Hybrid Codes

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Abstract—A hybrid code can simultaneously encode classical and quantum information into quantum digits such that the information is protected against errors when transmitted through a quantum channel. It is shown that a hybrid code has the remarkable feature that it can detect more errors than a comparable quantum code that is able to encode the classical and quantum information. Weight enumerators are introduced for hybrid codes that allow to characterize the minimum distance of hybrid codes. Surprisingly, the weight enumerators for hybrid codes do not obey the usual MacWilliams identity.

I. INTRODUCTION

A hybrid code can simultaneously encode classical and quantum information into quantum digits such that the information is protected against errors when transmitted through a quantum channel. We will show that hybrid codes have the remarkable feature that they can always detect more errors than quantum error detecting codes. So hybrid codes are in general preferable to quantum error detecting codes for the simultaneous transmission of classical and quantum information over a quantum channel.

In their seminal paper [2], Devetak and Shor characterized the set of admissible rate pairs for simultaneous transmission of classical and quantum information over a given quantum channel. They showed that time-sharing a quantum channel for the separate encoding of quantum and classical information is inferior to simultaneous transmission. This line of research was extended in various directions. For instance, Hsieh and Wilde [4] considered the problem of simultaneous transmission of classical and quantum information over an entanglement-assisted quantum channel. Yard, Hayden and Devetak [10] considered multi-access channels with two senders and one receiver to communicate both classical and quantum information to the receiver. There are more papers in quantum information theory about the simultaneous transmission of classical and quantum information, but the small selection that we have mentioned should convey the flavor of this line of research.

We need codes to transmit classical and quantum information over a quantum channel. Of course, we can always use a quantum error-correcting code for this purpose, and simply encode the classical information in some quantum bits. However, this fails to take advantage of gains promised by quantum information theorists. Surprisingly, the foundations of hybrid code have not yet been well developed. We are aware of a few notable exceptions. Kremsky, Hsieh, and Brun investigated early on entanglement-assisted hybrid stabilizer codes [8]. Beny, Kempf, and Kribs briefly sketched an operator-theoretic construction of hybrid codes [1], an approach that has much potential. More recently, Grassl, Lu, and Zeng [3] gave a number of hybrid code constructions, derived linear programming bounds for hybrid stabilizer codes, and found very remarkable examples of hybrid codes with good parameters.

In the next section, we define the notion of detectable errors of a hybrid code. We show that hybrid codes can detect more errors than comparable quantum codes. In Section III, we introduce weight enumerators for hybrid codes. As in the case of quantum codes, we have two weight enumerators. For one of the weight enumerators, we use the average of the Shor-Laflamme weight enumerators for the quantum codes that encode in the quantum information. We show that the two weight enumerators allow us to characterize the errors that can be detected and corrected by the hybrid code. In Section IV, we show the unexpected result that weight enumerators of a hybrid code do not satisfy the MacWilliams identity, but rather a relaxed version of the MacWilliams identity.

II. HYBRID CODES

Suppose that we want to simultaneously transmit classical and quantum messages. Our goal will be to encode them into the state of n quantum digits that have q-levels each, so that the encoded message can be transmitted over a quantum channel. In other words, an encoded message is a unit vector in the Hilbert space

$$H = \bigotimes_{k=1}^{n} \mathbf{C}^{q} \cong \mathbf{C}^{q^{n}}.$$

A hybrid code has the parameters $((n, K: M))_q$ if and only if it can simultaneously encode one of M different classical messages and a superposition of K orthogonal quantum states into n quantum digits with q levels. We can understand the hybrid code as a collection of M orthogonal K-dimensional quantum codes C_m that are indexed by the classical messages $m \in [M] := \{1, 2, ..., M\}$. If we want to transmit a classical message $m \in [M]$ and a quantum state φ , then we need to encode φ into the quantum code C_m .

The encoded states will be subject to errors when transmitted through a quantum channel. Our first task will be to characterize the errors that can be detected by the hybrid code. We will set up a projective measurement that either upon receipt of a state $|\psi\rangle$ in H either (a) returns ϵ to indicate that an error happened or (b) or claims that there is no error and returns a classical message m and a projection of $|\psi\rangle$ onto C_m .

Let P_m denote the orthogonal projector onto the quantum code C_m for all integers m in the range $1 \le m \le M$. For distinct integers a and b in the range $1 \le a, b \le M$, the quantum codes C_a and C_b are orthogonal, so $P_bP_a = 0$. It follows that the orthogonal projector onto $C = \bigoplus_{m=1}^M C_m$ is given by

$$P = P_1 + P_2 + \dots + P_M.$$

We define the orthogonal projection onto C^{\perp} by $P_{\epsilon} = 1 - P$.

For the hybrid code $\{C_m \mid m \in [M]\}$, we can define a projective measurement \mathcal{P} that corresponds to the set

$$\{P_1, P_2, \ldots, P_M, P_\epsilon\}$$

of projection operators that partition unity.

We can now define the concept of a detectable error. An error E is called *detectable* by the hybrid code $\{C_m \mid m \in [M]\}$ if and only if for each index a, b in the range $1 \le a, b \le M$, we have

$$P_b E P_a = \begin{cases} \lambda_{E,a} P_a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

for some scalar $\lambda_{E,a}$.

The motivation for calling an error E detectable is the following simple protocol. Suppose that we encode a classical message m and a quantum state into a state v_m of C_m , and transmit it through a quantum channel that imparts the error E. If the error is detectable, then measurement of the state $Ev_m = EP_mv_m$ with the projective measurement \mathcal{P} either (E1) returns ϵ , which signals that an error happened, or

(E2) returns m and corrects the error by projecting the state back onto a scalar multiple $\lambda_{E,m}v_m = P_m E P_m v_m$ of the state v_m .

The definition of a detectable error ensures that the measurement \mathcal{P} will never return an incorrect classical message d, since $P_d E P_m v_m = 0$ for all $d \neq m$, so the probability of detecting an incorrect message is zero. An error that is not detectable by the hybrid code can change the encoded classical information, the encoded quantum information, or both.

The next proposition shows that hybrid codes can always detect more errors than a comparable quantum code that encodes both classical and quantum information. This is remarkable given that the advantages are much less apparent when one considers minimum distance, see [3].

Let B(H) denote the set of linear operators on H.

Proposition 1. The subset \mathcal{D} of detectable errors in B(H) of an $((n, K:M))_q$ hybrid code form a vector space of dimension

$$\dim \mathcal{D} = q^{2n} - (MK)^2 + M_{\star}$$

In particular, an $((n, K:M))_q$ hybrid code with M > 1 can detect more errors than an $((n, KM))_q$ quantum code.

Proof. It is clear that any linear combination of detectable errors is detectable. If we choose a basis adapted to the orthogonal decomposition $H = C \oplus C^{\perp}$ with

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_M,$$

then an error E is represented by a matrix of the form

$$\left(\begin{array}{cc}A & R\\S & T\end{array}\right)$$

Since E is detectable, the $MK \times MK$ matrix A must satisfy

$$A = \lambda_{E,1} 1_K \oplus \lambda_{E,2} 1_K \oplus \cdots \oplus \lambda_{E,M} 1_K,$$

where 1_K denote a $K \times K$ identity matrix, but R, S, and T can be arbitrary. Therefore, the dimension of the vector space of detectable errors is given by $q^{2n} - (MK)^2 + M$. The vector space of detectable errors of an $((n, KM))_q$ quantum code has dimension $q^{2n} - (KM)^2 + 1$, which is strictly less than $q^{2n} - (MK)^2 + M$.

We conclude this section with a few remarks on sets of detectable and correctable errors. Detectable errors have many nice features. The set \mathcal{D} of all detectable errors of a hybrid code is a vector space that contains the identity operator, is closed under taking adjoints *, and is a closed subspace of B(H). Therefore, the set \mathcal{D} of detectable errors is an operator system of the the C^* -algebra B(H). This means that we can express every detectable error in \mathcal{D} as a linear combination of detectable errors that are positive operators. Indeed, an operator E in \mathcal{D} can be expressed as linear combination E = A + iB, where $A = \frac{1}{2}(E + E^*)$ and $B = \frac{i}{2}(E^* - E)$ are self-adjoint operators in \mathcal{D} . A self-adjoint operator X in \mathcal{D} can be expressed as the difference of the positive operators $\|X\|\mathbf{1}$ and $\|X\|\mathbf{1} - X$. In short, the set of detectable errors of a hybrid code has a quite well-behaved structure.

On the other hand, whenever we consider the correctability of errors, we must consider an entire set of errors rather than a single error. Depending on the set of errors that we would like to correct, a given error operator E might or might not be correctable. It is not difficult to show that a unital set \mathcal{E} of errors is correctable if and only if the set

$$\mathcal{E}^*\mathcal{E} = \{F^*E \mid E, F \in \mathcal{E}\}$$

of errors is detectable. In other words, all errors $E, F \in \mathcal{E}$ must satisfy

$$P_b F^* E P_a = \lambda_{F^* E, a} \left[a = b \right] P_a$$

for all $a, b \in [M]$, where [a = b] denotes the Iverson-Knuth bracket that is equal to 1 when the condition a = b is satisfied and 0 otherwise.

In the next section, we will introduce the notion of a weight of errors and introduce weight enumerators of hybrid codes.

III. WEIGHT ENUMERATORS

In this section, we define weight enumerators for an $((n, K : M)_q$ hybrid code

$$\mathcal{H} = \{ C_m \mid m \in [M] \}$$

Before we can define the weight enumerators, we will briefly recall the concept of a nice error basis (see [7], [6], [5] for further details), so that we can define a suitable notion of weight for the errors.

Let G be a group of order q^2 with identity element 1. A nice error basis on \mathbb{C}^q is a set $\mathcal{E} = \{\rho(g) \in \mathcal{U}(q) | g \in G\}$ of unitary matrices such that

- (i) $\rho(1)$ is the identity matrix,
- (ii) $\operatorname{Tr} \rho(g) = 0$ for all $g \in G \setminus \{1\}$,
- (iii) $\rho(g)\rho(h) = \omega(g,h) \rho(gh)$ for all $g, h \in G$,

where $\omega(g, h)$ is a nonzero complex number depending on $(g, h) \in G \times G$; the function $\omega \colon G \times G \to \mathbf{C}^{\times}$ is called the factor system of ρ . We call G the *index group* of the error basis \mathcal{E} . The nice error basis that we have introduced so far generalizes the Pauli basis to systems with $q \ge 2$ levels.

We can obtain a nice error basis \mathcal{E}_n on $H \cong \mathbb{C}^{q^n}$ by tensoring *n* elements of \mathcal{E} , so

$$\mathcal{E}_n = \mathcal{E}^{\otimes n} = \{ E_1 \otimes E_2 \otimes \cdots \otimes E_n \mid E_k \in \mathcal{E}, 1 \le k \le n \}.$$

The weight of an element in \mathcal{E}_n are the number of non-identity tensor components. We write wt(E) = d to denote that the element E in \mathcal{E}_n has weight d.

We can associate with a hybrid code $\ensuremath{\mathcal{H}}$ two weight enumerators

$$A(z) = \sum_{d=0}^{n} A_d z^d$$
 and $B(z) = \sum_{d=0}^{n} B_d z^d$

where the coefficients are given by

$$A_d = \frac{1}{K^2 M} \sum_{\substack{a,b=1\\\text{wt}(E)=d}}^M \sum_{\substack{E \in \mathcal{E}_n\\\text{wt}(E)=d}} |\text{tr}(P_b E P_a)|^2$$

and

$$B_d = \frac{1}{K^2 M} \sum_{\substack{a,b=1\\\text{wt}(E)=d}}^M \sum_{\substack{E \in \mathcal{E}_n\\\text{wt}(E)=d}} \operatorname{tr}((P_b E P_a)(P_b E P_a)^*) \operatorname{tr}(P_a).$$

We note that both sums can be considerably simplified, but we leave them in the current form for now, since that simplifies the proof of the next proposition. We call (A_0, A_1, \ldots, A_n) and (B_0, B_1, \ldots, B_n) the weight distributions of the hybrid code \mathcal{H} .

There is only one element in \mathcal{E}_n of weight 0, namely the identity matrix. The normalization constants are chosen such that $A_0 = B_0 = 1$.

Proposition 2. Let \mathcal{H} be a $((n, K : M))_q$ hybrid code with weight distributions A_d and B_d . Then the weight distributions satisfy the following properties.

- (a) The inequality $B_d \ge A_d \ge 0$ holds for all integers d in the range $0 \le d \le n$.
- (b) We have $A_d = B_d$ if and only if \mathcal{H} can detect all errors in \mathcal{E}_n of weight d.
- *Proof.* (a) Recall that the Cauchy-Schwarz inequality for operators $A, B \in B(H)$ is given by

$$\left|\operatorname{tr}(A^*B)\right|^2 \le \operatorname{tr}(A^*A)\operatorname{tr}(B^*B) \tag{1}$$

and equality holds precisely when A and B are linearly dependent.

If we apply this inequality to the term $|\operatorname{tr}(P_b E P_a)|^2$ in A_d , then we find that

$$|\operatorname{tr}(P_b E P_a)|^2 = |\operatorname{tr}((P_b E P_a) P_a)|^2$$

$$\leq \operatorname{tr}((P_b E P_a)(P_b E P_a)^*) \operatorname{tr}(P_a^* P_a)$$

$$= \operatorname{tr}((P_b E P_a)(P_b E P_a)^*) \operatorname{tr}(P_a)$$

Summing over all $a, b \in [M]$ and all error operators E of weight d and normalizing, we obtain $B_d \ge A_d \ge 0$.

(b) If \mathcal{H} can detect all errors of weight d in \mathcal{E}_n , then

$$A_d = \frac{1}{M} \sum_{a=1}^M \sum_{\substack{E \in \mathcal{E}_n \\ \operatorname{wt}(E) = d}} |\lambda_{E,a}|^2 = B_d$$

Conversely, if equality $A_d = B_d$ holds, then it follows that for all $a, b \in [M]$ and every error E in \mathcal{E}_n of weight d the Cauchy-Schwarz inequality

$$|\operatorname{tr}((P_b E P_a) P_a)|^2 \leq \operatorname{tr}((P_b E P_a) (P_b E P_a)^*) \operatorname{tr}(P_a^* P_a) \quad (2)$$

holds with equality. Therefore, $P_b E P_a$ and P_a are linearly dependent for all $a, b \in [M]$ and all E with wt(E) = d. We will distinguish between (i) the diagonal case a = b and (ii) the off-diagonal case $a \neq b$.

(i) If a = b, then we can deduce that for each a ∈ [M] and each error operator E of weight d there exists a scalar λ_{E,a} such that

$$P_b E P_a = \lambda_{E,a} P_a.$$

(ii) If $a \neq b$, then both sides of the inequality are equal to 0, since the left-hand side satisfies

$$|\operatorname{tr}((P_b E P_a))|^2 = |\operatorname{tr}(P_b E P_a P_b)|^2 = 0.$$

On the right-hand side, we have $tr(P_a) = K \neq 0$, so we can deduce that

$$\operatorname{tr}((P_b E P_a)(P_b E P_a)^*) = 0.$$

Since $tr(XX^*) = ||X||^2 = 0$ implies that X = 0, we can conclude that $P_b E P_a = 0$.

In other words, if $A_d = B_d$, then it follows from (i) and (ii) that every error operator E in \mathcal{E}_n of weight d is detectable by the hybrid code \mathcal{H} .

We can simplify the expressions for the coefficients A_d and B_d of the weight distributions of a hybrid code. The coefficients A_d take a particularly simple form, namely they are equal to the average of the Shor-Laflamme weights [9] of the quantum codes C_m with $m \in [M]$.

Lemma 3. The weight A_d of an $((n, K: M))_q$ hybrid code $\mathcal{H} = \{C_m \mid m \in [M]\}$ is obtained by averaging the Shor-Laflamme weights $A_d(C_m)$ of the quantum codes C_m . In other words,

$$A_d = \frac{1}{K^2 M} \sum_{a=1}^M \sum_{\substack{E \in \mathcal{E}_n \\ \operatorname{wt}(E) = d}} |\operatorname{tr}(P_a E)|^2$$

for all integers d in the range $0 \le d \le n$.

Proof. The proof of the previous proposition revealed that the and the corresponding weight enumerator off-diagonal terms in

$$A_d = \frac{1}{K^2 M} \sum_{\substack{a,b=1 \\ \text{wt}(E)=d}}^M \sum_{\substack{E \in \mathcal{E}_n \\ \text{wt}(E)=d}} |\text{tr}(P_b E P_a)|^2$$

vanish, since $|tr(P_b E P_a)|^2 = 0$ when $a \neq b$. The diagonal terms $|tr(P_a E P_a)|^2$ are equal to $|tr(P_a E)|^2$, which proves the claim.

We can also simplify the expression

$$B_d = \frac{1}{K^2 M} \sum_{\substack{a,b=1\\\text{wt}(E)=d}}^M \sum_{\substack{E \in \mathcal{E}_n\\\text{wt}(E)=d}} \operatorname{tr}((P_b E P_a)(P_b E P_a)^*) \operatorname{tr}(P_a),$$

a little bit by simplifying the argument of the first trace and noting that $\operatorname{tr} P_a = K$. Then we obtain

$$B_d = \frac{1}{KM} \sum_{\substack{a,b=1\\wt(E)=d}}^M \sum_{\substack{E \in \mathcal{E}_n\\wt(E)=d}} \operatorname{tr}(P_b E P_a E^*) \,.$$

Unlike in the case of the weights A_d , the off-diagonal terms $tr(P_b E P_a E^*)$ of the weight B_d do not necessarily vanish.

IV. MACWILLIAMS IDENTITIES?

Given that the Shor-Laflamme weights of quantum codes obey the quantum MacWilliams identities [9], it is natural to ask whether the weight enumerators A(z) and B(z) of a hybrid code also satisfy the MacWilliams identity

$$B(z) = \frac{K}{q^n} (1 + (q^2 - 1)z)^n A\left(\frac{1 - z}{1 + (q^2 - 1)z}\right)?$$

Since the weight A_d of an $((n, K : M))_q$ hybrid code is given by the average of the A-weights of the quantum codes C_m , it is natural to consider the average of the dual weights

$$A_d^{\perp} = \frac{1}{KM} \sum_{a=1}^M \sum_{\substack{E \in \mathcal{E}_n \\ \operatorname{wt}(E) = d}} \operatorname{tr}(P_a E P_a E^*).$$

We can define the weight enumerator

$$A^{\perp}(z) = \sum_{d=0}^{n} A_d^{\perp} z^d$$

This weight enumerator captures the diagonal part A_d^{\perp} of each weight B_d . By mimicking the proof of Shor and Laflamme [9] for the MacWilliams identity for quantum codes, it is possible to show that the average weight enumerators satisfy

$$A^{\perp}(z) = \frac{K}{q^n} (1 + (q^2 - 1)z)^n A\left(\frac{1 - z}{1 + (q^2 - 1)z}\right)$$

If we define the off-diagonal weights

$$C_d = \frac{1}{KM} \sum_{\substack{a,b=1\\a\neq b}}^M \sum_{\substack{E \in \mathcal{E}_n\\ \operatorname{wt}(E)=d}} \operatorname{tr}(P_b E P_a E^*)$$

$$C(z) = \sum_{d=0}^{n} C_d^{\perp} z^d,$$

then we can express the weight enumerator B(z) in the form

$$B(z) = A^{\perp}(z) + C(z).$$

The coefficients of C(z) satisfy $C_d \ge 0$. By Proposition 2, we have $C_d = 0$ when all errors of weight d are detectable by the hybrid code.

In terms of A(z), the weight enumerator B(z) is given by

$$B(z) = \frac{K}{q^n} (1 + (q^2 - 1)z)^n A\left(\frac{1 - z}{1 + (q^2 - 1)z}\right) + C(z).$$

Thus, the usual MacWilliams identity does not hold for hybrid codes, but a relaxed version does.

V. CONCLUSIONS

Many protocols in quantum communication require the transmission of both classical and quantum information. Devetak and Shor showed in [2] that a time-sharing approach for the transmission of classical and quantum information is in general inferior to a simultaneous transmission. The question is how to accomplish this task. We showed that hybrid codes always offer an advantage over a comparable quantum code, since they allow one to detect more errors. We introduced weight enumerators for hybrid codes that allow one to characterize the highest weight of errors that can be detected by the code.

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