

# Polar codes with a stepped boundary

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**Abstract:** We consider explicit polar constructions of block-length  $n \rightarrow \infty$  for the two extreme cases of code rates  $R \rightarrow 1$  and  $R \rightarrow 0$ . For code rates  $R \rightarrow 1$ , we design codes with complexity order of  $n \log n$  in code construction, encoding, and decoding. These codes achieve the vanishing output bit error rates on the binary symmetric channels with any transition error probability  $p \rightarrow 0$  and perform this task with a substantially smaller redundancy  $(1 - R)n$  than do other known high-rate codes, such as BCH codes or Reed-Muller (RM). We then extend our design to the low-rate codes that achieve the vanishing output error rates with the same complexity order of  $n \log n$  and an asymptotically optimal code rate  $R \rightarrow 0$  for the case of  $p \rightarrow 1/2$ .

**Keywords:** Polar codes; Reed-Muller codes; Boolean polynomials; successive cancellation decoding.

## I. INTRODUCTION

Below we consider the Plotkin recursive construction  $\mathbf{u}, \mathbf{u} + \mathbf{v}$  that repeatedly combines shorter codes to construct and decode the longer ones. RM codes  $\mathcal{R}(r, m)$  represent one Plotkin-type construction [1] of length  $n = 2^m$  and dimension  $k(r, m) = \sum_{i=0}^r \binom{m}{i}$  with parameters  $0 \leq r \leq m$ . Polar codes [3] introduce another recursive design. Both codes originate from the same full-space code  $\mathcal{R}(m, m)$  and filter it in two different ways. Namely, a code  $\mathcal{R}(r, m)$  maximizes the code rate among all codes that have the same distance  $2^{m-r}$  and are generated by the  $m$ -variate Boolean monomials. Polar codes use a more intricate optimization. First, the successive-cancellation decoding (SCD) of [2]-[6] performs step-by-step retrieval of information bits of code  $\mathcal{R}(m, m)$ . Analysis of SCD [6] shows that it yields both high and low-fidelity information bits for RM codes. Therefore, removing low-fidelity bits (by setting them as zeros) gives the better-performing subcodes of RM codes. For relatively short lengths of 512 or less, this was done in [5], [6]. In particular, it turns out that these subcodes achieve a nearly optimal (ML) performance on these lengths if SCD is combined with list decoding. For long codes with  $m \rightarrow \infty$ , the major breakthrough achieved in [3] shows that the subcodes of  $\mathcal{R}(m, m)$  that keep  $Rn$  most reliable bits are capacity achieving (CA) codes under SCD for any binary symmetric memoryless channel  $U$  and any code rate  $R \in (0, 1)$ . These polar codes also achieve a polynomial complexity of construction. Namely, for a channel  $U$  with capacity  $C$ , polar codes of code rate  $R > C - \epsilon$  have complexity [11] of order  $\text{poly}(a\epsilon^{-\mu})$  for any  $\epsilon > 0$ , where  $a = a(U)$  and  $\mu$  are some constants.

Below, we extend the above results for the special cases of  $R \rightarrow 1$  and  $R \rightarrow 0$ . In both cases, we consider code families that achieve a vanishing output bit error rate on a binary symmetric channel  $\text{BSC}(p)$  with a transition error probability  $p$  and capacity  $C = 1 - h(p)$ , where  $h(p)$  is a binary entropy. We say that a family of codes with  $n \rightarrow \infty$  and  $R \rightarrow 1$  is **strongly optimal** if the fraction  $\rho = 1 - R$  of redundant (parity-check) bits has the smallest possible order

$$1 - R \sim h(p) = p \log_2(e/p) + O(p^2)$$

A family of long codes is called **weakly optimal** if probability  $p \rightarrow 0$  and redundancy  $\rho$  have a similar decline rate

$$\log_2(1 - R) \sim \log_2 h(p) \sim \log_2 p \quad (1)$$

Our main result is as follows.

**Theorem 1.** *For any  $p \rightarrow 0$ , there exist weakly optimal codes of length  $n \rightarrow \infty$  that have a relative redundancy*

$$\rho \leq p \left( \log_2 \frac{1}{p} \right)^{\log_2 \log_2 \frac{1}{p}} \quad (2)$$

*and achieve a vanishing error probability on a binary symmetric channel  $\text{BSC}(p)$ . These codes can be constructed, encoded, and decoded with complexity of order  $n \ln n$ .*

Similarly, long codes of rate  $R \rightarrow 0$  are called strongly optimal if they achieve a vanishing output error rate on a  $\text{BSC}(p)$  with  $p \rightarrow 1/2$  and have the maximum possible order of code rate  $R \sim 1 - h(p) \sim (1 - 2p)^2 / \ln 4$ . We extend Theorem 1 and design strongly optimal codes of rate  $R \rightarrow 0$  and complexity  $n \ln n$ .

For a wide range of error probabilities  $p$ , codes of Theorem 1 outperform known codes of code rate  $R \rightarrow 1$ . For example, long primitive BCH codes require redundancy  $p \log_2 n$  to achieve a vanishing output error rate under the bounded-distance decoding on a  $\text{BSC}(p)$  if  $p = o(\log_2 n)$  [1]. However,  $R \rightarrow 0$  if  $p \log_2 n \rightarrow \infty$ . The recent breakthrough of [12] also shows that high-rate RM codes  $\mathcal{R}(m - 2r - 1, m)$  can correct the fraction of errors  $p \sim \binom{m}{r} / 2^m$  with polynomial complexity and low redundancy  $\rho \sim \binom{m}{2r+1} / 2^m$  if  $r = o(\sqrt{m / \log m})$ . This algorithm is still limited to the rapidly vanishing probabilities  $p$  unlike any  $p \rightarrow 0$  in Theorem 1. Note, however, that Theorem 1 achieves no improvements over BCH codes if probability  $p$  has an exponentially declining order  $p \leq 2^{-m^c}$  for any  $c > 0$ , nor does it give strongly optimal codes for  $R \rightarrow 1$ .

Sections II and III provide some background and address the common properties of RM and polar codes. Sections IV-VI introduce polarized design with a single boundary. We first design the weakly optimal codes of rates  $R \rightarrow 1$  and then extend them to the strongly optimal codes of rate  $R \rightarrow 0$ .

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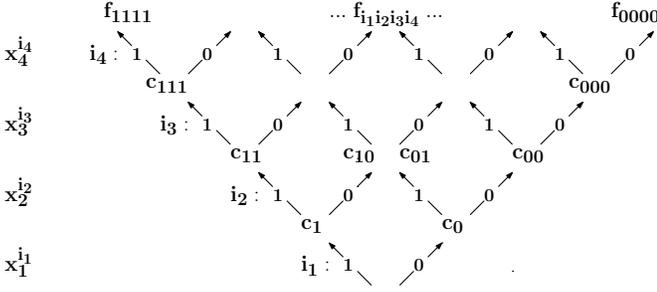


Fig. 1. Decomposition  $(c_0, c_0 + c_1)$  of RM code  $\mathcal{R}(4, 4)$

## II. RECURSIVE DESIGN OF RM AND POLAR CODES

Consider boolean polynomials  $f(x)$  of degree  $r$  or less in  $m$  binary variables  $x_1, \dots, x_m$ , where  $r \leq m$ . Vectors  $x = (x_1, \dots, x_m)$  will mark the positions of our code. Each map  $f(x) : \mathbb{F}_2^m \rightarrow \mathbb{F}_2$  generates a codeword  $\mathbf{c} = \mathbf{c}(f)$  of code  $\mathcal{R}(r, m)$ . We also use short notation  $\mathbf{x}_i|_j = (x_i, \dots, x_j)$  for  $i \leq j$ . Consider recursive decomposition

$$\begin{aligned} f(x) &= f_0(\mathbf{x}_2|m) + x_1 f_1(\mathbf{x}_2|m) = \dots \\ &= \sum_{i_1, \dots, i_\ell} x_1^{i_1} \dots x_\ell^{i_\ell} f_{i_1, \dots, i_\ell}(\mathbf{x}_{\ell+1}|m) \\ &= \dots = \sum_{i_1, \dots, i_m} f_{i_1, \dots, i_m} x_1^{i_1} \dots x_m^{i_m} \end{aligned} \quad (3)$$

The first step decomposes polynomial  $f(x)$  into polynomials  $f_0$  and  $f_1$  of degrees  $\deg f_0 \leq \min\{r, m-1\}$  and  $\deg f_1 \leq r-1$ . Then the codewords  $\mathbf{c}_0 = \mathbf{c}(f_0)$  and  $\mathbf{c}_1 = \mathbf{c}(f_1)$  belong to the codes  $\mathcal{R}(r, m-1)$  and  $\mathcal{R}(r-1, m-1)$  and form the codeword  $\mathbf{c} = \mathbf{c}_0, \mathbf{c}_0 + \mathbf{c}_1$  of code  $\mathcal{R}(r, m)$ . Similarly, any subsequent step  $\ell$  decomposes each polynomial with respect to  $x_\ell^{i_\ell}$  as follows

$$f_{i_1, \dots, i_{\ell-1}}(\mathbf{x}_\ell|m) = \sum_{i_\ell=0,1} f_{i_1, \dots, i_\ell}(\mathbf{x}_{\ell+1}|m) \cdot x_\ell^{i_\ell}$$

We then say that the  $\ell$ -level binary paths  $\xi_{1|\ell} = i_1, \dots, i_\ell$  decompose the original polynomial  $f(x)$  into sums of monomials  $x_1^{i_1} \dots x_\ell^{i_\ell} f_{i_1, \dots, i_\ell}(\mathbf{x}_{\ell+1}|m)$ . Finally, full paths  $\xi = i_1, \dots, i_m$  of step  $m$  define monomials  $x^\xi \equiv x_1^{i_1} \dots x_m^{i_m}$  with coefficients  $f_\xi = f_{i_1, \dots, i_m} = 0, 1$ . Note that each monomial  $x^\xi$  gives a codeword  $\mathbf{c}(x^\xi)$  of weight  $2^{m-w(\xi)}$ , where  $w(\xi)$  is the Hamming weight of the string  $\xi$ . RM codes  $\mathcal{R}(r, m)$  include only  $k(r, m)$  paths of weight  $w(\xi) \leq r$ .

In Fig. 1 we use this representation for the full code  $\mathcal{R}(4, 4)$ . Each decomposition step  $\ell = 1, \dots, 4$  is marked by the splitting monomial  $x_\ell^{i_\ell}$ . For example, path  $\xi = 0110$  gives the coefficient  $f_{0110}$  associated with the monomial  $x^\xi \equiv x_2 x_3$ .

Fig. 2 depicts code  $\mathcal{R}(2, 5)$ . Here we only include all paths  $\xi$  of weight  $w(\xi) \leq 2$ . Note that any two paths  $\xi_{1|\ell}$  entering some node have the same weight  $w$  and generate the same code  $\mathcal{R}(r-w, m-\ell)$  on their extensions. For example, path  $\xi = 01100$  proceeds from  $\mathcal{R}(2, 5)$  to the single bit  $\mathcal{R}(0, 0)$  via codes  $\mathcal{R}(2, 4)$ ,  $\mathcal{R}(1, 3)$ ,  $\mathcal{R}(0, 2)$ , and  $\mathcal{R}(0, 1)$ .

This design can be reformulated using a  $2 \times 2$  matrix

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then code  $\mathcal{R}(m, m)$  is generated by the Kronecker product  $G(m, m) = G^{\otimes m}$ . Each row of  $G^{\otimes m}$  is the map of the monomial  $x^\xi$  for some path  $\xi$ . Similarly, matrix  $G(r, m)$  is

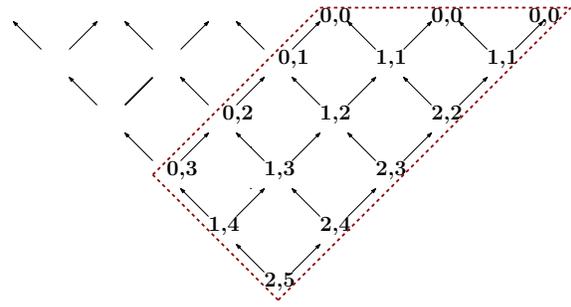


Fig. 2. Paths and nodes of RM code  $\mathcal{R}(2, 5)$

the map of all monomials  $x^\xi$  with paths  $\xi = i_1, \dots, i_m$  of weight  $w(\xi) \leq r$ .

Now consider a single path  $\xi$  that ends with an information bit  $f_{i_1, \dots, i_m} = 1$ . Encoding proceeds in the reverse order  $\ell = m, \dots, 1$ . We begin with a single bit codeword  $\mathbf{c}(\xi_{m+1|m}) = 1$ . In each step  $\ell$ , we use recursion and obtain the codeword

$$\mathbf{c}(\xi_\ell|m) = \begin{cases} \mathbf{c}(\xi_{\ell+1|m}), \mathbf{c}(\xi_{\ell+1|m}) & \text{if } i_\ell = 0 \\ \mathbf{0}, \mathbf{c}(\xi_{\ell+1|m}) & \text{if } i_\ell = 1 \end{cases} \quad (4)$$

of length  $2^{m-\ell+1}$ . Thus, any path  $\xi$  is encoded in the vector  $\mathbf{c} = \mathbf{c}(\xi)$  of length  $n$ . Also,  $\mathbf{c}(\xi) = \mathbf{0}$  if  $f_{i_1, \dots, i_m} = 0$ .

Now consider a subset of  $N$  paths  $T$ . Then we encode  $N$  information bits via their paths and obtain codewords  $\mathbf{c}(T) = \sum_{\xi \in T} \mathbf{c}(\xi)$ . These codewords form a linear code  $C(m, T)$ . Here at any level  $\ell$ , encoding adds two codewords of level  $\ell+1$  entering any node  $\xi_\ell|m$ . Thus, encoding (4) performs  $2^{m-\ell}$  operations on each of  $2^\ell$  nodes  $\xi_\ell|m$  and has the overall complexity of  $n \log_2 n$  over all levels  $\ell$ .

**Lemma 2.** Code  $C(m, T)$  has length  $2^m$ , dimension  $|T|$  and distance  $2^{m-r}$ , where  $r = \max\{w(\xi), \xi \in T\}$  is the weight of the heaviest path in  $T$ . Code  $\mathcal{R}(r, m)$  has the maximum code rate  $R$  among all codes  $C(m, T)$  of the distance  $2^{m-r}$ .

*Proof.* Let weight  $r$  be achieved on some path  $\psi \in T$ . Then code  $C(m, T)$  is generated by monomials  $x^\xi$  of degree  $r$  or less. Thus,  $C(m, T) \subseteq \mathcal{R}(r, m)$ . The monomial  $x^\psi$  has degree  $r$  and gives the minimum weight  $2^{m-r}$ .  $\square$

## III. RECURSIVE DECODING ALGORITHMS

Below, we use a map  $x \rightarrow (-1)^x$  for any  $x = 0, 1$  and consider a discrete memoryless channel (DMC)  $W$  with inputs  $\pm 1$ . Vector  $\mathbf{ab}$  will denote the component-wise product of vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c} = (\mathbf{u}, \mathbf{uv})$  will denote the codewords  $\mathbf{c}$  of a code  $\mathcal{R}(r, m)$  with symbols  $\pm 1$ . In particular,  $\mathbf{1}^n$  now represents a former all-zero codeword. For any codeword  $\mathbf{c}$ , let  $\mathbf{y}_0, \mathbf{y}_1$  be the two output halves corrupted by noise. We use double index  $i, j$  for any position  $j = 1, \dots, n/2$  in a half  $i = 0, 1$ . Define the posterior probability (PP)  $q_{i,j} = \Pr\{c_{i,j} = 1 \mid y_{i,j}\}$  that 1 is sent in position  $i, j$ . We will often replace  $q_{i,j}$  with two related quantities, which we call “the offsets”  $g_{i,j}$  and the likelihoods  $h_{i,j}$ :

$$g_{i,j} = 2q_{i,j} - 1, \quad h_{i,j} = q_{i,j} / (1 - q_{i,j}) \quad (5)$$

Thus, we will use vectors  $\mathbf{q} = (q_{i,j})$ ,  $\mathbf{g} = (g_{i,j})$  and  $\mathbf{h} = (h_{i,j})$ . For example, let  $W$  be a binary symmetric channel

BSC( $p$ ), where  $p = (1 - \epsilon)/2$ . Then any output  $y = \pm 1$  gives quantities  $g(y) = \epsilon y$  and  $h(y) = (1 + \epsilon y)/(1 - \epsilon y)$ .

The following recursive algorithm  $\Psi_r^m(\mathbf{q})$  of [2], [5] performs SCD of information bits in codes  $\mathcal{R}(r, m)$  or their subcodes  $C(m, T)$ . Here we relegate decoding of vector  $\mathbf{q}$  to two vectors  $\mathbf{q}^{(1)}$  and  $\mathbf{q}^{(0)}$  of length  $n/2$ . Vector  $\mathbf{q}^{(1)}$  consists of PP  $q_j^{(1)} \equiv \Pr\{v_j = 1 \mid q_{0,j}, q_{1,j}\}$  of symbols  $v_j$  in construction  $(\mathbf{u}, \mathbf{uv})$ . Simple recalculations [2] show that the offsets  $g_j^{(1)}$  of symbols  $v_j$  can be expressed as the products of two offsets  $g_{0,j}g_{1,j}$ . Thus, we obtain vectors  $\mathbf{g}^{(1)}$  and  $\mathbf{q}^{(1)}$  with symbols

$$g_j^{(1)} = g_{0,j}g_{1,j}, \quad q_j^{(1)} = (1 + g_j^{(1)})/2. \quad (6)$$

We may now apply some decoding algorithm  $\Psi_{r-1}^{m-1}$  to the vector  $\mathbf{q}^{(1)}$  and obtain a vector  $\tilde{\mathbf{v}} \in \mathcal{R}(r-1, m-1)$  of length  $n/2$ . Now we have two corrupted versions  $\mathbf{y}_0$  and  $\mathbf{y}_1\tilde{\mathbf{v}}$  of vector  $\mathbf{u}$ . We can then derive PP  $q_j^{(0)} = \Pr\{u_j = 1 \mid q_{0,j}, q_{1,j}, \tilde{v}_j\}$  of symbols  $u_j$  in the  $(\mathbf{u}, \mathbf{uv})$  construction. Indeed, any symbol  $u_j$  has likelihoods  $h_{0,j}$  and  $(h_{1,j})^{\tilde{v}_j}$  in the left and right halves, respectively. Then we combine the two likelihoods into their product:

$$h_j^{(0)} = h_{0,j} (h_{1,j})^{\tilde{v}_j}, \quad q_j^{(0)} = h_j^{(0)}/(1 + h_j^{(0)}) \quad (7)$$

Then we can apply some decoding  $\Psi_r^{m-1}$  to vector  $\mathbf{q}^{(0)}$  and obtain  $\tilde{\mathbf{u}} \in \mathcal{R}(r, m-1)$ .

Decomposition (6), (7) forms level  $\ell = 1$  of SCD, which can also be continued for vectors  $\mathbf{q}^{(1)}$  and  $\mathbf{q}^{(0)}$  on the codes  $\mathcal{R}(r-1, m-1)$  and  $\mathcal{R}(r, m-1)$ . Then levels  $\ell = 2, \dots, m$  are processed similarly, moving decoding along the paths of Fig. 1 or Fig. 2. Any incomplete path  $\xi_{1|\ell}$  begins with its  $\mathbf{v}$ -extension  $(\xi_{1|\ell}, 1)$ . Upon decoding, this path delivers its output  $\tilde{\mathbf{v}}$  to the  $\mathbf{u}$ -path  $(\xi_{1|\ell}, 0)$ . Thus, all paths are ordered lexicographically. Finally, the last step gives the likelihood  $q_\xi = \Pr\{f_\xi = 0 \mid \mathbf{y}_0, \mathbf{y}_1\}$  of one information bit  $f_\xi$  on the path  $\xi$ . We then choose the more reliable bit  $f_\xi$ . It is easy to verify [2] that  $m$  decomposition steps give complexity  $2n \log_2 n$ .

Any subcode  $C(m, T)$  is decoded similarly and assumes that all paths  $\xi \notin T$  are frozen and give information bits  $f_\xi \equiv 0$ . Let all  $N$  paths in  $T$  be ordered lexicographically as  $\xi^{(1)}, \dots, \xi^{(N)}$ . Then we have

Algorithm  $\Psi(m, T)$  for code  $C(m, T)$ .  
 Given: a vector  $\mathbf{q} = (q_{i,j})$  of PP.  
 Take  $s = 1, \dots, N$  and  $\ell = 1, \dots, m$ .  
 For path  $\xi^{(s)} = i_1^{(s)}, \dots, i_m^{(s)}$  in step  $\ell$  do:  
   Apply recalculations (6) if  $i_\ell^{(s)} = 1$   
   Apply recalculations (7) if  $i_\ell^{(s)} = 0$ .  
 Output the bit  $f_{\xi^{(s)}}$  for  $\ell = m$ .

#### IV. PATH ORDERING IN SC DECODING

Let a binary code  $C(m, T)$  be used over a symmetric DMC  $W$ . We now consider a code  $C_\xi$  defined by a single path  $\xi = (i_1, \dots, i_m)$  and estimate its decoding error probability  $P_\xi$ . Let a codeword  $1^n$  be transmitted over this path. We now

may assume that other paths give outputs  $\tilde{v}_j = 1$  in recursive recalculations (5)-(7). Then we re-arrange (5)-(7) as follows

$$g_j^{(1)} = g_{0,j}g_{1,j}, \quad g_j^{(0)} = (g_{0,j} + g_{1,j})/(1 + g_{0,j}g_{1,j}) \quad (8)$$

$$h_j^{(0)} = h_{0,j}h_{1,j}, \quad h_j^{(1)} = (1 + h_{0,j}h_{1,j})/(h_{0,j} + h_{1,j}) \quad (9)$$

From now on, we may consider recalculations (8) and (9) as the sequences of channel transformations applied to the original random variables (rv)  $g_{i,j}$  or  $h_{i,j}$ . In the end, we obtain a new memoryless channel  $W_\xi : X \rightarrow Y_\xi$  that outputs a single rv  $h(\xi)$  after  $m$  steps. For any parameter  $\lambda > 0$ , we also consider rv  $h^\lambda(\xi)$  and its expectation  $\mathbb{E}h^{-\lambda}(\xi)$ . Then the Chernoff upper bound gives

$$P_\xi \equiv \Pr\{h(\xi) < 1\} \leq \min_{\lambda > 0} \mathbb{E}h^{-\lambda}(\xi) = \min_{\lambda > 0} \mathbb{E}e^{-\lambda \ln h(\xi)}$$

Note that the quantity  $\mathbb{E}h^{-1/2}(\xi)$  is identical to the Bhat-tacharyya parameter

$$Z(W) = \sum_{y \in Y} \sqrt{W(y|0)} \sqrt{W(y|1)}$$

defined for a DMC channel  $W_\xi : X \rightarrow Y_\xi$ . For example, BSC( $p$ ) with  $p = (1 - g)/2$  gives

$$Z(W) = \mathbb{E}h^{-1/2}(\xi) = 2 \left(\frac{1+g}{2}\right)^{1/2} \left(\frac{1-g}{2}\right)^{1/2} = \sqrt{1 - g^2}$$

In a more general setting [7], we decompose a binary symmetric DMC  $W_\xi$  into some number  $k$  of binary symmetric channels BSC $_{\theta_i}(p_i)$  that have transition error probabilities  $p_i = (1 - g_i)/2$  and occur with some probability distribution  $\{\theta_i\}$ , where  $\sum_1^k \theta_i = 1$ . Then

$$Z(W_\xi) = \sum_i \theta_i \sqrt{1 - g_i^2} \quad (10)$$

Below we use the upper bound  $P_\xi \leq Z(W_\xi)$  employed by Arikan in [3]. It is also proved in [3] that a one step recursion  $(W, W) \rightarrow (W^{(1)}, W^{(0)})$  of (9) gives parameters  $Z(W^{(1)})$  and  $Z(W^{(0)})$  such that

$$1 - Z(W^{(1)}) \geq [1 - Z(W)]^2, \quad Z(W^{(0)}) = Z^2(W) \quad (11)$$

Now consider a compound channel  $W_\xi$  as a set of BSC $_{\theta_i}(p_i)$ . Then we can define the expectation of the offsets  $g_i > 0$ :

$$\mathcal{G}(W_\xi) = \sum_1^k \theta_i g_i$$

Note that  $\sqrt{1 - g^2}$  is a concave function. Also,  $\sqrt{1 - g^2} \geq 1 - g$  for any  $g \in [0, 1]$ . Thus, (10) yields two inequalities

$$1 - \mathcal{G}(W_\xi) \leq Z(W_\xi) \leq \sqrt{1 - [\mathcal{G}(W_\xi)]^2} \quad (12)$$

Given a one step recursion  $(W, W) \rightarrow (W^{(1)}, W^{(0)})$ , we can also take two independent identically distributed rv  $g_{0,j}$  and  $g_{1,j}$  in (8) and find the expectation of their product  $g_j^{(1)}$  for the channel  $W^{(1)}$ . Then we have two equalities

$$\mathcal{G}(W^{(1)}) = \mathcal{G}^2(W), \quad (13)$$

$$Z(W^{(0)}) = Z^2(W) \quad (14)$$

Below we replace notation  $Z(W_\xi)$  and  $\mathcal{G}(W_\xi)$  with  $Z(\xi)$  and  $\mathcal{G}(\xi)$ . Given a path  $\xi = (i_1, \dots, i_m)$ , we say that a path  $\eta =$



$\delta^2]$ . This allows us to completely compensate the relatively long chains of degrading channels  $1^{r_i}$  with short chains  $0^{\ell_i}$  of upgrading channels. In fact, we will improve the overall performance in each step of the boundary (21). It is this superiority of the chains  $0^{\ell_i}$  that yields small ratios  $\ell_i/r_i$  in our design and leads to a nearly optimal decline rate of redundancy  $\rho_{\mathcal{L}}$ . The exact calculations are given below.

Consider two functions  $f = f(n)$  and  $r = r(n)$  that have the same sign. Then we write  $f \lesssim r$  or  $f \gtrsim r$  if the asymptotic ratio  $\lambda = \lim_{n \rightarrow \infty} f/r$  is  $\lambda \in (0, 1)$  or  $\lambda \geq 1$ , respectively. We also write  $f \succ r$  if  $f > r^c$  for some  $c > 1$ . Finally, consider inequalities

$$\begin{aligned} -x - x^2 < \ln(1 - x) < -x, \quad x \in (0, 1/2) \\ 1 - x < -\ln x, \quad x \in (0, 1) \end{aligned} \quad (23)$$

which are tight as  $x \rightarrow 0$  and  $x \rightarrow 1$ , respectively. Using these inequalities, we can rewrite (12) as

$$\log Z(\xi) < \frac{1}{2} \log[-2 \ln \mathcal{G}(\xi)] \quad (24)$$

$$\ln \mathcal{G}(\xi) > -Z(\xi) - Z^2(\xi) \quad (25)$$

Below, we extensively use a recursion that employs inequalities (24) and (25). We will also see that  $Z(\xi) \rightarrow 0$  and  $\mathcal{G}(\xi) \rightarrow 1$  for the selected path  $\xi(\mathcal{L})$  of (21). In this case, we can also replace (24) and (25) with simpler inequalities  $\log Z(\xi) \lesssim \frac{1}{2} \log[-\ln \mathcal{G}(\xi)]$  and  $\ln \mathcal{G}(\xi) \gtrsim -Z(\xi)$ .

**Lemma 6.** *Codes  $\mathcal{R}(\mathcal{L})$  with a boundary (21) achieve an output bit error rate  $P_\eta \rightarrow 0$  for each path  $\eta(\mathcal{L})$  under SCD on a BSC( $p$ ) with  $p \rightarrow 0$ .*

*Proof.* Given the boundary  $\xi(\mathcal{L})$ , we will estimate the Bhat-tacharyya parameters

$$Z_{(i)} \equiv Z[1^{r_1} 0^{\ell_1} \dots 1^{r_i}], \quad Z^{(i)} \equiv Z[1^{r_1} 0^{\ell_1} \dots 1^{r_i} 0^{\ell_i}]$$

obtained in processing of each step  $i$ . We also use similar notation  $\mathcal{G}_{(i)}$  and  $\mathcal{G}^{(i)}$  for the offsets obtained in step  $i$ . The original channel BSC( $p$ ) gives parameter  $\mathcal{G} = 1 - 2p$ , where  $p \rightarrow 0$ . For the first segment  $1^{r_1}$ , equality (13) and the upper bound (12) give:

$$\begin{aligned} \mathcal{G}_{(1)} &= (1 - 2p)^{1/(64p)} \sim e^{-1/32} \\ Z_{(1)} &\lesssim \left(1 - e^{-1/16}\right)^{1/2} < 2^{-2} \end{aligned} \quad (26)$$

For the next segment  $0^{\ell_1}$ , equality (14) gives

$$Z^{(1)} = [Z_{(1)}]^{2^{\ell_1}} < 2^{-2 \log 1/p} = p^2$$

Then  $\mathcal{G}^{(1)} \geq 1 - Z^{(1)}$ , according to (12), and we proceed with the segment  $1^{r_2} 0^{\ell_2}$  using (13):

$$\begin{aligned} \mathcal{G}_{(2)} &\geq (1 - p^2)^{p^{-2/8}} \sim e^{-1/8} \\ Z_{(2)} &\lesssim \left(1 - e^{-1/4}\right)^{1/2} < 1/2 \\ Z^{(2)} &= [Z_{(2)}]^{2^{\ell_2}} < 2^{-\tau^2} = p^\tau \end{aligned}$$

Note that  $2^{r_i} = p^{-2^{i-1}}$  and  $2^{\ell_i} = \tau^{2^{i-1}}$  for  $i \geq 3$ . Now we use inequalities (24) and (25) to prove that parameters  $Z^{(i)}$  rapidly decline:

$$Z^{(i)} \leq p^{t_i}, \quad t_i = \tau^{2^i - i - 1} \quad (27)$$

Indeed,  $Z^{(2)}$  satisfies (27). We take  $Z^{(i-1)} \leq p^{t_{i-1}}$  and use induction on the  $i$ -th segment  $1^{r_i} 0^{\ell_i}$ . Then inequalities (24) and (25) give

$$\begin{aligned} \ln \mathcal{G}_{(i)} &\geq -2^{r_i} [p^{t_{i-1}} + p^{2^{t_{i-1}}}] \gtrsim -2^{r_i} p^{t_{i-1}} \\ \log Z_{(i)} &< \frac{1}{2} \log[-2 \ln \mathcal{G}_{(i)}] \lesssim \frac{1}{2} (t_{i-1} - r_i) \log p \end{aligned} \quad (28)$$

Note that  $r_i = o(t_{i-1})$ . Thus,  $\log Z_{(i)} \leq s_i \log p$ , where

$$s_i = t_{i-1}/\tau = \tau^{2^{i-1} - i - 1} = o(t_{i-1})$$

Then

$$\log Z^{(i)} = 2^{\ell_i} \log Z_{(i)} \leq \tau^{2^{i-1}} s_i \log p = t_i \log p \quad (29)$$

This proves (27) and gives  $P_\eta \leq Z^{(s)}$  for each path  $\eta$ .  $\square$

*Discussion.* Inequalities (28) and (29) show that the initial chains  $1^{r_i}$  and the subsequent chains  $0^{\ell_i}$  affect parameters  $Z_{(i)}$  and  $Z^{(i)}$  in a very different way. In particular, (28) shows that any chain  $1^{r_i}$  reduces the previous exponential order  $t_{i-1} = \log_p Z^{(i-1)}$  to  $t_{i-1}/2 - o(t_{i-1})$ . By contrast, the stretch  $0^{\ell_i}$  increases this order above  $2^{\ell_i} (t_{i-1}/\tau)$ . For this reason, good BSC( $p$ ) with  $p \rightarrow 0$  may overcompensate long chains  $1^{r_i}$  of degrading channels with the much shorter chains  $0^{\ell_i}$  of upgrading channels. Note also that equalities (13) and (14) are critical in our proof since they give exact estimates  $\mathcal{G}_{(i)}$  and  $Z^{(i)}$  in all intermediate steps of the segments  $1^{r_i}$  or  $0^{\ell_i}$ , without any loss in performance. To this end, note that inequalities (11) and (12) alone cannot furnish Lemma 6. For example, inequalities (11) replace estimate (26) with a loose bound  $Z_{(1)} \leq 1 - e^{-1/(32\sqrt{p})}$ . This bound will require a much longer path  $0^{\ell_1}$  to achieve a low quantity  $Z^{(2)}$ , which in turn increases redundancies  $\rho_1$  and  $\rho_{\mathcal{L}}$  above the bound (12) of the weakly optimal codes.

However, this particular construction fails to give the optimal redundancy  $\rho_{\text{opt}} \sim p \log 1/p$  or even reduce  $\rho_{\mathcal{L}}$  to the order of  $cp \log 1/p$  for some constant  $c > 1$ . Nor is it known if other low-complexity algorithms for polar or other codes can achieve  $\rho_{\text{opt}}$  for  $p \rightarrow 0$ . Note also that the single-boundary set  $\eta(\mathcal{L})$  of Lemma 6 does not form an optimized polar code since many other paths  $\eta$  also have a vanishing output error rate. For example, any initial segment  $1^r$  of length  $r < r_1$  gives rise to many paths  $\eta \notin \eta(\mathcal{L})$ . To reduce redundancy  $\rho_{\mathcal{L}}$ , one may consider a growing set  $\{\xi\}$  of boundary paths  $\xi$  and form an entire ‘‘envelope’’ of the descendant paths  $\eta(\xi)$ . Calculating the redundancy for this envelope-type boundary is another open problem that may be related to the Young diagrams.

## VI. LOW-RATE CODES WITH A STEPPED BOUNDARY

Consider a sequence of the BSCs( $p$ ) with  $p = (1 - \epsilon)/2$ , where  $\epsilon \rightarrow 0$  as length  $n \rightarrow \infty$ . Below we study capacity-achieving (CA) codes of rate  $R \sim C$  for the case of a vanishing capacity  $C = 1 - h(p) \sim \epsilon^2/\ln 4$ . It is proved

in [13] that RM codes  $\mathcal{R}(r, \mu)$  are CA codes under ML-decoding if  $r = o(\mu)$ . However, only codes  $\mathcal{R}(1, \mu)$  of length  $k = 2^\mu$  or their concatenations are known to be CA-codes of polynomial complexity. More specifically, consider a BSC( $p_*$ ) with capacity  $C \rightarrow 0$  and transition error probability

$$p_* = (1 - \epsilon_*)/2, \quad \epsilon_* = (C \ln 4)^{1/2} \quad (30)$$

According to [14], for any parameter  $\theta \in (0, 1)$ , codes  $\mathcal{R}(1, \mu)$  of code rate  $R = C(1 - \theta)$  achieve on BSC( $p_*$ ) the output bit error rate  $P_* \leq k^{-\theta}$  or less with complexity  $O(k \log k)$ .

To proceed with the low-rate codes, we need to substantially reduce the output error rate of (27). This is done in the following theorem, where we reduce the error rate  $P_\eta \leq Z^{(s)}$  at the expense of a slightly higher redundancy  $\rho_{\mathcal{L}}$ . Consider a boundary

$$\mathcal{L}_c = \{r_i = 2^{i-1} (\log 1/p) - c_i, \quad \ell_i = c2^{i-1} \log 1/p\} \quad (31)$$

where  $c_1 = 6$ ,  $c_i = 0$  for  $i \geq 2$ , and  $c \in (0, 1)$  is a parameter. This boundary has length

$$m = \sum_{i=1}^s m_i = (c+1)(2^s - 1)(\log 1/p) - 6 \quad (32)$$

**Lemma 7.** *Codes  $\mathcal{R}(\mathcal{L}_c)$  with boundary (31) have redundancy  $\rho_{\mathcal{L}} \rightarrow 0$  as  $p \rightarrow 0$ . These codes perform SCD with an output bit error rate  $P_\eta$ , where for each path  $\eta$ ,*

$$\log P_\eta \lesssim -2^{2-s} p^{-c(2^s-1)} \quad (33)$$

*Proof.* Note that  $\ell_i/m_i = c/(c+1)$ . For  $i \geq 2$ , let

$$c_1 \equiv h(\ell_i/m_i) = h[c/(c+1)] < 1.$$

Then  $\rho_i \leq 2^{-m_i(1-c_1)} = o(\rho_{i-1})$  for  $p \rightarrow 0$ , and

$$\rho_{\mathcal{L}} \sim \rho_1 \leq 64p^{(1+c)(1-c_1)} \rightarrow 0$$

Also,  $2^{r_i} = p^{-2^{i-1}}$  and  $2^{\ell_i} = p^{-c2^{i-1}}$  for  $i \geq 2$ . Next, we estimate parameters  $Z_{(i)}$  and  $Z^{(i)}$  and follow the proof of Lemma 6. Given the same length  $r_1$ , we again obtain  $Z_{(1)} < 1/4$  of (26). The next segment  $0^{\ell_1}$  gives

$$Z^{(1)} = [Z_{(1)}]^{2^{\ell_1}} < 2^{-2^{p-c}}$$

Then the segment  $1^{r_2} 0^{\ell_2}$  yields estimates

$$\ln \mathcal{G}_{(2)} \gtrsim -p^{-2} Z^{(1)} \gtrsim -p^{-2} 2^{-2^{p-c}} \quad (34)$$

$$\log Z_{(2)} \lesssim \frac{1}{2} \log [-2 \ln \mathcal{G}_{(2)}] \lesssim -p^{-c}$$

$$\log Z^{(2)} \lesssim -2^{\ell_2} p^{-c} \lesssim -p^{-3c}$$

Now we prove that parameters  $Z^{(i)}$  rapidly decline:

$$\log Z^{(i)} \lesssim -2^{2-i} p^{-c(2^i-1)} \quad (35)$$

Indeed,  $Z^{(2)}$  satisfies (35). We take  $Z^{(i-1)}$  of (35) and proceed with the  $i$ -th segment  $1^{r_i} 0^{\ell_i}$ . We proceed similarly to (34),

$$\ln \mathcal{G}_{(i)} \gtrsim -2^{r_i} Z^{(i-1)}$$

$$\log Z_{(i)} < \frac{1}{2} \log [-2 \ln \mathcal{G}_{(i)}] \lesssim \frac{1}{2} r_i + \frac{1}{2} \log Z^{(i-1)}$$

Since  $r_i = o(\log Z^{(i-1)})$ , we obtain :

$$\log Z^{(i)} = 2^{\ell_i} \log Z_{(i)} \lesssim 2^{\ell_i-1} \log Z^{(i-1)}$$

which gives (35) and proves the theorem.  $\square$

We will now combine codes  $\mathcal{R}(1, \mu)$  with the high-rate polar codes of Lemma 7 to obtain new CA codes.

Note that code  $\mathcal{R}(1, \mu)$  is defined by a boundary path  $\xi^{(0)} = 1^1 0^{\mu-1}$ . We then combine  $\xi^{(0)}$  with the boundary  $\mathcal{L}_c$  of (31) and obtain the extended boundary

$$\mathcal{L}_{ext} = \{r_0 = 1, \ell_0 = \mu - 1, \mathcal{L}_c\} \quad (36)$$

Lemma 4 shows that  $\mathcal{L}_{ext}$  generates the direct product  $\mathcal{R}_{ext}$  of  $s+1$  RM codes  $\mathcal{R}(r_i, m_i)$ . Thus, code  $\mathcal{R}_{ext}$  has code rate  $R$  and length  $N$ , where

$$R = R(1, \mu) R_{\mathcal{L}_c} \sim (\mu + 1)/2^\mu$$

$$N = kn, \quad k = 2^\mu, \quad n = 2^m$$

Codes  $\mathcal{R}_{ext}$  also represent a simple concatenated construction, which first uses  $\mu + 1$  arbitrary codewords of the code  $\mathcal{R}(\mathcal{L}_c)$  and forms an  $(\mu + 1) \times n$  matrix. Then each column of this matrix is encoded into the code  $\mathcal{R}(1, \mu)$ . The result is an  $k \times n$  matrix, which represents a codeword formed by the inner code of length  $k$  and  $s$  outer codes of length  $n$ . Below we take  $\mu, m \rightarrow \infty$ . Below we take  $p_o = 2^{-\mu\theta}$  in (31).

**Theorem 8.** *Let codes  $\mathcal{R}_{ext}$  of code rate  $C(1 - \theta)$  with an  $s$ -step boundary (36) be used on a BSC( $p_*$ ) of capacity  $C \rightarrow 0$ . For any  $\theta \in (0, 1)$ , codes  $\mathcal{R}_{ext}$  have decoding complexity  $O(N \log N)$  in length  $N = nk$  and achieve a bit error probability  $P_\eta$  such that*

$$\log P_\eta \prec -2^{2-s} n^{c/(c+1)} \quad (37)$$

*Proof.* Decoding of codes  $\mathcal{R}_{ext}$  can be expressed as SCD; below we also describe it as concatenated decoding of inner codes. We take codes  $\mathcal{R}(1, \mu)$  of rate  $\epsilon^2/\ln 4$  as  $\theta \rightarrow 0$ . Then  $R_{ext} \sim (1 - \theta)\epsilon^2/\ln 4$  as  $m \rightarrow \infty$ . Given a received  $2^\mu \times 2^m$  matrix, we first perform ML decoding of each column of length  $2^\mu$  into the code  $\mathcal{R}(1, \mu)$ . The resulting  $(\mu + 1) \times 2^m$  matrix contains errors with probability  $p_o$  or less. Each row is decoded into the code  $\mathcal{R}(\mathcal{L})$  using SCD on a BSC( $p_o$ ). Note that for any  $\theta \in (0, 1)$ ,

$$m = (2^s - 1)(c + 1)\mu\theta - c_1 \quad (38)$$

Then (33) gives the bit error rate (37):

$$\log P_\eta \lesssim -2^{2-s} 2^{c\mu\theta(2^s-1)} \quad (39)$$

Thus, codes  $\mathcal{R}_{ext}$  are CA codes. Inner and outer decodings have the complexity  $nk \log k$  and  $\mu n \log n$  bounded by  $N \log N$ .  $\square$

*Discussion.* According to (39), the order  $\log P_\eta$  depends exponentially on the margin  $\theta$  between the code rate  $R$  and channel capacity  $C$ . Below, we compare the performance of codes  $\mathcal{R}(1, \mu)$  and  $\mathcal{R}_{ext}$  for the same code rate  $R \sim C(1 - \theta)$ , and define the minimum code length  $k$  or  $N$  that enables a given output bit error  $P$ . Here we consider an asymptotic case with parameters  $c \rightarrow 1$  and  $P \rightarrow 0$ . For codes  $\mathcal{R}(1, \mu)$ , we have  $k \sim P^{-1/\theta}$ . For codes  $\mathcal{R}_{ext}$ , we use notation

$A = 2^s\theta$ . Then parameters (38) and (39) yield asymptotic approximations

$$n = 2^m \asymp k^{2A}, \log P \asymp -n^{1/2} \asymp -k^A \quad (40)$$

(here  $f \asymp r$  if  $\log f \sim \log r$ ). Recall that the outer codes  $\mathcal{R}(\mathcal{L}_c)$  require a vanishing input error rate  $k^{-\theta}$ , in which case  $k = B^{1/\theta}$  for some  $B \rightarrow \infty$ . Then  $N = k^{2A+1} \asymp B^{1/\theta}(\log^2 P)$ . Thus, codes  $\mathcal{R}_{ext}$  can improve the trade-off  $k \sim P^{-1/\theta}$  of the inner codes  $\mathcal{R}(1, \mu)$  only for the declining error rates  $P = o(1)$ . We further note that this is the case for all other known concatenated constructions. In particular, consider a classic concatenation that uses the inner codes  $\mathcal{R}(1, \mu)$  and the outer RS codes of the same length  $2^\mu$  and code rate  $R \rightarrow 1$ . It can be verified that this construction requires the overall length  $N_1 \asymp \max\{\theta^2 \log^2 P, B^{2/\theta}\}$  given the same inner length  $k = B^{1/\theta}$ . One possible advantage of codes  $\mathcal{R}_{ext}$  over classic concatenation is the extra parameter  $s$  that allows the outer code length  $n$  arbitrarily exceed the inner length  $k$  in (38). In particular, we have inequality  $N \lesssim N_1$  for both cases  $B^{1/\theta} < \log^2 P$  and  $B^{1/\theta} > \log^2 P$ . Thus, construction of Theorem 8 allows us to shorten the length  $N_1$  of the classical concatenated construction. More generally, it is an important problem to find low-complexity codes of code rate  $R \rightarrow 0$  that can achieve the vanishing error rates at the shorter lengths of order  $N \sim 2^{c/\theta}$  for some  $c \in (0, 1)$ .

## VII. CONCLUDING REMARKS

In this paper, we address explicit constructions of polar codes that are nearly optimal for the extreme cases of a BSC( $p$ ) with  $p \rightarrow 0$  and  $p \rightarrow 1/2$ . In case of  $p \rightarrow 0$ , we obtain weakly optimal codes of rate  $R \rightarrow 1$ , whose redundancy order  $\log \rho$  declines at the optimal rate (2). For the low-rate codes, we obtain the optimal decline of code rate  $R \rightarrow 0$ . These simple constructions are completely defined by a single  $s$ -step boundary path  $\xi(\mathcal{L})$  that only depends on transition error probability  $p$ . In turn, this boundary defines all other paths  $\eta$ , which form other sequences of upgrading-degrading channels included in code construction. An important point is that the boundary  $\mathcal{L}$  consists of the consecutive chains of upgrading or degrading channels, with a growing length of each segment. For this reason, these single-boundary codes can be considered as direct products of  $s$  Reed-Muller codes. One way to amplify this design is to consider polar codes that include multiple overlapping boundaries  $\mathcal{L}_1, \dots, \mathcal{L}_k$  and admit all descendant paths  $\eta$  that satisfy at least one boundary restriction. Another interesting problem is to extend this design to other code rates and consider the explicit constructions that admit the finite-length stretches of the upgrading-degrading channels.

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**Appendix. Proof of Lemma 3.** To prove Lemma 3, we will assume that any channel  $W_\xi$  satisfies the "symmetry" condition ([15], p. 628). This condition (expressed in terms of log likelihoods in [15]) implies that the likelihoods  $h$  of transmitted symbols have the probability density function (pdf)  $p(x) \equiv p_h(x)$  such that

$$p(x)/p(x^{-1}) = x, \quad \forall x \in (0, \infty). \quad (41)$$

Condition (41) can be used for many conventional channels; in particular, for a BSC( $p$ ) or an AWGN channel. It is also proven in [15] that the "symmetry" condition is left intact by transformations (9). Namely, both rv  $h_j^{(0)}$  and  $h_j^{(1)}$  in (9) satisfy condition (41) if so do rv  $h_{0,j}$  and  $h_{1,j}$ .

Next, we consider an output  $h(a)$  of the prefix path  $a$ . Let  $h_1, h_2, h_3$ , and  $h_4$  denote 4 independent ID rv, which represent 4 different outputs  $h(a)$  of the prefix  $a$ . We need to calculate the outputs  $h_{01} \equiv h(a01)$  and  $h_{10} \equiv h(a10)$  and prove that  $Eh_{01}^{-\lambda} \leq Eh_{10}^{-\lambda}$  if  $\lambda \in [0, 1]$ . An equivalent formulation is to prove inequality  $Ef_{01}^\lambda \leq Ef_{10}^\lambda$  given *inverse* likelihoods  $f_i = h_i^{-1}$ ,  $f_{01} = h_{01}^{-1}$  and  $f_{10} = h_{10}^{-1}$ . Correspondingly, we consider the 4-dimensional space  $\mathbb{R}_+^4$  formed by vectors  $F = (f_1, f_2, f_3, f_4)$  with positive coordinates. For extended subpaths  $a01$  and  $a10$ , recalculations (9) give the rv outputs

$$f_{01} = \frac{f_1 f_2 + f_3 f_4}{1 + f_1 f_2 f_3 f_4},$$

$$f_{10} = \frac{(f_1 + f_2)(f_3 + f_4)}{(1 + f_1 f_2)(1 + f_3 f_4)}.$$

Below we also consider another rv

$$u_{01} = \frac{(f_1 + f_2)(f_3 + f_4)}{2(1 + f_1 f_2 f_3 f_4)}$$

and prove two inequalities

$$\mathbb{E}f_{01}^\lambda \leq \mathbb{E}u_{01}^\lambda \leq \mathbb{E}f_{10}^\lambda, \quad \lambda \in [0, 1]. \quad (42)$$

To prove the left inequality, note that  $u_{01} = (f'_{01} + f''_{01})/2$ , where

$$f'_{01} = \frac{f_1 f_3 + f_2 f_4}{1 + f_1 f_2 f_3 f_4}, \quad f''_{01} = \frac{f_1 f_4 + f_2 f_3}{1 + f_1 f_2 f_3 f_4}.$$

The variables  $f'_{01}$  and  $f''_{01}$  are obtained from  $f_{01}$  by replacements  $f_2 \Leftrightarrow f_3$  and  $f_2 \Leftrightarrow f_4$  respectively. Then independent and ID rv  $f_i$  give equalities

$$\mathbb{E}f_{01}^\lambda = \mathbb{E}(f'_{01})^\lambda = \mathbb{E}(f''_{01})^\lambda$$

Since  $x^\lambda$  is a concave function of any  $x > 0$  for  $\lambda \in [0, 1]$ ,

$$\frac{(f'_{01})^\lambda}{2} + \frac{(f''_{01})^\lambda}{2} \leq \left( \frac{f'_{01} + f''_{01}}{2} \right)^\lambda = u_{01}^\lambda \quad (43)$$

and  $\mathbb{E}f_{01}^\lambda \leq \mathbb{E}u_{01}^\lambda$ .

To compare the expectations  $\mathbb{E}u_{01}^\lambda$  and  $\mathbb{E}f_{10}^\lambda$ , we combine each vector  $F \equiv F_0 \in \mathbb{R}_+^4$  with three other vectors (which may also coincide with  $F$ ):

$$\begin{aligned} F_1 &= (f_1^{-1}, f_2^{-1}, f_3, f_4), & F_2 &= (f_1^{-1}, f_2^{-1}, f_3^{-1}, f_4^{-1}), \\ F_3 &= (f_1, f_2, f_3^{-1}, f_4^{-1}) \end{aligned}$$

We also consider the *orbit*  $\mathbf{T} = \{F_0, F_1, F_2, F_3\}$  of vector  $F_0 \in \mathbb{R}_+^4$ . Clearly, the whole space  $\mathbb{R}_+^4$  is now partitioned into non-intersecting orbits  $\mathbf{T}$ . Below we use notation

$$\alpha = f_1 f_2, \quad \beta = f_3 f_4, \quad A = (f_1 + f_2)(f_3 + f_4).$$

It can be readily verified that the rv  $f_{10}(\mathbf{T})$  does not change on the orbit  $\mathbf{T}$ :

$$f_{10}(F_i) = \frac{A}{(1 + \alpha)(1 + \beta)}, \quad i = 0, \dots, 3, \quad (44)$$

while  $u_{01}(\mathbf{T})$  takes two values

$$\begin{aligned} u_{01}(F_0) &= u_{01}(F_2) = \frac{A}{2(1 + \alpha\beta)} \\ u_{01}(F_1) &= u_{01}(F_3) = \frac{A}{2(\alpha + \beta)} \end{aligned}$$

Let  $p = p(F_0)$  denote the pdf of the 4-dimensional rv  $F_0 \in \mathbb{R}_+^4$ , which consists of inverse likelihoods. According to (41), the pdfs of other orbit points are

$$p(F_2) = \alpha\beta p, \quad p(F_1) = \alpha p, \quad p(F_3) = \beta p \quad (45)$$

Then simple recalculations using equalities (44) and (45) give

$$\begin{aligned} \mathbb{E}f_{10}^\lambda(\mathbf{T}) &= pA^\lambda [(1 + \alpha)(1 + \beta)]^{1-\lambda}, \\ \mathbb{E}u_{01}^\lambda(\mathbf{T}) &= pA^\lambda 2^{-\lambda} \left[ (1 + \alpha\beta)^{1-\lambda} + (\alpha + \beta)^{1-\lambda} \right] \end{aligned}$$

Since  $x^{1-\lambda}$  is a concave function, we have inequality

$$\frac{(1 + \alpha\beta)^{1-\lambda}}{2} + \frac{(\alpha + \beta)^{1-\lambda}}{2} \leq \left[ \frac{(1 + \alpha)(1 + \beta)}{2} \right]^{1-\lambda} \quad (46)$$

which proves the right inequality in (42). The second case with  $\lambda \in [1, \infty)$  is studied similarly. Now both  $x^\lambda$  and  $x^{1-\lambda}$  are a convex functions of  $x > 0$ . Then inequalities (43) and (46) change their sign and we have inequality (17).  $\square$