# Low-Power Cooling Codes with Efficient Encoding and Decoding 

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#### Abstract

A class of low-power cooling (LPC) codes, to control simultaneously both the peak temperature and the average power consumption of interconnects, was introduced recently. An ( $n, t, w)$-LPC code is a coding scheme over $n$ wires that (A) avoids state transitions on the $t$ hottest wires (cooling), and (B) limits the number of transitions to $w$ in each transmission (low-power).

A few constructions for large LPC codes that have efficient encoding and decoding schemes, are given. In particular, when $w$ is fixed, we construct LPC codes of size $(n / w)^{w-1}$ and show that these LPC codes can be modified to correct errors efficiently. We further present a construction for large LPC codes based on a mapping from cooling codes to LPC codes. The efficiency of the encoding/decoding for the constructed LPC codes depends on the efficiency of the decoding/encoding for the related cooling codes and the ones for the mapping.


## I. INTRODUCTION

Power and heat dissipation have emerged as first-order design constraints for chips, whether targeted for battery-powered devices or for high-end systems. High temperatures have dramatic negative effects on interconnect performance. Power-aware design alone is insufficient to address the thermal challenges, since it does not directly target the spatial and temporal behavior of the operating environment. For this reason, thermally-aware approaches have emerged as one of the most important domains of research in chip design today. Numerous techniques have been proposed to reduce the overall power consumption of on-chip buses (see [3] which uses coding techniques and the references therein using non-coding techniques). However, all the non-coding techniques do not directly address peak temperature minimization.

Recently, Chee et al. [3] introduced several efficient coding schemes to directly control the peak temperature and the average power consumption. Among others, low-power cooling (LPC) codes are of particular interest as they control both the peak temperature and the average power consumption simultaneously. Specifically, an

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$(n, t, w)$-LPC code is a coding scheme for communication over a bus consisting of $n$ wires, if the scheme has the following two features:
(A) every transmission does not cause state transitions on the $t$ hottest wires;
(B) the number of state transitions on all the wires is at most $w$ in every transmission.

LPC codes have both features, while cooling codes control only the peak temperature.

Definition 1. For $n$ and $t$, an $(n, t)$-cooling code $\mathbb{C}$ of size $M$ is defined as a collection $\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}\right\}$, where $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}$ are disjoint subsets of $\{0,1\}^{n}$ satisfying the following property: for any set $S \subseteq[n]$ of size $|S|=t$ and for $i \in[M]$, there exists a vector $\boldsymbol{u} \in \mathcal{C}_{i}$ such that $\operatorname{supp}(\boldsymbol{u}) \cap S=\varnothing$. We refer to $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}$ as codesets and the vectors in them as codewords.

Using partial spreads, Chee et al. [3] constructed LPC codes with efficient encoding and decoding schemes. When $t \leq 0.687 n$ and $w \geq(n-t) / 2$, these codes achieve optimal asymptotic rates. However, when $w$ is small, i.e. low-power coding is used, the code rates are small and Chee et al. proposed another construction based on decomposition of the complete hypergraph into perfect matchings. While the construction results in LPC codes of large size, usually efficient encoding and decoding algorithms are not known.

In this work, we focus on this regime ( $w$ small) and construct LPC codes with efficient encoding and decoding schemes. Specifically, our contributions are as follows.
(I) We propose a method that takes a linear erasure code as input and constructs an LPC code. Using this method, we then construct a family of LPC codes of size $(n / w)^{w-1}$ which attains the asymptotic upper bound $O\left(n^{w-1}\right)$ when $w$ is fixed. We also use this method to construct a class of LPC codes of size $(n / w)^{w-e-1}$ which is able to correct $e$ transmission errors.
(II) We propose efficient encoding/decoding schemes for the LPC codes of the given construction. In particular, for the above family of LPC codes, we demonstrate encoding with $O(n)$ multiplications over $\mathbb{F}_{q}$ and decoding with $O\left(w^{3}\right)$ multiplications over $\mathbb{F}_{q}$, where $q=n / w$. Furthermore, the related class of LPC codes is able to correct $e$ errors with complexity $O\left(n^{3}\right)$.
(III) A definition for a new family of low-power cooling codes, called constant-power cooling (CPC in short) codes, which have the same weight for all the codewords. All our previous constructions can be applied to obtain such codes.
(IV) A recursive construction for a class of $(n q, t q, w)$-CPC codes (and also ( $n q, t q, w)$-LPC codes) from $(n, t, w)$ CPC codes (and a special type of ( $n, t, w$ )-LPC codes).
(V) A construction for a class of $(n, t, w)$-LPC codes based on a mapping from $(m, t)$-cooling codes. This mapping send all the binary words of length $m$ into a Hamming ball of radius $w$ in $\mathbb{F}_{q}^{n}$ such that each coordinate of the obtained words in the Hamming ball is dominated by a coordinate of the binary words of length $m$. This property guarantees that the cooling property of the $(m, t)$-cooling code is preserved in the low-power cooling code.

Our main target in this paper are cooling codes, but these codes, or more precisely their definition by codesets, might have other applications too. One such application is in the design of WOM (Once Write Memory) codes
which are very important in coding for flash memories (see [5] and references therein). This application of codesets into construction of WOM codes was written in detail in [5] and is described in short as follows. In a WOM code one is trying to write binary information words of length $k$ into a memory of length $n$, where the information is written only in positions where there are zeroes. The goal is to write as many rounds as possible until there is no way to distinguish between some of the written words. Each information word will be identified by a codeset, the codeword taken from the appropriate codeset should have ones on all positions where the memory has ones and hence the ones in complement of the codeword should have empty intersection with the ones of the memory. As was mentioned, this application to WOM codes was considered in [5]. We believe that other applications will arise in the future.

The rest of this paper is organized as follows. In Section $\Pi$ we present some necessary definitions for our exposition, some of the known results, and new upper bounds on the sizes of low-power cooling codes and constant-power cooling codes. Finally, the known constructions are presented and a new one is suggested. Section [III suggests a construction for CPC codes based on non-binary linear codes in general and on MDS codes in particular. For these codes efficient encoding and decoding algorithms are derived. We continue in Section IV] and add error-correction capabilities for such codes and provide efficient algorithms also in this case. The construction that was used in Section $\overline{I V}$ is modified in Section $\bar{V}$ to provide a recursive construction for $(n q, t q, w)$-CPC codes (and related $(n q, t q, w)$-LPC codes) from ( $n, t, w$ )-CPC codes (and some special $(n, t, w)$ CPC codes). While in Section IIII the constructions are for $t \leq n / w-1$, in Section $\nabla$ the construction is effective for larger $t$. In Section VI a method to transfer an ( $m, t$ )-cooling code to an ( $n, t, w$ )-LPC code is given. This method is based on a special injection from the set of all binary words of length $m$ into binary words of length $n$ and weight at most $w$. A product construction using this method implies codes with efficient encoding and decoding algorithms. We further analyse and compare between this construction and constructions in previous works.

## II. Upper Bounds and Known Results

Given a positive integer $n$, the set $\{1,2, \ldots, n\}$ is abbreviated as $[n]$. The Hamming weight of a vector $x \in \mathbb{F}_{q}^{n}$, denoted $w t(x)$, is the number of nonzero positions in $\boldsymbol{x}$, while the support of $\boldsymbol{x}$ is defined as $\operatorname{supp}(\boldsymbol{x}) \triangleq\{i \in[n]$ : $\left.x_{i} \neq 0\right\}$.

A $q$-ary code $\mathcal{C}$ of length $n$ is a subset of $\mathbb{F}_{q}^{n}$. If $\mathcal{C}$ is a subspace of $\mathbb{F}_{q}^{n}$, it is called a linear code. An $[n, k, d]_{q}$ code is a linear code with dimension $k$ and minimum Hamming distance $d$.

Definition 2. For $n, t$ and $w$ with $t+w \leq n$, an ( $n, t, w$ )-low-power cooling (LPC) code $\mathbb{C}$ of size $M$ is defined as a collection of codesets $\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}\right\}$, where $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}$ are disjoint subsets of $\left\{\boldsymbol{u} \in\{0,1\}^{n}: w t(u) \leq w\right\}$ satisfying the following property: for any set $S \subseteq[n]$ of size $|S|=t$ and for $i \in[M]$, there exists a vector $\boldsymbol{u} \in \mathcal{C}_{i}$ such that $\operatorname{supp}(u) \cap S=\varnothing$.

In this paper, we focus on a class of $(n, t, w)$-LPC codes where every transmission results in exactly $w$ state transitions. We call such codes $(n, t, w)$-constant-power cooling (CPC) codes. In particular, let $J(n, w) \triangleq\left\{\boldsymbol{u} \in\{0,1\}^{n}\right.$ :
$\mathrm{wt}(\boldsymbol{u})=w\}$. Then an $(n, t, w)$-CPC code is an $(n, t, w)$-LPC code such that $\mathcal{C}_{i} \subseteq J(n, w)$ for each $i \in[M]$.

## A. Set Systems

For a finite set $X$ of size $n, 2^{X}$ denotes the collection of all subsets of $X$, i.e., $2^{X} \triangleq\{A: A \subseteq X\}$. A set system of order $n$ is a pair $(X, \mathcal{B})$, where $X$ is a finite set of $n$ points, $\mathcal{B} \subseteq 2^{X}$, and the elements of $\mathcal{B}$ are called blocks. Two set systems $\left(X, \mathcal{B}_{1}\right)$ and ( $X, \mathcal{B}_{2}$ ) with the same point set are called disjoint if $\mathcal{B}_{1} \cap \mathcal{B}_{2}=\varnothing$, i.e. they don't have any block in common.

A partial parallel class of a set system $(X, \mathcal{B})$ is a collection of pairwise disjoint blocks. If a partial parallel class partitions the point set $X$, it is called parallel class. A set system $(X, \mathcal{B})$ is called resolvable if the block set $\mathcal{B}$ can be partitioned into parallel classes.

There is a canonical one-to-one correspondence between the set of all codes of length $n$ and the set of all set systems of order $n$ : the coordinates of vectors in $\{0,1\}^{n}$ correspond to the points in $[n]$, and each vector $\boldsymbol{u} \in\{0,1\}^{n}$ corresponds to the block defined by $\operatorname{supp}(u)$. Thus we may speak of the set system of a code or the code of a set system. By abuse of notation we sometimes do not distinguish between the two different notations and this can be readily observed in the text.

## B. Upper Bounds

Given a $t$-subset $S$ and a vector $\boldsymbol{u} \in\{0,1\}^{n}$, we shall say that $\boldsymbol{u}$ avoids $S$ if $\operatorname{supp}(u) \cap S=\varnothing$. The following bounds on LPC codes and CPC codes are easily derived.

Theorem 3. Let $\mathbb{C}$ be an ( $n, t, w)$-LPC code of size $M$, then

$$
M \leq \sum_{i=0}^{w}\binom{n-t}{i} .
$$

Furthermore, if $\mathbb{C}$ is an $(n, t, w)$-CPC code, then

$$
M \leq\binom{ n-t}{w}
$$

Proof. For any given $t$-subset $S$ of [ $n$ ], each codeset should have at least one codeword which avoids $S$. The number of words with weight $i$ which avoid $S$ is $\binom{n-t}{i}$ and hence there are no more than $\binom{n-t}{w}$ codesets in an $(n, t, w)$-CPC code and no more than $\sum_{i=0}^{w}\binom{n-t}{i}$ codesets in an $(n, t, w)$-LPC code.

Theorem 3 implies that both CPC codes and LPC codes share the same asymptotic upper bound $O\left(n^{w}\right)$ on the number of codewords. The upper bound of Theorem 3 can be improved for some parameters. For this purpose, we need to define and to introduce some results on Turán systems.
Let $n \geq k \geq r$, and let $X$ be a finite set with $n$ distinct elements. The set $\binom{X}{r}$ is the collection of all $r$-subsets of $X$. A Turán $(n, k, r)$-system is a set system $(X, \mathcal{B})$, where $|X|=n$ and $\mathcal{B} \subseteq\binom{X}{r}$ is the set of blocks such that each $k$-subset of $X$ contains at least one of the blocks. The Turán number $T(n, k, r)$ is the minimum number of blocks in such a system. This number is determined only for $r=2$ and some sporadic examples (see [13], [16] and references therein). De Caen [7] proved the following general lower bound on $T(n, k, r)$.

$$
\begin{equation*}
T(n, k, r) \geq \frac{n-k+1}{n-r+1} \cdot \frac{\binom{n}{r}}{\binom{k-1}{r-1}} \tag{1}
\end{equation*}
$$

The following proposition is an immediate result from the definition of Turán systems.
Proposition 4. A family of codesets $\left\{\mathcal{C}_{1}, \mathfrak{C}_{2}, \ldots, \mathfrak{C}_{M}\right\}$ is an $(n, t, w)-C P C$ code if and only if the set system of each $\mathfrak{C}_{i}$ is a Turán $(n, n-t, w)$-system and these $M$ set systems are pairwise disjoint.

Combining the bound in (1) and Proposition 4, we have the following upper bound on the size of CPC codes.
Theorem 5. If $\mathbb{C}$ is an $(n, t, w)$-CPC code of size $M$, then

$$
M \leq \frac{n-w+1}{t+1}\binom{n-t-1}{w-1}
$$

Proof. By (1) we have that

$$
\begin{equation*}
T(n, n-t, w) \geq \frac{t+1}{n-w+1} \cdot \frac{\binom{n}{w}}{\binom{n-1-1}{w-1}} \tag{2}
\end{equation*}
$$

By Proposition 4 we have that

$$
\begin{equation*}
M \leq \frac{\binom{n}{w}}{T(n, n-t, w)} \tag{3}
\end{equation*}
$$

Combining (2) and (3) yield that

$$
M \leq \frac{n-w+1}{t+1}\binom{n-t-1}{w-1}
$$

Corollary 6. If $\mathbb{C}$ is an $(n, t, w)$-LPC code of size $M$, then

$$
M \leq \sum_{i=0}^{w-1}\binom{n}{i}+\frac{n-w+1}{t+1}\binom{n-t-1}{w-1}
$$

Proof. If we consider an $(n, t, w)$-CPC code $\mathbb{C}$, then to form an $(n, t, w)$-LPC code we can add to $\mathbb{C}$ at most $\sum_{i=0}^{w-1}\binom{n}{i}$ codesets, each one contains exactly one codeword of weight less than $w$.

When $t=\Theta(n)$, we have that $(n-w+1) /(t+1)=O(1)$, and so the upper bound for $(n, t, w)$-CPC codes is improved from $O\left(n^{w}\right)$ implied by Theorem 3 to $O\left(n^{w-1}\right)$ implied by Theorem 5

For an $(n, t, w)$-LPC code, we have by Corollary 6 that the size of such a code is at most

$$
\sum_{i=0}^{w-1}\binom{n}{i}+\frac{n-w+1}{t+1}\binom{n-t-1}{w-1}
$$

which is also $O\left(n^{w-1}\right)$ when $t$ and $n$ are of the same order of magnitude.

## C. Some Known Constructions

Chee et. al [3] provided the following construction of LPC/CPC codes.

Proposition 7 (Decomposition of Complete Hypergraphs). If $n=(t+1) w$ then there exists an ( $n, t, w$ )-CPC code of size $\binom{n-1}{w-1}$.

When $w$ is fixed, we have that $t$ and $n$ are of the same order of magnitude and the above construction attains the asymptotic upper bound $O\left(n^{w-1}\right)$. Unfortunately, usually no efficient encoding and decoding methods are known for this construction and generally the only known encoding method involves listing all the $\binom{n-1}{w-1}$ codesets. The exceptions are for small $n$ or when $w$ is very small, e.g. when $w=2$ or $w=3$ [1], [8].

Chee et. al [3] also proposed the following constructions of LPC codes which have efficient coding schemes.

(i) If $t+1 \leq m / 2$, then there exists an $\left(m s, t, m w^{\prime}\right)$-LPC code of size $q^{m-t-1}$.
(ii) If $t+1 \leq m \leq q+1$, then there exists an $\left(m s, t, m w^{\prime}\right)$-LPC code of size $q^{m-t}$.

Proposition 9 (Sunflower Construction). Let $r+t \leq(n+s) / 2$. If a linear $[n, s, w+1]_{2}$ code exists and a linear $[n-t, r, w+1]_{2}$ code does not exist, then there exists an ( $n, t, w$ )-LPC code of size $2^{n-t-r}$.

Finally, Proposition 4 suggests a new method to construct ( $n, t, w$ )-CPC codes. We just have to find a set with large number of pairwise disjoint Turán $(n, n-t, w)$-systems. One work in this direction was done in [11] where pairwise disjoint Turán $(n, w+1, w)$-systems were considered. Another possible construction based on Proposition 4 is to consider complements of pairwise disjoint Steiner systems. Such pairwise disjoint systems were considered in [2], [17] and for Steiner quadruple systems which will be used in the sequel in [9].

## iII. An Efficient Construction for Constant-Power Cooling Codes

In this section, we present a new construction of CPC codes which has efficient encoding and decoding algorithms. Asymptotically, the codes obtained by the construction attain the bound of Theorem 5 As was mentioned before, there are three types of constructions for LPC codes in [3]. The first one is based on decomposition of the complete hypergraph, the second one is a concatenation method based on $q$-ary cooling codes, and the third one is a Sunflower Construction. The construction in this section, is an explicit construction for CPC codes which combines the advantages of the first two types of constructions. We first rephrase the construction based on decomposition of the complete hypergraph in terms of set systems. The construction is based on the following generalization of Proposition 7

Proposition 10. Let $(X, \mathcal{B})$ be a set system of order $n$, where $\mathcal{B}$ is partitioned into $M$ partial parallel classes $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{M}$. If $\mathcal{B} \subseteq\binom{X}{w}$ and each $\mathcal{P}_{i}$ has at least $t+1$ blocks, then the codesets $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{M}$ form an $(n, t, w)$-CPC code.

Proof. By definition, each codeword of a codeset has weight $w$. Hence, to show that $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{M}$ form an $(n, t, w)$-CPC code, we only have to prove that given a $t$-subset $S$ of $X$ with the list of hottest wires and a codeset $\mathcal{P}_{i}, 1 \leq i \leq M$, there exists a block $B \in \mathcal{P}_{i}$ such that $B \cap S=\varnothing$. Since $\mathcal{P}_{i}$ is a partial parallel class with at least $t+1$ codewords, it follows that $S$ intersects at most $t$ blocks of $\mathcal{P}_{i}$. Hence, there exists a block $B \in \mathcal{P}_{i}$ such that $B \cap S=\varnothing$.

The complete $k$-uniform hypergraph $G=(V, E)$ has a vertex set $V$ with $n \geq k$ vertices, and each subset of $\binom{V}{k}$ is connected by an hyperedge. The decomposition of $G$ is a partition of the set of edges $E$ in $G$ into disjoint perfect matching. In other words, a partition into vertex-disjoint sets of edges, where each vertex of $V$ appears exactly once in each set of the partition. The celebrated Baranyai's theorem [18, p. 536] asserts that such a decomposition always exists if $k$ divides the number of vertices in $V$. Therefore, since a decomposition of the complete $k$-uniform hypergraph with vertex set $X$ is a resolvable set system $\left(X,\binom{X}{k}\right)$, if $k$ divides $|X|$, we recover Proposition 7

## A. CPC Codes Based on Linear Codes

Let $\mathcal{C}$ be an $[N, K, D]_{q}$ code. Using the codewords of $\mathcal{C}$, we will show how to construct a set system with $q^{K-1}$ partial parallel classes, each one has blocks of the same size, and as a consequence Proposition 10 yields a CPC code $\mathbb{D}$. To equip $\mathbb{D}$ with efficient encoding and decoding schemes, we utilize the erasure-correcting algorithms of the linear code $\mathcal{C}$. These schemes are discussed in detail in Section III-B

For a set of coordinates $T$ and a vector $\sigma \in \mathbb{F}_{q}^{|T|}$, we say that $\sigma$ appears $\lambda$ times in $\mathcal{C}$ at $T$ if there are $\lambda$ codewords in $\mathcal{C}$ whose restriction on $T$ is $\sigma$. Since any two codewords of $\mathcal{C}$ differ in at least $D$ coordinates, it follows that they agree in at most $N-D$ positions. Hence, we have the following observations.

Lemma 11. Let $\mathcal{C}$ be an $[N, K, D]_{q}$ code.
(i) For any $(N-D+1)$-subset of coordinates $T$ and any $\sigma \in \mathbb{F}_{q}^{N-D+1}, \sigma$ appears in at most one codeword of $\mathcal{C}$ at $T$.
(ii) For any $(N-D)$-subset of coordinates $T$ and any $\boldsymbol{\tau} \in \mathbb{F}_{q}^{N-D}, \boldsymbol{\tau}$ appears in at most $q$ codewords of $\mathcal{C}$ at $T$.

Proof.
(i) If $\sigma \in \mathbb{F}_{q}^{N-D+1}$ appears twice in codewords of $\mathcal{C}$ at an $(N-D+1)$-subset of coordinates $T$, then the two related codewords have distance at most $D-1$, a contradiction.
(ii) If $\boldsymbol{\tau} \in \mathbb{F}_{q}^{N-D}$ appears in $q+1$ codewords of $\mathcal{C}$ at an $(N-D)$-subset of coordinates $T$, then let $t$ be a coordinate not in $T$. In at least two of the related codewords coordinate $t$ has the same symbol. We add this symbol to $\boldsymbol{\tau}$ to obtain $\sigma \in \mathbb{F}_{q}^{N-D+1}$ which appears in two codewords of $\mathcal{C}$ at the $(N-D+1)$-subset $T \cup\{t\}$, contradicting claim (i) of this lemma.

For a code $\mathcal{C}$ and a subset of coordinates $T$, let $\left.\mathcal{C}\right|_{T}$ denotes the set of codewords restricted to the coordinates of $T$, i.e., the projection of $\mathcal{C}$ into the set of coordinates indexed by $T$. For a word $\boldsymbol{u}$, let $\left.\boldsymbol{u}\right|_{T}$ denotes the restriction of $\boldsymbol{u}$ to the coordinates of $T$. Finally, for a matrix $\boldsymbol{G}$, let $\left.G\right|_{T}$ denotes the submatrix of $\boldsymbol{G}$ obtained from the columns indexed by $T$.

Lemma 12. Let $\mathcal{C}$ be an $[N, K, D]_{q}$ code. If $G$ is a generator matrix of $\mathcal{C}$, then every $K \times(N-D)$ submatrix of $G$ has rank either $K$ or $K-1$. Furthermore, there exists a $K \times(N-D)$ submatrix of $G$ whose rank is $K-1$.

Proof. Let $T$ be a subset of $N-D$ coordinate positions and assume the corresponding $K \times(N-D)$ submatrix $\left.G\right|_{T}$ has rank $r$, where $r \leq K$. Let $\phi_{T}$ be the linear map from $\mathbb{F}_{q}^{K}$ to $\mathbb{F}_{q}^{N-D}$ defined by $\phi_{T}(\boldsymbol{x})=\left.\boldsymbol{x} \boldsymbol{G}\right|_{T}$. Clearly, the
dimension of the kernel of $\phi_{T}$ is $K-r$. Hence, the all-zeroes vector of length $N-D$ appears in $q^{K-r}$ codewords of $\left.\mathcal{C}\right|_{T}$. By Lemma 11 the all-zeroes vector appears in at most $q$ codewords of $\left.\mathcal{C}\right|_{T}$ at $T$ which implies that $K-r \leq 1$ and therefore $r \geq K-1$.

Let $\boldsymbol{u}$ be a codeword of $\mathcal{C}$ with minimum weight $D, T$ be the subset of $[N]$ not in the support of $\boldsymbol{u}$, i.e. $T=[N] \backslash \operatorname{supp}(\boldsymbol{u})$. Let $\boldsymbol{x}$ be the information vector of length $K$ such that $\boldsymbol{u}=\boldsymbol{x} \boldsymbol{G}$. Since $\boldsymbol{u}$ has weight $D$, it follows that $|\operatorname{supp}(\boldsymbol{u})|=D$ and hence the size of $T$ is $N-D$. Since $\boldsymbol{u}$ has zeroes in the coordinates of $T$, it follows that for the $K \times(N-D)$ submatrix $\left.G\right|_{T}$ of $G$ we have that $\left.x \boldsymbol{G}\right|_{T}=\mathbf{0}$. Therefore, the rank of $\boldsymbol{G}_{T}$ is at most $K-1$.

Thus the rank of $\left.G\right|_{T}$ is at least $K-1$ by the first part of the proof and at most $K-1$ by the second part of the proof, which implies that the rank of $\left.G\right|_{T}$ is $K-1$.

We are now ready to present the new efficient construction for CPC codes.

Construction 1. Let $\mathcal{C}$ be an $[N, K, D]_{q}$ code and $G$ be a generator matrix of $\mathcal{C}$, where the last $N-D$ columns of $G$ form a $K \times(N-D)$ submatrix of $G$ whose rank is $K-1$.

- Partition the codewords of $\mathcal{C}$ into disjoint codesets $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}$ such that two codewords $u$ and $v$ are in the same codeset if and only if they agree on their last $N-D$ symbols.
- For each $i \in[M]$, truncate the codewords in $\mathcal{C}_{i}$ to length $w$ by removing their last $N-w$ symbols. In other words, set $\mathcal{C}_{i}^{\prime} \triangleq\left\{\left.\boldsymbol{u}\right|_{[w]}: u \in \mathcal{C}_{i}\right\}$ for each $i \in[M]$.
- For each $i \in[M]$ construct the set system $\left(X, \mathcal{D}_{i}\right)$, where $X=\mathbb{F}_{q} \times[w]$ and

$$
\mathcal{D}_{i}=\left\{\left(x_{j}, j\right): x=x_{1} x_{2} \cdots x_{w} \in \mathcal{C}_{i}^{\prime}, j \in[w]\right\}
$$

Theorem 13. If $N-D+1 \leq w \leq D$, then the collection of codesets $\mathbb{D}=\left\{\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{M}\right\}$ is an $(n, t, w)$-CPC code of size $M=q^{K-1}$, where $n=q w$ and $t=q-1$.

Proof. Clearly, by the definition of the construction we have that $n=q w$ and each codeword has weight $w$. Hence, to complete the proof it is sufficient to show that $M=q^{K-1}$, the $M$ codesets are pairwise disjoint, and for any $t$-subset $S$ of coordinates from $X$ and each $i \in[M]$, there exists a codeword $\boldsymbol{u}$ in the codeset $\mathcal{D}_{i}$ such that $\operatorname{supp}(\boldsymbol{u}) \cap S=\varnothing$.

1) Let $G^{\prime}$ be the $K \times(N-D)$ submatrix of $G$ formed from the last $N-D$ columns of $G$. Consider the linear map $\phi$ from $\mathbb{F}_{q}^{K}$ to $\mathbb{F}_{q}^{N-D}$ defined by $\phi(\boldsymbol{x})=x \boldsymbol{G}^{\prime}$. Since the rank of $G$ is $K-1$, it follows that the image of $\phi$ has dimension $K-1$ and the kernel of $\phi$ has dimension one. It follows that $M=q^{K-1}$ and $\left|\mathcal{C}_{i}\right|=q$ for each $i \in[M]$.
2) The minimum distance of $\mathcal{C}$ is $D$ and hence each two codewords of $\mathcal{C}$ can agree in at most $N-D$ coordinates, i.e. they differ in any subset of $N-D+1$ coordinates. Since $w \geq N-D+1$ and the codewords of $\mathcal{C}$ were shortened in their last $N-w$ coordinates, it follows that all the shortened codewords of $\mathcal{C}$ are distinct. Thus $\mathcal{C}_{1}^{\prime}, \mathfrak{C}_{2}^{\prime}, \ldots, \mathfrak{C}_{M}^{\prime}$ are pairwise disjoint and $\left|\mathfrak{C}_{i}^{\prime}\right|=\left|\mathcal{C}_{i}\right|=q$. Now, it can be easily verified by the definition of $\mathcal{D}_{i}$ that $\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{M}$ are pairwise disjoint and $\left|\mathcal{D}_{i}\right|=q$ for each $i \in[M]$.
3) Each two codewords of $\mathcal{C}_{i}$ agree on their last $N-D$ coordinates and since their distance is at least $D$, it follows that they differ in the first $D$ coordinates. Since $w \leq D$, this implies that any two codewords of $\mathcal{C}_{i}^{\prime}$ differ in all their $w$ coordinates. Hence, by the definition of $\mathcal{D}_{i}$ it implies that each two codewords of $\mathcal{D}_{i}$ differ in their nonzero coordinates. Therefore, the codewords in $\mathcal{D}_{i}$ are pairwise disjoint, i.e., $\mathcal{D}_{i}$ is a partial parallel class. We also have that $\left|\mathcal{D}_{i}\right|=q$ for each $i \in[M]$. Hence, $\mathcal{D}_{i}$ is a parallel class and as in the proof of Proposition 10 we have that for any $t$-subset $S, \mathcal{D}_{i}$ has a codeword $u$ such that $\operatorname{supp}(u) \cap S=\varnothing$.

Thus, the the required claims were proved and hence the collection of codesets $\mathbb{D}=\left\{\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{M}\right\}$ is a $(q w, q-1, w)$-CPC code of size $M=q^{K-1}$.

For a given $[N, K, D]_{q}$ code $\mathcal{C}$ and its generator matrix $G$ in Construction 1, we need to find a minimum weight codeword in $\mathcal{C}$ in order to determine a $K \times(N-D)$-submatrix of $G$ with rank $K-1$, i.e., to find a permutation of the columns of $G$ such that the last $N-D$ coordinates of $G$ will have rank $K-1$. Finding the minimum distance of a code is an NP-hard problem and the decision problem is NP-complete [19]. Therefore, we focus on certain families of codes where it is computationally easy to find minimum weight codewords. One such family is the maximum distance separable (MDS) codes. Recall that a linear $[N, K, D]_{q}$ code is an MDS code if $D=N-K+1$ [14, Ch.11]. If the code $\mathcal{C}$ in Construction 1 is an MDS code, then every $K$ columns of $G$ are linearly independent and hence each $K \times(N-D)$ submatrix of $G$ has rank $K-1$ since $N-D=K-1$. Therefore, we may use any $N-D$ coordinate as the last $N-D$ coordinates of $\mathcal{C}$. It is well known that MDS codes exist for the following parameters.

Theorem 14 (see [14, Ch.11]). Let $q$ be a prime power. If $D \geq 3$, then there exists an $[N, K, D]_{q} M D S$ code if $N \leq q+1$ for all $q$ and $2 \leq K \leq q-1$, except when $q$ is even and $K \in\{3, q-1\}$, in which case $N \leq q+2$.

Setting $N=q+1, K=w, D=q-w+2$, and using an $[N, K, D]_{q}$ MDS code as the code $\mathcal{C}$ in Construction 1, we have that $w \leq D=q-w+2$, i.e., $q \geq 2 w-2$. Hence, Theorem 13 yields the following corollary.

Corollary 15. Let $n, t$ and $w$ be positive integers. If $q=n / w$ is a prime power and $q \geq 2 w-2$, then there exists an $(n, q-1, w)-C P C$ code of size $(n / w)^{w-1}$.

In Corollary 15, when $w$ is fixed, $t=q-1$ has the same order of magnitude as $n$. Hence, the codes constructed in this case asymptotically attain the upper bound $O\left(n^{w-1}\right)$. We also note that for some parameters, these CPC codes are much larger than the LPC codes provided by Propositions 8 and 9

Example 16. By choosing $n=96, w=6$ and $t=15$, Corollary 15 yields a $(96,15,6)$-CPC code of size $16^{5}=2^{20}$.
In contrast, suppose we use Proposition 8 to construct a $(96, t, 6)$-LPC code with $t \leq 15$. The largest size $16^{5}=2^{20}$ is obtained by choosing $m=6, t=1, s=16, w^{\prime}=1$, and $q=16$. The resulting $(96,1,6)$-LPC code has the same size as the CPC obtained by Corollary 15, but the cooling capability of the former is clearly much weaker. Proposition 9 on the other hand, yields a $(96,15,6)$-LPC code of size $2^{16}$ by choosing $s=81$ and $r=65$. This code has similar parameters, but its size is much smaller.

Example 17. If we choose a $[17,8,9]_{9}$ code (see [12]). If $w=9$ and $t=8$ in Construction 11, then we obtain an $(81,8,9)$-CPC code of size $9^{7} \approx 2^{22.189}$.
In contrast, the largest $(81,8,9)$-LPC code obtained from Proposition 8 has size $9 \approx 2^{3.17}$ by choosing $m=s=q=9, w^{\prime}=1$. Proposition 9, on the other hand, yields an $(81,8,9)$-LPC of size $2^{21}$ by choosing $s=54$ and $r=52$.

## B. Encoding and Decoding Schemes

We continue in this subsection and discuss the encoding and decoding schemes for the code $\mathbb{D}$ obtained in Construction 1. Let $G$ be a generator matrix of the $[N, K, D]_{q}$ code $\mathcal{C}$, where the last $N-D$ columns of $G$ form a $K \times(N-D)$ submatrix $G^{\prime}$ whose rank is $K-1$. Furthermore, w.l.o.g. we assume that $G$ has the form

$$
G=\left(\begin{array}{cc}
\boldsymbol{A} & \boldsymbol{I}_{K-1} \\
\boldsymbol{\beta}_{K} & 0 \cdots 0
\end{array}\right)
$$

where $I_{K-1}$ is the identity matrix of order $K-1$.
Each codeset in $\mathbb{D}$ will be identified by the unique vector from $\mathbb{F}_{q}^{K-1}$. This is possible since the number of codesets is $q^{K-1}$. For $\sigma \in \mathbb{F}_{q}^{K-1}$, let $\mathcal{C}_{\sigma}$ be the set of $q$ codewords from $\mathcal{C}$ whose suffix of length $K-1$ is $\sigma$. Furthermore, let $\mathcal{C}_{\boldsymbol{\sigma}}^{\prime}$ and $\mathcal{D}_{\boldsymbol{\sigma}}$ be the derived codesets as defined in Construction 1

Given a $t$-subset $S$ of $\mathbb{F}_{q} \times[w]$ and a word $\sigma=\left(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{K-1}\right) \in \mathbb{F}_{q}^{K-1}$, our objective for encoding of Construction 1 is to find a codeword $u \in \mathcal{D}_{\boldsymbol{\sigma}}$ such that $\operatorname{supp}(u) \cap S=\varnothing$. Let $\beta_{i}$ be the $i$-th row of $G$. Let

$$
\boldsymbol{r}=\left.\boldsymbol{\sigma} \boldsymbol{A}\right|_{[w]}=\left.\sum_{i=1}^{K-1} \sigma_{i} \boldsymbol{\beta}_{i}\right|_{[w]},
$$

and hence the codeset $\mathcal{C}_{\boldsymbol{\sigma}}^{\prime}$ is

$$
\mathfrak{C}_{\boldsymbol{\sigma}}^{\prime}=\left\{\boldsymbol{r}+\left.\lambda \boldsymbol{\beta}_{K}\right|_{[w]}: \lambda \in \mathbb{F}_{q}\right\} .
$$

The codeset $\mathcal{D}_{\boldsymbol{\sigma}}$ is derived from $\mathcal{C}_{\boldsymbol{\sigma}}^{\prime}$ as indicated in Construction 1 , and hence we can consider the intersection of each one of the $q$ blocks in $\mathcal{D}_{\boldsymbol{\sigma}}$ with $S$ to find the block $B$ such that $B \cap S=\varnothing$.

Hence, for the encoding, $O(n)$ multiplications over $\mathbb{F}_{q}$ are required to find $\mathcal{D}_{\sigma}$. During this computation we can also check whether each codeword of $\mathcal{D}_{\boldsymbol{\sigma}}$ has nontrivial intersection with $B$ or not. Therefore, there is no need for further computations to find $B$.

For the decoding, suppose that we have a codeword $\left\{\left(x_{1}, 1\right),\left(x_{2}, 2\right), \ldots,\left(x_{w}, w\right)\right\}$. By our choice we have that $w \geq N-D+1$ which implies that $D-1 \geq N-w$ and hence we can correct any $N-w$ erasures in any codeword of $\mathcal{C}$. Hence, the $N-w$ erasures in $\left(x_{1}, x_{2}, \ldots, x_{w}, ?, ?, \ldots, ?\right)$ can be recovered and the last $K-1$ symbols, $x_{N-K+2}, x_{N-K+3}, \ldots, x_{N}$ are the information symbols. In particular, if the code $\mathcal{C}$ is a Reed-Solomon code, then by using Lagrange interpolation, $O\left(w^{3}\right)$ multiplications are enough to perform the decoding, e.g. [15].

## IV. Error-Correcting CPC Codes

In this section we consider CPC codes that can correct transmission errors (' 0 ' received as ' 1 ', or ' 1 ' received as ' 0 '). An $(n, w, t)$-CPC which can correct up to $e$ errors will be called an $(n, t, w, e)$-CPECC (constant weight
power error-correcting cooling) code. First, Construction 1 will be used to produce CPECC codes by examining the minimum distance of the constructed codes.

Theorem 18. If the code $\mathcal{C}$ used for Construction $\mathbb{1}$ is an $[N, K, D]_{q}$ code, then the code $\mathbb{D}$ obtained by Construction $\mathbb{1}$ is an $(n, t, w, e)$-CPECC code of size $M=q^{K-1}$, where $n=q w, t=q-1$, and $e \geq w+D-N-1$.

Proof. All the parameters of the code except for $e=w+D-N-1$ were proved in Theorem 13 Since the minimum distance of $\mathcal{C}$ is $D$ and the code $\mathcal{C}$ was punctured in the last $N-w$ coordinates to obtain the code $\mathfrak{C}^{\prime}$ (the union of the codesets $\mathfrak{C}_{i}^{\prime}, 1 \leq i \leq M$ ), it follows that the minimum distance of $\mathfrak{C}^{\prime}$ is at least $D-(N-w)$. By the definition of $\mathcal{D}^{\prime}$ (the union of the codesets $\mathcal{D}_{i}^{\prime}, 1 \leq i \leq M$ ) we have that if $u, u^{\prime} \in \mathcal{C}^{\prime}$ differ in $\ell$ coordinates, then the related codewords in $\mathcal{D}$ differ in $2 \ell$ positions. Hence, the minimum distance of $\mathcal{D}$ is at least $2(D+w-N)$ and thus the number of errors that it can correct is $e \geq w+D-N-1$.

Next, an algorithm which demonstrates the error-correction for an ( $n, t, w, e$ )-CPECC code will be given. For simplicity, we will focus on a special example, where our starting point is a Reed-Solomon code $\mathcal{C}$ (which is of course an MDS code), where $K=N-D+1=w-e$.

Construction 2. Let $w$ and $e$ be positive integers and $q$ be a prime power such that $q \geq 2 w-e-1$. Let $a_{1}, a_{2}, \ldots, a_{w}, b_{1}, b_{2}, \ldots, b_{w-e-1}$ be $2 w-e-1$ distinct elements of $\mathbb{F}_{q}$.

- For each polynomial $f(X) \in \mathbb{F}_{q}[X]$, define the following block on the point set $\mathbb{F}_{q} \times[w]$,

$$
C_{f}=\left\{\left(f\left(a_{j}\right), j\right): j \in[w], \operatorname{deg}(f) \leq w-e-1\right\} .
$$

- For each $\boldsymbol{\sigma}=\left(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{w-e-1}\right) \in \mathbb{F}_{q}^{w-e-1}$, let

$$
\mathcal{E}_{\boldsymbol{\sigma}}=\left\{C_{f}: f \in \mathbb{F}_{q}[X], \operatorname{deg}(f) \leq w-e-1, f\left(b_{i}\right)=\sigma_{i} \text { for each } i \in[w-e-1]\right\} .
$$

Theorem 19. The code $\mathbb{E}=\left\{\mathcal{E}_{\boldsymbol{\sigma}}: \boldsymbol{\sigma} \in \mathbb{F}_{q}^{w-e-1}\right\}$ is an $(n, t, w, e)$-CPECC code of size $q^{w-e-1}$, where $n=q w$ and $t=q-1$.

Proof. It is an immediate observation from the definition of the point set $\mathbb{F}_{q} \times[w]$ and the codeword $C_{f}$ that each codeword has length $q w$ and weight $w$. The rest of the proof has four steps. In the first one we will prove that for each $\boldsymbol{\sigma}, \boldsymbol{\sigma}^{\prime} \in F_{q}^{w-e-1}, \varepsilon_{\sigma}$ and $\varepsilon_{\sigma^{\prime}}$ are disjoint whenever $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^{\prime}$. In the second step we will prove that for each $\boldsymbol{\sigma} \in \mathbb{F}_{q}^{w-e-1}$ the blocks in $\mathcal{E}_{\boldsymbol{\sigma}}$ are pairwise disjoint. As a result, by a simple counting argument in the third step it will be proved that $\mathbb{E}$ has $q^{w-e-1}$ codesets, each one has parallel class of size $q$, and as a consequence $\mathbb{E}$ is a $(q w, q-1, w)$-CPC code. In the last step we will find the minimum Hamming distance of $\mathbb{E}$ and as a result the number of errors $e$ that it can correct.

1) Assume that there exist two codewords $C_{f} \in \mathcal{E}_{\boldsymbol{\sigma}}$ and $C_{g} \in \mathcal{E}_{\boldsymbol{\sigma}^{\prime}}$ such that $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^{\prime}$ and $C_{f}=C_{g}$. Then $f$ and $g$ agree on at least $w$ points and since the degrees of the polynomials are less than $w$, it follows that $f=g$. It implies that $\sigma_{i}=f\left(b_{i}\right)=g\left(b_{i}\right)=\sigma_{i}^{\prime}$ for all $i \in[w-e-1]$ and hence $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{\prime}$, a contradiction. Thus, $\varepsilon_{\boldsymbol{\sigma}}$ and $\varepsilon_{\sigma^{\prime}}$ are disjoint whenever $\sigma \neq \sigma^{\prime}$.
2) Assume that the blocks $C_{f}$ and $C_{g}$ in $\mathcal{E}_{\boldsymbol{\sigma}}$, where $f \neq g$, intersect at the point $\left(x, i_{0}\right)$ for some $x \in \mathbb{F}_{q}$ and $i_{0} \in[w]$. It implies that $f\left(a_{i_{0}}\right)=g\left(a_{i_{0}}\right)$ and since $C_{f}, C_{g} \in \mathcal{E}_{\boldsymbol{\sigma}}$, it follows that $f\left(b_{i}\right)=g\left(b_{i}\right)$ for each $i \in[w-e-1]$. Therefore, $f$ and $g$ agree on at least $w-e$ points. Since the degrees of $f$ and $g$ are at most $w-e-1$, it follows that $f=g$, a contradiction. Therefore, the blocks in $\mathcal{E}_{\sigma}$ are pairwise disjoint. Recall that each block has size $w$ and the size of the point set of these blocks $\mathbb{F}_{q} \times[w]$ is $q w$. Hence, each codeset $\mathcal{E}_{\boldsymbol{\sigma}}$ contains at most $q$ blocks.
3) The number of distinct polynomials in $\mathbb{F}_{q}[X]$ whose degrees are at most $w-e-1$ is $q^{w-e}$. Each polynomial induces exactly one codeword in $\mathbb{E}$. Hence, $\mathbb{E}$ contains exactly $q^{w-e}$ distinct codewords. Since there are $q^{w-e-1}$ codesets and each one contains at most $q$ codewords, it follows that each one contains exactly $q$ codewords. The length of a codeword is $q w$ and the weight of a codeword is $w$ which implies that each codeset is a parallel class. Thus, by Proposition 10, $\mathbb{E}$ is a $(q w, q-1, w)$-CPC code.
4) Finally, for any two distinct codewords $C_{f}$ and $C_{g}$, where $f$ and $g$ have degree at most $w-e-1$, we have that $\left|C_{f} \cap C_{g}\right| \leq w-e-1$ since larger intersection implies that $f=g$. Therefore, the Hamming distance between $C_{f}$ and $C_{g}$ is at least $2 e+2$. Thus, the code $\mathbb{E}$ has minimum Hamming distance at least $2 e+2$ and it can correct $e$ errors.

Thus, $\mathbb{E}$ is an $(n, t, w, e)$-CPECC code of size $q^{w-e-1}$, where $n=q w$ and $t=q-1$.
The encoding scheme in Section III-B can be easily adapted for the encoding of the CPECC code $\mathbb{E}$. Algorithm 1 illustrates the decoding scheme for the $(n, t, w, e)$-CPECC code $\mathbb{E}$ obtained in Construction 2

```
Algorithm 1 Error-Correction for the CPECC codes in Construction 2
Input: a binary word \(u \subset \mathbb{F}_{q} \times[w] \quad\{\) the word received after the transmission of a codeword \(\}\)
Output: a message \(\sigma \in \mathbb{F}_{q}^{w-e-1} \quad\{\) the information word that was sent \(\}\)
    for each \(i \in[w]\) do
        if \(\left|Y_{i} \triangleq\{(y, i):(y, i) \in \boldsymbol{u}\}\right|=1\) then
            \(y_{i} \leftarrow y\), where \((y, i)\) is the unique pair in \(Y_{i}\);
        else
            \(y_{i} \leftarrow\) '?';
    \(\hat{\boldsymbol{y}} \leftarrow\left(y_{1}, y_{2}, \ldots, y_{w}\right)\);
    apply the decoding algorithm for Reed-Solomon codes on \(\hat{\boldsymbol{y}}\)
    The output of the algorithm is a polynomial \(L(x)\) of degree \(w-e-1\);
    \(\boldsymbol{\sigma} \leftarrow\left(L\left(b_{1}\right), L\left(b_{2}\right), \ldots, L\left(b_{w-e-1}\right)\right) ;\)
    return \(\sigma\);
```

Theorem 20. Suppose that the codeword $c \in \mathbb{E}$ obtained in Construction 2 was submitted and the word $\boldsymbol{u}$ was received from $c$ with at most e errors. Then, Algorithm 11 returns the word $\boldsymbol{\sigma} \in \mathbb{F}_{q}^{w-e-1}$ such that $c \in \mathcal{E}_{\boldsymbol{\sigma}}$.

Proof. Using the notation of Algorithm 1, let $i \in[w], Y_{i} \triangleq\{(y, i):(y, i) \in \boldsymbol{u}\}$, and $e^{\prime}=\left|\left\{i:\left|Y_{i}\right| \neq 1\right\}\right|$. If $\left|Y_{i}\right|=0$ then an erasure occurred and this is reflected as an erasure in $y_{i}$. If $\left|Y_{i}\right|>1$ then we also know that an error has occurred for at least one coordinate $(y, i)$. This will be also reflected as an erasure in $y_{i}$. Hence, at least $e^{\prime}$ erasure errors are reflected in $\hat{\boldsymbol{y}}$ as a result of at least $e^{\prime}$ errors in these $Y_{i}^{\prime}$ s. For the remaining $w-e^{\prime} Y_{i}^{\prime}$ 's, while
there may be errors, we know that each of these $Y_{i}^{\prime}$ 's contains either no errors or two errors. Thus, the number of other erroneous $Y_{i}^{\prime}$ s is at most $\left\lfloor\left(e-e^{\prime}\right) / 2\right\rfloor$.

The vector $\hat{\boldsymbol{y}}$ is obtained by mapping the subsets $Y_{1}, \Upsilon_{2}, \ldots, \Upsilon_{w}$ to the elements of $\mathbb{F}_{q} \cup\{?\}$. The word $\hat{y}$ was obtained from a codeword $\boldsymbol{x}_{f}$ of a Reed-Solomon code of length $N=w$, dimension $K=w-e$, and minimum Hamming distance $D=N-K+1=e+1$. An error-correction algorithm for such a code is capable of correcting $e^{\prime}$ erasures and at most $\left\lfloor\left(e-e^{\prime}\right) / 2\right\rfloor$ errors as required by Algorithm 1

Using the Berlekamp-Welch algorithm [20] we can correct the errors with $O\left(q^{3}\right)$ operations [20], and hence, Algorithm 1 has complexity $O\left(n^{3}\right)$.

## V. Recursive Construction

All the $(n, t, w)$-CPC codes obtained from Proposition 7 and Construction 1 have $t=n / w-1$. In this section, we present a recursive construction that yields $(n, t, w)$-CPC codes which will be designed especially for larger values of $t$.

For this purpose, recall the conditions of Proposition 10 . Let $(X, \mathcal{B})$ be a set system with a point set of size $n$, where $\mathcal{B} \subseteq\binom{X}{w}$ and $\mathcal{B}$ can be partitioned into $M$ partial parallel classes $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{M}$ to form a code $\mathbb{E}$ with $M$ codesets. Suppose further that each partial parallel class $\mathcal{P}_{i}$ has exactly $q$ blocks. Let $S$ be a $t$-subset of $X$ and $\mathcal{P}_{i}$ be a given partial parallel class. If $t \geq q$, it might not be possible to choose a block/codeword in $\mathcal{P}_{i}$ which avoids $S$. However, by the pigeonhole principle, we can find such a block/codeword which intersects $S$ in at most $\lfloor t / q\rfloor$ elements. Given a $\left(w,\lfloor t / q\rfloor, w^{\prime}\right)$-LPC code $\mathbb{C}$ it is possible to substitute it instead of each block/codeword of $\mathcal{B}$ and break up each codeword into codewords of weight at most $w^{\prime}$. This will enable to find a block/codeword of weight $w^{\prime}$ which avoids $S$. The following construction is based on this idea, where the code $\mathbb{E}$ is constructed similarly to the code in Construction 2.

Construction 3. Let $q \geq n+w-1$ be a prime power and let $a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{w-1}$ be $n+w-1$ distinct elements of $\mathbb{F}_{q}$.

- Consider the point set $\mathbb{F}_{q} \times[n]$ and let

$$
\mathcal{B}=\left\{C_{f} \triangleq\left\{\left(f\left(a_{j}\right), j\right): j \in[n]\right\}: f \in \mathbb{F}_{q}[x], \operatorname{deg}(f) \leq w-1\right\}
$$

Note that the size of each block $C_{f}$ is $n$.

- For each $\boldsymbol{\sigma}=\left(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{w-1}\right) \in \mathbb{F}_{q}^{w-1}$, let

$$
\mathcal{E}_{\boldsymbol{\sigma}}=\left\{C_{f}: f \in \mathbb{F}_{q}[X], \operatorname{deg}(f) \leq w-1, f\left(b_{i}\right)=\sigma_{i} \text { for each } i \in[w-1]\right\}
$$

Similarly to the proof of Theorem 19 one can show that $\mathcal{B}$ is partitioned by $\mathcal{E}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma} \in \mathbb{F}_{q}^{w-1}$ into $q^{w-1}$ parallel classes, each one of size $q$. Label the parallel classes and their blocks by $\mathcal{P}_{i}=\left\{B_{i j}: j \in[q]\right\}$ for $i \in\left[q^{w-1}\right]$.

- Let $\mathbb{D}$ be an $(n, t, w)$-CPC code of size $m$, where $t \geq n / w$.
- Each block $B_{i j}$ is replaced by the codewords of each codeset of $\mathbb{D}$ by using any bijection between the set of points of $B_{i j}$ and the point set of $\mathbb{D}$. Therefore, each codeword in $\mathbb{D}$ corresponds to a $w$-subset of $B_{i j}$
and from each block $B_{i j}$ we construct codewords for $m$ new codesets. These sets of codewords from the $m$ codesets will be denoted by $\mathcal{E}_{i j \ell}$ for each $\ell \in[m]$.
- For $(i, \ell) \in\left[q^{w-1}\right] \times[m]$, the codeset $\mathcal{E}_{i \ell}$ is defined by $\mathcal{E}_{i \ell} \triangleq \bigcup_{j=1}^{q} \mathcal{E}_{i j \ell}$.

Along the same lines of the proof in Theorem 19 one can prove that
Theorem 21. The code $\left\{\mathcal{P}_{i}: 1 \leq i \leq q^{w-1}\right\}$ is an $(n q, q-1, n)-C P C$ code.
Theorem 22. The code $\mathbb{E}=\left\{\mathcal{E}_{i \ell}: i \in\left[q^{w-1}\right], \ell \in[m]\right\}$ is an $(n q, t q, w)$-CPC code of size $m q^{w-1}$.
Proof. The size of $\mathbb{E}$, the length of its codewords and their weight follow immediately from the definition of the codewords in $\mathbb{E}$.

Given a $(t q)$-subset $S \subset \mathbb{F}_{q} \times[n]$ and a codeset $\mathcal{E}_{i \ell},(i, \ell) \in\left[q^{w-l}\right] \times[m]$, we should find a codeword $\boldsymbol{u} \in \mathcal{E}_{i \ell}$ such that $\operatorname{supp}(u) \cap S=\varnothing$. Since $\mathcal{E}_{i \ell}$ was constructed from the $q$ blocks of $\mathcal{P}_{i}$ in which the codewords of the $\ell$-th codeset of $\mathbb{D}$ were substituted, we have to find first a block $B_{i j} \in \mathcal{P}_{i}$ which contains a subset $S^{\prime}$ of $S$ whose size is at most $t$. Such a block exists since the number of blocks in $\mathcal{P}_{i}$ is $q$ and $S$ has size $t q$. Since $\mathcal{E}_{i j \ell}$ is a codeset in an $(n, t, w)$-CPC code, we can find a block $\boldsymbol{u}$ in $\mathcal{E}_{i j \ell}$ which avoids $S^{\prime}$. As a consequence $\operatorname{supp}(\boldsymbol{u}) \cap S=\varnothing$ as required.

To complete the proof we have to show that all the codesets of $\mathbb{E}$ are pairwise disjoint, i.e. $\mathcal{E}_{i \ell}$ and $\mathcal{E}_{i^{\prime} \ell^{\prime}}$ are disjoint whenever $(i, \ell) \neq\left(i^{\prime}, \ell^{\prime}\right)$. To this end, it suffices to show $\mathcal{E}_{i j \ell}$ and $\mathcal{E}_{i^{\prime} j^{\prime} \ell^{\prime}}$ are disjoint for any $j, j^{\prime} \in[q]$. If $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$, it can be verified that $\left|B_{i j} \cap B_{i^{\prime} j^{\prime}}\right| \leq w-1$ since intersection of size $w$ will imply that the related polynomials are equal. Hence, since each $\varepsilon_{i j \ell}$ is a collection of $w$-subsets of $B_{i j}$, we have that $\mathcal{E}_{i j \ell}$ and $\mathcal{E}_{i^{\prime} j^{\prime} \ell^{\prime}}$ are disjoint. If $(i, j)=\left(i^{\prime}, j^{\prime}\right)$ then $\mathcal{E}_{i j \ell}$ and $\mathcal{E}_{i j \ell^{\prime}}$ are from the same $(n, t, w)$-CPC code and therefore they are disjoint.

Construction 3 can be applied also on $(n, t, w)$-LPC code (instead of ( $n, t, w$ )-CPC code). The only condition is that there is no codeset in which there are codewords of different weight. Also, when there are codewords of weight $w^{\prime}<w$ in the codeset, the whole construction should work with $w^{\prime}$ instead of $w$, e.g. the degree of the polynomial must be at most $w^{\prime}-1$.

Corollary 23. Let $q$ be a prime power. If $t+w \leq n$ and $q \geq n+w-1$, then
(i) there exists an $(n q, t q, w)-C P C$ code of size $q^{w-1}$;
(ii) there exists an $(n q, t q, w)$-LPC code of size $\sum_{i=0}^{w-1} q^{i}$.

Proof. (i) the first claim follows from the fact that we can use an $(n, t, w)$-CPC code with exactly one codeset which contains all the $w$-subsets of the related $n$-set.
(ii) the second claim follows from the fact we can apply Construction 3 and claim (i) on any $w^{\prime} \leq w$ and obtain disjoint codes that can be combined together.

Example 24. We compare certain CPC codes obtained from Construction 3 and Corollary 23 with the LPC codes obtained from Proposition 9 .
(i) Consider the set of five disjoint $3-(10,4,1)$ designs constructed by Etzion and Hartman [9]. By taking the complements of the blocks we obtain a $(10,3,6)$-CPC code of size five. Applying Construction 3 with $q=16$, we obtain a $(160,48,6)$-CPC code of size $5 \cdot 16^{5} \approx 2^{22.322}$.
In contrast, Proposition 9 yields a $(160,48,6)$-LPC code of size $2^{17}$ by setting $s=137$ and $r=95$.
(ii) Setting $n=9, t=2, w=7$, and $q=16$ in Corollary 23 yields a $(144,32,7)$-LPC code of size $\sum_{i=0}^{6} 16^{i} \approx 2^{24.093}$. In contrast, Proposition 9 yields a $(144,32,7)$-LPC code of size $2^{18}$ by setting $s=121$ and $r=94$.

In the regime where $w$ is fixed and $t$ has order of magnitude as $n$, we show that the codes obtained in this section are asymptotically larger than those obtained from Proposition 9. The CPC codes obtained from Construction 3 and Corollary 23 attain the asymptotic upper bound $O\left((n q)^{w-1}\right)$ when $w$ is fixed. In contrast, if we apply Proposition 9 with $s=n q-\left\lceil\log _{2}\left(\sum_{i=0}^{w-1}\binom{n q-1}{i}\right\rceil\right.$ ) (the Gilbert-Varshamov lower bound) and $r=$ $n q-t q-\left\lfloor\log _{2}\left(\sum_{i=0}^{\frac{w}{2}}\binom{n q-t q}{i}\right)\right.$ (the Hamming upper bound), we obtain an $(n q, t q, w)$-LPC code of smaller size $O\left((n q)^{w / 2}\right)$, or $o\left((n q)^{w-1}\right)$.

## VI. LPC CODES FROM COOLING CODES

In this section we use a novel method to transform cooling codes into low-power cooling codes, while preserving the efficiency of the cooling codes. The construction is based on an injective mapping called domination mapping which was defined as follows in [6]

The Hamming ball of radius $w$ in $\{0,1\}^{n}$ is the set $\mathcal{B}(n, w)$ of all words of weight at most $w$. Explicitly, $\mathcal{B}(n, w) \triangleq\left\{\boldsymbol{y} \in\{0,1\}^{n}: \operatorname{wt}(\boldsymbol{y}) \leq w\right\}$. Given $m \leq n$, we are interested in injective mappings $\varphi$ from $\{0,1\}^{m}$ into $\mathcal{B}(n, w)$ that establish a certain domination relationship between positions in $x \in\{0,1\}^{m}$ and positions in its image $\boldsymbol{y}=\varphi(\boldsymbol{x})$. Specifically, one should be able to "switch off" every position $j \in[n]$ in $\boldsymbol{y}$ (that is, ensure that $y_{j}=0$ ) by switching off a corresponding position $i \in[m]$ in $x$ (that is, setting $x_{i}=0$ ). More precisely, let $G=([m] \cup[n], E)$ be a bipartite graph with $m$ left vertices and $n$ right vertices. If $G$ has no isolated right vertices, we refer to $G$ as a domination graph.

Definition 25. Given an injective map $\varphi:\{0,1\}^{m} \rightarrow \mathcal{B}(n, w)$ and a graph $G=([m] \cup[n], E)$, we say that $\varphi$ is a G-domination mapping, or G-dominating in brief, if

$$
\begin{aligned}
& \forall\left(x_{1}, x_{2}, \ldots, x_{m}\right) \in\{0,1\}^{m}, \forall(i, j) \in E: \\
& \\
& \text { if } \varphi\left(x_{1}, x_{2}, \ldots, x_{m}\right)=\left(y_{1}, y_{2}, \ldots, y_{n}\right) \text { and } x_{i}=0, \text { then } y_{j}=0
\end{aligned}
$$

We say that $\varphi$ is an $(m, n, w)$-domination mapping if there exists a domination graph $G=([m] \cup[n], E)$, such that $\varphi$ is $G$-dominating.

Properties of domination mappings, bounds on their parameters, constructions, and existence theorems were given in [6]. For our purpose we need some results from [6] and some which will be developed in the sequel.

The first one taken from [6] restricts the structure of the domination graph.

Lemma 26. The domination graph $G=([m] \cup[n], E)$ of an $(m, n, w)$-domination mapping has a subgraph with no isolated vertices and the degrees of the right vertices is exactly one.

In view of Lemma 26 we will assume in the sequel that our domination graphs have no isolated vertices and all the right vertices have degree exactly one. We will define the neighbourhood of a vertex $v$ is $G$ as the set of vertices adjacent to $v$ and denote it by $N(v)$. The following lemma is an immediate consequence of these observations and definition.

Lemma 27. If $U \subset[n]$ is a set of right vertices of $G$ then $N(U) \triangleq\{N(\boldsymbol{u}): \boldsymbol{u} \in U\}$ is a set of vertices in [m] and $|N(U)| \leq|U|$.

Next, the obvious connection between domination mappings, cooling codes, and low-power cooling codes is given in the following theorem.

Theorem 28. If there exists an $(m, t)$-cooling code $\mathbb{C}=\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{M}\right\}$ and an $(m, n, w)$-domination mapping $\varphi$ then the code $\mathbb{C}^{\prime}=\left\{\mathfrak{C}_{1}^{\prime}, \mathfrak{C}_{2}^{\prime}, \ldots, \mathcal{C}_{M}^{\prime}\right\}$, where

$$
\mathcal{C}_{i}^{\prime} \triangleq\left\{\varphi(\boldsymbol{x}): x \in \mathcal{C}_{i}\right\}, \text { for each } 1 \leq i \leq M
$$

in an $(n, t, w)$-LPC code.

Proof. The length $n$ and the weight which is smaller from or equal to $w$ for the codewords of $\mathbb{C}$ are immediate consequences from the definition of the $(m, n, w)$-domination mapping. Now, suppose we are given a $t$-subset $S^{\prime} \subset[n]$ and a codeset $\mathcal{C}_{i}^{\prime}$ for some $1 \leq i \leq M$. To complete the proof we have to show that there exists a codeword $u^{\prime} \in \mathcal{C}_{i}^{\prime}$ such that $\operatorname{supp}\left(\boldsymbol{u}^{\prime}\right) \cap S^{\prime}=\varnothing$. The $t$-subset $S^{\prime}$ can be viewed as a set of right vertices in the domination graph $G=([m] \cup[n], E)$. By Lemma 27, for the set of neighbours of $S^{\prime} \subset[n], S \triangleq N\left(S^{\prime}\right) \subset[m]$, we have that $|S| \leq\left|S^{\prime}\right|$ and hence $|S| \leq t$. Since $\mathbb{C}$ is an $(m, t)$-cooling code, it follows that there exists a codeword $\boldsymbol{u}$ in $\mathcal{C}_{i}$ such that $\operatorname{supp}(\boldsymbol{u}) \cap S=\varnothing$ which implies by the domination property that $\operatorname{supp}(\varphi(\boldsymbol{u})) \cap S^{\prime}=\varnothing$.

A product construction for domination mappings was presented in [6].
Let $\varphi_{1}:\{0,1\}^{m_{1}} \rightarrow \mathcal{B}\left(n_{1}, w_{1}\right)$ and $\varphi_{2}:\{0,1\}^{m_{2}} \rightarrow \mathcal{B}\left(n_{2}, w_{2}\right)$ be arbitrary domination mappings. Then their product $\varphi=\varphi_{1} \times \varphi_{2}$ is a mapping from $\{0,1\}^{m_{1}+m_{2}}$ into $\mathcal{B}\left(n_{1}+n_{2}, w_{1}+w_{2}\right)$ defined as follows:

$$
\varphi\left(x_{1}, x_{2}\right)=\left(\varphi_{1}\left(x_{1}\right), \varphi_{2}\left(x_{2}\right)\right)
$$

where $x_{1} \in\{0,1\}^{m_{1}}, x_{2} \in\{0,1\}^{m_{2}}$, and $(\cdot, \cdot)$ stands for string concatenation. That is, in order to find the image of a word $\boldsymbol{x} \in\{0,1\}^{m_{1}+m_{2}}$ under $\varphi$, we first parse $\boldsymbol{x}$ as $\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}\right)$, then apply $\varphi_{1}$ and $\varphi_{2}$ to the two parts.

Theorem 29. If $\varphi_{1}$ is an $\left(m_{1}, n_{1}, w_{1}\right)$-domination mapping and $\varphi_{2}$ is an $\left(m_{2}, n_{2}, w_{2}\right)$-domination mapping, then their product $\varphi=\varphi_{1} \times \varphi_{2}$ is an $\left(m_{1}+m_{2}, n_{1}+n_{2}, w_{1}+w_{2}\right)$-domination mapping.

The idea in Theorem 29 can be generalized as follows to a large number of domination mappings.
Theorem 30. Let $\varphi_{i}$ be an $\left(m_{i}, n_{i}, w_{i}\right)$-domination mapping for each $1 \leq i \leq \ell$, and let $\left(x_{1}, x_{2}, \ldots, x_{\ell}\right)$ be a binary word, where the length of $x_{i}$ is $m_{i}$, for each $1 \leq i \leq \ell$. The mapping $\varphi$, defined by

$$
\varphi\left(x_{1}, x_{2}, \ldots, x_{\ell}\right)=\left(\varphi_{1}\left(x_{1}\right), \varphi_{2}\left(x_{2}\right), \ldots, \varphi_{\ell}\left(x_{\ell}\right)\right),
$$

is also an ( $m, n, w$ )-domination mapping for $m=\sum_{i=1}^{\ell} m_{i}, n=\sum_{i=1}^{\ell} n_{i}$, and $w=\sum_{i=1}^{\ell} w_{i}$.
Domination mappings are not difficult to find (at least for small parameters). For example, in [6] ( $2,3,1$ )-domination mapping, ( $9,15,3$ )-domination mapping, and ( $12,20,4$ )-domination mapping were presented. These three domination mappings have also efficient encoding and decoding procedures.
Certainly, one can use an ( $m, n, w$ )-domination mapping to form an ( $n, t, w$ )-LPC code from an ( $m, t$ ) cooling code. The only question is whether there is an efficient encoding and decoding schemes for the constructed LPC code. Such encoding and decoding schemes should be based on efficient encoding and decoding schemes for both the related cooling code and the related domination mapping. For large parameters such coding procedures might not exist. Hence, it is better to use the product constructions using many domination mappings with small parameters, but with efficient encoding and decoding schemes. Our next construction for ( $n, t, w$ )-LPC codes is based on this idea. For demonstration we will use a specific family of $(n, t, w)$-LPC code, but the same idea will work on any set of parameters that can be obtained from domination mappings with small parameters by using Theorem 30 We will describe the construction via its encoding scheme.

Construction 4. Assume we are given $w \geq 6, m=3 w=9 \alpha+12 \beta, n=5 w=15 \alpha+20 \beta, t$, and an $(m, t)-$ cooling code $\mathbb{C}$ with $2^{k}$ codesets. We will construct an $(n, t, w)$-LPC code $\mathbb{C}^{\prime}$. Let $\boldsymbol{u}$ be the information word of length $k$ and let $\mathbb{C}_{\boldsymbol{u}}$ be its related codeset in $\mathbb{C}$. The encoder partitions the set of $m$ coordinates into $\alpha+\beta$ subsets, $\alpha$ subsets of size 9 and $\beta$ subsets of size 12. Similarly, it partitions the set $n$ coordinates of the codewords from $\mathbb{C}^{\prime}$ into $\alpha+\beta$ subsets, $\alpha$ subsets of size 15 and $\beta$ subsets of size 20 . Let $\varphi_{1}$ and $\varphi_{2}$ be a $(9,15,3)$-domination mapping and a ( $12,20,4$ )-domination mapping, respectively. Let $\varphi$ be the ( $m, n, w$ )-domination mapping implied by the product construction of Theorem 30 on $\alpha$ copies of $\varphi_{1}$ and $\beta$ copies of $\varphi_{2}$. Let $T$ be a $t$-subset of [ $n$ ] and let $T^{\prime}=N(T)$ be a $t^{\prime}$-subset of $[m]$, where $t^{\prime} \leq t$ by Lemma [27. The encoder finds the vector $v$ in $\mathbb{C}_{u}$ related to the set $T^{\prime}$, i.e. $v$ has zeroes in the coordinates of $T^{\prime}$, as required. Finally the encoder parse $v$ into $v_{1} v_{2} \ldots v_{\alpha} v_{1}^{\prime} v_{2}^{\prime} \ldots v_{\beta}^{\prime}$, where $v_{i}$ is of length 9 and $v_{i}^{\prime}$ is of length 12 . By using the encodings of the mappings $\varphi_{1}$ and $\varphi_{2}$, the encoder maps $v_{1} v_{2} \ldots v_{\alpha} v_{1}^{\prime} v_{2}^{\prime} \ldots v_{\beta}^{\prime}$ to the word $\varphi\left(v_{1} v_{2} \ldots v_{\alpha} v_{1}^{\prime} v_{2}^{\prime} \ldots v_{\beta}^{\prime}\right)$ of the ( $n, t, w$ )-LPC code, where each $v_{i}$ is mapped by $\varphi_{1}$ to a word of length 15 and each $v_{i}^{\prime}$ is mapped by $\varphi_{2}$ to a word of length 20 .
The decoder is applied in reverse order to generate the information word of length $k$ from a word of length $n$ of the ( $n=5 w, t, w$ )-LPC code $\mathbb{C}^{\prime}$, by first generating a word, of length $m$, from $\mathbb{C}$ and after that using the decoder of $\mathbb{C}$ to find the information word of length $m$.

Note that Construction 4 can be viewed as a modification of the Concatenation construction (See Proposition 8 ). Construction 4 has an advantage on the Concatenation of Proposition 8 and other constructions with larger size
for the same weight $w$ and the same number of hottest wires $t$.
How good are the codes constructed by using the domination mappings. They are incomparable with the other codes which were constructed in previous sections due to their parameters. But, they can be easily compared with the codes obtained in Proposition 8. We will consider some examples by using three of the most simple (and less powerful) domination mappings, a (2,3,1)-domination mapping, a (3,9,15)-domination mapping and a $(4,12,20)$-domination mapping (note that the last two were used in Construction (4).

To this end we will describe the simple and effective construction of cooling codes given in [3]. This construction is based on spreads (or partial spreads) which will be defined next.

Loosely speaking, a partial $\tau$-spread of the vector space $\mathbb{F}_{q}^{n}$ is a collection of disjoint $\tau$-dimensional subspaces of $\mathbb{F}_{q}^{n}$. Formally, a collection $V_{1}, V_{2}, \ldots, V_{M}$ of $\tau$-dimensional subspaces of $\mathbb{F}_{q}^{n}$ is said to be a partial $\tau$-spread of $\mathbb{F}_{q}^{n}$ if

$$
\begin{aligned}
& V_{i} \cap V_{j}=\{\mathbf{0}\} \text { for all } i \neq j, \\
& \mathbb{F}_{q}^{n} \supseteq V_{1} \cup V_{2} \cup \cdots \cup V_{M} .
\end{aligned}
$$

If the $\tau$-dimensional subspaces form a partition of $\mathbb{F}_{q}^{n}$ then the partial $\tau$-spread is called a $\tau$-spread. It is well known that such $\tau$-spreads exist if and only if $\tau$ divides $n$, in which case $M=\left(q^{n}-1\right) /\left(q^{\tau}-1\right)>q^{n-\tau}$. For the case where $\tau$ does not divide $n$, partial $\tau$-spreads with $M \geq q^{n-\tau}$ have been constructed in [10, Theorem 11].

Theorem 31. Let $V_{1}, V_{2}, \ldots, V_{M}$ be a partial $(t+1)$-spread of $\mathbb{F}_{2}^{n}$, and define the code $\mathbb{C}=\left\{V_{1}^{*}, V_{2}^{*}, \ldots, V_{M}^{*}\right\}$, where $V_{i}^{*}=V_{i} \backslash\{\mathbf{0}\}$ for all $i$. Then $\mathbb{C}$ is an $(n, t)$-cooling code of size $M \geq 2^{n-t-1}$.

Assume first that we want to construct an $(3 w, t, w)$-LPC code from a $(2 w, t)$-cooling code. We consider the trivial $(2,3,1)$-domination mapping and use a $(t+1)$-spread over $\mathbb{F}_{2}^{2 w}$, where $2(t+1) \leq 2 w$ to obtain a $(3 w, t, w)$ LPC code $\mathbb{C}$ of size $\frac{2^{2 w}-1}{2^{t+1}-1}>2^{2 w-t-1}$ (from a ( $2 w, t$ )-cooling code) for any $t+1 \leq w$. Assume now that we want to form a comparable code using Proposition 8 There are a few options that can be taken as parameters in Proposition 8

1) Assume first that we take $w^{\prime}=1, s=3$, and $m=w$, in Proposition 8 As a consequence we can take $q=4$, and hence the size of the $(3 w, t, w)$-LPC code obtained by Proposition 8 will be $2^{2 w-2 t-1}$, for $t \leq 3$ and $t+1 \leq w / 2$, which is clearly much smaller than $\mathbb{C}$. Moreover, $t$ is at most the minimum between 3 and $\frac{w}{2}-1$ compared to $t \leq w-1$ for the code $\mathbb{C}$ based on the ( $2,3,1$ )-domination mapping.
2) A different choice in Proposition 8 is $w^{\prime}=3, s=9$, and $m=w / 3$. As a consequence we can take $q=128$, and hence the size of the $(3 w, t, w)$-LPC code obtained by Proposition 8 will be $2^{7 w / 3-7 t-7}$ for any $t+1 \leq w / 6$, but not larger than 9 . Hence, $t$ is much smaller compared to $t \leq w-1$ for the code $\mathbb{C}$ based on the $(2,3,1)$-domination mapping. As for the size of the code, for the same $t$ the code $\mathbb{C}$ is larger if $t \geq \frac{w}{18}-1$
We continue with our constructed $(5 w, t, w)$-LPC code $\mathbb{C}$ from a $(3 w, t)$-cooling code discussed in Construction 4. It has $2^{3 w-t-1}$ codesets and it can handle any $t \leq 3 w / 2-1$ (if the spread construction of Theorem 31 is
used). Assume now that we want to form a comparable code by using Proposition 8. There are a few options that can be taken as parameters in Proposition 8
3) Assume first that we take $w^{\prime}=3, s=15$, and $m=w / 3$, in Proposition 8 As a consequence we can take $q=2^{9}$, and hence the size of the $(5 w, t, w)$-LPC code obtained by Proposition 8 will be $2^{3 w-9 t-9}$, for any $t+1 \leq w / 6$ but not larger than 15 , which is clearly much smaller than $\mathbb{C}$ for both size and $t$.
4) A different choice in Proposition 8 is $w^{\prime}=4, s=20$, and $m=w / 4$. As a consequence we can take $q=2^{12}$, and hence the size of the $(5 w, t, w)$-LPC code obtained by Proposition 8 will be $2^{3 w-12 t-12}$ for any $t+1 \leq w / 8$, but not larger than 20 , which is clearly much smaller than $\mathbb{C}$ for both size and $t$.

It should be noted that $q$ can be sometimes slightly larger than the one given in the example. This won't make much difference in the comparison, but the computation in a large field size which is not a power of 2 is more messy.

We conclude that the codes obtained by this new method have in most cases larger size and better capabilities than the best codes obtained by previous known constructions.

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