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# A Tunable Measure for Information Leakage

Jiachun Liao, Oliver Kosut, Lalitha Sankar School of Electrical, Computer and Energy Engineering, Arizona State University Email: {jiachun.liao,lalithasankar,okosut}@asu.edu

Abstract—A tunable measure for information leakage called maximal  $\alpha$ -leakage is introduced. This measure quantifies the maximal gain of an adversary in refining a tilted version of its prior belief of any (potentially random) function of a dataset conditioning on a disclosed dataset. The choice of  $\alpha$  determines the specific adversarial action ranging from refining a belief for  $\alpha = 1$  to guessing the best posterior for  $\alpha = \infty$ , and for these extremal values this measure simplifies to mutual information (MI) and maximal leakage (MaxL), respectively. For all other  $\alpha$  this measure is shown to be the Arimoto channel capacity. Several properties of this measure are proven including: (i) quasiconvexity in the mapping between the original and disclosed datasets; (ii) data processing inequalities; and (iii) a composition property.

### I. INTRODUCTION

Information leakage metrics seek to quantify an adversary's ability of inferring information about one quantity from another. Mutual information (MI) is a classic measure for quantifying information and often used to measure information secrecy [1] or leakage in data publishing settings [2], [3]. More recently, Issa *et al.* introduced a measure, called *maximal leakage* (MaxL), for a guessing adversary that quantifies the maximal multiplicative gain of an adversary, with access to a disclosed dataset, to guess *any* (*possible random*) *function* of the original dataset [4].

Information leakage measures can be viewed through the lens of adversarial inference capabilities, and therefore, quantified via a loss function that the adversary seeks to minimize. The choice of a loss function provides a concrete measure of the gain in adversarial inference capability. For example, the definition of MaxL can be interpreted in terms of an adversary seeking to minimize the 0-1 loss function, which induces the adversary towards a hard decision, i.e., a maximum likelihood estimator. On the other hand, when MI is used as a leakage measure, the underlying loss function is the *logarithmic loss* (log-loss) function [5]-[7], which models a (soft decision) belief-refining adversary. These two models capture two extremal actions of adversaries. Can these measures be viewed through the same framework? In this paper, we introduce a tunable measure, called *maximal*  $\alpha$ -leakage, for information leakages, which encompasses MI (for  $\alpha = 1$ ) and MaxL (for  $\alpha = \infty$ ) and allows continuous interpolation between the two extremes. The parameter  $\alpha$  can be viewed as a tunable

Flavio P. Calmon School of Engineering and Applied Sciences Harvard University Email: fcalmon@g.harvard.edu

parameter that determines how much weight the adversary gives to its posterior belief.

In this paper, we define two tunable measures for information leakages in Section III:  $\alpha$ -leakage (Definition 4) and maximal  $\alpha$ -leakage (Definition 5). In Section III, we prove that the  $\alpha$ -leakage can be expressed as Arimoto mutual information (A-MI) (Theorem 1), and the maximal  $\alpha$ -leakage is equivalent to the supremum of A-MI and Sibson mutual information (S-MI) (Theorem 2) over all distributions of the original dataset. In Section IV, we prove several important properties of the maximal  $\alpha$ -leakage.

### **II. PRELIMINARIES**

We begin by reviewing Rényi entropy and divergence [8].

**Definition 1.** Given a discrete distribution  $P_X$  over a finite alphabet  $\mathcal{X}$ , the Rényi entropy of order  $\alpha \in (0, 1) \cup (1, \infty)$  is defined as

$$H_{\alpha}(P_X) = \frac{\alpha}{1-\alpha} \log \|P_X\|_{\alpha}.$$
 (1)

Let  $Q_X$  be a discrete distribution over  $\mathcal{X}$ . The Rényi divergence (between  $P_X$  and  $Q_X$ ) of order  $\alpha \in (0, 1) \cup (1, \infty)$  is defined as

$$D_{\alpha}(P_X \| Q_X) = \frac{1}{\alpha - 1} \log \left( \sum_x \frac{P_X(x)^{\alpha}}{Q_X(x)^{\alpha - 1}} \right).$$
(2)

Both of the two quantities are defined by their continuous extension for  $\alpha = 1$  or  $\infty$ .

The  $\alpha$ -leakage and max  $\alpha$ -leakage metrics can be expressed in terms of Sibson mutual information (S-MI) [9] and Arimoto mutual information (A-MI) [10]. These quantities generalize the usual notion of MI. We review these definitions next.

**Definition 2.** Let discrete random variables  $(X, Y) \sim P_{XY}$ with  $P_X$  and  $P_Y$  as the marginal distributions, respectively, and  $Q_Y$  be an arbitrary marginal distribution of Y. The Sibson mutual information (S-MI) of order  $\alpha \in (0,1) \cup (1,\infty)$  is defined as

$$I_{\alpha}^{S}(X;Y) \triangleq \inf_{Q_{Y}} D_{\alpha}(P_{XY} || P_{X} \times Q_{Y}),$$
(3)

$$= \frac{\alpha}{\alpha - 1} \log \sum_{y} \left( \sum_{x} P_X(x) P_{Y|X}(y|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(4)

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The Arimoto mutual information (A-MI) of order  $\alpha \in (0,1) \cup (1,\infty)$  is defined as

$$I^{A}_{\alpha}(X;Y) \triangleq H_{\alpha}(X) - H_{\alpha}(X|Y)$$
(5)

$$= \frac{\alpha}{\alpha - 1} \log \sum_{y} \left( \frac{\sum\limits_{x} P_X(x)^{\alpha} P_{Y|X}(y|x)^{\alpha}}{\sum\limits_{x} P_X(x)^{\alpha}} \right)^{\frac{1}{\alpha}}, \quad (6)$$

where  $H_{\alpha}(X|Y)$  is Arimoto conditional entropy of X given Y defined as

$$H_{\alpha}(X|Y) = \frac{\alpha}{1-\alpha} \log \sum_{y} \left( \sum_{x} P_X(x)^{\alpha} P_{Y|X}(y|x)^{\alpha} \right)^{\frac{1}{\alpha}}.$$
 (7)

All of these quantities are defined by their continuous extension for  $\alpha = 1$  or  $\infty$ .

# III. INFORMATION LEAKAGE MEASURES

In this section, we formally define the tunable leakage measures:  $\alpha$ -leakage and maximal  $\alpha$ -leakage.

Let X, Y and U be three discrete random variables with finite supports  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{U}$ , respectively. Let  $\hat{X}$  be an estimator of X and  $P_{\hat{X}|Y}$  indicate a strategy for estimating X given Y. We denote the probability of correctly estimating X = x given Y = y as

$$P_c(P_{\hat{X}|Y}, x, y) = P_{\hat{X}|Y}(x|y) = \mathbb{P}(\hat{X} = x|x, y).$$
(8)

Let X and Y represent the original data and disclosed data, respectively, and let U represent an arbitrary (potentially random) function of X that the adversary (a curious or malicious user of the disclosed data Y) is interested in learning. In [11], Issa *et al.* introduced MaxL to qualify the maximal gain in an adversary's ability of guessing U by knowing Y. We review the definition below.

**Definition 3** ([11, Def. 1]). *Given a joint distribution*  $P_{XY}$ , *the maximal leakage from* X *to* Y *is* 

$$\mathcal{L}_{MaxL}(X \to Y) \triangleq \sup_{U-X-Y} \log \frac{\max_{u} \mathbb{E}\left[\mathbb{P}(\hat{U} = u|Y)\right]}{\max_{u} \mathbb{P}(\tilde{U} = u)}.$$
 (9)

where both estimators  $\hat{U}$  and  $\tilde{U}$  take values from the same arbitrary finite support as U.

**Remark 1.** Note that from (8), the numerator of the logarithmic term in (9) can be explicitly written as

$$\max_{u} \mathbb{E}\left[\mathbb{P}(\hat{U}=u|Y)\right] = \max_{u} \sum_{y} P_{Y}(y) P_{\hat{U}|Y}(u|y).$$
(10)

In Definition 3, U represents any (possibly random) function of X. The numerator represents the maximal probability of correctly guessing U based on Y, while the denominator represents the maximal probability of correctly guessing U without knowing Y. Thus, MaxL quantifies the maximal gain (in bits) in guessing any possible function of X when an adversary has access to Y. We now present  $\alpha$ -leakage and maximal  $\alpha$ -leakage (under the assumptions of discrete random variables and finite supports). The  $\alpha$ -leakage measures *various* aspects of the leakage (ranging from the probability of correctly guessing to the posteriori distribution) about data X from the disclosed Y.

**Definition 4** ( $\alpha$ -Leakage). Given a joint distribution  $P_{XY}$  and an estimator  $\hat{X}$  with the same support as X, the  $\alpha$ -leakage from X to Y is defined as

$$\mathcal{L}_{\alpha}(X \to Y) \triangleq \frac{\alpha}{\alpha - 1} \log \frac{\max_{\substack{P_{\hat{X}|Y}}} \mathbb{E}\left[\mathbb{P}(\hat{X} = X|X, Y)^{\frac{\alpha - 1}{\alpha}}\right]}{\max_{\substack{P_{\hat{X}}}} \mathbb{E}\left[\mathbb{P}(\hat{X} = X|X)^{\frac{\alpha - 1}{\alpha}}\right]}$$
(11)

for  $\alpha \in (1,\infty)$  and by the continuous extension of (11) for  $\alpha = 1$  and  $\infty$ .

From (8), the numerator of the logarithmic term in (11) can be explicitly written as

$$\max_{P_{\hat{X}|Y}} \mathbb{E}\left[\mathbb{P}(\hat{X} = X|X, Y)^{\frac{\alpha-1}{\alpha}}\right]$$
$$= \max_{P_{\hat{X}|Y}} \sum_{xy} P_{XY}(xy) P_{\hat{X}|Y}(x|y)^{\frac{\alpha-1}{\alpha}}.$$
(12)

Analogous to the analysis for MaxL in Remark 1,  $\alpha$ -leakage quantifies the multiplicative increase in the expected reward for correctly inferring X when an adversary has access to Y.

Whereas  $\alpha$ -leakage captures how much an adversary can learn about X from Y, we also wish to quantify the information leaked about *any function* of X through Y. To this end, we define maximal  $\alpha$ -leakage below.

**Definition 5** (Maximal  $\alpha$ -Leakage). Given a joint distribution  $P_{XY}$  on finite alphabets  $\mathcal{X} \times \mathcal{Y}$ , the maximal  $\alpha$ -leakage from X to Y is defined as

$$\mathcal{L}_{\alpha}^{max}(X \to Y) \triangleq \sup_{U = X - Y} \mathcal{L}_{\alpha}(U \to Y)$$
(13)

where  $\alpha \in [1, \infty]$ , U represents any function of X and takes values from an arbitrary finite alphabet.

**Remark 2.** Note that the optimal  $P_{\hat{X}}^*$  of the maximization in the denominator of the logarithmic term in (11) minimizes the expectation of the following loss function

$$\ell(x, P_{\hat{X}}) = \frac{\alpha}{\alpha - 1} \left( 1 - P_{\hat{X}}(x)^{1 - \frac{1}{\alpha}} \right), \tag{14}$$

for each  $\alpha \in (1, \infty)$ . The limit of the loss function in (14) leads to the log-loss (for  $\alpha = 1$ ) and 0-1 loss (for  $\alpha = \infty$ ) functions, respectively. In addition, for  $\alpha = 1$  and  $\infty$ , the maximal  $\alpha$ -leakage simplifies to MI and MaxL, respectively. These comments are formalized in the following theorems.

The following theorem simplifies the expression of the  $\alpha$ -leakage in (11) by solving the two maximizations in the logarithmic term.

**Theorem 1.** For  $\alpha \in [1, \infty]$ ,  $\alpha$ -leakage defined in (11) simplifies to

$$\mathcal{L}_{\alpha}(X \to Y) = I^{A}_{\alpha}(X;Y) \quad \alpha \in [1,\infty].$$
(15)

The proof hinges on solving the optimal estimations  $P^*_{\hat{X}|Y}$ and  $P^*_{\hat{X}}$  in (11) for knowing Y or not, respectively, as

$$P_{\hat{X}|Y}^{*}(x|y) = \frac{P_{X|Y}(x|y)^{\alpha}}{\sum_{x} P_{X|Y}(x|y)^{\alpha}} \qquad (x,y) \in \mathcal{X} \times \mathcal{Y} \quad (16a)$$
$$P_{Y}(x)^{\alpha}$$

$$P_{\hat{X}}^*(x) = \frac{P_X(x)^{\alpha}}{\sum_x P_X(x)^{\alpha}} \qquad \qquad x \in \mathcal{X}, \quad (16b)$$

and therefore, the logarithm of the ratio in (11) simplifies to A-MI. A detailed proof is in Appendix A. Making use of the conclusion in Theorem 1, the following theorem gives equivalent expressions for the maximal  $\alpha$ -leakage.

**Theorem 2.** For  $\alpha \in [1, \infty]$ , the maximal  $\alpha$ -leakage defined in (13) simplifies to  $\mathcal{L}^{max}_{\alpha}(X \to Y)$ 

$$= \begin{cases} \sup_{P_{\tilde{X}}} I^{S}_{\alpha}(\tilde{X};Y) = \sup_{P_{\tilde{X}}} I^{A}_{\alpha}(\tilde{X};Y) & \alpha \in (1,\infty] \quad (17a)\\ I(X;Y) & \alpha = 1 \quad (17b) \end{cases}$$

where  $P_{\tilde{X}}$  has the same support as  $P_X$ .

Note that the maximal  $\alpha$ -leakage is essentially the Arimoto channel capacity (with a support-set constrained input distribution) for  $\alpha \geq 1$  [10]. This theorem is proved by first applying Theorem 1 to write the maximal  $\alpha$ -leakage as

$$\mathcal{L}^{\max}_{\alpha}(X \to Y) = \sup_{U = X - Y} I^{\mathsf{A}}_{\alpha}(U;Y) \quad \alpha \in [1,\infty].$$
(18)

Subsequently, using the facts that A-MI and S-MI have the same supremum [12, Thm. 5] and that S-MI satisfies data processing inequality [12, Thm. 3], we upper bound the supremum of (18) by  $\sup_{P_{\tilde{x}}} I^{S}_{\alpha}(X;Y)$ , and then, show that the upper bound can be achieved by a specific U with H(X|U) = 0. A detailed proof can be found in Appendix **B**.

### IV. PROPERTIES OF MAXIMAL $\alpha$ -Leakage

In this section, we will prove that maximal  $\alpha$ -leakage has several properties that one would expect any reasonable leakage measure to have, including: (i) quasi-convexity in the conditional distribution  $P_{Y|X}$ ; (ii) data processing inequalities; and (iii) a composition property.

These properties are proved in the following theorem, which makes use of the equivalent form of maximal alpha-leakage found in Theorem 2, as well as known properties of S-MI from [9], [12], [13].

# **Theorem 3.** For $\alpha \in [1, \infty]$ , maximal $\alpha$ -leakage

- 1. is quasi-convex in  $P_{Y|X}$ ;
- 2. is monotonically non-decreasing in  $\alpha$ ;
- 3. satisfies data processing inequalities: let random variables X, Y, Z form a Markov chain, i.e., X - Y - Z, then

$$\mathcal{L}^{max}_{\alpha}(X \to Z) \le \mathcal{L}^{max}_{\alpha}(X \to Y) \tag{19a}$$

$$\mathcal{L}^{max}_{\alpha}(X \to Z) \le \mathcal{L}^{max}_{\alpha}(Y \to Z).$$
(19b)

$$\mathcal{L}^{max}_{\alpha}(X \to Y) \ge 0 \tag{20}$$

with equality if and only if X is independent of Y, and

$$\mathcal{L}_{\alpha}^{max}(X \to Y) \le \begin{cases} \log |\mathcal{X}| & \alpha > 1\\ H(P_X) & \alpha = 1 \end{cases}$$
(21)

with equality if X is a deterministic function of Y.

- 5.  $\mathcal{L}_{\alpha}^{\max}(X \to Y) \leq I_{\infty}^{s}(P_X, P_{Y|X})$  with equality if  $P_{Y|X}$ has either 0 or the maximal leakage in Part 4; 6.  $\mathcal{L}_{\alpha}^{\max}(X \to Y) \geq I_{\alpha}^{s}(P_X^{(u)}, P_{Y|X})$ , where  $P_X^{(u)}$  indicates
- the uniform distribution of X, i.e.,

$$\mathcal{L}_{\alpha}^{max}(X \to Y) \ge \frac{\alpha}{\alpha - 1} \log \frac{\sum\limits_{y \in \mathcal{Y}} \left(\sum\limits_{x \in \mathcal{X}} P_{Y|X}(y|x)^{\alpha}\right)^{\frac{1}{\alpha}}}{|\mathcal{X}|^{\frac{1}{\alpha}}}.$$
 (22)

The equality holds if either  $P_{Y|X}$  is symmetric<sup>1</sup> or  $P_{Y|X}$ has 0 leakage.

A detailed proof is in Appendix C.

**Remark 3.** Note that both MI and MaxL are convex in  $P_{Y|X}$ so that  $\mathcal{L}_1^{max}(X \to Y)$  and  $\mathcal{L}_{\infty}^{max}(X \to Y)$  are convex in  $P_{Y|X}$ .

Consider two disclosed versions  $Y_1$  and  $Y_2$  of X. The following theorem upper bounds the maximal  $\alpha\text{-leakage}$  to an adversary who has access to both  $Y_1$  and  $Y_2$  simultaneously.

Theorem 4 (Composition Theorem). Given a Markov chain  $Y_1 - X - Y_2$ , we have  $(\alpha \in [1, \infty])$ 

$$\mathcal{L}^{\max}_{\alpha}(X \to Y_1, Y_2) \le \sum_{i \in \{1,2\}} \mathcal{L}^{\max}_{\alpha}(X \to Y_i).$$
(23)

This composition theorem allows composing multiple releases under a total leakage constraint. A detailed proof is in Appendix D.

### V. CONCLUDING REMARKS

Via  $\alpha$ - and maximal  $\alpha$ -leakage, we have introduced novel tunable measures for information leakage. These measures can find direct applications in privacy and secrecy problems. The choice of restricting either specific variables or all possible functions of a dataset determines the choice of  $\alpha$ - and maximal  $\alpha$ -leakage measures, respectively. Future work includes characterizing privacy-utility tradeoffs for these measures and evaluating existing privacy mappings against these metrics.

### APPENDIX A **PROOF OF THEOREM 1**

The expression (11) can be explicitly written as

$$\mathcal{L}_{\alpha}(X \to Y) = \lim_{\alpha' \to \alpha} \frac{\alpha'}{\alpha' - 1} \\ \log \left( \frac{\max_{P_{\hat{X}|Y}} \sum_{xy} P_{XY}(xy) \left( P_{\hat{X}|Y}(x|y) \right)^{\frac{\alpha' - 1}{\alpha'}}}{\max_{P_{\hat{X}}} \sum_{x} P_{X}(x) P_{\hat{X}}(x)^{\frac{\alpha' - 1}{\alpha'}}} \right).$$
(24)

To simplify the expression in (24), we need to solve the two maximizations in the logarithm. First, we concentrate on the

<sup>1</sup>All rows of  $P_{Y|X}$  are permutations of other rows, and so are columns.

maximization in the denominator of the logarithm in (24) and the one in the numerator can be solved following the same analysis. The maximization in the denominator can be equivalently written as

$$\max_{P_{\hat{X}}} \quad \sum_{x \in \mathcal{X}} P_X(x) P_{\hat{X}}(x)^{1 - \frac{1}{\alpha'}}$$
(25a)

s.t. 
$$\sum_{x \in \mathcal{X}} P_{\hat{X}}(x) = 1$$
(25b)

$$P_{\hat{X}}(x) \ge 0$$
 for all  $x \in \mathcal{X}$  (25c)

For  $\alpha' \in [1, \infty)$ , the problem in (25) is a convex program. Therefore, by using Karush-Kuhn-Tucker (KKT) conditions, we obtain the optimal value of (25) as

$$\max_{P_{\hat{X}}} \sum_{x \in \mathcal{X}} P_X(x) P_{\hat{X}}(x)^{\frac{\alpha'-1}{\alpha'}} = \left(\sum_{x \in \mathcal{X}} P_X(x)^{\alpha'}\right)^{\frac{1}{\alpha'}}, \quad (26)$$

with the optimal solution  $P^*_{\hat{X}}$  as

$$P_{\hat{X}}^{*}(x) = \frac{P_X(x)^{\alpha'}}{\sum\limits_{x \in \mathcal{X}} P_X(x)^{\alpha'}} \quad \text{for all } x \in \mathcal{X}$$
(27)

Similarly, we attain the optimal solution  $P^*_{\hat{X}|Y}$  of the maximization in the numerator of the logarithm in (24) as

$$P_{\hat{X}|Y}^{*}(x|y) = \frac{P_{X|Y}(x|y)^{\alpha'}}{\sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\alpha'}}$$
(28)

for all  $x \in \mathcal{X}, y \in \mathcal{Y}$ , and therefore, we have

$$\max_{P_{\hat{X}|Y}} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(xy) P_{\hat{X}|Y}(x|y)^{\frac{\alpha'-1}{\alpha'}}$$
$$= \sum_{y \in \mathcal{Y}} P_Y(y) \left( \sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\alpha'} \right)^{\frac{1}{\alpha'}}.$$
(29)

Thus, for  $\alpha \in [1, \infty)$ , we have

$$\mathcal{L}_{\alpha}(X \to Y) = \lim_{\alpha' \to \alpha} \frac{\alpha'}{\alpha' - 1} \log \left( \frac{\sum_{y} P_{Y}(y) \left(\sum_{x} P_{X|Y}(x|y)^{\alpha'}\right)^{\frac{1}{\alpha'}}}{\left(\sum_{x} P_{X}(x)^{\alpha'}\right)^{\frac{1}{\alpha'}}} \right), \quad (30)$$

i.e., A-MI of order  $\alpha \in [1, \infty)$  in (6).

Note that if  $\alpha = \infty$ , the optimal solution in (27) is  $\frac{0}{0}$ . We go back to the expression in (11) and observe that if  $\alpha = \infty$ , the expression  $\mathcal{L}_{\infty}(X \to Y)$  becomes

$$\mathcal{L}_{\infty}(X \to Y) = \log \left( \frac{\max \sum_{P_{\hat{X}|Y}} P_{XY}(xy) P_{\hat{X}|Y}(x|y)}{\max \sum_{P_{\hat{X}}} \sum_{x} P_{X}(x) P_{\hat{X}}(x)} \right).$$
(31)

Since the largest convex combinations is the maximal involved value, the optimal values of the two maximizations in (31) are

$$\max_{P_{\hat{X}|Y}} \sum_{xy} P_{XY}(xy) P_{\hat{X}|Y}(x|y)$$
$$= \sum_{y} P_{Y}(y) \max_{x} P_{X|Y}(x|y)$$
(32a)

$$\max_{P_{\hat{X}}} \sum_{x} P_X(x) P_{\hat{X}}(x) = \max_{x} P_X(x).$$
(32b)

Therefore, for  $\alpha = \infty$ , we have

$$\mathcal{L}_{\infty}(X \to Y) = \log \left( \frac{\sum\limits_{y \in \mathcal{Y}} P_Y(y) \max\limits_{x} P_{X|Y}(x|y)}{\max\limits_{x} P_X(x)} \right), \quad (33)$$

which is exactly the A-MI of order  $\infty$ . Therefore,  $\alpha$ -leakage can be equivalently expressed as  $I^{A}_{\alpha}(X;Y)$  for  $\alpha \in [1,\infty]$ .

# APPENDIX B Proof of Theorem 2

From Theorem 1, we have for  $\alpha \in [1, \infty]$ ,

$$\mathcal{L}^{\max}_{\alpha}(X \to Y) = \sup_{U = X - Y} I^{\mathsf{A}}_{\alpha}(U; Y).$$
(34)

If  $\alpha = 1$ , we have

$$\mathcal{L}_1^{\max}(X \to Y) = \sup_{U = X - Y} I(U;Y) \le I(X;Y)$$
(35)

where the inequality is from data processing inequalities of MI [14, Thm 2.8.1].

If  $\alpha = \infty$ , we have

$$\mathcal{L}_{\infty}^{\max}(X \to Y) = \sup_{U-X-Y} \log \frac{\sum\limits_{y}^{\sum} P_Y(y) \max\limits_{u} P_{U|Y}(u|y)}{\max\limits_{u} P_U(u)}, \quad (36)$$

which is exactly the expression of MaxL, and therefore, we have [11, Thm. 1]

$$\mathcal{L}_{\infty}^{\max}(X \to Y) = \log \sum_{y} \max_{x} P_{Y|X}(y|x).$$
(37)

For  $\alpha \in (1,\infty)$ , we provide an upper bound for  $\mathcal{L}^{\max}_{\alpha}(X \to Y)$ , and then, give an achievable scheme as follows.

**Upper Bound**: We have an upper bound of  $\mathcal{L}^{\max}_{\alpha}(X \to Y)$  as

$$\mathcal{L}_{\alpha}^{\max}(X \to Y) = \sup_{U-X-Y} I_{\alpha}^{A}(U;Y)$$
(38a)

$$\leq \sup_{P_{\tilde{X}|\tilde{U}}:P_{\tilde{X}|\tilde{U}}(\cdot|u) \ll P_{X}} \sup_{P_{\tilde{U}}} I^{A}_{\alpha}(\tilde{U};Y)$$
(38b)

$$= \sup_{P_{\tilde{X}|\tilde{U}}: P_{\tilde{X}|\tilde{U}}(\cdot|u) \ll P_{X}} \sup_{P_{\tilde{U}}} I_{\alpha}^{\mathbf{S}}(\tilde{U};Y)$$
(38c)

$$= \sup_{P_{\tilde{X}} \ll P_{X}} I^{\mathbf{S}}_{\alpha}(\tilde{X};Y)$$
(38d)

$$= \sup_{P_{\tilde{X}} \ll P_{X}} I^{\mathcal{A}}_{\alpha}(\tilde{X};Y)$$
(38e)

) where  $P_{\tilde{X}} \ll P_X$  means the alphabet of  $P_{\tilde{X}}$  is a subset of that of  $P_X$ . The inequality in (38b) holds because the supremum

of A-MI over all  $P_{\tilde{U},\tilde{X}}$  on  $\mathcal{U} \times \mathcal{X}$  is no less than that (in (38a)) over these  $P_{U,X}$  constrained by the  $P_X$ . The equations in (38c) and (38e) result from that A-MI and S-MI of order  $\alpha > 0$  have the same supremum [12, Thm. 5]; and (38d) obeys the data processing inequalities [12, Thm. 3].

**Lower bound**: We lower bound (34) by consider a random variable U such that U - X - Y is a Markov chain and H(X|U) = 0. Specifically, let the alphabet  $\mathcal{U}$  consist of  $\mathcal{U}_x$ , a collection of U mapped to a  $x \in \mathcal{X}$ , i.e.,  $\mathcal{U} = \bigcup_{x \in \mathcal{X}} \mathcal{U}_x$  with  $U = u \in \mathcal{U}_x$  if and only if X = x. Therefore, for the specific variable U, we have

$$P_{Y|U}(y|u) = \begin{cases} P_{Y|X}(y|x) & \text{ for all } u \in \mathcal{U}_x \\ 0 & \text{ otherwise.} \end{cases}$$
(39)

Construct a probability distribution  $P_{\tilde{X}}$  over  $\mathcal{X}$  from  $P_U$  as

$$P_{\tilde{X}}(x) = \frac{\sum_{u \in \mathcal{U}_x} P_U^{\alpha}(u)}{\sum_{x \in \mathcal{X}} \sum_{u \in \mathcal{U}_x} P_U^{\alpha}(u)} \quad \text{for all } x \in \mathcal{X}.$$
(40)

Thus,

$$\begin{split} I^{A}_{\alpha}(U;Y) \\ = & \frac{\alpha}{\alpha - 1} \log \frac{\sum\limits_{y \in \mathcal{Y}} \left( \sum\limits_{x \in \mathcal{X}} \sum\limits_{u \in \mathcal{U}_{x}} P_{Y|U}(y|u)^{\alpha} P_{U}(u)^{\alpha} \right)^{\frac{1}{\alpha}}}{\left( \sum\limits_{x \in \mathcal{X}} \sum\limits_{u \in \mathcal{U}_{x}} P_{U}(u)^{\alpha} \right)^{\frac{1}{\alpha}}} \\ = & \frac{\alpha}{\alpha - 1} \log \frac{\sum\limits_{y \in \mathcal{Y}} \left( \sum\limits_{x \in \mathcal{X}} P_{Y|X}(y|x)^{\alpha} \sum\limits_{u \in \mathcal{U}_{x}} P_{U}(u)^{\alpha} \right)^{\frac{1}{\alpha}}}{\left( \sum\limits_{x \in \mathcal{X}} \sum\limits_{u \in \mathcal{U}_{x}} P_{U}(u)^{\alpha} \right)^{\frac{1}{\alpha}}} \\ = & \frac{\alpha}{\alpha - 1} \log \left( \sum\limits_{y \in \mathcal{Y}} \left( \sum\limits_{x \in \mathcal{X}} P_{Y|X}(y|x)^{\alpha} P_{\tilde{X}}(x)^{\alpha} \right)^{\frac{1}{\alpha}} \right) \\ = & I^{S}_{\alpha}(\tilde{X};Y) \end{split}$$

Therefore,

$$\mathcal{L}_{\alpha}^{\max}(X \to Y) = \sup_{U-X-Y} I_{\alpha}^{A}(U;Y)$$

$$\geq \sup_{U:U-X-Y,H(X|U)=0} I_{\alpha}^{A}(U;Y) \quad (41a)$$

$$= \sup_{P_{\tilde{X}} \ll P_{X}} I_{\alpha}^{S}(\tilde{X};Y), \quad (41b)$$

where (41b) is because for any  $P_{\tilde{X}} \ll P_X$ , it can be obtained through (40) by appropriately choosing  $P_U$ . Therefore, combining (38) and (41), we obtain (17a).

# APPENDIX C PROOF OF THEOREM 3

The proof of part 1: We know that for  $\alpha \ge 1$ ,  $I_{\alpha}^{S}(X;Y)$  is quasi-convex  $P_{Y|X}$  for given  $P_X$  [14, Thm. 2.7.4], [13, Thm. 10]. In addition, the supreme of a set of quasi-convex functions is also quasi-convex, i.e., let function f(a, b) is quasi-convex in b, such that  $\sup_a f(a, b)$  is also quasi-convex in b [15]. Therefore, the maximal  $\alpha$ -leakage in (17) is quasi-convex  $P_{Y|X}$  for given  $P_X$ .

The proof of part 2: Let  $\beta > \alpha \ge 1$ , and  $P_{X\alpha}^* = \arg \sup_{P_X} I_{\alpha}^{S}(P_X, P_{Y|X})$  for given  $P_{Y|X}$ , such that

$$\mathcal{L}^{\max}_{\alpha}(X \to Y) = I^{\mathsf{S}}_{\alpha}(P^*_{X\alpha}, P_{Y|X}) \tag{42a}$$

$$\leq I_{\beta}^{\mathsf{S}}(P_{X\alpha}^*, P_{Y|X}) \tag{42b}$$

$$\leq \sup_{P_X} I_{\beta}^{\mathsf{S}}(P_X, P_{Y|X}) \tag{42c}$$

$$= \mathcal{L}_{\beta}^{\max}(X \to Y) \tag{42d}$$

where (42b) results from that  $I_{\alpha}^{S}$  is non-decreasing in  $\alpha$  for  $\alpha > 0$  [13, Thm. 4], and the equality in (42c) holds if and only if  $P_{X\alpha}^{*} = \arg \sup_{P_{X}} I_{\beta}(P_{X}, P_{Y|X})$ .

The proof of part 3: Let random variables X, Y and Z form the Markov chain X - Y - Z. Making use of that S-MI of order  $\alpha > 1$  satisfies data processing inequalities [12, Thm. 3], i.e.,

$$I^{\mathbf{s}}_{\alpha}(X;Z) \le I^{\mathbf{s}}_{\alpha}(X;Y) \tag{43a}$$

$$I^{\mathbf{S}}_{\alpha}(X;Z) \le I^{\mathbf{S}}_{\alpha}(Y;Z), \tag{43b}$$

we prove that maximal  $\alpha$ -leakage satisfies data processing inequalities as follows.

We first prove (19a). Let  $P_X^* = \arg \sup_{P_X} I_\alpha^{S}(P_X, P_{Z|X})$ . For the Markov chain X - Y - Z, we have

$$\mathcal{L}^{\max}_{\alpha}(X \to Z) = I^{\mathsf{S}}_{\alpha}(P^*_X, P_{Z|X}) \tag{44a}$$

$$\leq I_{\alpha}^{\mathbf{s}}(P_X^*, P_{Y|X}) \tag{44b}$$

$$\leq \sup_{P_Y} I^{\mathfrak{s}}_{\alpha}(P_X, P_{Y|X}) \tag{44c}$$

$$= \mathcal{L}_{\alpha}^{\max}(X \to Y) \tag{44d}$$

where the inequality in (44b) results from (43a). Similarly, the inequality in (19b) can be proved directly from (43b).

The proof of part 4: For  $\alpha \in (1, \infty]$ , referring to (4) and (17a) we have

$$\mathcal{L}_{\alpha}^{\max}(X \to Y) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{y} \left( \sum_{x} P_X(x) P_{Y|X}(y|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(45a)

$$\geq \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{y} \left( \sum_{x} P_X(x) P_{Y|X}(y|x) \right)^{\frac{\alpha}{\alpha}}$$
(45b)

$$=\sup_{P_X} \frac{\alpha}{\alpha - 1} \log 1 = 0, \tag{45c}$$

where (45b) results from applying Jensens inequality to the convex function  $f: t \to t^{\alpha}$   $(t \ge 0)$ , such that the equality holds if and only if given any  $y \in \mathcal{Y}$ ,  $P_{Y|X}(y|x)$  are the same for all  $x \in \mathcal{X}$ , such that

$$P_{Y|X}(y|x) = P_Y(y) \quad x \in \mathcal{X}, y \in \mathcal{Y}$$
(46)

which means X and Y are independent, i.e.,  $P_{Y|X}$  is a rank-1 row stochastic matrix. For  $\alpha = 1$ , we have

$$\mathcal{L}_1^{\max}(X \to Y) = I(X;Y) \ge 0, \tag{47}$$

with equalities if and only if X is independent of Y [14]. Let  $P_{X \leftarrow Y}$  be an conditional probability matrix with only one non-zero entry in each column, and indicate the only non-zero entries by  $x_y$ , i.e.,  $x_y = \arg_x P_{X \leftarrow Y}(y|x) > 0$  for all  $y \in \mathcal{Y}$ . For  $\alpha = \infty$ , we have

$$\mathcal{L}_{\infty}^{\max}(P_{X \leftarrow Y}) = \log \sum_{y \in \mathcal{Y}} P_{X \leftarrow Y}(y|x_y) = \log |\mathcal{X}|, \quad (48)$$

which is exactly the upper bound of MaxL [16, Lem. 1] and absolutely an upper bound of maximal  $\alpha$  leakage due to its monotonicity in  $\alpha$ .

For  $\alpha \in (1, \infty)$ , from (4) and (17a) we have

$$\mathcal{L}_{\alpha}^{\max}(P_{X \leftarrow Y}) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{y \in \mathcal{Y}} \left( P_X^{\frac{1}{\alpha}}(x_y) P_{X \leftarrow Y}(y|x_y) \right)$$
(49a)

$$= \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{x \in \mathcal{X}} P_X^{\frac{1}{\alpha}}(x);$$
(49b)

in addition, since the function maximized in (49b) is symmetric and concave in  $P_X$ , it is Schur-concave in  $P_X$ , and therefore, the optimal distribution of X achieving the supreme in (49b) is uniform. Thus,

$$\mathcal{L}^{\max}_{\alpha}(P_{X \leftarrow Y}) = \log |\mathcal{X}| \quad \text{ for } \alpha \in (1, \infty).$$
 (50)

For  $\alpha = 1$ , referring to (17b) we have

$$\mathcal{L}_{1}^{\max}(X \to Y) = \sum_{y \in \mathcal{Y}} P_{X}(x_{y}) P_{X \leftarrow Y}(y|x_{y}) \log \frac{P_{X \leftarrow Y}(y|x_{y})}{P_{X}(x_{y}) P_{X \leftarrow Y}(y|x_{y})}$$
(51a)

$$= \sum_{y \in \mathcal{Y}} P_X(x_y) P_{X \leftarrow Y}(y|x_y) \log \frac{1}{P_X(x_y)}$$
(51b)

$$=\sum_{x\in\mathcal{X}} P_X(x)\log\frac{1}{P_X(x)} = H(P_X),$$
(51c)

which is exactly the upper bound of I(X; Y).

Therefore, if X is a deterministic function of Y, maximal  $\alpha$ leakage achieves its maximal value  $\log |\mathcal{X}|$  for  $\alpha > 1$ , and  $H(P_X)$  for  $\alpha = 1$ .

**The proof of part 5**: The upper bound is directly from the fact that maximal  $\alpha$ -leakage is non-decreasing in  $\alpha$ . In addition, from the results in part 4, we know that if  $P_{Y|X}$  has 0 or the maximal leakage in in part 4, the upper bound is tight.

**The proof of part 6**: Given  $P_{Y|X}$ , the lower bound is actually the S-MI of order  $\alpha$  for the uniform distribution of X. Due to the concavity of  $I_{\alpha}^{S}(P_X, P_{Y|X})$  ( $\alpha \ge 1$ ) in  $P_X$  [13, Thm. 8] <sup>2</sup>, we know that  $I_{\alpha}^{S}(P_X, P_{Y|X})$  is Schur concave in  $P_X$ for any symmetric  $P_{Y|X}$ . Therefore the uniform distribution of X maximizes (17a) and its S-MI is exactly the maximal  $\alpha$ - leakage [13, Col. 9]<sup>3</sup>. The  $P_{Y|X}$  in part 4 with zero leakage make the lower bound tight.

# APPENDIX D Proof of Theorem 4

Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be the alphabets of  $Y_1$  and  $Y_2$ , respectively. For any  $(y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ , due to the Markov chain  $Y_1 - X - Y_2$ , the corresponding entry of the conditional probability matrix of  $(Y_1, Y_2)$  given X is

$$P(y_1, y_2|x) = P(y_1|x)P(y_2|x, y_1) = P(y_1|x)P(y_2|x).$$
 (52)

Therefore, for  $\alpha \in (1, \infty)$ 

$$\mathcal{L}_{\alpha}^{\max}(X \to Y_{1}, Y_{2})$$

$$= \sup_{P_{X}} \frac{\alpha}{\alpha - 1} \log \sum_{y_{1}, y_{2} \in \mathcal{Y}_{1} \times \mathcal{Y}_{2}} \left( \sum_{x \in \mathcal{X}} P_{X}(x) P_{Y_{1}, Y_{2}|X}(y_{1}, y_{2}|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$

$$= \sup_{P_{X}} \frac{\alpha}{\alpha - 1} \log \sum_{y_{1}, y_{2} \in \mathcal{Y}_{1} \times \mathcal{Y}_{2}} \left( \sum_{x \in \mathcal{X}} P_{X}(x) P_{Y_{1}|X}(y_{1}|x)^{\alpha} P_{Y_{2}|X}(y_{2}|x)^{\alpha} \right)^{\frac{1}{\alpha}}.$$
(53b)

Let  $K(y_1) = \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x)^{\alpha}$ , for all  $y_1 \in \mathcal{Y}_1$ , such that we can construct a set of distributions over  $\mathcal{X}$  as

$$P_{\tilde{X}}(x|y_1) = \frac{P_X(x)P_{Y_1|X}(y_1|x)^{\alpha}}{K(y_1)}.$$
(54)

Therefore, from (53b),  $\mathcal{L}^{\max}_{\alpha}(X \to Y_1, Y_2)$  can be rewritten as

$$\mathcal{L}_{\alpha}^{\max}(X \to Y_{1}, Y_{2})$$

$$= \sup_{P_{X}} \frac{\alpha}{\alpha - 1} \log \sum_{y_{1}, y_{2} \in \mathcal{Y}_{1} \times \mathcal{Y}_{2}} \left( \sum_{x \in \mathcal{X}} K(y_{1}) P_{\tilde{X}}(x|y_{1}) \right)$$

$$P_{Y_{2}|X}(y_{2}|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(55)

$$= \sup_{P_{X}} \frac{\alpha}{\alpha - 1} \log \sum_{y_{1}, y_{2} \in \mathcal{Y}_{1} \times \mathcal{Y}_{2}} \left( \sum_{x \in \mathcal{X}} P_{X}(x) \right)^{\frac{1}{\alpha}} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_{1}) P_{Y_{2}|X}(y_{2}|x)^{\alpha} \right)^{\frac{1}{\alpha}} (56)$$
$$= \sup_{P_{X}} \frac{\alpha}{\alpha - 1} \log \sum_{y_{1} \in \mathcal{Y}_{1}} \left( \sum_{x \in \mathcal{X}} P_{X}(x) P_{Y_{1}|X}(y_{1}|x)^{\alpha} \right)^{\frac{1}{\alpha}} (57)$$
$$\sum_{y_{2} \in \mathcal{Y}_{2}} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_{1}) P_{Y_{2}|X}(y_{2}|x)^{\alpha} \right)^{\frac{1}{\alpha}} (57)$$

<sup>3</sup>Let  $f(\mathbf{x})$  be a function which is Schur concave in a vector variable  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{x}_1$  and  $\mathbf{x}_2$  be two decreasing-ordered vectors in the domain of  $f(\mathbf{x})$ . If  $\mathbf{x}_1$  majors  $\mathbf{x}_2$ , i.e.,  $\sum_1^k x_{1i} \ge \sum_1^k x_{2i}$  (for all  $k \le n$ ) and  $\sum_1^n x_{1i} = \sum_1^n x_{2i}$ , then  $f(\mathbf{x}_1) \le f(\mathbf{x}_2)$ .

<sup>&</sup>lt;sup>2</sup>The concavity of  $I_{\alpha}^{S}(P_{X}, P_{Y|X})$  is based on the fact that a conditional Rényi divergence is concave in  $P_{X}$  [13].

$$\leq \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \left( \sum_{y_1 \in \mathcal{Y}_1} \left( \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x)^{\alpha} \right)^{\frac{1}{\alpha}} \right)$$
$$\max_{y_1 \in \mathcal{Y}_1} \sum_{y_2 \in \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_1) P_{Y_2|X}(y_2|x)^{\alpha} \right)^{\frac{1}{\alpha}} \right)$$
(58)

$$= \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \left( \sum_{y_1 \in \mathcal{Y}_1} \left( \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x)^{\alpha} \right)^{\frac{1}{\alpha}} \right)$$

$$\sum_{y_1 \in \mathcal{Y}_1} \left( \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(50)

$$\sum_{y_2 \in \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_1^*) P_{Y_2|X}(y_2|x)^{\alpha} \right) \qquad (59)$$

$$\leq \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \left( \sum_{y_1 \in \mathcal{Y}_1} \left( \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x)^{\alpha} \right) + \sup_{P_{\tilde{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_2 \in \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x) P_{Y_2|X}(y_2|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
(60)

$$=\mathcal{L}_{\alpha}^{\max}(X \to Y_1) + \mathcal{L}_{\alpha}^{\max}(X \to Y_2).$$
(61)

where  $y_1^*$  in (59) is the optimal  $y_1$  achieving the maximum in (58). Therefore, the equality in (58) holds if and only if, for all  $y_1 \in \mathcal{Y}_1$ ,

$$\sum_{y_2 \in \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_1) P_{Y_2|X}(y_2|x)^{\alpha} \right)^{\frac{1}{\alpha}}$$
$$= \sum_{y_2 \in \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} P_{\tilde{X}}(x|y_1^*) P_{Y_2|X}(y_2|x)^{\alpha} \right)^{\frac{1}{\alpha}}; \quad (62)$$

and the equality in (60) holds if and only if the optimal solutions  $P_X^*$  and  $P_{\tilde{X}}^*$  of the two maximizations in (60) satisfy, for all  $x \in \mathcal{X}$ ,

$$P_{\tilde{X}}^{*}(x) = \frac{P_{X}^{*}(x)P_{Y_{1}|X}^{\alpha}(y_{1}^{*}|x)}{\sum_{x \in \mathcal{X}} P_{X}(x)P_{Y_{1}|X}^{\alpha}(y_{1}^{*}|x)}.$$
(63)

Now we consider  $\alpha = 1$ . For  $Y_1 - X - Y_2$ , we have

$$I(Y_2; X|Y_1) \le I(Y_2; X).$$
 (64)

From Theorem 2, there is

$$\mathcal{L}_1^{\max}(X \to Y_1, Y_2)$$

$$=I(X;Y_1) + I(X;Y_2|Y_1)$$
(65a)

$$\leq I(X;Y_1) + I(X;Y_2) \tag{65b}$$

$$=\mathcal{L}_1^{\max}(X \to Y_1) + \mathcal{L}_1^{\max}(X \to Y_2).$$
(65c)

For  $\alpha = \infty$ , we also have

$$\mathcal{L}_{\infty}^{\max}(X \to Y_1, Y_2) = \log \sum_{y_1, y_2 \in \mathcal{Y}_1 \times \mathcal{Y}_2} \max_{x \in \mathcal{X}} P(y_1|x) P(y_2|x)$$
(66a)

$$\leq \log \sum_{y_1, y_2 \in \mathcal{Y}_1 \times \mathcal{Y}_2} \left( \max_{x \in \mathcal{X}} P(y_1|x) \right) \left( \max_{x \in \mathcal{X}} P(y_2|x) \right)$$
(66b)

$$= \log \sum_{y_1 \in \mathcal{Y}_1} \max_{x \in \mathcal{X}} P(y_1|x) + \log \sum_{y_2 \in \mathcal{Y}_2} \max_{x \in \mathcal{X}} P(y_2|x) \quad (66c)$$

$$=\mathcal{L}_{\infty}^{\max}(X \to Y_1) + \mathcal{L}_{\infty}^{\max}(X \to Y_2).$$
(66d)

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# REFERENCES

- C. E. Shannon, "Communication theory of secrecy systems," *The Bell System Technical Journal*, vol. 28, no. 4, pp. 656–715, Oct. 1949.
- [2] L. Sankar, S. R. Rajagopalan, and H. V. Poor, "Utility-privacy tradeoffs in databases: An information-theoretic approach," *IEEE Trans. on Inform. For. and Sec.*, vol. 8, no. 6, pp. 838–852, 2013.
- [3] F. P. Calmon, M. Varia, and M. Médard, "On information-theoretic metrics for symmetric-key encryption and privacy," in *Proc. 52nd Annual Allerton Conf. on Commun., Control, and Comput.*, 2014.
- [4] I. Issa, S. Kamath, and A. B. Wagner, "An operational measure of information leakage," in 2016 Annual Conference on Information Science and Systems, CISS 2016, Princeton, NJ, USA, March 16-18, 2016, 2016, pp. 234–239. [Online]. Available: http://dx.doi.org/10.1109/CISS.2016.7460507
- [5] N. Merhav and M. Feder, "Universal prediction," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2124–2147, Oct 1998.
- [6] T. A. Courtade and R. D. Wesel, "Multiterminal source coding with an entropy-based distortion measure," in *IEEE International Symposium on Information Theory Proceedings*, July 2011, pp. 2040–2044.
- [7] F. du Pin Calmon and N. Fawaz, "Privacy against statistical inference," in 50th Annual Allerton Conference on Communication, Control, and Computing, 2012.
- [8] A. Rényi, "On measures of entropy and information," in *Proceedings* of the Fourth Berkeley Symposium on Mathematical Statistics and *Probability*. The Regents of the University of California, 1961, pp. 547–561.
- [9] R. Sibson, "Information radius," Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, vol. 14, no. 2, pp. 149–160, 1969.
- [10] S. Arimoto, "Information measures and capacity of order  $\alpha$  for discrete memoryless channels," in *Colloquia mathematica Societatis János Bolyai*, Kestheley, Hungary, 1975, p. 41C52.
- [11] I. Issa, S. Kamath, and A. B. Wagner, "An operational measure of information leakage," in 2016 Annual Conference on Information Science and Systems (CISS), 2016.
- [12] S. Verdú, "α-mutual information," in 2015 Information Theory and Applications Workshop (ITA), 2015.
- [13] S.-W. Ho and S. Verdú, "Convexity/concavity of rényi entropy and α-mutual information," in 2015 IEEE International Symposium on Information Theory (ISIT), 2015.
- [14] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. Wiley-Interscience, 2006.
- [15] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2014.
- [16] J. Liao, L. Sankar, F. P. Calmon, and V. Y. F. Tan, "Hypothesis testing under maximal leakage privacy constraints," in rXiv:1701.07099 [cs.IT], 2017.