENDIX B

Proof of Theorem 2

Fix a code $\Phi=(\varphi,\psi,\mathcal{C})$ such that $P_e(\Phi)\leq 1-\varepsilon.$

Recall that we allow φ to be stochastic. Let $W_{\varphi}(c|x)$ be the probability such that $x \in \mathcal{X}$ is mapped to $c \in \mathcal{C}$. Let

$$\Gamma(x) \triangleq \{ c \in \mathcal{C} : W_{\varphi}(c|x) > 0, x = \psi(c) \}$$
(41)

and

$$\gamma(x) \triangleq \sum_{c \in \Gamma(x)} W_{\varphi}(c|x).$$
(42)

Note that, since $P_e(\Phi) \leq \varepsilon$, we have

$$\sum_{x \in \mathcal{X}} P(x)\gamma(x) \ge 1 - \varepsilon.$$
(43)

Further, we have

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right]$$

$$=\sum_{x\in\mathcal{X}}P(x)\sum_{c\in\mathcal{C}}W_{\varphi}(c|x)\exp\{\lambda\|c\|\log 2\}$$
(44)

$$\geq \sum_{x \in \mathcal{X}} P(x) \sum_{c \in \Gamma(x)} W_{\varphi}(c|x) \exp\{\lambda \|c\| \log 2\}.$$
(45)

From Jensen's inequality, it is not hard to see that

$$\sum_{c \in \Gamma(x)} W_{\varphi}(c|x) \exp\{\lambda \|c\| \log 2\}$$

$$\geq \gamma(x) \exp\left\{\lambda \sum_{c \in \Gamma(x)} \frac{W_{\varphi}(c|x)}{\gamma(x)} \|c\| \log 2\right\}$$
(46)

$$\geq \gamma(x) \exp\left\{\lambda \sum_{c \in \Gamma(x)} \frac{W_{\varphi}(c|x)}{\gamma(x)} \bar{\ell}(x)\right\}$$
(47)

$$=\gamma(x)\exp\left\{\lambda\bar{\ell}(x)\right\}\tag{48}$$

where

$$\bar{\ell}(x) \triangleq \min_{c \in \Gamma(x)} \|c\| \log 2.$$
(49)

Substituting (48) into (45), we have

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right] \ge \sum_{x\in\mathcal{X}} P(x)\gamma(x)\exp\{\lambda\bar{\ell}(x)\}.$$
(50)

Let $Q(x) = P(x)\gamma(x)$. Then, from (43), we have $Q \in \mathcal{B}^{\varepsilon}(P)$. Let $\mathcal{A} = \{x : Q(x) > 0\}$. Then, (50) can be written as

$$\mathbb{E}_P\left[\exp\{\lambda\ell(X)\}\right] \ge \sum_{x\in\mathcal{A}} Q(x) \exp\left\{\lambda\bar{\ell}(x)\right\}.$$
(51)

On the other hand, from the definition of the set $\Gamma(x)$, we can see that $\Gamma(x) \cap \Gamma(x') = \emptyset$ for all $x, x' \in \mathcal{A}$ such that $x \neq x'$, and thus we have

$$\sum_{x \in \mathcal{A}} \exp\{-\bar{\ell}(x)\} \le 1.$$
(52)

Now, let us consider the problem of minimizing $\sum_{x \in \mathcal{A}} Q(x) \exp \{\lambda \bar{\ell}(x)\}$ subject to (52). As shown in Example 1 in Section 3 of [18], the minimum is achieved by

$$\bar{\ell}(x) = -\log \frac{[Q(x)]^{1/(1+\lambda)}}{\sum_{x' \in \mathcal{A}} [Q(x')]^{1/(1+\lambda)}}, \quad x \in \mathcal{A}.$$
(53)

In other words, (51) can be rewritten as

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell(X)\}\right] \geq \sum_{x\in\mathcal{A}} Q(x) \exp\left\{-\lambda\log\frac{[Q(x)]^{1/(1+\lambda)}}{\sum_{x'\in\mathcal{A}}[Q(x')]^{1/(1+\lambda)}}\right\}$$
(54)

$$=\left[\sum_{x\in\mathcal{A}} [Q(x)]^{1/(1+\lambda)}\right]^{(1+\lambda)}$$
(55)

$$\geq \left[r_{1/(1+\lambda)}^{\varepsilon}(P)\right]^{(1+\lambda)} \tag{56}$$

where the last inequality follows from the fact $Q \in \mathcal{B}^{\varepsilon}(P)$. By the definition of the smooth Rényi entropy, we have (12).

October 23, 2018

APPENDIX C

PROOF OF PROPOSITION 1

Let

$$\hat{Q}^*(i) \triangleq \begin{cases} P(i), & i = 1, 2, \dots, k^*(\varepsilon) - 1, \\ 0, & i > k^*(\varepsilon). \end{cases}$$
(57)

Then, from Theorem 1 (A) of [7], we have

$$\lambda H_{1/(1+\lambda)}^{\varepsilon+\gamma_{\varepsilon}}(P) = (1+\lambda) \log\left(\sum_{i \in \mathcal{X}} [Q^*(i)]^{1/(1+\lambda)}\right).$$
(58)

Now, let us substitute ε (resp. Q) in the proof of Theorem 1 with $\varepsilon + \gamma_{\varepsilon}$ (resp. \hat{Q}^*). Note that $\gamma(x) = \hat{Q}^*(x)/P(x)$ satisfies

$$\gamma(x) = \begin{cases} 1, & i = 1, 2, \dots, k^*(\varepsilon) - 1, \\ 0, & i > k^*(\varepsilon). \end{cases}$$
(59)

Thus, the encoder constructed in the proof of Theorem 1 becomes deterministic. Hence, we can obtain the proposition. \Box

APPENDIX D

PROOF OF (20)

Fix $\alpha \in (0,1)$ and $\varepsilon \in [0,1)$, and then, choose $\varepsilon' > 0$ so that $\varepsilon + \varepsilon' < 1$. From Lemma 2 of [2], we have

$$\frac{1}{n}H_{\alpha}^{\varepsilon}(P_{X^{n}}) \geq \frac{1}{n}H_{0}^{\varepsilon+\varepsilon'}(P_{X^{n}}) - \frac{\log(1/\varepsilon')}{n(1-\alpha)}.$$
(60)

On the other hand, it is known that $H_0^{\varepsilon+\varepsilon'}(P_{X^n})$ can be written as

$$H_0^{\varepsilon+\varepsilon'}(P_{X^n}) = \min_{\substack{\mathcal{A}\subseteq\mathcal{X}\\P(\mathcal{A})\geq 1-\varepsilon-\varepsilon'}} \log|\mathcal{A}|,$$
(61)

where $|\mathcal{A}|$ is the cardinality of \mathcal{A} , and thus, $H_0^{\varepsilon+\varepsilon'}(P_{X^n}) \ge 0$. So, taking the inferior limit of both sides of (60), we have (20).

APPENDIX E

PROOF OF THEOREM 3

Direct Part: At first, we consider the case where

$$H_{1/(1+\lambda)}^{\varepsilon}(\boldsymbol{X}) > 0.$$
(62)

In this case, for all sufficiently small $\delta > 0$ and sufficiently large n, we have

$$2^{2\lambda} \exp\left\{\lambda H^{\varepsilon+\delta}_{1/(1+\lambda)}(P_{X^n})\right\} > \varepsilon 2^{\lambda}.$$
(63)

Hence, from Theorem 1, there exists $\{\Phi_n\}_{n=1}^\infty$ such that

$$P_{e}(\Phi_{n}) \leq \varepsilon + \delta, \quad n = 1, 2, \dots, \tag{64}$$

and, for sufficiently large n,

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell_{n}(X^{n})\}\right] \leq 2 \times 2^{2\lambda} \exp\left\{\lambda H_{1/(1+\lambda)}^{\varepsilon+\delta}(P_{X^{n}})\right\}.$$
(65)

Eq. (65) gives

$$\lim_{n \to \infty} \sup_{n} \frac{1}{n} \log \mathbb{E}_{P} \left[\exp\{\lambda \ell_{n}(X^{n})\} \right]$$

$$\leq \lambda \limsup_{n \to \infty} \frac{1}{n} H_{1/(1+\lambda)}^{\varepsilon + \delta}(P_{X^{n}}).$$
(66)

By using the *diagonal line argument* (see [6]), we can conclude that $\lambda H^{\varepsilon}_{1/(1+\lambda)}(X)$ is ε -achievable.

If $H^{\varepsilon}_{1/(1+\lambda)}(\boldsymbol{X}) = 0$ then (65) is replaced with

$$\mathbb{E}_{P}\left[\exp\{\lambda\ell_{n}(X^{n})\}\right] \leq \max\left\{2 \times 2^{2\lambda} \exp\left\{\lambda H_{1/(1+\lambda)}^{\varepsilon+\delta}(P_{X^{n}})\right\}, 2 \times \varepsilon 2^{\lambda}\right\}$$
(67)

In this case, we can also prove that 0 is ε -achievable in the same way as the case $H^{\varepsilon}_{1/(1+\lambda)}(X) > 0$.

Converse Part: Suppose that E is ε -achievable and fix $\delta > 0$ arbitrarily. Then there exists $\{\Phi_n\}_{n=1}^{\infty}$ such that, for sufficiently large n,

$$P_{e}(\Phi_{n}) \le \varepsilon + \delta \tag{68}$$

and

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_P \left[\exp\{\lambda \ell_n(X^n)\} \right] \le E.$$
(69)

On the other hand, from Theorem 2, for sufficiently large n such that (68) holds,

$$\mathbb{E}_P\left[\exp\{\lambda\ell_n(X^n)\}\right] \ge \exp\left\{\lambda H_{1/(1+\lambda)}^{\varepsilon+\delta}(P_{X^n})\right\}.$$
(70)

Combining (69) with (70), we have

$$E \ge \lambda \limsup_{n \to \infty} \frac{1}{n} H_{1/(1+\lambda)}^{\varepsilon+\delta}(P_{X^n}).$$
(71)

Since $\delta > 0$ is arbitrary, letting $\delta \downarrow 0$, we have $E \ge \lambda H_{1/(1+\lambda)}^{\varepsilon}(\mathbf{X})$. \Box

APPENDIX F

PROOF OF THEOREM 4

A. Lemmas

Before proving the theorem, we introduce some lemmas.

Lemma 1. Fix $\gamma > 0$ arbitrarily. Then, there exists an integer n_0 so that for all $n \ge n_0$ and all $i = 1, 2, \ldots, m$,

$$\Pr\left\{\frac{1}{n}\log\frac{1}{P_{X^n}(X^n)} \ge H(X_i) - \gamma\right\} \ge A_{i+1} - \gamma.$$
(72)

Proof: For each $k = 1, 2, \ldots, m$, let

$$\mathcal{S}_{k}^{n} \triangleq \left\{ x^{n} : \frac{1}{n} \log \frac{1}{P_{X_{k}^{n}}(x^{n})} \ge H(X_{k}) - \frac{\gamma}{2} \right\}.$$
(73)

Since i.i.d. sources satisfy the AEP [23], we can choose n_1 such that

$$\sum_{x^n \in \mathcal{S}_k^n} P_{X_k^n}(x^n) \ge 1 - \frac{\gamma}{2}, \quad \forall n \ge n_1, \forall k = 1, 2, \dots, m.$$

$$(74)$$

Moreover, we can choose $n_0 \ge n_1$ so that

$$-\frac{1}{n}\log\gamma \le \frac{\gamma}{2}, \quad \forall n \ge n_0.$$
(75)

Then, for all $n \ge n_0$, any $i = 1, 2, \ldots, m$, and any $k = 1, 2, \ldots, i$,

$$\tilde{\mathcal{S}}_{i}^{n} \triangleq \left\{ x^{n} : \frac{1}{n} \log \frac{1}{P_{X^{n}}(x^{n})} \ge H(X_{i}) - \gamma \right\}$$
(76)

and

$$\mathcal{T}_k^n \triangleq \left\{ x^n : P_{X_k^n}(x^n) \le \gamma P_{X^n}(x^n) \right\}$$
(77)

satisfy that

$$\tilde{\mathcal{S}}_{i}^{n} \cup \mathcal{T}_{k}^{n} \supseteq \left\{ x^{n} : \frac{1}{n} \log \frac{\gamma}{P_{X_{k}^{n}}(x^{n})} \ge H(X_{i}) - \gamma \right\}$$
(78)

$$=\left\{x^{n}:\frac{1}{n}\log\frac{1}{P_{X_{k}^{n}}(x^{n})}\geq H(X_{i})-\gamma-\frac{1}{n}\log\gamma\right\}$$
(79)

$$\supseteq \left\{ x^n : \frac{1}{n} \log \frac{1}{P_{X_k^n}(x^n)} \ge H(X_i) - \frac{\gamma}{2} \right\}$$
(80)

$$\supseteq \mathcal{S}_k^n. \tag{81}$$

Thus, we have

$$\sum_{x^n \in \tilde{\mathcal{S}}_i^n} P_{X^n}(x^n) \ge \sum_{k=1}^i \alpha_k \sum_{x^n \in \tilde{\mathcal{S}}_i^n} P_{X^n_k}(x^n)$$
(82)

$$\geq \sum_{k=1}^{i} \alpha_k \sum_{x^n \in \mathcal{S}_k^n} P_{X_k^n}(x^n) - \sum_{k=1}^{i} \alpha_k \sum_{x^n \in \mathcal{T}_k^n} P_{X_k^n}(x^n)$$
(83)

$$\geq A_{i+1}(1-\gamma/2) - \sum_{k=1}^{i} \alpha_k \sum_{x^n \in \mathcal{T}_k^n} \frac{\gamma}{2} P_{X^n}(x^n)$$
(84)

$$\geq A_{i+1}(1-\gamma/2) - \frac{\gamma}{2} \tag{85}$$

$$\geq A_{i+1} - \gamma. \tag{86}$$

Lemma 2. Fix $\gamma > 0$ arbitrarily. Then, there exists an integer n_0 so that for all $n \ge n_0$ and all $i = 1, 2, \ldots, m$,

$$\Pr\left\{\frac{1}{n}\log\frac{1}{P_{X^n}(X^n)} \le H(X_i) + \gamma\right\} \ge 1 - A_i - \gamma.$$
(87)

Proof: For each $k = 1, 2, \ldots, m$, let

$$\mathcal{S}_{k}^{n} \triangleq \left\{ x^{n} : \frac{1}{n} \log \frac{1}{P_{X_{k}^{n}}(x^{n})} \le H(X_{k}) + \frac{\gamma}{2} \right\}.$$
(88)

Since i.i.d. sources satisfy the AEP [23], we can choose n_1 such that

$$\sum_{(x^n, y^n) \in \mathcal{S}_k^n} P_{X_k^n Y_k^n}(x^n, y^n) \ge 1 - \gamma, \quad \forall n \ge n_1, \forall k = 1, 2, \dots, m.$$
(89)

Moreover, we can choose $n_0 \ge n_1$ so that

$$-\frac{1}{n}\log\alpha_k \le \frac{\gamma}{2}, \quad \forall n \ge n_0, \forall k = 1, 2, \dots, m.$$
(90)

Hence, for all $n \ge n_0$ and any i,

$$\tilde{\mathcal{S}}_{i}^{n} \triangleq \left\{ x^{n} : \frac{1}{n} \log \frac{1}{P_{X^{n}}(x^{n})} \le H(X_{k}) + \gamma \right\}$$
(91)

satisfies that

$$\tilde{\mathcal{S}}_{i}^{n} \supseteq \left\{ x^{n} : \frac{1}{n} \log \frac{1}{\alpha_{k} P_{X_{k}^{n}}(x^{n})} \le H(X_{i}) + \gamma \right\}$$
(92)

$$=\left\{x^{n}:\frac{1}{n}\log\frac{1}{P_{X_{k}^{n}}(x^{n})}\leq H(X_{i})+\gamma+\frac{1}{n}\log\alpha_{k}\right\}$$
(93)

$$\supseteq S_k^n. \tag{94}$$

Thus, we have

$$\sum_{x^n \in \tilde{\mathcal{S}}_i^n} P_{X^n}(x^n) \ge \sum_{k=i}^m \alpha_k \sum_{x^n \in \tilde{\mathcal{S}}_i^n} P_{X^n_k}(x^n)$$
(95)

$$\geq \sum_{k=i}^{m} \alpha_k \sum_{x^n \in \mathcal{S}_k^n} P_{X_k^n}(x^n) \tag{96}$$

$$\geq (1 - A_i)(1 - \gamma) \tag{97}$$

$$\geq 1 - A_i - \gamma. \tag{98}$$

Lemma 3. Fix $\gamma > 0$ so that $H(X_j) - \gamma > H(X_{j+1}) + \gamma$ for all j = 1, 2, ..., m - 1. Then, for sufficiently large n and any i = 1, 2, ..., m,

$$\alpha_i - 2\gamma \le \Pr\left\{ \left| \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} - H(X_i) \right| \le \gamma \right\} \le \alpha_i + 2\gamma.$$
(99)

October 23, 2018

DRAFT

Proof: From Lemmas 1 and 2, we have

$$\Pr\left\{ \left| \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} - H(X_i) \right| \le \gamma \right\}$$
$$= \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \le H(X_i) + \gamma \right\} - \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} < H(X_i) - \gamma \right\}$$
(100)

$$\geq \{1 - A_i - \gamma\} - \{1 - (A_{i+1} - \gamma)\}$$
(101)

$$=\alpha_i - 2\gamma \tag{102}$$

and

$$\Pr\left\{ \left| \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} - H(X_i) \right| \le \gamma \right\}$$
$$= \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \le H(X_i) + \gamma \right\} - \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} < H(X_i) - \gamma \right\}$$
(103)

$$\leq \Pr\left\{\frac{1}{n}\log\frac{1}{P_{X^{n}}(X^{n})} < H(X_{i-1}) - \gamma\right\} - \Pr\left\{\frac{1}{n}\log\frac{1}{P_{X^{n}}(X^{n})} \le H(X_{i+1}) + \gamma\right\}$$
(104)

$$\leq \{1 - (A_i - \gamma)\} - \{1 - A_{i+1} - \gamma\}$$
(105)

$$=\alpha_i + 2\gamma. \tag{106}$$

B. Proof of Theorem 4

To prove the theorem, it is sufficient to show that, for ε satisfying $A_i < \varepsilon < A_{i+1},$

$$\limsup_{n \to \infty} \frac{1}{n} H^{\varepsilon}_{\alpha}(P_{X^n}) \le H(X_i)$$
(107)

and

$$\liminf_{n \to \infty} \frac{1}{n} H^{\varepsilon}_{\alpha}(P_{X^n}) \ge H(X_i).$$
(108)

Proof of (107): Fix $\gamma > 0$ sufficiently small so that $H(X_j) - \gamma > H(X_{j+1}) + \gamma$ for all j = 1, 2, ..., m - 1and that $A_i + 2m\gamma < \varepsilon$. For j = 1, 2, ..., m, let

$$\mathcal{T}_n(j) \triangleq \left\{ x^n : \left| \frac{1}{n} \log \frac{1}{P_{X^n}(x^n)} - H(X_j) \right| \le \gamma \right\}.$$
(109)

Note that $\mathcal{T}_n(j) \cap \mathcal{T}_n(\hat{j}) = \emptyset \ (j \neq \hat{j})$. Further, from Lemma 3, we have

$$\Pr\left\{X^{n} \in \bigcup_{j=i}^{m} \mathcal{T}_{n}(j)\right\} = \sum_{j=i}^{m} \Pr\left\{X^{n} \in \mathcal{T}_{n}(j)\right\}$$
(110)

$$\geq \sum_{j=i}^{m} \left(\alpha_j - 2\gamma \right) \tag{111}$$

$$\geq 1 - A_i - 2m\gamma \tag{112}$$

$$\geq 1 - \varepsilon. \tag{113}$$

From (113), we can see that

$$Q_n(x^n) \triangleq \begin{cases} P_{X^n}(x^n), & \text{if } x^n \in \bigcup_{j=i}^m \mathcal{T}_n(j) \\ 0, & \text{otherwise} \end{cases}$$
(114)

satisfies $Q_n \in \mathcal{B}^{\varepsilon}(P_{X^n})$. Thus, from the definition of $r_{\alpha}^{\varepsilon}(P_{X^n})$,

$$r_{\alpha}^{\varepsilon}(P_{X^n}) \le \sum_{x^n \in \mathcal{X}^n} [Q_n(x^n)]^{\alpha}$$
(115)

$$= \sum_{j=i}^{m} \sum_{x^{n} \in \mathcal{T}_{n}(j)} [P_{X^{n}}(x^{n})]^{\alpha}$$
(116)

$$\leq \sum_{j=i}^{m} |\mathcal{T}_n(j)| \exp\{-\alpha n(H(X_j) - \gamma)\}$$
(117)

$$\leq \sum_{j=i}^{m} \exp\{n(H(X_j) + \gamma)\} \exp\{-\alpha n(H(X_j) - \gamma)\}$$
(118)

$$= \sum_{j=i}^{m} \exp\{n[(1-\alpha)H(X_j) + (1+\alpha)\gamma]\}$$
(119)

$$\leq m \exp\{n[(1-\alpha)H(X_i) + (1+\alpha)\gamma]\}.$$
(120)

Hence, we have

$$\frac{1}{n}H_{\alpha}^{\varepsilon}(P_{X^{n}}) \le H(X_{i}) + \frac{1+\alpha}{1-\alpha}\gamma + \frac{1}{n}\log m$$
(121)

and thus

$$\limsup_{n \to \infty} \frac{1}{n} H_{\alpha}^{\varepsilon}(P_{X^n}) \le H(X_i) + \frac{1+\alpha}{1-\alpha}\gamma.$$
(122)

Since we can choose $\gamma > 0$ arbitrarily small, we have (107).

Proof of (108): If $H(X_i) = 0$ then (108) is apparent, since (20) holds. So, we assume $H(X_i) > 0$.

Fix $\gamma > 0$ sufficiently small so that $H(X_j) - \gamma > H(X_{j+1}) + \gamma$ for all j = 1, 2, ..., m-1 and that $A_i + 6m\gamma < \varepsilon < A_{i+1} - 6m\gamma$. We assume that n is sufficiently large so that $\exp\{-n[H(X_i) - \gamma]\} \le m\gamma$. Let us define $\mathcal{T}_n(j)$ as in (109). Note that

$$P_{X^n}(x^n) < P_{X^n}(\hat{x}^n), \quad x^n \in \mathcal{T}_n(j), \hat{x}^n \in \mathcal{T}_n(\hat{j}), j < \hat{j}.$$
 (123)

Let $S_n \triangleq \bigcup_{j=1}^m \mathcal{T}_n(j)$ and $\overline{S}_n \triangleq \mathcal{X}^n \setminus S_n$. Then, from Lemma 3, we have

$$P_{X^n}(\overline{\mathcal{S}}_n) \le 2m\gamma. \tag{124}$$

Let us sort the sequences in \mathcal{X}^n so that

$$P_{X^n}(x_1^n) \ge P_{X^n}(x_2^n) \ge P_{X^n}(x_3^n) \ge \dots$$
 (125)

Then, let $\mathcal{A}_n \triangleq \{x_1^n, x_2^n, \dots, x_{k^*-1}^n\}$ and $\mathcal{A}_n^+ \triangleq \mathcal{A}_n \cup \{x_{k^*}^n\}$ where k^* is the integer satisfying

$$\sum_{k=1}^{k^*} P_{X^n}(x_k^n) \ge 1 - \varepsilon \tag{126}$$

and

$$\sum_{k=1}^{k^*-1} P_{X^n}(x_k^n) < 1 - \varepsilon.$$
(127)

We first show that

$$x_{k^*}^n \in \overline{\mathcal{S}}_n \text{ or } x_{k^*}^n \in \bigcup_{j=1}^i \mathcal{T}_n(j).$$
 (128)

From Lemma 3, we have

$$\Pr\left\{X^n \in \bigcup_{j=i+1}^m \mathcal{T}_n(j)\right\} \le \sum_{j=i+1}^m (\alpha_j + 2\gamma)$$
(129)

 $\leq 1 - A_{i+1} + 2m\gamma \tag{130}$

$$\leq 1 - \varepsilon - 4m\gamma. \tag{131}$$

Since $P(\mathcal{A}_n^+) \geq 1 - \varepsilon$ holds, from (124) and (131), we have

$$\mathcal{A}_{n}^{+} \cap \left[\bigcup_{j=1}^{i} \mathcal{T}_{n}(j)\right] \neq \emptyset.$$
(132)

From (123) and (132), we can obtain (128).

We next notice that, from (123) and the assumption that n is sufficiently large, we have $P_{X^n}(x^n) \le \exp\{-n[H(X_i) - \gamma]\} \le m\gamma$ for all $x^n \in \bigcup_{j=1}^i \mathcal{T}_n(j)$. Combining this fact with (124) and (128), we can see that

$$P_{X^n}(\mathcal{A}_n \cap \mathcal{S}_n) \ge 1 - \varepsilon - m\gamma - P_{X^n}(\overline{\mathcal{S}}_n)$$
(133)

$$\geq 1 - \varepsilon - 3m\gamma. \tag{134}$$

Thus, from (131) and (134), we have

$$\Pr\left\{X^{n} \in \mathcal{A}_{n} \cap \left[\bigcup_{j=1}^{i} \mathcal{T}_{n}(j)\right]\right\} \ge m\gamma.$$
(135)

Moreover, since (123) holds, (135) implies that

$$P_{X^n}(\mathcal{A}_n \cap \mathcal{T}_n(i)) \ge \beta \triangleq \min\{m\gamma, \alpha_i\}$$
(136)

and thus

$$|\mathcal{A}_n \cap \mathcal{T}_n(i)| \ge \beta \exp\{n[H(X_i) - \gamma]\}.$$
(137)

Hence, we have

$$\sum_{x^n \in \mathcal{A}_n \cap \mathcal{T}_n(i)} [P_{X^n}(x^n)]^{\alpha} \ge \beta \exp\{n[(1-\alpha)H(X_i) - (1+\alpha)\gamma]\}.$$
(138)

October 23, 2018

Now we use the result of Koga [7]. Theorem 1 (A) of [7] tells us that

$$H_{\alpha}^{\varepsilon}(P_{X^n}) \ge \frac{1}{1-\alpha} \log \left(\sum_{k=1}^{k^*-1} [P_{X^n}(x_k^n)]^{\alpha} \right)$$
(139)

$$= \frac{1}{1-\alpha} \log \left(\sum_{x^n \in \mathcal{A}_n} [P_{X^n}(x^n)]^{\alpha} \right).$$
(140)

By combining this with (138), we have

$$\frac{1}{n} H^{\varepsilon}_{\alpha}(P_{X^n}) \ge \frac{1}{n(1-\alpha)} \log \left(\sum_{x^n \in \mathcal{A}_n \cap \mathcal{T}_n(i)} [P_{X^n}(x^n)]^{\alpha} \right)$$
(141)

$$\geq \frac{1}{n(1-\alpha)} \log \left(\beta \exp\{n[(1-\alpha)H(X_i) - (1+\alpha)\gamma]\}\right)$$
(142)

$$= H(X_i) - \frac{1+\alpha}{1-\alpha}\gamma + \frac{\log\beta}{n(1-\alpha)}.$$
(143)

Thus, we have

$$\liminf_{n \to \infty} \frac{1}{n} H_{\alpha}^{\varepsilon}(P_{X^n}) \ge H(X_i) - \frac{1+\alpha}{1-\alpha}\gamma.$$
(144)

Since we can choose $\gamma > 0$ arbitrarily small, we have (108).

ACKNOWLEDGMENT

The author would like to thank Prof. Mitsugu Iwamoto for informing him about [14].

REFERENCES

- [1] R. Renner and S. Wolf, "Smooth Rényi entropy and applications," in Proc. IEEE ISIT 2004, 2004, p. 232.
- [2] —, "Simple and tight bounds for information reconciliation and privacy amplification," in Advances in cryptology-ASIACRYPT 2005.
 Springer, 2005, pp. 199–216.
- [3] A. Rényi, "On measures of entropy and information," in Proc. 4th Berkeley Symp. on Math. Stat. and Prob., 1961, pp. 547-561.
- [4] H. Koga and H. Yamamoto, "Asymptotic properties on codeword lengths of an optimal FV code for general sources," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1546–1555, Apr. 2005.
- [5] S. Verdú and T. S. Han, "A general formula for channel capacity," IEEE Trans. Inf. Theory, vol. 40, no. 4, pp. 1147–1157, Jul. 1994.
- [6] T. S. Han, Information-spectrum methods in information theory. New York: Springer-Verlag, 2002.
- [7] H. Koga, "Characterization of the smooth Rényi entropy using majorization," in *Proc. 2013 IEEE Information Theory Workshop (ITW2013)*, 2013.
- [8] T. Uyematsu, "A new unified method for fixed-length source coding problems of general sources," *IEICE Trans. Fundamentals*, vol. E93-A, no. 11, pp. 1868–1877, 2010.
- [9] T. Uyematsu and S. Kunimatsu, "A new unified method for intrinsic randomness problems of general sources," in *Information Theory* Workshop (ITW), 2013 IEEE, Sept 2013, pp. 624–628.
- [10] S. Saito and T. Matsushima, "On the achievable overflow threshold of variable-length coding using smooth max-entropy for general sources," in *Proc. of the 38th Symposium on Information Theory and Its Applications (SITA2015)*, Okayama, Japan, Nov. 2015, pp. 142–146, in Japanese.
- [11] N. Datta and R. Renner, "Smooth entropies and the quantum information spectrum," *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2807–2815, 2009.
- [12] L. Wang, R. Colbeck, and R. Renner, "Simple channel coding bounds," in Proc. 2009 IEEE International Symposium on Information Theory (ISIT2009), 2009, pp. 1804–1808.

- [13] N. A. Warsi, "One-shot bounds for various information theoretic problems using smooth min and max Rényi divergence," in Proc. 2013 IEEE Information Theory Workshop (ITW2013), 2013.
- [14] L. Campbell, "A coding theorem and Rényi's entropy," Information and control, vol. 8, no. 4, pp. 423–429, 1965.
- [15] F. Jelinek, "Buffer overflow in variable length coding of fixed rate sources," IEEE Trans. Inf. Theory, vol. 14, no. 3, pp. 490-501, 1968.
- [16] H. Shimokawa, "Rényi's entropy and error exponent of source coding with countably infinite alphabet," in 2006 IEEE International Symposium on Information Theory, 2006, pp. 1831–1835.
- [17] N. Merhav, "On optimum strategies for minimizing exponential moments of a given cost function," *Communications in Information and Systems*, vol. 11, no. 4, pp. 343–368, 2011.
- [18] —, "On optimum strategies for minimizing the exponential moments of a given cost function." [Online]. Available: http://arxiv.org/abs/1103.2882
- [19] I. Kontoyiannis and S. Verdú, "Optimal lossless data compression: Non-asymptotics and asymptotics," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 777–795, 2014.
- [20] T. Courtade and S. Verdú, "Cumulant generating function of codeword lengths in optimal lossless compression," in *Information Theory* (ISIT), 2014 IEEE International Symposium on, June 2014, pp. 2494–2498.
- [21] V. Kostina, Y. Polyanskiy, and S. Verdú, "Variable-length compression allowing errors," in *Information Theory (ISIT), 2014 IEEE International Symposium on*, June 2014, pp. 2679–2683.
- [22] —, "Variable-length compression allowing errors (extended)," arXiv:1402.0608.
- [23] T. M. Cover and J. A. Thomas, Elements of Information Theory, 2nd ed. John Wiley & Sons, Inc., 2006.