# How to Quantize $n$ Outputs of a Binary Symmetric Channel to $n-1$ Bits? 

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#### Abstract

Suppose that $Y^{n}$ is obtained by observing a uniform Bernoulli random vector $X^{n}$ through a binary symmetric channel with crossover probability $\alpha$. The "most informative Boolean function" conjecture postulates that the maximal mutual information between $Y^{n}$ and any Boolean function $\mathrm{b}\left(X^{n}\right)$ is attained by a dictator function. In this paper, we consider the "complementary" case in which the Boolean function is replaced by $f:\{0,1\}^{n} \rightarrow\{0,1\}^{n-1}$, namely, an $n-1$ bit quantizer, and show that $I\left(f\left(X^{n}\right) ; Y^{n}\right) \leq(n-1) \cdot(1-h(\alpha))$ for any such $f$. Thus, in this case, the optimal function is of the form $f\left(x^{n}\right)=\left(x_{1}, \ldots, x_{n-1}\right)$.


## I. Introduction

Let $X^{n}$ be an $n$-dimensional binary vector uniformly distributed over $\{0,1\}^{n}$, and let $Y^{n}$ be the output of passing $X^{n}$ through a binary symmetric channel (BSC) with crossover probability $\alpha \in[0,1 / 2]$. In other words, $Y^{n}=X^{n} \oplus Z^{n}$, where $Z^{n}$ is a sequence of $n$ independent and identically distributed (i.i.d.) Bernoulli $(\alpha)$ random variables, statistically independent of $X^{n}$. The following conjecture [1] have recently received considerable attention.

Conjecture 1: For any Boolean function b : $\{0,1\}^{n} \rightarrow$ $\{0,1\}$, we have $I\left(\mathrm{~b}\left(X^{n}\right) ; Y^{n}\right) \leq 1-h(\alpha)$, where $h(\alpha) \triangleq$ $-\alpha \log _{2} \alpha-(1-\alpha) \log _{2}(1-\alpha)$ is the binary entropy function.

Since the dictator function $\mathrm{b}\left(X^{n}\right)=X_{i}$ (for any $1 \leq i \leq$ $n$ ) achieves this upper bound with equality, then intuitively Conjecture 1 postulates that the dictator function is the most "informative" one-bit quantization of $X^{n}$ in terms of achieving the maximal $I\left(\mathrm{~b}\left(X^{n}\right) ; Y^{n}\right)$. Clearly, by the symmetry of the pair $\left(X^{n}, Y^{n}\right)$ we have that for any function $I\left(\mathrm{~b}\left(X^{n}\right) ; Y^{n}\right)=$ $I\left(X^{n} ; \mathrm{b}\left(Y^{n}\right)\right)$, so we can equivalently think of the problem at hand as seeking the optimal one-bit quantizer of $n$ outputs of the channel. Despite attempts in various directions [1][7], Conjecture 1 remains open in general. However, for the "very noisy" case, where $\alpha>1 / 2-\delta$, for some $\delta>0$ independent of $n$, the validity of the conjecture was established by Samorodnitsky [8].

In this paper, we consider the "complementary" case in which the Boolean function in Conjecture 1 is replaced by an $n-1$ bit quantizer. Our main result is the following.

Theorem 1: For any function $f:\{0,1\}^{n} \rightarrow\{0,1\}^{n-1}$ we

[^0]have
\[

$$
\begin{equation*}
I\left(f\left(X^{n}\right) ; Y^{n}\right) \leq(n-1) \cdot(1-h(\alpha)) \tag{1}
\end{equation*}
$$

\]

and this bound is attained with equality by, e.g., $f\left(x^{n}\right)=$ $\left(x_{1}, \ldots, x_{n-1}\right)$.

One may wonder whether for any $f:\{0,1\}^{n} \rightarrow\{0,1\}^{k}$ we have $I\left(f\left(X^{n}\right) ; Y^{n}\right) \leq k \cdot(1-h(\alpha))$. However, for $k=R n$ with $0<R<1$, the problem essentially reduces to remote source coding under log-loss distortion measure, for which the maximal value of $I\left(f\left(X^{n}\right) ; Y^{n}\right) / n$ (as a function of $R, \alpha$ ) can be determined up to $o(n)$ terms. Indeed, [3], [9] characterizes this quantity which turns out to be greater than $R \cdot(1-h(\alpha))$. Conjecture 1 as well as Theorem 1 deal with the extreme cases of $k=1$ and $k=n-1$, respectively, where neglecting the $o(n)$ terms leads to non-informative characterization of the maximal $I\left(f\left(X^{n}\right) ; Y^{n}\right)$, and therefore [3], [9] do not suffice.

Theorem 1 can be generalized to a stronger statement concerning the entire class of binary-input memoryless outputsymmetric (BMS) channels.

Definition 1 (BMS channels): A memoryless channel with binary input $X$ and output $Y$ is called binary-input memoryless output-symmetric (BMS) if there exists a sufficient statistic $g(Y)=\left(X \oplus Z_{T}, T\right)$ for $X$, where $\left(T, Z_{T}\right)$ are statistically independent of $X$, and $Z_{T}$ is a binary random variable with $\operatorname{Pr}\left(Z_{T}=1 \mid T=t\right)=t$.

Corollary 1 ( [10]): Let $X^{n}$ be an $n$-dimensional binary vector uniformly distributed over $\{0,1\}^{n}$, and let $Y^{n}$ be the output of passing $X^{n}$ through a BMS with capacity $C$. Then for every $f:\{0,1\}^{n} \rightarrow\{0,1\}^{n-1}$, we have

$$
I\left(f\left(X^{n}\right) ; Y^{n}\right) \leq(n-1) \cdot C
$$

and this bound is attained with equality by, e.g., $f\left(x^{n}\right)=$ $\left(x_{1}, \ldots, x_{n-1}\right)$.

Proof of Corollary [7. Let $W$ be a BMS channel with capacity $C=1-h(\alpha)$. Let $Y_{W}^{n}$ and $Y_{\mathrm{BSC}}^{n}$ be the outputs corresponding to the channel $W$ and a BSC with crossover probability $\alpha$, respectively, when the input to both channels is $X^{n}$. Define $[n] \triangleq\{1,2, \ldots, n\}$. For any function $f:\{0,1\}^{n} \rightarrow[M]$, we can write

$$
\begin{gathered}
I\left(f\left(X^{n}\right) ; Y_{W}^{n}\right)=I\left(f\left(X^{n}\right), X^{n} ; Y_{W}^{n}\right)-I\left(X^{n} ; Y_{W}^{n} \mid f\left(X^{n}\right)\right) \\
=I\left(X^{n} ; Y_{W}^{n}\right)+I\left(f\left(X^{n}\right) ; Y_{W}^{n} \mid X^{n}\right)-I\left(X^{n} ; Y_{W}^{n} \mid f\left(X^{n}\right)\right)
\end{gathered}
$$

$$
\begin{aligned}
= & I\left(X^{n} ; Y_{W}^{n}\right) \\
& -\sum_{m=1}^{M} \operatorname{Pr}\left(f\left(X^{n}\right)=m\right) I\left(X^{n} ; Y_{W}^{n} \mid f\left(X^{n}\right)=m\right)
\end{aligned}
$$

We proceed by noting that $I\left(X^{n} ; Y_{W}^{n}\right)=I\left(X^{n} ; Y_{\mathrm{BSC}}^{n}\right)=n C$ as the capacity achieving input distribution of both channels is Bernoulli $(1 / 2)$. Furthermore, recall the fact that the BSC is the least capable among all BMS channels with the same capacity [11, page 116], [12, Lemma 7.1]. To wit, for any input $U^{n}$, the corresponding outputs of $W$ and the BSC will satisfy $I\left(U^{n} ; Y_{\mathrm{BSC}}^{n}\right) \leq I\left(U^{n} ; Y_{W}^{n}\right)$. This implies that

$$
I\left(X^{n} ; Y_{W}^{n} \mid f\left(X^{n}\right)=m\right) \geq I\left(X^{n} ; Y_{\mathrm{BSC}}^{n} \mid f\left(X^{n}\right)=m\right)
$$

for all $m=1, \ldots, M$. Thus, we get that for any function $f$,

$$
\begin{equation*}
I\left(f\left(X^{n}\right) ; Y_{W}^{n}\right) \leq I\left(f\left(X^{n}\right) ; Y_{\mathrm{BSC}}^{n}\right) \tag{2}
\end{equation*}
$$

The corollary now follows by invoking Theorem 1 .

## II. Proof of Theorem 1

Since the vector $Y^{n}$ is uniformly distributed over $\{0,1\}^{n}$, we have

$$
\begin{equation*}
I\left(f\left(X^{n}\right) ; Y^{n}\right)=n-H\left(Y^{n} \mid f\left(X^{n}\right)\right) \tag{3}
\end{equation*}
$$

Our goal is therefore to lower bound $H\left(Y^{n} \mid f\left(X^{n}\right)\right)$.
Consider the function $f:\{0,1\}^{n} \rightarrow\left[2^{n-1}\right]$, and define the sets

$$
f^{-1}(j) \triangleq\left\{x^{n} \in\{0,1\}^{n}: f\left(x^{n}\right)=j\right\}, j=1, \ldots, 2^{n-1}
$$

which form a disjoint partition of $\{0,1\}^{n}$. Further, define the sizes of these sets as
$m_{j} \triangleq\left|f^{-1}(j)\right|=\sum_{x^{n} \in\{0,1\}^{n}} \mathbb{1}\left\{f\left(x^{n}\right)=j\right\}, j=1, \ldots, 2^{n-1}$,
and assume without loss of generality that $m_{j}>0$, for all $j$. To see why this assumption is valid, first note that there must exist some $i$, for which $m_{i} \geq 2$. Let $f^{-1}(i)=\left\{x_{i_{1}}^{n}, \ldots, x_{i_{m_{i}}}^{n}\right\}$. Now if there exists some $j \neq i$, such that $m_{j}=0$, we can define a new function $\tilde{f}:\{0,1\}^{n} \rightarrow\left[2^{n-1}\right]$ where $\tilde{f}^{-1}(i)=$ $\left\{x_{i_{1}}^{n}, \ldots, x_{i_{m_{i}-1}}^{n}\right\}, \tilde{f}^{-1}(j)=\left\{x_{i_{m_{i}}}^{n}\right\}$, and $\tilde{f}^{-1}(t)=f^{-1}(t)$, for all $t \neq i, j$. For this function we must have

$$
\begin{aligned}
H\left(Y^{n} \mid f\left(X^{n}\right)\right) & \geq H\left(Y^{n} \mid f\left(X^{n}\right), \mathbb{1}\left\{X^{n}=x_{i_{m_{i}}}\right\}\right) \\
& =H\left(Y^{n} \mid \tilde{f}\left(X^{n}\right)\right)
\end{aligned}
$$

and consequently $I\left(f\left(X^{n}\right) ; Y^{n}\right) \leq I\left(\tilde{f}\left(X^{n}\right) ; Y^{n}\right)$.
Next, for every $m=0,1, \ldots, 2^{n}$ define the quantity

$$
\begin{equation*}
\lambda(m) \triangleq \sum_{j=1}^{2^{n-1}} \mathbb{1}\left\{m_{j}=m\right\} \tag{4}
\end{equation*}
$$

which counts the number of sets $f^{-1}(j)$ with cardinality $m$, in the partition induced by the function $f 1$ The next proposition expresses $\lambda(1)$ in terms of $\{\lambda(m)\}_{m \geq 2}$.
${ }^{1}$ In fact, since we have already assumed that $m_{j}>0$ for all $j$, we have that $\lambda(0)=0$ and $\lambda(m)=0$ for $m>2^{n}-\left(2^{n-1}-1\right)$.

Proposition 1: For any $f:\{0,1\}^{n} \rightarrow\left[2^{n-1}\right]$ with $m_{j}>0$ for all $j$, we have that

$$
\begin{equation*}
\lambda(1)=\sum_{m \geq 3}(m-2) \lambda(m) \tag{5}
\end{equation*}
$$

Intuitively, this proposition states that since the average size of the sets $f^{-1}(j)$ is 2 , then every set $f^{-1}(j)$ of cardinality $m>2$, must be compensated for by $(m-2)$ sets of cardinality 1.

Proof: Using the definition of $\lambda(m)$ in (4), and the fact that $\left\{f^{-1}(j)\right\}$ forms a disjoint partition of $\{0,1\}^{n}$, we have

$$
\begin{align*}
\sum_{m=0}^{2^{n}} \lambda(m) & =2^{n-1}  \tag{6}\\
\sum_{m=0}^{2^{n}} m \lambda(m) & =2^{n} \tag{7}
\end{align*}
$$

Multiplying (6) by 2 and equating it with the left-hand side of (7), we get

$$
\sum_{m=0}^{2^{n}} 2 \lambda(m)=\sum_{m=0}^{2^{n}} m \lambda(m)
$$

which implies

$$
2 \lambda(0)+\lambda(1)=\sum_{m \geq 3}(m-2) \lambda(m)
$$

Invoking our assumption that $\lambda(0)=0$ gives the desired result.
Definition 2 (Minimal entropy of a noisy subset): For a family of vectors $S \subset\{0,1\}^{n}$ let $U_{S}$ be a random vector uniformly distributed over $S$, and let $Z^{n}$ be a sequence of $n$ i.i.d. Bernoulli $(\alpha)$ random variables, statistically independent of $U_{S}$. For $m=1, \ldots, 2^{n}$, we define the quantity

$$
\begin{equation*}
H_{m}^{n}(\alpha) \triangleq \min _{S \subset\{0,1\}^{n}:|S|=m} H\left(U_{S} \oplus Z^{n}\right) \tag{8}
\end{equation*}
$$

Some properties of $H_{m}^{n}(\alpha)$ will be studied in the next section. In particular, we will prove the following lemma.

Lemma 1: For any $2<m<2^{n}$,

$$
\begin{equation*}
\frac{m-2}{2 m-2} H_{1}^{n}(\alpha)+\frac{m}{2 m-2} H_{m}^{n}(\alpha) \geq H_{2}^{n}(\alpha) \tag{9}
\end{equation*}
$$

We can now write

$$
\begin{aligned}
& H\left(Y^{n} \mid f\left(X^{n}\right)\right)=\sum_{j=1}^{2^{n-1}} \operatorname{Pr}\left(f\left(X^{n}\right)=j\right) H\left(Y^{n} \mid f\left(X^{n}\right)=j\right) \\
& =\sum_{j=1}^{2^{n-1}} \operatorname{Pr}\left(X^{n} \in f^{-1}(j)\right) H\left(Y^{n} \mid X^{n} \in f^{-1}(j)\right) \\
& =2^{-n} \sum_{j=1}^{2^{n-1}}\left|f^{-1}(j)\right| H\left(U_{f^{-1}(j)} \oplus Z^{n}\right)
\end{aligned}
$$

$$
\begin{align*}
& \geq 2^{-n} \sum_{j=1}^{2^{n-1}} m_{j} H_{m_{j}}^{n}(\alpha) \\
& =2^{-n} \sum_{m=1}^{2^{n}} m \lambda(m) H_{m}^{n}(\alpha) \\
& =2^{-n}\left(\lambda(1) H_{1}^{n}(\alpha)+2 \lambda(2) H_{2}^{n}(\alpha)+\sum_{m \geq 3}^{2^{n}} m \lambda(m) H_{m}^{n}(\alpha)\right) \\
& =2^{-n}\left(2 \lambda(2) H_{2}^{n}(\alpha)\right. \\
& \left.\quad \quad+\sum_{m \geq 3}^{2^{n}}(m-2) \lambda(m) H_{1}^{n}(\alpha)+m \lambda(m) H_{m}^{n}(\alpha)\right)  \tag{10}\\
& =2^{-n}\left(2 \lambda(2) H_{2}^{n}(\alpha)\right. \\
& \left.+\sum^{2^{n}}(2 m-2) \lambda(m)\left[\frac{m-2}{2 m-2} H_{1}^{n}(\alpha)+\frac{m}{2 m-2} H_{m}^{n}(\alpha)\right]\right) \\
& \geq H_{2}^{n}(\alpha) \cdot 2^{-n} \sum_{m=1}^{2^{n}}(2 m-2) \lambda(m)  \tag{11}\\
& =H_{2}^{n}(\alpha), \tag{12}
\end{align*}
$$

where in (10) follows from Proposition 1 in (11) we have used Lemma (1) and (12) follows from (6) and (7). Proposition 4 , stated and proved in the next section, shows that $H_{2}^{n}(\alpha)=$ $1+(n-1) h(\alpha)$. Combining this with (3) and (12) establishes the desired result.

## III. Properties of $H_{m}^{n}(\alpha)$

The main goal of this section is to prove Lemma 1 To this end, we establish some properties of the function $H_{m}^{n}(\alpha)$, which may be of independent interest.

Proposition 2 (Monotonicity in $m$ ): The function $H_{m}^{n}(\alpha)$ is monotonically non-decreasing as a function of $m$.

Proof: It is suffice to show that for any natural number $1 \leq m<2^{n}$ it holds that $H_{m}^{n}(\alpha) \leq H_{m+1}^{n}(\alpha)$. To this end, let $S=\left\{s_{1}, \ldots, s_{m+1}\right\} \subset\{0,1\}^{n}$ be a family of $m+1$ vectors, and let $S_{-i} \triangleq S \backslash\left\{s_{i}\right\}$, for $i=1, \ldots, m+1$. Clearly, $\left|S_{-i}\right|=m$ for all $i$. Furthermore, the random vector $U_{S}$ can be generated by first drawing a random variable $A \sim \operatorname{Uniform}([m+1])$ and then drawing a statistically independent random vector uniformly over $S_{-A}$. Thus, for any $S \subset\{0,1\}^{n}$ of size $m+1$ we have that

$$
\begin{aligned}
H\left(U_{S} \oplus Z^{n}\right) & \geq H\left(U_{S} \oplus Z^{n} \mid A\right) \\
& =\frac{1}{m+1} \sum_{a=1}^{m+1} H\left(U_{S_{-a}} \oplus Z^{n}\right) \geq H_{m}^{n}(\alpha)
\end{aligned}
$$

and in particular $H_{m}^{n}(\alpha) \leq H_{m+1}^{n}(\alpha)$.
We define the partial order " $\leq$ " on the hypercube $\{0,1\}^{n}$ as $y \leq x$ iff $y_{i} \leq x_{i}$, for all $i=1, \ldots, n$.

Definition 3 (Monotone sets): A set $S \subset\{0,1\}^{n}$ is monotone if $x \in S$ implies $y \in S$, for all $y \leq x$.

Let $\mathcal{M}_{m}^{n} \triangleq\left\{S \subset\{0,1\}^{n}:|S|=m, S\right.$ is monotone $\}$. We will prove the following result.

Lemma 2 (Sufficiency of monotone sets):

$$
H_{m}^{n}(\alpha)=\min _{S \in \mathcal{M}_{m}^{n}} H\left(U_{S} \oplus Z^{n}\right)
$$

Remark 1: Theorem 3 in [1] states that among all boolean functions, $I\left(\mathrm{~b}\left(X^{n}\right) ; Y^{n}\right)$ is maximized by functions for which the induced set $\mathrm{b}^{-1}(0)$ is monotone 2 While this statement is closely related to our Lemma 2, it does not imply it, although the proof technique is somewhat similar.

The proof of Lemma 2 is based on applying a procedure called shifting [13]-[15].

Definition 4 (Shifting): For a set of binary vectors $S \subset$ $\{0,1\}^{n}$ the shifting procedure is defined as follows. For $i \in[n]$ and $x \in\{0,1\}^{n}$ write $x-i$ for the vector obtained by setting $x_{i}=0$, and define

$$
S_{i} \triangleq\left\{x \in S: x_{i}=1, x-i \notin S\right\}
$$

Find the smallest $i$ such that $S_{i} \neq \emptyset$. If there is no such $i$ then we are done. Otherwise, replace $S$ with the set $\left(S \backslash S_{i}\right) \cup$ ( $S_{i}-i$ ), where $S_{i}-i \triangleq\left\{x-i: x \in S_{i}\right\}$, and repeat. The output of this process is a monotone set, denoted by $S_{\text {shifted }}$, with cardinality $\left|S_{\text {shifted }}\right|=|S|$.

The proof of Lemma 2 hinges on the following result.
Lemma 3: Let $S \subset\{0,1\}^{n}$ be some subset of vectors, and $\bar{S} \subset\{0,1\}^{n}$ be the result of applying one iteration of the shifting procedure, say, on the first coordinate. Let $P_{Y \mid X}$ be some discrete memoryless channel with binary input, and let $Y^{n}$ be its output when the input is $U_{S}$ and $\bar{Y}^{n}$ be its output when the input is $U_{\bar{S}}$. For every $\omega \in \mathcal{Y}^{n-1}$ we have that $\operatorname{Pr}\left(Y_{2}^{n}=\omega\right)=\operatorname{Pr}\left(\bar{Y}_{2}^{n}=\omega\right)$, and
$\left|\operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid \bar{Y}_{2}^{n}=\omega\right)-\frac{1}{2}\right| \geq\left|\operatorname{Pr}\left(U_{S, 1}=1 \mid Y_{2}^{n}=\omega\right)-\frac{1}{2}\right|$.

Proof of Lemma 3. Let $S_{2}^{n}$ be the projection of $S$ onto the coordinates $\{2, \ldots, n\}$, and note that the projection of $\bar{S}$ onto these coordinates is also $S_{2}^{n}$, as the shifting operations does not effect these coordinates. Consequently, $U_{S, 2}^{n}$ and $\bar{U}_{S, 2}^{n}$ have the same distribution, and therefore $Y_{2}^{n}$ and $\bar{Y}_{2}^{n}$ have the same distribution.

Next, for any vector $\omega \in \mathcal{Y}^{n-1}$, we have

$$
\begin{aligned}
& \operatorname{Pr}\left(U_{S, 1}=1 \mid Y_{2}^{n}=\omega\right) \\
& \quad=\sum_{x \in S_{2}^{n}} \operatorname{Pr}\left(U_{S, 1}=1, U_{S, 2}^{n}=x \mid Y_{2}^{n}=\omega\right) \\
& \quad=\sum_{x \in S_{2}^{n}} \operatorname{Pr}\left(U_{S, 1}=1 \mid U_{S, 2}^{n}=x\right) \operatorname{Pr}\left(U_{S, 2}^{n}=x \mid Y_{2}^{n}=\omega\right)
\end{aligned}
$$

[^1]The fact that $U_{S, 2}^{n}$ and $U_{\bar{S}, 2}^{n}$ have the same distribution, implies that $P_{U_{S, 2}^{n} \mid Y_{2}^{n}}=P_{U_{S, 2}^{n} \mid \bar{Y}_{2}^{n}}$, and therefore

$$
\begin{aligned}
& \operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid \bar{Y}_{2}^{n}=\omega\right) \\
& \quad=\sum_{x \in S_{2}^{n}} \operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid U_{\bar{S}, 2}^{n}=x\right) \operatorname{Pr}\left(U_{S, 2}^{n}=x \mid Y_{2}^{n}=\omega\right)
\end{aligned}
$$

We partition the set $S_{2}^{n}$ into three subsets:

- $A \triangleq\left\{x \in S_{2}^{n}:[0 x] \in S,[1 x] \in S\right\}$
- $B \triangleq\left\{x \in S_{2}^{n}:[0 x] \notin S,[1 x] \in S\right\}$
- $C \triangleq\left\{x \in S_{2}^{n}:[0 x] \in S,[1 x] \notin S\right\}$
and we note that

$$
\operatorname{Pr}\left(U_{S, 1}=1 \mid U_{S, 2}^{n}=x\right)= \begin{cases}1 / 2 & x \in A \\ 1 & x \in B \\ 0 & x \in C\end{cases}
$$

Letting

$$
\begin{aligned}
& a_{\omega} \triangleq \operatorname{Pr}\left(U_{S, 2}^{n} \in A \mid Y_{2}^{n}=\omega\right), \\
& b_{\omega} \triangleq \operatorname{Pr}\left(U_{S, 2}^{n} \in B \mid Y_{2}^{n}=\omega\right), \\
& c_{\omega} \triangleq \operatorname{Pr}\left(U_{S, 2}^{n} \in C \mid Y_{2}^{n}=\omega\right),
\end{aligned}
$$

we get

$$
\operatorname{Pr}\left(U_{S, 1}=1 \mid Y_{2}^{n}=\omega\right)=\frac{a_{\omega}}{2}+b_{\omega}
$$

By the definition of the shifting procedure in Definition 4,

$$
\operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid U_{\bar{S}, 2}^{n}=x\right)= \begin{cases}1 / 2 & x \in A \\ 0 & x \in B \\ 0 & x \in C\end{cases}
$$

Thus,

$$
\operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid \bar{Y}_{2}^{n}=\omega\right)=\frac{a_{\omega}}{2}
$$

We can use this to see that $\operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid \bar{Y}_{2}^{n}=\omega\right)$ is more biased than $\operatorname{Pr}\left(U_{S, 1}=1 \mid Y_{2}^{n}=\omega\right)$. Indeed

$$
\begin{aligned}
& \left(\frac{1}{2}-\operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid \bar{Y}_{2}^{n}=\omega\right)\right)^{2} \\
& \quad-\left(\frac{1}{2}-\operatorname{Pr}\left(U_{S, 1}=1 \mid Y_{2}^{n}=\omega\right)\right)^{2} \\
& =\left(\frac{1}{2}\left(1-a_{\omega}\right)\right)^{2}-\left(\frac{1}{2}\left(1-a_{\omega}\right)-b_{\omega}\right)^{2} \\
& =b_{\omega}\left(1-a_{\omega}\right)-b_{\omega}^{2}=b_{\omega} c_{\omega} \geq 0
\end{aligned}
$$

as desired.
Corollary 2 (Shifting decreases output entropy): Let $S \subset$ $\{0,1\}^{n}$ be some subset of vectors, and $\bar{S} \subset\{0,1\}^{n}$ be the result of applying one iteration of the shifting procedure, say, on the first coordinate. Let $Z^{n}$ be a sequence of $n$ i.i.d. Bernoulli $(\alpha)$ random variables, statistically independent of $U_{S}$ and $U_{\bar{S}}$. Then,

$$
\begin{equation*}
H\left(U_{\bar{S}} \oplus Z^{n}\right) \leq H\left(U_{S} \oplus Z^{n}\right) \tag{13}
\end{equation*}
$$

Proof: By the chain rule,
$H\left(U_{S} \oplus Z^{n}\right)=H\left(U_{S, 2}^{n} \oplus Z_{2}^{n}\right)+H\left(U_{S, 1} \oplus Z_{1} \mid U_{S, 2}^{n} \oplus Z_{2}^{n}\right)$, and

$$
\begin{aligned}
H\left(U_{\bar{S}} \oplus Z^{n}\right) & =H\left(U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}\right)+H\left(U_{\bar{S}, 1} \oplus Z_{1} \mid U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}\right) \\
& =H\left(U_{S, 2}^{n} \oplus Z_{2}^{n}\right)+H\left(U_{\bar{S}, 1} \oplus Z_{1} \mid U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}\right)
\end{aligned}
$$

where the last equality follows from the fact that $P_{U_{S, 2}^{n}} \oplus Z_{2}^{n}=$ $P_{U_{S, 2}^{n} \oplus Z_{2}^{n}}$ due to Lemma 3 Thus, it suffices to show that

$$
H\left(U_{\bar{S}, 1} \oplus Z_{1} \mid U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}\right) \leq H\left(U_{S, 1} \oplus Z_{1} \mid U_{S, 2}^{n} \oplus Z_{2}^{n}\right)
$$

For any $\omega \in\{0,1\}^{n-1}$ let $\alpha_{\omega} \triangleq \operatorname{Pr}\left(U_{S, 1}=1 \mid U_{S, 2}^{n} \oplus Z_{2}^{n}=\omega\right)$ and $\beta_{\omega} \triangleq \operatorname{Pr}\left(U_{\bar{S}, 1}=1 \mid U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}=\omega\right)$. Then, we get

$$
\begin{align*}
& H\left(U_{\bar{S}, 1} \oplus Z_{1} \mid U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}\right) \\
& =\sum_{\omega \in\{0,1\}^{n-1}} \operatorname{Pr}\left(U_{\bar{S}, 2}^{n} \oplus Z_{2}^{n}=\omega\right) h\left(\alpha * \beta_{\omega}\right) \\
& =\sum_{\omega \in\{0,1\}^{n-1}} \operatorname{Pr}\left(U_{S, 2}^{n} \oplus Z_{2}^{n}=\omega\right) h\left(\alpha * \beta_{\omega}\right) \\
& \leq \sum_{\omega \in\{0,1\}^{n-1}} \operatorname{Pr}\left(U_{S, 2}^{n} \oplus Z_{2}^{n}=\omega\right) h\left(\alpha * \alpha_{\omega}\right) \\
& =H\left(U_{S, 1} \oplus Z_{1} \mid U_{S, 2}^{n} \oplus Z_{2}^{n}\right) \tag{14}
\end{align*}
$$

where $a * b \triangleq a \cdot(1-b)+(1-a) \cdot b$ for any $a, b \in[0,1]$, the second equality follows since $P_{U_{S, 2}^{n}}^{n} \oplus Z_{2}^{n}=P_{U_{S, 2}^{n}} \oplus Z_{2}^{n}$, and the inequality is because $\beta_{\omega}$ is more biased than $\alpha_{\omega}$, by Lemma 3 ,

Applying Corollary 2 recursively, we see that for any $S \subset$ $\{0,1\}^{n}$ we have

$$
\begin{equation*}
H\left(U_{S_{\text {shifted }}} \oplus Z^{n}\right) \leq H\left(U_{S} \oplus Z^{n}\right) \tag{15}
\end{equation*}
$$

In fact, it is easy to extend the above argument to show that for any BMS channel with inputs $U_{S}$ and $U_{S_{\text {shifted }}}$ and corresponding outputs $Y^{n}$ and $\tilde{Y}^{n}$, respectively, we get $H\left(\tilde{Y}^{n}\right) \leq$ $H\left(Y^{n}\right)$. Inequality (15) immediately establishes Lemma 2

We now turn to finding $H_{m}^{n}(\alpha)$ for $m=1,2,3,4$.
Proposition 3: $H_{1}^{n}(\alpha)=n \cdot h(\alpha)$.
Proof: For any vector $u \in\{0,1\}^{n}$ we have that $H(u \oplus$ $\left.Z^{n}\right)=H\left(Z^{n}\right)=n \cdot h(\alpha)$.

Proposition 4: $H_{2}^{n}(\alpha)=1+(n-1) \cdot h(\alpha)$.
Proof: By Lemma2 it is suffice to minimize $H\left(U_{S} \oplus Z^{n}\right)$ over $S \in \mathcal{M}_{2}^{n}$. It is easy to see that $\mathcal{M}_{2}^{n}$ consists of a single set $S^{*}=\left\{\left[\begin{array}{llll}1 & 0 & \cdots & 0\end{array}\right],\left[\begin{array}{llll}0 & 0 & \cdots & 0\end{array}\right]\right\}$, up to permuting the order of coordinates. Thus, direct calculation gives

$$
\begin{equation*}
H_{2}^{n}(\alpha)=H\left(U_{S^{*}} \oplus Z^{n}\right)=1+(n-1) \cdot h(\alpha) \tag{16}
\end{equation*}
$$

Proposition 5:

$$
\begin{align*}
H_{3}^{n}(\alpha) & =h\left(\frac{1}{3} * \alpha\right)+\left(\frac{2}{3} * \alpha\right) h\left(\frac{1-\alpha^{2}}{2-\alpha}\right) \\
& +\left(\frac{1}{3} * \alpha\right) h\left(\frac{1-\alpha+\alpha^{2}}{1+\alpha}\right)+(n-2) h(\alpha) \tag{17}
\end{align*}
$$

$$
\begin{equation*}
\geq h\left(\frac{1}{3} * \alpha\right)+\frac{1}{3} h(\alpha)+\frac{2}{3}+(n-2) h(\alpha) \tag{18}
\end{equation*}
$$

Proof: By Lemma2, it is suffice to minimize $H\left(U_{S} \oplus Z^{n}\right)$ over $S \in \mathcal{M}_{3}^{n}$. It is easy to see that $\mathcal{M}_{3}^{n}$ consists of a single set $S^{*}=\left\{\left[\begin{array}{lllll}1 & 0 & 0 & \cdots & 0\end{array}\right],\left[\begin{array}{lllll}0 & 1 & 0 & \cdots & 0\end{array}\right],\left[\begin{array}{lllll}0 & 0 & 0 & \cdots & 0\end{array}\right]\right\}$, up to permuting the order of coordinates. Thus, (17) is obtained by direct calculation of $H\left(U_{S^{*}} \oplus Z^{n}\right)$. To obtain the lower bound (18) we write

$$
\begin{aligned}
& H_{3}^{n}(\alpha)=H\left(U_{S^{*}} \oplus Z^{n}\right) \\
& =H\left(U_{S_{1}^{*}} \oplus Z_{1}\right)+H\left(U_{S_{2}^{*}} \oplus Z_{2} \mid U_{S_{1}^{*}} \oplus Z_{1}\right)+H\left(Z_{3}^{n}\right) \\
& \geq H\left(U_{S_{1}^{*}} \oplus Z_{1}\right)+H\left(U_{S_{2}^{*}} \oplus Z_{2} \mid U_{S_{1}^{*}}\right)+H\left(Z_{3}^{n}\right) \\
& =h\left(\frac{1}{3} * \alpha\right)+\frac{1}{3} h(\alpha)+\frac{2}{3}+(n-2) h(\alpha) .
\end{aligned}
$$

Proposition 6: $H_{4}^{n}(\alpha)=2+(n-2) \cdot h(\alpha)$.
Proof: By Lemma2, it is suffice to minimize $H\left(U_{S} \oplus Z^{n}\right)$ over $S \in \mathcal{M}_{4}^{n}$. It is easy to see that $\mathcal{M}_{4}^{n}$ consists of two sets

$$
\begin{aligned}
& \mathcal{C} \triangleq\left\{\left[\begin{array}{lllll}
1 & 1 & 0 & \cdots & 0
\end{array}\right],\left[\begin{array}{lllll}
1 & 0 & 0 & \cdots & 0
\end{array}\right],\right. \\
& \left.\left[\begin{array}{lllll}
0 & 1 & 0 & \cdots & 0
\end{array}\right],\left[\begin{array}{lllll}
0 & 0 & 0 & \cdots & 0
\end{array}\right]\right\}, \\
& \mathcal{B} \triangleq\left\{\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & \cdots & 0
\end{array}\right],\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & \cdots & 0
\end{array}\right],\right. \\
& \left.\left[\begin{array}{llllll}
0 & 0 & 1 & 0 & \cdots & 0
\end{array}\right],\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right]\right\},
\end{aligned}
$$

up to permuting the order of coordinates. In particular, $\mathcal{C}$ is the 2 -dimensional cube padded by $(n-2)$ zeros, whereas $\mathcal{B}$ is the 3 -dimensional Hamming ball of radius 1 , padded by $n-3$ zeros. Thus,

$$
H_{4}^{n}(\alpha)=\min \left\{H\left(U_{\mathcal{C}} \oplus Z^{n}\right), H\left(U_{\mathcal{B}} \oplus Z^{n}\right)\right\}
$$

It is easy to verify that $H\left(U_{\mathcal{C}} \oplus Z^{n}\right)=2+(n-2) \cdot h(\alpha)$. We show that $H\left(U_{\mathcal{B}} \oplus Z^{n}\right) \geq 2+(n-2) \cdot h(\alpha)$. Indeed,

$$
\begin{align*}
& H\left(U_{\mathcal{B}} \oplus Z^{n}\right) \\
& =H\left(U_{\mathcal{B}, 1}^{2} \oplus Z_{1}^{2}\right)+H\left(U_{\mathcal{B}, 3} \oplus Z_{3} \mid U_{\mathcal{B}, 1}^{2} \oplus Z_{1}^{2}\right)+H\left(Z_{4}^{n}\right) \\
& \geq H\left(U_{\mathcal{B}, 1}^{2} \oplus Z_{1}^{2}\right)+H\left(U_{\mathcal{B}, 3} \oplus Z_{3} \mid U_{\mathcal{B}, 1}^{2}\right)+(n-3) \cdot h(\alpha) \\
& =H\left(U_{\mathcal{B}, 1}^{2} \oplus Z_{1}^{2}\right)+\frac{1}{2}+\frac{h(\alpha)}{2}+(n-3) \cdot h(\alpha) \tag{19}
\end{align*}
$$

Direct calculation gives

$$
\begin{equation*}
H\left(U_{\mathcal{B}, 1}^{2} \oplus Z_{1}^{2}\right)=\frac{3}{2}+\frac{h(\alpha)}{2} \tag{20}
\end{equation*}
$$

which together with (19) shows that $H\left(U_{\mathcal{B}} \oplus Z^{n}\right) \geq 2+(n-$ $2) \cdot h(\alpha)$.

We are now in a position to prove Lemma 1 .
Proof of Lemma 7 . For any $m \geq 4$ we have that $H_{m}^{n}(\alpha) \geq H_{4}^{n}(\alpha)>H_{1}^{n}(\alpha)$, which implies that

$$
\begin{aligned}
& \frac{m-2}{2 m-2} H_{1}^{n}(\alpha)+\frac{m}{2 m-2} H_{m}^{n}(\alpha) \geq \frac{H_{1}^{n}(\alpha)+H_{m}^{n}(\alpha)}{2} \\
& \quad \geq \frac{H_{1}^{n}(\alpha)+H_{4}^{n}(\alpha)}{2}=1+(n-1) \cdot h(\alpha)=H_{2}^{n}(\alpha)
\end{aligned}
$$

It then remains to verify (9) for $m=3$. Using the lower bound (18) for $H_{3}^{n}(\alpha)$, it suffices to verify that

$$
\begin{align*}
\frac{1}{4} n h(\alpha) & +\frac{3}{4}\left[h\left(\frac{1}{3} * \alpha\right)+\frac{1}{3} h(\alpha)+\frac{2}{3}+(n-2) h(\alpha)\right] \\
& \geq 1+(n-1) h(\alpha) \tag{21}
\end{align*}
$$

which is equivalent to

$$
\begin{equation*}
3 \cdot h((1 / 3) * \alpha)-2-h(\alpha) \geq 0 \tag{22}
\end{equation*}
$$

Let $g(\alpha) \triangleq 3 \cdot h\left(\frac{1}{3} * \alpha\right)-2-h(\alpha)$. It is easy to check that $g(0)>0$ and that $g(1 / 2)=0$. Thus, it suffices to show that $g(\alpha)$ is monotonically decreasing as a function of $\alpha$, namely, that $\mathrm{d} g(\alpha) / \mathrm{d} \alpha<0$, for any $\alpha \in(0,1 / 2)$. We have

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} \alpha} g(\alpha) & =-\log _{2}\left(\frac{\frac{1}{3} * \alpha}{\frac{2}{3} * \alpha}\right)+\log _{2}\left(\frac{\alpha}{1-\alpha}\right)  \tag{23}\\
& =\log _{2}\left(\frac{2 \alpha-\alpha^{2}}{1-\alpha^{2}}\right) \tag{24}
\end{align*}
$$

which is negative for all $\alpha \in(0,1 / 2)$.

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[^1]:    ${ }^{2}$ In fact, (1) Theorem 3] provides a stronger statement about the structure of the induced $\mathrm{b}^{-1}(0)$.

