Compute-and-Forward over Block-Fading Channels Using Algebraic Lattices

Shanxiang Lyu, Antonio Campello and Cong Ling
Department of EEE, Imperial College London
London, SW7 2AZ, United Kingdom
Email: s.lyu14, a.campello, c.ling@imperial.ac.uk

Jean-Claude Belfiore
Mathematical and Algorithmic Sciences Lab
France Research Center
Huawei Technologies
belfiore@telecom-paristech.fr

***Abstract*—Previous approaches to compute-and-forward (C&F) are mostly based on quantizing channel coefficients to integers. In this work, we investigate the C&F strategy over block fading channels using Construction A over rings, so as to allow better quantization for the channels. Advantages in decoding error probabilities and computation rates are demonstrated, and the construction is shown to outperform the C&F strategy over the integers $\mathbb{Z}$.**

***Index Terms*—algebraic lattice, block fading, compute and forward, Construction A.**

I. Introduction

Building upon the property that lattice codes are closed under integer combinations of codewords, the compute-and-forward (C&F) relaying protocol proposed by Nazer and Gaspar [1] has become a popular physical layer network coding framework. The protocol has been extended in several directions. Since $\mathbb{Z}$ may not be the most suitable space to quantize the actual channel, one line of work is to use more compact rings. If the message space and the lattice cosets are both $\mathcal{O}$-modules where $\mathcal{O}$ refers to a ring, the linear labeling technique in [2] enables the decoding of a ring combination of lattice codewords. It has also been shown that using Eisenstein integers $\mathbb{Z}[\omega]$ [3], [4] or rings from quadratic number fields [5] can have better computation rates for some complex channels than Gaussian integers $\mathbb{Z}[i]$.

The second line of work is to incorporate more realistic channel models such as MIMO and block fading. MIMO C&F and integer forcing (IF) linear receivers were studied in [6], [7]. Block fading was investigated in [8], [9]. Reference [8] analyzed the computation rates and argued that the rationale of decoding an integer combination of lattice codewords still works to some extent in block fading channels. Actual implementation of this idea based on root-LDA lattices was later investigated in [9], where full diversity was shown for two-way relay channels and multiple-hop line networks. As the channel coefficients in different fading blocks are not the same, it seems natural to employ different integer coefficients across different blocks so as to enjoy better quantizing performance, rather than approaches of [8], [9] that fix the integer coefficients for the whole duration of a codeword. However, the resulted combination may no longer be a lattice codeword, which draws us into a dilemma.

In [10], it was briefly suggested that number-field constructions as in [5], [11] could be advantageous for C&F in a block-fading scenario. Here we provide a detailed analysis on its decoding error performance and rates. Specifically, with these codes, the coefficients of an equation belong to a ring, whereas $\mathbb{Z}$ is only a special case where its conjugates are the same. This type of lattices naturally suits block fading channels as algebraic lattice codes can be capacity-achieving for compound block fading channels [11]. The contribution of this work is to demonstrate the error and rates advantages of algebraic lattices for C&F in block fading channels, and to present a practical algorithm to find equations with high rates.

The rest of this paper is organized as follows. In Section II, we review some background about C&F and algebraic number theory. In Sections III and IV, we present our coding scheme and the analysis of error probability and achievable rates, respectively. Section V gives a search algoirthm, and the last section provides some simulation results.

Due to the space limit, we omit some technical proofs, especially those of the closure of an algebraic lattice under $\mathcal{O}$-linear combinations and of quantization goodness of algebraic lattices. These will be provided in a forthcoming journal paper.

Notation: Matrices and column vectors are denoted by uppercase and lowercase boldface letters. $x(i)$ and $X(i, j)$ refer to scalars of $\mathbf{x}$ and $\mathbf{X}$ with indexes i and i, j. The set of all $n \times n$ matrices with determinant ± 1 and integer coefficients will be denoted by $\mathrm{GL}_n(\mathbb{Z})$. We denote $\log^+(x) = \max(\log(x), 0)$.

II. Preliminaries

A. Compute and forward

Consider a general real-valued AWGN network [1] with L source nodes and M relays. We assume that each source node l is operating at the same rate and define the message rate as $R_{\text{mes}} = \frac{1}{n}\log(|W|)$, where W is the message space. A message $\mathbf{w}_l \in W$ is encoded, via a function $\mathcal{E}(\cdot)$, into a point $\mathbf{x}_l \in \mathbb{R}^T$, satisfying the power constraint $\|\mathbf{x}_l\|^2 \leq TP$, where T is the block length and P denotes the signal to noise ratio (SNR). The received signal at one relay is given by

$$\mathbf{y} = \sum_{l=1}^{L} h_l \mathbf{x}_l + \mathbf{z},$$

where the channel coefficients $\{h_l\}$ remain constant over the whole time frame, and $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_T)$.

In the C&F scheme [1], $\mathbf{x}_l$ is a lattice point representative of a coset in the quotient Λ_f/Λ_c, where Λ_f and Λ_c are called the *fine* and *coarse* lattices. Instead of directly decoding the messages, a relay searches for an integer combination of $\mathbf{w}_l$, $l = 1, \ldots, L$. To this purpose, the relay first estimates a linear combination of lattice codewords $\hat{\mathbf{x}} = [\mathcal{Q}(\alpha\mathbf{y})] \bmod \Lambda_c = \sum_{l=1}^{L} a_l \mathbf{x}_l$, where $\alpha \in \mathbb{R}$ is a minimum mean square error (MMSE) constant, and $\mathcal{Q}(\cdot)$ is a nearest neighbor quantizer to Λ_f. For certain coding schemes, there exists an isomorphic mapping $g(\cdot)$ between the lattice cosets Λ_f/Λ_c and the message space W, $g(\Lambda_f/\Lambda_c) \cong W$, which enables the relay to forward a message $\mathbf{u} = g(\hat{\mathbf{x}})$ in the space W, explicitly given by

$$\mathbf{u} = \sum_{l=1}^{L} g(a_l)\mathbf{w}_l, \tag{1}$$

the decoding error event of a relay given $\mathbf{h} \in \mathbb{R}^L$ and $\mathbf{a} \in \mathbb{Z}^L$ as $[\mathcal{Q}(\alpha\mathbf{y})] \bmod \Lambda_c \neq \sum_{l=1}^{L} a_l \mathbf{x}_l$ for optimized α. A computation rate is said to be achievable at a given relay if there exists a coding scheme such that the probability of decoding error tends to zero as $T \to \infty$. The achievable computation rates by the C&F protocol are given in the following theorem.

Theorem 1. *[1] The following computation rate is achievable:*

$$R_{\mathrm{comp}}(\mathbf{h}, \mathbf{a}) = \frac{1}{2} \max_{\alpha \in \mathbb{R}} \log^{+} \left(\frac{P}{|\alpha|^2 + P \left\| \alpha\mathbf{h} - \mathbf{a} \right\|^2} \right).$$

B. Number fields and algebraic lattices

A number field is a field extension $\mathbb{K} = \mathbb{Q}(\zeta)$ that defines a minimum field containing both $\mathbb{Q}$ and a primitive element ζ. The degree of the minimum polynomial of ζ, denoted by n, is called the degree of $\mathbb{K}$. Any element in $\mathbb{K}$ can be represented by using the power basis $\{1, \zeta, ..., \zeta^{n-1}\}$, so that if $c \in \mathbb{K}$, then $c = c_1 + c_2\zeta + \ldots + c_n\zeta^{n-1}$ with $c_i \in \mathbb{Q}$. A number is called an algebraic integer if its minimal polynomial has integer coefficients. Let $\mathbb{S}$ be the set of algebraic integers, then the integer ring is $\mathcal{O}_{\mathbb{K}} = \mathbb{K} \cap \mathbb{S}$. For instance, $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ is a quadratic field, its power basis is $\{1, \sqrt{5}\}$, and an integral basis for $\mathcal{O}_{\mathbb{K}}$ is $\{1, \frac{1+\sqrt{5}}{2}\}$.

An ideal $\mathfrak{I}$ of $\mathcal{O}_{\mathbb{K}}$ is a nonempty subset of $\mathcal{O}_{\mathbb{K}}$ that has the following properties. 1) $c_1 + c_2 \in \mathfrak{I}$ if $c_1, c_2 \in \mathfrak{I}$; 2) $c_1 c_2 \in \mathfrak{I}$ if $c_1 \in \mathfrak{I}$, $c_2 \in \mathcal{O}_{\mathbb{K}}$. Every ideal of $\mathcal{O}_{\mathbb{K}}$ can be decomposed into a product of prime ideals. Let p be a rational prime, we have $p\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i}$ in which e_i is the ramification index of prime ideal $\mathfrak{p}_i$. The inertial degree of $\mathfrak{p}_i$ is defined as $r_i = [\mathcal{O}_{\mathbb{K}}/\mathfrak{p}_i : \mathbb{Z}/p\mathbb{Z}]$, and it satisfies $\sum_{i=1}^{g} e_i r_i = n$. Each prime ideal $\mathfrak{p}_i$ is said to be lying above p.

We follow [5], [11], [12] to build lattices by construction A over rings. Choose $\mathfrak{p}$ lying above p with inertial degree r, so that $\mathcal{O}_{\mathbb{K}}/\mathfrak{p} \cong \mathbb{F}_{p^r}$. Let $\mathbf{G}$ be a generator matrix of a (T, t) linear code over $\mathbb{F}_{p^r}$ and $t < T$. An algebraic lattice $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ is generated via the following procedures.

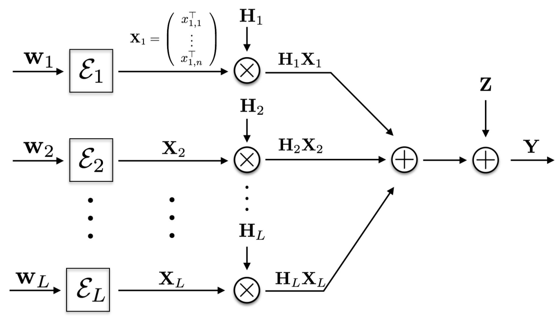

Fig. 1. The block fading model at one relay.

1) Construct a codebook $\mathcal{C} = \left\{\mathbf{x} = \mathbf{G}\mathbf{c} \mid \mathbf{c} \in \mathbb{F}_{p^r}^{t}\right\}$ with multiplication over $\mathbb{F}_{p^r}$.

2) Define a component-wise ring isomorphism $\mathcal{M} : \mathbb{F}_{p^r} \to \mathcal{O}_{\mathbb{K}}/\mathfrak{p}$, so that $\mathcal{C}$ is mapped to the coset leaders of $\mathcal{O}_{\mathbb{K}}/\mathfrak{p}^T$ defined by $\Lambda^* \triangleq \mathcal{M}(\mathcal{C})$.

3) Expand Λ^* by tiling $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C}) = \Lambda^* + \mathfrak{p}^T$.

Since $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ is an $\mathcal{O}_{\mathbb{K}}$-module of rank T, a summation over $\mathcal{O}_{\mathbb{K}}$ is closed in this group.

III. Algebraic coding for block fading channels

For a block fading scenario consisting of n blocks and block length T, the received message in a relay written in a matrix format is

$$\mathbf{Y} = \sum_{l=1}^{L} \mathbf{H}_l \mathbf{X}_l + \mathbf{Z}, \tag{2}$$

where the channel state information (CSI) $\mathbf{H}_l = \mathrm{diag}(h_{l,1}, ..., h_{l,n})$ is available at the relay, $\mathbf{X}_l = [\mathbf{x}_{l,1}, ..., \mathbf{x}_{l,n}]^{\top} \in \mathbb{R}^{n \times T}$ denotes a transmitted codeword, and $\mathbf{Z} = [\mathbf{z}_1, ..., \mathbf{z}_n]^{\top}$ with $\mathbf{z}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_T)$ being Gaussian noise. A diagram for this block fading channel model is shown in Fig. 1. In this figure, each $\mathbf{X}_l$ consists of codes over multiple frequency carriers or multiple antennas. If our channel matrices $\{\mathbf{H}_l\}$ are not restricted to be diagonal, then the general model is called MIMO C&F [6].

In our transmission scheme, an $\mathcal{O}_{\mathbb{K}}$-module of rank T is built first, where the degree of $\mathbb{K}$ matches the size of the block fading channel. The coding lattice is however not $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ as that of [5], but rather its canonical embedding into the Euclidean space defined as $\Lambda^{\mathbb{Z}}(\mathcal{C})$, which is a free $\mathbb{Z}$-module of rank nT. The canonical embedding is $\sigma : \mathbb{K} \to \mathbb{R}^n$, where $\sigma(x) = (\sigma_1(x), ..., \sigma_n(x))$ and all the embeddings are real. $\sigma_1(x), ..., \sigma_n(x)$ are also called the conjugates of x, and the algebraic norm of x is $\mathrm{Nr}(x) = \prod_{i=1}^{n} \sigma_i(x)$. The generator matrix of $\Lambda^{\mathbb{Z}}(\mathcal{C})$ can be found in [12, Prop. 1].

First we construct a pair of linear codes $(\mathcal{C}_f, \mathcal{C}_c)$ to build the coding lattice $\Lambda_f^{\mathbb{Z}}$ and the shaping lattice $\Lambda_c^{\mathbb{Z}}$. Define $\mathcal{C}_f = \left\{\mathbf{G}_f\mathbf{w} \mid \mathbf{w} \in \mathbb{F}_{p^r}^{l_f}\right\}$ and $\mathcal{C}_c = \left\{\mathbf{G}_c\mathbf{w} \mid \mathbf{w} \in \mathbb{F}_{p^r}^{l_c}\right\}$, where $\mathbf{G}_f \in \mathbb{F}_{p^r}^{T \times l_f}$ and $\mathbf{G}_c \in \mathbb{F}_{p^r}^{T \times l_c}$ is contained in the first

l_c columns of $\mathbf{G}_f$. Then the fine and coarse lattices are given by $\Lambda_f^{\mathcal{O}_{\mathbb{K}}} = \mathcal{M}(\mathcal{C}_f) + \mathfrak{p}^T$ and $\Lambda_c^{\mathcal{O}_{\mathbb{K}}} = \mathcal{M}(\mathcal{C}_c) + \mathfrak{p}^T$. For the time being, a candidate lattice code $\widetilde{\mathbf{x}}_l$ belongs to $\Lambda_f^{\mathcal{O}_{\mathbb{K}}} \cap \mathcal{V}(\Lambda_c^{\mathcal{O}_{\mathbb{K}}})$. Since $[\mathbb{K} : \mathbb{Q}] = n$, we generate a transmitted vector by the canonical embedding, i.e., $\mathbf{x}_l = \gamma\sigma(\widetilde{\mathbf{x}}_l) \in \mathbb{R}^{nT}$, and γ denotes a scaling constant such that the second moment of the shaping lattice $\gamma\mathcal{V}(\Lambda_c^{\mathbb{Z}})$ has a power smaller than P. Now we have $\mathbf{x}_l \in \gamma\Lambda_f^{\mathbb{Z}} \cap \gamma\mathcal{V}(\Lambda_c^{\mathbb{Z}})$. By rearranging $\mathbf{x}_l$ into $\mathbf{X}_l$, it represents the row composition of the conjugates of $\widetilde{\mathbf{x}}_l$, i.e.,

$$\mathbf{X}_l = \gamma \begin{bmatrix} \sigma_1(\widetilde{\mathbf{x}}_l^\top) \\ \sigma_2(\widetilde{\mathbf{x}}_l^\top) \\ \vdots \\ \sigma_n(\widetilde{\mathbf{x}}_l^\top) \end{bmatrix}.$$

Similar to [5, Thm. 5], there exists an isomorphism between $\gamma\Lambda_f^{\mathbb{Z}}/\gamma\Lambda_c^{\mathbb{Z}}$ and the message space W. The equivalent lattices of $\Lambda_f^{\mathbb{Z}}$ and $\Lambda_c^{\mathbb{Z}}$ have volumes $p^{(T-l_f)r}\gamma^{nT}\Delta_{\mathbb{K}}^{T/2}$ and $p^{(T-l_c)r}\gamma^{nT}\Delta_{\mathbb{K}}^{T/2}$ ($\Delta_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$), so the message rate at every node is $R_{\text{mes}} = \frac{(l_f-l_c)r}{T}\log(p)$.

IV. Error probability and rate analysis

The following lemma is the crux of our decoding algorithm, which means the rows of $\mathbf{X}_l$ are not only closed in $\gamma\Lambda_f^{\mathbb{Z}}$ under $\mathbb{Z}$-linear combinations, but more generally under $\mathcal{O}_{\mathbb{K}}$-linear combinations.

Lemma 1. *Let $a_l \in \mathcal{O}_{\mathbb{K}}$, and $\mathbf{A}_l = \mathrm{diag}(\sigma_1(a_l), ..., \sigma_n(a_l))$ for $1 \le l \le L$, The physical layer codewords are closed under the action of ring elements, i.e., $\sum_{l=1}^L \left(\mathbf{A}_l\mathbf{X}_l\right) \in \gamma\Lambda_f^{\mathbb{Z}}$.*

Based on Lemma 1, the decoder for block fading channel (2) extracts an algebraic combination of lattice codewords:

$$\mathbf{BY} = \underbrace{\sum_{l=1}^L \mathbf{A}_l\mathbf{X}_l}_{\text{effective codeword}} + \underbrace{\mathbf{B}\sum_{l=1}^L \mathbf{H}_l\mathbf{X}_l - \sum_{l=1}^L \mathbf{A}_l\mathbf{X}_l + \mathbf{BZ}}_{\text{effective noise}}, \tag{3}$$

where $\mathbf{B} = \mathrm{diag}(b_1, ..., b_n)$, $b_i \in \mathbb{R}$ is a constant diagonal matrix, to be optimized in the sequel. The following proposition uses a union bound argument to evaluate the decoding error probability w.r.t. model (3), whose proof can be found in the appendix.

Proposition 1. *Let $\mathbf{a} = [a_1, \ldots, a_L]^\top \in \mathcal{O}_{\mathbb{K}}^L$ and keep the notation as above. The error probability of minimum-distance lattice decoding associated to coefficient vector $\mathbf{a}$ is upper bounded as*

$$P_e(\mathbf{B}, \mathbf{a}) \le \sum_{\mathbf{x}\in\Lambda_f^{\mathbb{Z}}\setminus\Lambda_c^{\mathbb{Z}}} \frac{1}{2}\exp\left(-\frac{n\Big(d_{n,T}(\gamma\mathbf{x})\Big)^{1/n}}{8\sum_{j=1}^n \nu_{\text{eff},\,j}^2}\right), \tag{4}$$

where

$$\nu_{\text{eff},\,j}^2 = |b_j|^2 + P\left\|b_j\mathbf{h}_j - \sigma_j(\mathbf{a})\right\|^2,$$

$$\mathbf{h}_j \triangleq [H_1(j,\,j), \ldots, H_L(j,\,j)]^\top \in \mathbb{R}^L,$$

$$\sigma_j(\mathbf{a}) = [A_1(j,\,j), \ldots, A_L(j,\,j)]^\top,$$

and

$$d_{n,T}(\mathbf{x}) \triangleq \prod_{j=1}^n \left(\sum_{i=(j-1)T+1}^{jT} x(i)^2\right)$$

is the block-wise product distance of a lattice point $\mathbf{x}$.

Further define the minimum block-wise product distance of a lattice as $d_{\min}(\Lambda) \triangleq \min_{\mathbf{x}\in\Lambda\setminus\mathbf{0}} d_{n,T}(\mathbf{x})$. It follows from (4) that the decoding error probability is dictated by $d_{\min}(\gamma\Lambda_f^{\mathbb{Z}})$ and the power of the effective noise. The first advantage of coding over algebraic lattices is to bring a lower bound to $d_{\min}(\gamma\Lambda_f^{\mathbb{Z}})$. To be concise, we have $\mathrm{Nr}(x(i)) \in \mathbb{Z}$ for $x(i) \in \mathcal{O}_{\mathbb{K}}$, so that for a $\gamma\mathbf{x} \in \gamma\Lambda_f^{\mathbb{Z}} \neq \mathbf{0}$, it yields

$$\begin{aligned} d_{n,T}(\gamma\mathbf{x}) &= \gamma^{2n}\prod_{j=1}^n \Big(\sum_{i=(j-1)T+1}^{jT} x(i)^2\Big), \\ &\ge \gamma^{2n}\prod_{j=1}^n \Big(T\Big(\prod_{i=(j-1)T+1}^{jT} x(i)^2\Big)^{1/T}\Big), \\ &= \gamma^{2n}T^n\Big(\prod_{i=1}^T \mathrm{Nr}(x(i))^2\Big)^{1/T} \ge \gamma^{2n}T^n. \end{aligned}$$

The second advantage of our scheme is that it often yields smaller effective noise power due to finer quantization than $\mathbb{Z}^L$. This will be reflected by the computation rate analysis. According to the proof of Proposition 1, the nub to obtain the computation rate hinges on decoding the fine lattice under the block-wise additive noise. Define $\sigma_{\mathrm{GM}}^2 = \left(\prod_{j=1}^n \nu_{\text{eff},\,j}^2\right)^{1/n}$. It follows from [11, Thm. 2] that the decoding error probability vanishes if $\mathrm{Vol}(\gamma\Lambda_f^{\mathbb{Z}})^{2/(nT)}/\sigma_{\mathrm{GM}}^2 > 2\pi e$. From quantization goodness [5], we have that $P/\mathrm{Vol}(\gamma\Lambda_c^{\mathbb{Z}})^{2/(nT)} < 1/(2\pi e)$. Therefore, any computation rate up to

$$\frac{1}{T}\log\left(\frac{\mathrm{Vol}(\gamma\Lambda_c^{\mathbb{Z}})}{\mathrm{Vol}(\gamma\Lambda_f^{\mathbb{Z}})}\right) \le \frac{n}{2}\log\left(\frac{P}{\sigma_{\mathrm{GM}}^2}\right)$$

is achievable. In order to relate the achievable rate to the successive minima of a lattice, we define $\sigma_{\mathrm{AM}}^2 = \left(\sum_{j=1}^n \nu_{\text{eff},\,j}^2\right)/n$ and use the fact that $\sigma_{\mathrm{AM}}^2 \ge \sigma_{\mathrm{GM}}^2$ to attain the following result.

Proposition 2. *With properly chosen lattice codebooks, given channels $\{\mathbf{H}_l\}$ and the desired quantization coefficient $\mathbf{a}$ in a relay, the computation rate of the arithmetic mean (AM) decoder is given by*

$$R_{\mathrm{AM}}(\{\mathbf{H}_l\}, \mathbf{a}) = \max_{\mathbf{B}} \frac{n}{2}\log^+\left(\frac{nP}{\|\mathbf{B}\|^2 + P\sum_{l=1}^L \|\mathbf{BH}_l - \mathbf{A}_l\|^2}\right). \tag{5}$$

Denote the denominator inside (5) as $n\sigma_{\mathrm{AM}}^2 = \|\mathbf{B}\|^2 + P\sum_{l=1}^L \|\mathbf{BH}_l - \mathbf{A}_l\|^2$. By assuming $\mathbf{a}$ to be fixed, the MMSE principle for optimizing $n\sigma_{\text{AM}}^2$ is to pick the diagonal elements of $\mathbf{B}$ in the following way:

$$b_j = \frac{P\sigma_j(\mathbf{a})^\top \mathbf{h}_j}{P\left\|\mathbf{h}_j\right\|^2+1}. \quad (6)$$

Plugging (6) back yields

$$n\sigma_{\text{AM}}^2 = \sum_{j=1}^{n}\left(P\left\|\frac{P\mathbf{h}_j\sigma_j(\mathbf{a})^\top\mathbf{h}_j}{P\left\|\mathbf{h}_j\right\|^2+1}-\sigma_j(\mathbf{a})\right\|^2+\left(\frac{P\sigma_j(\mathbf{a})^\top\mathbf{h}_j}{P\left\|\mathbf{h}_j\right\|^2+1}\right)^2\right).$$

Further define a Gram matrix

$$\mathbf{M}_j = \mathbf{I} - \frac{P}{P\left\|\mathbf{h}_j\right\|^2+1}\mathbf{h}_j\mathbf{h}_j^\top,$$

then the computation rate of our AM decoder becomes

$$R_{\text{AM}}\left(\{\mathbf{M}_j\},\mathbf{a}\right) = \frac{n}{2}\log^+\left(\frac{n}{\sum_{j=1}^{n}\sigma_j(\mathbf{a})^\top\mathbf{M}_j\sigma_j(\mathbf{a})}\right). \quad (7)$$

Its achievable rate is therefore maximized by optimizing $\mathbf{a} \in \mathcal{O}_{\mathbb{K}}^L$. Since $\mathbb{Z}^L \subseteq \mathcal{O}_{\mathbb{K}}^L$, the achievable rate in (7) is no smaller than that of $\mathbb{Z}$-lattices.

V. Search Algorithm

The optimization target in (7) is to find $\mathbf{a} \in \mathcal{O}_{\mathbb{K}}^L$ to reach the minimum of $f(\mathbf{a}) \triangleq \sum_{j=1}^{n}\sigma_j(\mathbf{a})^\top\mathbf{M}_j\sigma_j(\mathbf{a})$. Our approach is to take advantage of the generator matrix of $\mathcal{O}_{\mathbb{K}}$, so that $f(\mathbf{a})$ represents the square distance of a lattice vector, and (7) is turned into a shortest vector problem (SVP). Let $\{\phi_1, ..., \phi_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$, then its generator matrix is given by

$$\Phi = \begin{bmatrix} \sigma_1(\phi_1) & \cdots & \sigma_1(\phi_n) \\ \sigma_2(\phi_1) & \cdots & \sigma_2(\phi_n) \\ \vdots & \vdots & \vdots \\ \sigma_n(\phi_1) & \cdots & \sigma_n(\phi_n) \end{bmatrix}.$$

With Cholesky decomposition $\mathbf{M}_j = \bar{\mathbf{M}}_j^\top\bar{\mathbf{M}}_j$, we have $f(\mathbf{a}) = \sum_{j=1}^{n}\left\|\bar{\mathbf{M}}_j\sigma_j(\mathbf{a})\right\|^2$. The lattice associated with $f(\mathbf{a})$ is indeed a $\mathbb{Z}$-submodule of $\mathbb{R}^{nL}$, with a generator matrix $\bar{\Phi} = \mathbf{M}_{\text{mix}}\Phi_{\text{mix}}$,

$$\mathbf{M}_{\text{mix}} = \begin{bmatrix} \bar{\mathbf{M}}_1 & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \bar{\mathbf{M}}_n \end{bmatrix},$$

and $\Phi_{\text{mix}} = \mathbf{U}(\mathbf{I}_L \otimes \Phi)$ where $\mathbf{U} \in \text{GL}_{nL}(\mathbb{Z})$ is a row-shuffling operation. For instance, when $n = 2$, $L = 2$, we can visualize Φ_{mix} as

$$\Phi_{\text{mix}} = \begin{bmatrix} \sigma_1(\phi_1) & \sigma_1(\phi_2) & 0 & 0 \\ 0 & 0 & \sigma_1(\phi_1) & \sigma_1(\phi_2) \\ \sigma_2(\phi_1) & \sigma_2(\phi_2) & 0 & 0 \\ 0 & 0 & \sigma_2(\phi_1) & \sigma_2(\phi_2) \end{bmatrix}.$$

Finally, it yields $f(\mathbf{a}) = f(\tilde{\mathbf{a}}) = \left\|\bar{\Phi}\tilde{\mathbf{a}}\right\|^2$, with $\tilde{\mathbf{a}} \in \mathbb{Z}^{nL}$. Many algorithms are now available to solve SVP over the $\mathbb{Z}$ lattice $\mathcal{L}(\bar{\Phi})$, e.g., the classic sphere decoding algorithm [13] can help to obtain this solution with reasonable complexity.

The explicit structure of lattice basis $\bar{\Phi}$ facilitates estimating the bounds of rates via different number fields. Denote the first successive minimum of $\mathcal{L}(\bar{\Phi})$ by λ_1, we have $\lambda_1 < \sqrt{nL}\left|\det\left(\bar{\Phi}\right)\right|^{1/(nL)}$ according to Minkowski's first theorem [14, P. 12]. We claim that a smaller discriminant $\Delta_{\mathbb{K}}$ can contribute to a sharper bound for it, so that $\mathbb{Q}(\sqrt{5})$ should be the best real quadratic number field to use. Specifically, $|\det(\bar{\Phi})| = |\det(\mathbf{M}_{\text{mix}})||\det(\Phi_{\text{mix}})|$, and since $|\det(\Phi_{\text{mix}})| = |\det(\Phi)|^L = (\Delta_{\mathbb{K}})^{L/2}$due to the unimodularity of $\mathbf{U}$, it yields $|\det(\bar{\Phi})| = |\det(\mathbf{M}_{\text{mix}})|\,(\Delta_{\mathbb{K}})^{L/2}$. The relation to channel capacity can also be obtained by using Sylvester's Theorem to expand each $|\det(\bar{\mathbf{M}}_i)|$ as Ref. [15] did to the static Gaussian MAC.

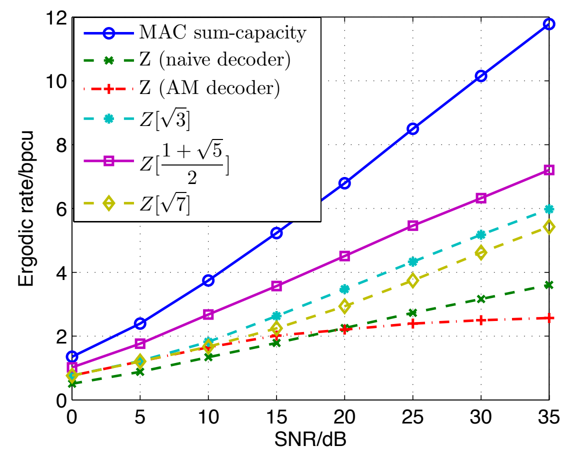

Fig. 2. Comparison of achievable rates with different rings.

VI. Numerical Results

In this section, we will numerically verify the validness of the AM computation rate (7). In the example, we let $n = 2$, $L = 2$, $\mathbf{h}_1$ and $\mathbf{h}_2$ chosen from $\mathcal{N}(0, 1)$ entries, and compare the average achievable rates (ergodic rates) of 2000 Monte Carlo runs.

In Fig. 2, we plot the rates of AM decoders with quantization coefficients in $\mathbb{Z}$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and $\mathbb{Z}[\sqrt{7}]$, respectively. The MAC sum-capacity is provided as the upper bound of decoding two equations. The rate of an oblivious transmitter [6] that neglects the advantage of multiple antennas is also included in the figure, denoted as $\mathbb{Z}$ (naive decoder).

We can observe from Fig. 2 that the degree of freedom (DOF) of the MAC sum-capacity is 2, the DOF's of non-trivial rings are 1 (and they are optimal because decoding two equations suffices to reach the DOF 2), and that of the naive decoder is only $1/2$. The performance of $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is better than those of other rings. The AM decoder with the $\mathbb{Z}$ restriction seems quite sub-optimal, as it becomes inferior to the naive decoder in high SNR.

Appendix A
Proof of Proposition 1

Proof: We first follow [1] to find the effective noise. With chosen $\mathbf{B}$ and $\mathbf{A}_l$, it first computes $\mathbf{S} = \mathbf{B}\mathbf{Y} + \sum_{l=1}^{L} \mathbf{A}_l \mathbf{D}_l$, where $\mathbf{D}_l$ is the dither from a source node which is uniformly distributed on the Voronoi region $\gamma\mathcal{V}_{\Lambda_c^{\mathbb{Z}}}$. To get an estimate of the lattice equation $\mathbf{V} = \sum_{l=1}^{L} \mathbf{A}_l \mathbf{X}_l \mod \gamma\Lambda_c^{\mathbb{Z}}$, $\mathbf{S}$ is first quantized w.r.t. the fine lattice $\gamma\Lambda_f^{\mathbb{Z}}$ denoted by $\mathcal{Q}(\cdot)$ and then modulo the coarse lattice $\gamma\Lambda_c^{\mathbb{Z}}$. Since

$$[\mathcal{Q}(\mathbf{S})] \mod \gamma\Lambda_c^{\mathbb{Z}} = [\mathcal{Q}([\mathbf{S}] \mod \gamma\Lambda_c^{\mathbb{Z}})] \mod \gamma\Lambda_c^{\mathbb{Z}},$$

if the effective noise of $[\mathbf{S}] \mod \gamma\Lambda_c^{\mathbb{Z}}$ falls within the Voronoi region of the fine lattice, then the noise effect can be canceled. Now we show that $[\mathbf{S}] \mod \gamma\mathcal{V}_{\Lambda_c^{\mathbb{Z}}}$ is equivalent to $\mathbf{V}$ pluses a block-wise noise. Denote $\Theta_l = \mathbf{B}\mathbf{H}_l - \mathbf{A}_l$ and $\bar{\mathbf{X}}_l = [\mathbf{X}_l + \mathbf{D}_l] \mod \gamma\Lambda_c^{\mathbb{Z}}$, then

$$[\mathbf{S}] \mod \gamma\Lambda_c^{\mathbb{Z}} = [\mathbf{V} + \underbrace{\sum_{l=1}^{L} (\Theta_l \bar{\mathbf{X}}_l) + \mathbf{B}\mathbf{Z}]}_{\mathbf{Z}_{\text{eff}}} \mod \gamma\Lambda_c^{\mathbb{Z}}.$$

As each block of $\bar{\mathbf{X}}_l$ is uniformly distributed, the probability density function (PDF) of the jth row of $\mathbf{Z}_{\text{eff}}$ can be shown to be upper bounded by a Gaussian $\mathcal{N}(\mathbf{0}, \nu_{\text{eff},\,j}^2 \mathbf{I}_T)$, where

$$\nu_{\text{eff},\,j}^2 = |b_j|^2 + P \left\| b_j \mathbf{h}_j - \sigma_j(\mathbf{a}) \right\|^2, \tag{8}$$

It turns out to be a non-AWGN lattice decoding problem, whose decoding error probability is $P_e(\mathbf{B}, \mathbf{a}) = \sum_{\mathbf{V}' \in \{\mathbf{V} + \gamma\Lambda_f^{\mathbb{Z}}\} \setminus \{\mathbf{V} + \gamma\Lambda_c^{\mathbb{Z}}\}} \Pr(\mathbf{V} \to \mathbf{V}')$ which equals

$$\begin{aligned} &\sum_{\mathbf{V}-\mathbf{V}' \in \gamma\Lambda_f^{\mathbb{Z}} \setminus \gamma\Lambda_c^{\mathbb{Z}}} \Pr\Big(\left\| \mathbf{V} + \mathbf{Z}_{\text{eff}} - \mathbf{V}' \right\|^2 \leq \left\| \mathbf{Z}_{\text{eff}} \right\|^2 \Big), \\ =& \sum_{\mathbf{V}-\mathbf{V}' \in \gamma\Lambda_f^{\mathbb{Z}} \setminus \gamma\Lambda_c^{\mathbb{Z}}} \Pr\Big(\sum_{j=1}^{n} (\left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2 + 2(\mathbf{v}_j - \mathbf{v}_j')^\top \mathbf{z}_{\text{eff},\,j}) \leq 0 \Big), \end{aligned} \tag{9}$$

in which $\mathbf{v}_j^\top$, $\mathbf{v}_j'^\top$ and $\mathbf{z}_{\text{eff},\,j}^\top$ are the jth rows of $\mathbf{V}$, $\mathbf{V}'$ and $\mathbf{Z}_{\text{eff}}$, respectively. Further define $\varUpsilon \triangleq \sum_{j=1}^{n} 2(\mathbf{v}_j' - \mathbf{v}_j)^\top \mathbf{z}_{\text{eff},\,j}$. Similar to the analysis of (8), the PDF of $\varUpsilon$ is upper bounded by a zero mean Gaussian with variance $\sum_{j=1}^{n} 4\nu_{\text{eff},\,j}^2 \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2$. It then follows from the property of a Q function $Q_g(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\Big(-\frac{u^2}{2} \Big) \mathrm{d}u$ that the summation term of (9) can be written as

$$\begin{aligned} \Pr(\mathbf{V} \to \mathbf{V}') &\leq Q_g \left(\frac{\sum_{j=1}^{n} \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2}{2\sqrt{\sum_{j=1}^{n} \nu_{\text{eff},\,j}^2 \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2}} \right), \\ &\overset{(a)}{\leq} \frac{1}{2} \exp \left(-\frac{\Big(\sum_{j=1}^{n} \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2 \Big)^2}{8 \sum_{j=1}^{n} \nu_{\text{eff},\,j}^2 \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2} \right), \\ &\overset{(b)}{\leq} \frac{1}{2} \exp \left(-\frac{n \Big(\prod_{j=1}^{n} \left\| \mathbf{v}_j - \mathbf{v}_j' \right\|^2 \Big)^{1/n}}{8 \sum_{j=1}^{n} \nu_{\text{eff},\,j}^2} \right), \end{aligned} \tag{10}$$

where (a) has used the bound $Q_g(x) \leq 1/2 \exp(-x^2/2)$, (b) comes after using $\nu_{\text{eff},\,j}^2 \leq \sum_{j=1}^{n} \nu_{\text{eff},\,j}^2$ and the AM-GM inequality. The relaxation in (b) serves the purpose of bounding the error probability via the block-wise product distance of our algebraic lattice. Plugging (8) into (10) proves the proposition. ∎

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