# Generalized Reed-Solomon Codes with Sparsest and Balanced Generator Matrices 

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#### Abstract

We prove that for any positive integers $n$ and $k$ such that $n \geq k \geq 1$, there exists an $[n, k]$ generalized Reed-Solomon (GRS) code that has a sparsest and balanced generator matrix (SBGM) over any finite field of size $q \geq n+\left\lceil\frac{k(k-1)}{n}\right\rceil$, where sparsest means that each row of the generator matrix has the least possible number of nonzeros, while balanced means that the number of nonzeros in any two columns differ by at most one. Previous work by Dau et al (ISIT'13) showed that there always exists an MDS code that has an SBGM over any finite field of size $q \geq\binom{ n-1}{k-1}$, and Halbawi et al (ISIT'16, ITW'16) showed that there exists a cyclic Reed-Solomon code (i.e., $n=q-1$ ) with an SBGM for any prime power $q$. Hence, this work extends both of the previous results.


## I. Introduction

Maximum distance separable (MDS) codes, especially Reed-Solomon (RS) codes, with constrained generator matrices are recently attracting attention for their applications in the scenarios where encoding is performed in a distributed way [1]- [11]. Examples of such scenarios include wireless sensor networks [1], cooperative data exchange [3], [4], [11], and simple multiple access networks [2], [6]. An interesting problem of this topic is how to construct MDS codes that have a sparsest and balanced generator matrix (SBGM), where sparsest means that each row of the generator matrix has the least possible number of nonzeros, while balanced means that the number of nonzeros in any two columns differ by at most one [1]. More specifically, in an SBGM of an $[n, k]$ MDS code, the weight of each row is $n-k+1$ and the weight of each column is either $\left\lfloor\frac{k(n-k+1)}{n}\right\rfloor$ or $\left\lceil\frac{k(n-k+1)}{n}\right\rceil$.

In general, for every MDS code we can easily find a sparsest generator matrix. The difficulty of this problem is to ensure that a sparsest generator matrix is also balanced. In [1], it was shown that there always exists an MDS code with an SBGM over any finite field of size $q>\binom{n-1}{k-1}$ for any $n \geq k \geq 1$. The authors in [9] constructed an $[n, k]$ cyclic Reed-Solomon code (i.e., $n=q-1$ ) that has an SBGM for any prime power $q$ and any $k$ such that $1 \leq k \leq n$. However, it was left as an open problem whether there exists an $[n, k]_{q}$ generalized Reed-Solomon (GRS) code with an SBGM for $k \leq n<q-1$. In this paper, we extends the results in [1], [9] by proving that for any positive integers $n$ and $k$ such that $n \geq k \geq 1$, there exists an $[n, k]$ generalized Reed-Solomon code that has an SBGM over any finite field $\mathbb{F}_{q}$ of size $q \geq n+\left\lceil\frac{k(k-1)}{n}\right\rceil$.

## A. Related Work

MDS codes with more general constraints on the support of their generator matrices were studied in [2], [4], [5]. A conjecture, called GM-MDS Conjecture, was proposed in [5] stating that given any $k \times n$ binary matrix $M=\left(m_{i, j}\right)$ that satisfies the so-called MDS Condition, there exists an $[n, k]_{q}$ MDS code for any prime power $q \geq n+k-1$ that has a generator matrix $G=\left(g_{i, j}\right)$ satisfying $g_{i, j}=0$ whenever $m_{i, j}=0$, where the MDS Condition requests that for any $r \in\{1,2, \cdots, k\}$, the union of the supports of any $r$ rows of $M$ has size at least $n-k+r{ }^{11}$ Unfortunately, the GM-MDS Conjecture is proved to be true only for some very special cases, that is, a) the rows of $M$ are divided into three groups such that the rows within each group have the same support [2]; or b) the supports of any two rows of $M$ intersect with at most one element [5]; or c) $k \leq 5$ [10].

## II. Preliminary

For any positive integer $n,[n]:=\{1,2, \cdots, n\}$; if $n \leq 0$, $[n]$ is the empty set. For any set $A,|A|$ is the size (i.e., the number of elements) of $A$. We denote by $\mathbb{F}_{q}$ the field with $q$ elements, where $q \geq 2$ is a prime power. The support of a row/column vector over $\mathbb{F}_{q}$ is the set of its nonzero coordinates and the weight of a row/column vector is the size of its support.
A multiset $S$ with underlying set $\left\{s_{1}, s_{2}, \cdots, s_{L}\right\}$ is a set of ordered pairs $S=\left\{\left(s_{1}, n_{1}\right),\left(s_{2}, n_{2}\right), \cdots,\left(s_{L}, n_{L}\right)\right\}$, where each $n_{i} \geq 0$ is an integer, called the multiplicity of $s_{i}$ and denoted by $n_{i}=\operatorname{mult}_{S}\left(s_{i}\right)$. We also denote $S$ as $S=\left\{s_{1}, \cdots, s_{1}, s_{2}, \cdots, s_{2}, \cdots, s_{L}, \cdots, s_{L}\right\}$, where each $s_{i}$ appears $n_{i}$ times in $S$ and is also called an element of $S$. The size $|S|$ of $S$ is the sum of the multiplicities of its different elements, i.e., $|S|=\sum_{i=1}^{L} n_{i}$. Any subset $S_{0}$ of $\left\{s_{1}, s_{2}, \cdots, s_{L}\right\}$ can be viewed as a multiset such that $\operatorname{mult}_{S_{0}}\left(s_{i}\right)=1$ if $s_{i} \in S_{0}$, and mult $S_{S_{0}}\left(s_{i}\right)=0$ if $s_{i} \notin S_{0}$. If $S^{\prime}=\left\{\left(s_{1}, m_{1}\right),\left(s_{2}, m_{2}\right), \cdots,\left(s_{L}, m_{L}\right)\right\}$ is another multiset, not necessarily $S^{\prime} \neq S$, the union of $S$ and $S^{\prime}$, denoted by $S \sqcup S^{\prime}$, is $\left\{\left(s_{1}, n_{1}+m_{1}\right),\left(s_{2}, n_{2}+m_{2}\right), \cdots,\left(s_{L}, n_{L}+m_{L}\right)\right\}$.
Let $\mathcal{P}_{k}[x]$ denote the set of polynomials in $\mathbb{F}_{q}[x]$ of degree less than $k$, including the zero polynomial, where $x$ is an indeterminant. Then $\mathcal{P}_{k}[x]$ is a $k$-dimensional vector space over $\mathbb{F}_{q}$ according to the usual addition and multiplication of polynomials. Let $q \geq n \geq k$ and $a_{1}, a_{2}, \cdots, a_{n}$ be $n$ distinct

[^0]elements of $\mathbb{F}_{q}$. The $[n, k]$ generalized Reed-Solomon (GRS) code defined by $a_{1}, a_{2}, \cdots, a_{n}$ is [?]:
$$
\mathcal{C}=\left\{\left(f\left(a_{1}\right), f\left(a_{2}\right), \cdots, f\left(a_{n}\right)\right) ; f \in \mathcal{P}_{k}[x]\right\} .
$$

The code $\mathcal{C}$ is an MDS code, i.e., the minimum distance of $\mathcal{C}$ is $d=n-k+1$. A generator matrix $G$ of $\mathcal{C}$ is said to be sparsest and balanced if $G$ satisfies the following two conditions:
(P1) Sparsest condition: the weight of each row of $G$ is exactly $n-k+1$;
(P2) Balanced condition: the weight of each column of $G$ is either $\left\lfloor\frac{k(n-k+1)}{n}\right\rfloor$ or $\left\lceil\frac{k(n-k+1)}{n}\right\rceil$.
A GRS code that has a sparsest and balanced generator matrix (SBGM) is simply called a sparsest and balanced GRS code.

## III. Existence of Sparsest and Balanced GrS Codes

In this section, we prove that there always exists a sparsest and balanced $[n, k]$ GRS code for any $n \geq k \geq 1$. Formally, we have the following theorem.

Theorem 1: For any $n \geq k \geq 1$, there exists an $[n, k]$ generalized Reed-Solomon code that has a sparsest and balanced generator matrix over any field $\mathbb{F}_{q}$ of size $q \geq n+\left\lceil\frac{k(k-1)}{n}\right\rceil$.

Clearly, $[1,1, \cdots, 1]$ is an SBGM of the $[n, 1]$ GRS code; and the identity matrix is an SBGM of the $[n, n]$ GRS code. Hence, in the following, we only need to consider the case of

$$
n>k \geq 2
$$

Before proving Theorem 1 we first prove two lemmas.
First, let $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ be an $n$-tuple of distinct indeterminants. For each subset $Z$ of $[n]$ and $0 \leq \ell \leq|Z|$, let $s_{Z}^{(\ell)}(\boldsymbol{\alpha})$ be the $\ell$ th elementary symmetric polynomial with respect to $\left\{\alpha_{j} ; j \in Z\right\}$. That is,

$$
s_{Z}^{(0)}(\boldsymbol{\alpha})=1
$$

and for $1 \leq \ell \leq|Z|$,

$$
s_{Z}^{(\ell)}(\boldsymbol{\alpha})=\sum_{U \subseteq Z \text { and }|U|=\ell}\left(\prod_{j \in U} \alpha_{j}\right) .
$$

Then we have the following lemma.
Lemma 1: Suppose $n>k \geq 2$. There exists a $k \times n$ binary matrix $W=\left(w_{i, j}\right)$ satisfying the following four conditions:
(i) The weight of each row of $W$ is $k-1$;
(ii) The weight of each column of $W$ is either $\left\lfloor\frac{k(k-1)}{n}\right\rfloor$ or $\left\lceil\frac{k(k-1)}{n}\right\rceil$;
(iii) $\xi(\boldsymbol{\alpha}) \not \equiv 0$, where

$$
\xi(\boldsymbol{\alpha})=\left|\begin{array}{cccc}
s_{Z_{1}}^{(0)}(\boldsymbol{\alpha}) & s_{Z_{2}}^{(0)}(\boldsymbol{\alpha}) & \cdots & s_{Z_{k}}^{(0)}(\boldsymbol{\alpha})  \tag{1}\\
s_{Z_{1}}^{(1)}(\boldsymbol{\alpha}) & s_{Z_{2}}^{(1)}(\boldsymbol{\alpha}) & \cdots & s_{Z_{k}}^{(1)}(\boldsymbol{\alpha}) \\
s_{Z_{1}}^{(2)}(\boldsymbol{\alpha}) & s_{Z_{2}}^{(2)}(\boldsymbol{\alpha}) & \cdots & s_{Z_{k}}^{(2)}(\boldsymbol{\alpha}) \\
\cdots & \cdots & \cdots & \cdots \\
s_{Z_{1}}^{(k-1)}(\boldsymbol{\alpha}) & s_{Z_{2}}^{(k-1)}(\boldsymbol{\alpha}) & \cdots & s_{Z_{k}}^{(k-1)}(\boldsymbol{\alpha})
\end{array}\right|
$$

and $Z_{i}$ is the support of the $i$ th row of $W, \forall i \in[k]$;
(iv) The degree of each $\alpha_{i}$ in $\xi(\boldsymbol{\alpha})$ is at most $\left\lceil\frac{k(k-1)}{n}\right\rceil$.

Proof: First, consider $n \geq k(k-1)$. In this case, we have $\left\lceil\frac{k(k-1)}{n}\right\rceil=1$. Let $W=\left(w_{i, j}\right)$ such that $w_{i, j}=1$ for each $i \in[k]$ and each $(i-1)(k-1)+1 \leq j \leq i(k-1)$, and $w_{i, j}=0$ otherwise. Then we have $Z_{i}=\{j \in[n] ;(i-1)(k-1)+1 \leq$ $j \leq i(k-1)\}$, and $Z_{1}, Z_{2}, \cdots, Z_{k}$ are mutually disjoint. It is easy to check that $W$ satisfies conditions (i) - (iv).

In the following, we consider the case that $k(k-1)>n$. Since we have assumed $n>k \geq 2$, then we always have

$$
k(k-1)>n>k \geq 2
$$

For convenience, we write $k(k-1)$ as

$$
\begin{equation*}
k(k-1)=a n+r \tag{2}
\end{equation*}
$$

where $0 \leq r \leq n-1$, and let

$$
\begin{align*}
\boldsymbol{\delta} & =\left(\delta_{1}, \delta_{2}, \cdots, \delta_{n}\right) \\
& =(\overbrace{a+1, \cdots, a+1}^{r(a+1)^{\prime} \mathrm{s}}, \tag{3}
\end{align*}, \overbrace{a, \cdots, a)}^{n-r} .
$$

Here we point out some simple facts about $a$ and $\delta$. First, since $k(k-1)>n>k \geq 2$, if $r=0$, then

$$
2 \leq a=\left\lfloor\frac{k(k-1)}{n}\right\rfloor=\left\lceil\frac{k(k-1)}{n}\right\rceil<k-1
$$

if $0<r \leq n-1$, then

$$
1 \leq a=\left\lfloor\frac{k(k-1)}{n}\right\rfloor<a+1=\left\lceil\frac{k(k-1)}{n}\right\rceil \leq k-1
$$

So we always have

$$
\begin{equation*}
\delta_{j} \in\left\{\left\lfloor\frac{k(k-1)}{n}\right\rfloor,\left\lceil\frac{k(k-1)}{n}\right\rceil\right\} . \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
2 \leq a+1 \leq k-1 \tag{5}
\end{equation*}
$$

Moreover, by (2) and (3), we have

$$
\begin{equation*}
\sum_{j=1}^{n} \delta_{j}=(a+1) r+a(n-r)=a n+r=k(k-1) \tag{6}
\end{equation*}
$$

Construction of $\boldsymbol{W}$ : The binary matrix $W$ is constructed by the following three steps.

Step 1. List the elements of the multiset

$$
S=\{(1, k-1),(2, k-2), \cdots,(k-1,1)\}
$$

in a sequence

$$
\begin{align*}
\bar{S} & =\underbrace{1,2, \cdots, k-1}_{c_{1}, c_{2}, \cdots, c_{K}}, \underbrace{1,2, \cdots, k-2}, \cdots, \underbrace{1,2}, 1  \tag{7}\\
& 1
\end{align*}
$$

where

$$
\begin{equation*}
K=\sum_{\ell=1}^{k-1} \ell=\frac{k(k-1)}{2} \tag{8}
\end{equation*}
$$

Then construct subsets $S_{1}, S_{2}, \cdots, S_{n}$ of $[k]$ by Algorithm 1 .
Step 2. List the elements of the multiset

$$
T=\{(k, k-1),(k-1, k-2), \cdots,(2,1)\}
$$

in a sequence

$$
\begin{align*}
\bar{T} & =k, \underbrace{k-1, k}, \underbrace{k-2, k-1, k}, \cdots, \underbrace{2,3, \cdots, k} \\
& =e_{1}, e_{2}, \cdots, e_{K} \tag{9}
\end{align*}
$$

and let

$$
\begin{equation*}
\boldsymbol{\theta}=\left(\theta_{1}, \theta_{2}, \cdots, \theta_{n}\right)=\boldsymbol{\delta}-\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right) \tag{10}
\end{equation*}
$$

Then construct subsets $T_{1}, T_{2}, \cdots, T_{n}$ of $[k]$ by Algorithm 2 .
Step 3. Let $W$ be the $k \times n$ binary matrix such that for each $j \in[n], Y_{j}=S_{j} \cup T_{j}$ is the support of the $j$ th column of $W$.

## Algorithm 1

```
Input: \(\bar{S}=c_{1}, c_{2}, \cdots, c_{K}\), and \(\boldsymbol{\delta}=\left(\delta_{1}, \delta_{2}, \cdots, \delta_{n}\right)\);
Output: \(S_{1}, S_{2}, \cdots, S_{n}\);
Initialization: \(L=0, j=0\);
    while \(L<K\) do
        \(j=j+1\);
        if \(L<\sum_{\ell=a+1}^{k-1} \ell\) then
            \(S_{j}=\left\{c_{L+1}, \cdots, c_{L+\delta_{j}}\right\} ;\)
        else if \(\sum_{\ell=m+1}^{k-1} \ell \leq L<\sum_{\ell=m}^{k-1} \ell\) for some
        \(m \in[a]=\{1,2, \cdots, a\}\) then
            \(S_{j}=\left\{c_{L+1}, \cdots, c_{L+m}\right\} ;\)
        end if
        \(L=L+\left|S_{j}\right| ;\)
    end while
    if \(j<n\)
        \(S_{j+1}=\cdots=S_{n}=\emptyset ;\)
    end if
```


## Algorithm 2

```
Input: \(\bar{T}=e_{1}, e_{2}, \cdots, e_{K}\), and \(\boldsymbol{\theta}=\left(\theta_{1}, \theta_{2}, \cdots, \theta_{n}\right)\);
Output: \(T_{1}, T_{2}, \cdots, T_{n}\);
Initialization: \(L=0, j=1\);
    while \(L<K\) do
        \(j=j+1 ;\)
        \(T_{j}=\left\{e_{L+1}, \cdots, e_{L+\theta_{j}}\right\} ;\)
        \(L=L+\left|T_{j}\right| ;\)
    end while
```

Two examples of our construction are given in Section IV. Moreover, we have the following three claims.

Claim 1. For each $j \in[n], S_{j}$ is a subset of $[k]$ and, when viewed as multisets, we have $\sqcup_{j=1}^{\lambda_{1}} S_{j}=S$, where $\lambda_{1}$ is the value of $j$ at the end of the while loop of Algorithm 1.

Claim 2. For each $j \in[n], T_{j}$ is a subset of $[k]$ and $S_{j} \cap T_{j}=$ $\emptyset$. Moreover, when viewed as multisets, we have $\sqcup_{j=1}^{n} T_{j}=T$.

Claim 3. Let $X^{*}=\left\{\left(1,\left|S_{1}\right|\right),\left(2,\left|S_{2}\right|\right), \cdots,\left(\lambda_{1},\left|S_{\lambda_{1}}\right|\right)\right\}$. Then there exist a unique $\sigma^{*} \in \mathscr{S}_{k}$ and a unique $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$ such that $X^{*}=X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}$, where $\mathscr{S}_{k}$ denotes the permutation group on $[k]$ and, for each $\sigma \in \mathscr{S}_{k}, \mathcal{X}_{\sigma}$ denotes the set of all tuples $\left(X_{1}, X_{2}, \cdots, X_{k}\right)$ such that $X_{i} \subseteq Z_{i}$ and $\left|X_{i}\right|=\sigma(i)-1, i=1,2, \cdots, k$.

The proof of Claims $1-3$ are given in Appendices A C, respectively.

Note that for each $i \in[k]$, mult ${ }_{S \sqcup T}(i)=k-1$. Then by Claims 1 and 2, each $i \in[k]$ is contained by $k-1$ sets in the collection $\left\{Y_{1}, Y_{2}, \cdots, Y_{n}\right\}$, where $Y_{j}(j \in[n])$ is the support of the $j$ th column of $W$ by our construction. So each row of $W$ has weight $k-1$, hence condition (i) is satisfied.

For each $j \in[n]$, by (4), $\delta_{j} \in\left\{\left\lfloor\frac{k(k-1)}{n}\right\rfloor,\left\lceil\frac{k(k-1)}{n}\right\rceil\right\}$. So by Claims 1, 2 and Algorithm 2, the weight of the $j$ th column of $W$ is $\left|Y_{j}\right|=\left|S_{j}\right|+\left|T_{j}\right|=\delta_{j} \in\left\{\left\lfloor\frac{k(k-1)}{n}\right\rfloor,\left\lceil\frac{k(k-1)}{n}\right\rceil\right\}$, hence condition (ii) is satisfied.

For any multiset $X=\left\{\left(1, \ell_{1}\right),\left(2, \ell_{2}\right), \cdots,\left(n, \ell_{n}\right)\right\}$, let

$$
\boldsymbol{\alpha}^{X}:=\prod_{j=1}^{n} \alpha_{j}^{\ell_{j}}
$$

Then from (1), we have

$$
\begin{align*}
\xi(\boldsymbol{\alpha}) & =\sum_{\sigma \in \mathscr{S}_{k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} s_{Z_{i}}^{(\sigma(i)-1)}(\boldsymbol{\alpha}) \\
& =\sum_{\sigma \in \mathscr{S}_{k}} \operatorname{sgn}(\sigma) \sum_{\left(X_{1}, X_{2}, \cdots, X_{k}\right) \in \mathcal{X}_{\sigma}} \boldsymbol{\alpha}^{X_{1}} \boldsymbol{\alpha}^{X_{2}} \cdots \boldsymbol{\alpha}^{X_{k}} \\
& =\sum_{\sigma \in \mathscr{S}_{k}} \operatorname{sgn}(\sigma) \sum_{\left(X_{1}, X_{2}, \cdots, X_{k}\right) \in \mathcal{X}_{\sigma}} \boldsymbol{\alpha}^{X_{1} \sqcup X_{2} \sqcup \cdots \sqcup X_{k}} . \tag{11}
\end{align*}
$$

where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation $\sigma$. By Claim 3, there exist a unique $\sigma^{*} \in \mathscr{S}_{k}$ and a unique $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$ such that $X^{*}=X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}$. So by 111, $\operatorname{sgn}\left(\sigma^{*}\right) \boldsymbol{\alpha}^{X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}}$ is a non-zero monomial in $\xi(\boldsymbol{\alpha})$. Hence, $\xi(\boldsymbol{\alpha}) \not \equiv 0$ and condition (iii) is satisfied.

Note that $X_{i} \subseteq Z_{i}, \forall i \in[k]$, and each column of $W$ has weight either $\left\lfloor\frac{k(k-1)}{n}\right\rfloor$ or $\left\lceil\frac{k(k-1)}{n}\right\rceil$, i.e., each $j \in[n]$ is contained by at most $\left\lceil\frac{k(k-1)}{n}\right\rceil$ sets in $\left\{Z_{1}, Z_{2}, \cdots, Z_{k}\right\}$, where $Z_{i}$ is the support of the $i$ th row of $W$. So in (11), the degree of $\alpha_{j}$ in each $\boldsymbol{\alpha}^{X_{1} \sqcup X_{2} \sqcup \cdots \sqcup X_{k}}$ is at most $\left\lceil\frac{k(k-1)}{n}\right\rceil$. Hence, the degree of $\alpha_{j}$ in $\xi(\boldsymbol{\alpha})$ is at most $\left\lceil\frac{k(k-1)}{n}\right\rceil$. Hence, condition (iv) is satisfied, which completes the proof.

Lemma 2: Suppose $\xi\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ is a nonzero polynomial over the field $\mathbb{F}_{q}$ such that the degree of each $\alpha_{i}$ is at most $m(m \geq 1)$. If $q \geq n+m$, then there exist distinct $a_{1}, a_{2}, \cdots, a_{n} \in \mathbb{F}_{q}$ such that $\xi\left(a_{1}, a_{2}, \cdots, a_{n}\right) \neq 0$.

Proof: Similar to the Schwartz-Zippel Theorem, this lemma can be proved by induction on the number of indeterminants $n$. First, for $n=1, \xi\left(\alpha_{1}\right)$ has at most $m$ zeros in $\mathbb{F}_{q}$ because the degree of $\alpha_{1}$ is at most $m$. So there exists an $a_{1} \in \mathbb{F}_{q}$ such that $\xi\left(a_{1}\right) \neq 0$, provided that $q \geq 1+m$.

Now assume that $n>1, q \geq n+m$ and the induction hypothesis is true for polynomials of up to $n-1$ indeterminants. Consider the polynomial $\xi\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$. Without loss of
generality, assume the degree of $\alpha_{1}$ in $\xi$ is $t(1 \leq t \leq m)$. Then we can factor out $\alpha_{1}$ and obtain

$$
\xi\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\sum_{i=0}^{t} \alpha_{1}^{i} \xi_{i}\left(\alpha_{2}, \cdots, \alpha_{n}\right)
$$

where $\xi_{t}\left(\alpha_{2}, \cdots, \alpha_{n}\right) \not \equiv 0$. Clearly, the degree of each $\alpha_{i}(2 \leq i \leq n)$ in $\xi_{t}$ is at most $m$. The induction hypothesis implies that there exist distinct $a_{2}, \cdots, a_{n} \in \mathbb{F}_{q}$ such that $\xi_{t}\left(a_{2}, \cdots, a_{n}\right) \neq 0$. Then the polynomial

$$
\eta\left(\alpha_{1}\right)=\xi\left(\alpha_{1}, a_{2}, \cdots, a_{n}\right)=\sum_{i=0}^{t} \alpha_{1}^{i} \xi_{i}\left(a_{2}, \cdots, a_{n}\right) \not \equiv 0
$$

and has degree $t$. Note that $q \geq n+m \geq n+t$. There exists an $a_{1} \in \mathbb{F}_{q} \backslash\left\{a_{2}, \cdots, a_{n}\right\}$ such that

$$
\xi\left(a_{1}, a_{2}, \cdots, a_{n}\right)=\eta\left(a_{1}\right) \neq 0
$$

This completes the induction.
Now we are able to prove Theorem 1.
Proof of Theorem 17. Let $W$ be a $k \times n$ binary matrix satisfying conditions (i) - (iv) of Lemma 1 By Lemma 2 , if $q \geq n+\left\lceil\frac{k(k-1)}{n}\right\rceil$, there exist distinct $a_{1}, a_{2}, \cdots, a_{n} \in \mathbb{F}_{q}$ such that $\xi\left(a_{1}, a_{2}, \cdots, a_{n}\right) \neq 0$.

For each $i \in[k]$, let

$$
\begin{equation*}
f_{i}(x)=\prod_{j \in Z_{i}}\left(x-a_{j}\right) \tag{12}
\end{equation*}
$$

where $Z_{i}$ is the support of the $i$ th row of $W$. Clearly, $f_{1}(x)$, $f_{2}(x), \cdots, f_{k}(x) \in \mathcal{P}_{k}[x]$. Moreover, $f_{1}(x), f_{2}(x), \cdots, f_{k}(x)$ are linearly independent in $\mathcal{P}_{k}[x]$, which can be proved as follows. By (12), we have

$$
\begin{aligned}
f_{i}(x) & =\prod_{j \in Z_{i}}\left(x-a_{j}\right) \\
& =x^{k-1}+\sum_{\ell=1}^{k-1}\left[\sum_{U \subseteq Z_{i},|U|=\ell}\left(\prod_{j \in U} a_{j}\right)\right](-1)^{\ell} x^{k-1-\ell} \\
& =x^{k-1}+\sum_{\ell=1}^{k-1} s_{Z_{i}}^{(\ell)}\left(a_{1}, a_{2}, \cdots, a_{n}\right)(-1)^{\ell} x^{k-1-\ell}
\end{aligned}
$$

for each $i \in[k]$. Denote $c_{i, \ell}:=s_{Z_{i}}^{(\ell)}\left(a_{1}, a_{2}, \cdots, a_{n}\right)$ and

$$
\begin{aligned}
& C= \\
& {\left[\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
-c_{1,1} & -c_{2,1} & \cdots & -c_{k, 1} \\
\cdots & \cdots & \cdots & \cdots \\
(-1)^{k-1} c_{1, k-1} & (-1)^{k-1} c_{2, k-1} & \cdots & (-1)^{k-1} c_{k, k-1}
\end{array}\right]}
\end{aligned}
$$

Then $f_{1}(x), f_{2}(x), \cdots, f_{k}(x)$ are linearly independent in $\mathcal{P}_{k}[x]$ if and only if $\operatorname{det}(C) \neq 0$. From (1), we can easily see that

$$
\xi\left(a_{1}, a_{2}, \cdots, a_{n}\right)=(-1)^{1+2+\cdots+(k-1)} \operatorname{det}(C)
$$

Since $\xi\left(a_{1}, a_{2}, \cdots, a_{n}\right) \neq 0$, then $\operatorname{det}(C) \neq 0$. Hence, $f_{1}(x), f_{2}(x), \cdots, f_{k}(x)$ are linearly independent in $\mathcal{P}_{k}[x]$.

Now, let $\mathcal{C}$ be the GRS code defined by $a_{1}, a_{2}, \cdots, a_{n}$ and

$$
G=\left(\begin{array}{cccc}
f_{1}\left(a_{1}\right) & f_{1}\left(a_{2}\right) & \cdots & f_{1}\left(a_{n}\right) \\
f_{2}\left(a_{1}\right) & f_{2}\left(a_{2}\right) & \cdots & f_{2}\left(a_{n}\right) \\
\cdots & \cdots & \cdots & \cdots \\
f_{k}\left(a_{1}\right) & f_{k}\left(a_{2}\right) & \cdots & f_{k}\left(a_{n}\right)
\end{array}\right) .
$$

Since $f_{1}(x), f_{2}(x), \cdots, f_{k}(x)$ are linearly independent in $\mathcal{P}_{k}[x]$, then $G$ is a generator matrix of $\mathcal{C}$.

By assumption, $W$ satisfies conditions (i) and (ii) of Lemma (1) that is, the weight of each row of $W$ is $k-1$ and the weight of each column of $W$ is either $\left\lfloor\frac{k(k-1)}{n}\right\rfloor$ or $\left\lceil\frac{k(k-1)}{n}\right\rceil$. Moreover, by (12), for each $i \in[k]$ and $j \in[n], f_{i}\left(a_{j}\right)=0$ if and only if $j \in Z_{i}$, that is $w_{i, j}=1$ (because $Z_{i}$ is the support of the $i$ th row of $W$ ). So according to the construction of $G$, the number of zeros in every row of $G$ is $k-1$ and the number of zeros in every column of $G$ is either $\left\lfloor\frac{k(k-1)}{n}\right\rfloor$ or $\left\lceil\frac{k(k-1)}{n}\right\rceil$. Equivalently, the number of ones in every row of $G$ is $n-k+1$ and the number of ones in every column of $G$ is either $\left\lfloor\frac{k(n-k+1)}{n}\right\rfloor$ or $\left\lceil\frac{k(n-k+1)}{n}\right\rceil$. So $G$ satisfies conditions (P1) and (P2), hence is an SBGM of $\mathcal{C}$.

## IV. Examples of the Construction

As an illustration of our construction, consider the following two examples, which reflect two typical cases of the output of Algorithm 1.

Example 1: Let $k=7$ and $n=10$. Then $k(k-1)=4 n+2$. So $a=4, r=2,\left\lfloor\frac{k(k-1)}{n}\right\rfloor=4$ and $\left\lceil\frac{k(k-1)}{n}\right\rceil=5$. According to (3), we have

$$
\boldsymbol{\delta}=(5,5,4,4,4,4,4,4,4,4)
$$

and according to (7), we have

$$
\bar{S}=\underbrace{1,2,3,4,5,6}, \underbrace{1,2,3,4,5}, \underbrace{1,2,3,4}, \underbrace{1,2,3}, \underbrace{1,2}, 1 .
$$

By Algorithm $1, \bar{S}$ is divided into $S_{1}, \cdots, S_{6}$ as follows:

$$
\begin{equation*}
\underbrace{1,2,3,4,5}_{S_{1}}, \underbrace{6,1,2,3,4}_{S_{2}}, \underbrace{5,1,2,3}_{S_{3}}, \underbrace{4,1,2,3}_{S_{4}}, \underbrace{1,2}_{S_{5}}, \underbrace{1}_{S_{6}} \tag{13}
\end{equation*}
$$

and $S_{7}=\cdots=S_{n}=\emptyset$. Hence,

$$
\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right)=(5,5,4,4,2,1,0,0,0,0)
$$

and according to 10), we have

$$
\begin{aligned}
\boldsymbol{\theta} & =\boldsymbol{\delta}-\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right) \\
& =(0,0,0,0,2,3,4,4,4,4)
\end{aligned}
$$

Moreover, according to (9), we have

$$
\bar{T}=7, \underbrace{6,7}, \underbrace{5,6,7}, \underbrace{4,5,6,7}, \underbrace{3,4,5,6,7}, \underbrace{2,3,4,5,6,7} .
$$

Then by Algorithm 2, we have $T_{1}=\cdots=T_{4}=\emptyset$ and $\bar{T}$ is divided into $T_{5}, \cdots, T_{10}$ as follows:

$$
\underbrace{7,6}_{T_{5}}, \underbrace{7,5,6}_{T_{6}}, \underbrace{7,4,5,6}_{T_{7}}, \underbrace{7,3,4,5}_{T_{8}}, \underbrace{6,7,2,3}_{T_{9}}, \underbrace{4,5,6,7}_{T_{10}} .
$$

So we obtain

$$
W=\left[\begin{array}{llllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
$$

We can easily check that Claims 1 and 2 are true. We now check that Claim 3 is true. From (13), we have $\lambda_{1}=6$ and

$$
\begin{aligned}
X^{*} & =\left\{\left(1,\left|S_{1}\right|\right),\left(2,\left|S_{2}\right|\right), \cdots,\left(6,\left|S_{6}\right|\right)\right\} \\
& =\{(1,5),(2,5),(3,4),(4,4),(5,2),(6,1)\}
\end{aligned}
$$

Suppose

$$
X^{*}=X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}
$$

for some $\sigma^{*} \in \mathscr{S}_{k}$ and some $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$. We show that $\sigma^{*}$ and $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right)$ are unique as follows.

First, note that for each $j \in\{1,2,3,4\}$, mult $X^{*}(j)$ equals the weight of the $j$ th column of $W$. Then considering the first four columns of $W$, we have $\{1,2,3,4\} \subseteq\left(\cap_{i=1}^{3} X_{i}^{*}\right)$, $\{1,2,4\} \subseteq X_{4}^{*},\{1,3\} \subseteq X_{5}^{*}$ and $\{2\} \subseteq X_{6}^{*}$. So it must be that $\sigma^{*}(7)=1$ and $X_{7}^{*}=\emptyset$. Recursively, we obtain $\sigma^{*}(6)=2$ and $X_{6}^{*}=\{2\} ; \sigma^{*}(5)=3$ and $X_{5}^{*}=\{1,3\} ; \sigma^{*}(4)=4$ and $X_{4}^{*}=\{1,2,4\}$. And hence, we have $\sigma^{*}(i) \in\{5,6,7\}$ for each $i \in\{1,2,3\}$, and mult $X_{7}^{*} \sqcup X_{6}^{*} \sqcup X_{5}^{*} \sqcup X_{4}^{*}(j)=0$ for $j=5,6$.

Further, consider the first five columns of $W$. Since $\operatorname{mult}_{X_{7}^{*} \sqcup X_{6}^{*} \sqcup X_{5}^{*} \sqcup X_{4}^{*}}(5)=0$, then mult $X_{1}^{*} \sqcup X_{2}^{*} \sqcup X_{3}^{*}(5)=$ mult $_{X^{*}}(5)=2$ and $\{1,2,3,4,5\} \subseteq\left(X_{1}^{*} \cap X_{2}^{*}\right)$. So $\sigma^{*}(3)=5$ and $X_{3}^{*}=\{1,2,3,4\}$. Similarly, considering the first six columns of $W$, we can obtain $\sigma^{*}(2)=6$ and $X_{2}^{*}=$ $\{1,2,3,4,5\}$. And finally, we can obtain $\sigma^{*}(1)=7$ and $X_{1}^{*}=\{1,2,3,4,5,6\}$.

Hence, $\sigma^{*} \in \mathscr{S}_{k}$ and $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$ are uniquely determined. That is, Claim 3 is true.

As discussed in the proof of Lemma 1 W satisfies conditions (i) - (iv) of Lemma 1

Example 2: Let $k=7$ and $n=13$. Then $k(k-1)=3 n+3$. So $a=3, r=3,\left\lfloor\frac{k(k-1)}{n}\right\rfloor=3$ and $\left\lceil\frac{k(k-1)}{n}\right\rceil=4$. According to (3), we have

$$
\boldsymbol{\delta}=(4,4,4,3,3,3,3,3,3,3,3,3,3)
$$

and according to (7), we have

$$
\bar{S}=\underbrace{1,2,3,4,5,6}, \underbrace{1,2,3,4,5}, \underbrace{1,2,3,4}, \underbrace{1,2,3}, \underbrace{1,2}, 1 .
$$

By Algorithm $1, \bar{S}$ is divided into $S_{1}, \cdots, S_{7}$ as follows:

$$
\begin{equation*}
\underbrace{1,2,3,4}_{S_{1}}, \underbrace{5,6,1,2}_{S_{2}}, \underbrace{3,4,5,1}_{S_{3}}, \underbrace{2,3,4}_{S_{4}}, \underbrace{1,2,3}_{S_{5}}, \underbrace{1,2}_{S_{6}}, \underbrace{1}_{S_{7}} . \tag{14}
\end{equation*}
$$

And $S_{8}=\cdots=S_{n}=\emptyset$. Hence,

$$
\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right)=(4,4,4,3,3,2,1,0,0,0,0,0,0)
$$

and according to 10,

$$
\begin{aligned}
\boldsymbol{\theta} & =\boldsymbol{\delta}-\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right) \\
& =(0,0,0,0,0,1,2,3,3,3,3,3,3) .
\end{aligned}
$$

Moreover, according to (9), we have

$$
\bar{T}=7, \underbrace{6,7}, \underbrace{5,6,7}, \underbrace{4,5,6,7}, \underbrace{3,4,5,6,7}, \underbrace{2,3,4,5,6,7} .
$$

Then by Algorithm 2, we have $T_{1}=\cdots=T_{5}=\emptyset$ and $\bar{T}$ is divided into $T_{6}, \cdots, T_{13}$ as follows:

$$
\underbrace{7}_{T_{6}}, \underbrace{6,7}_{T_{7}}, \underbrace{5,6,7}_{T_{8}}, \underbrace{4,5,6}_{T_{9}}, \underbrace{7,3,3}_{T_{10}}, \underbrace{5,6,7}_{T_{11}}, \underbrace{2,3,4}_{T_{12}}, \underbrace{5,6,7}_{T_{13}} .
$$

So we obtain

$$
W=\left[\begin{array}{lllllllllllll}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{array}\right]
$$

We can check that Claims 1 and 2 are true. Moreover, let

$$
\begin{aligned}
X^{*} & =\left\{\left(1,\left|S_{1}\right|\right),\left(2,\left|S_{2}\right|\right), \cdots,\left(7,\left|S_{7}\right|\right)\right\} \\
& =\{(1,4),(2,4),(3,4),(4,3),(5,3),(6,2),(7,1)\}
\end{aligned}
$$

and suppose $X^{*}=X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}$ for some $\sigma^{*} \in \mathscr{S}_{k}$ and some $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$. Then similar to Example 11 we can obtain $\sigma^{*}(i)=k-i+1, \forall i \in[k]$, and $X_{7}^{*}=\emptyset$, $X_{6}^{*}=\{2\}, X_{5}^{*}=\{2,3\}, X_{4}^{*}=\{1,3,4\}, X_{3}^{*}=\{1,3,4,5\}$, $X_{2}^{*}=\{1,2,4,5,6\}, X_{1}^{*}=\{1,2,3,5,6,7\}$. So both $\sigma^{*} \in \mathscr{S}_{k}$ and $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$ are unique and Claim 3 is true.

## V. Conclusion

We show that for any $n \geq k \geq 1$, there exists an $[n, k]$ sparsest and balanced GRS code over any field $\mathbb{F}_{q}$ with size $q \geq n+\left\lceil\frac{k(k-1)}{n}\right\rceil$. It is still an open problem whether $[n, k]$ sparsest and balanced GRS codes exist when the field size $q$ satisfies $n+1<q<n+\left\lceil\frac{k(k-1)}{n}\right\rceil$.

## Appendix A <br> Proof of Claim 1

In this appendix, we are to prove Claim 1.
By (3) and (5), we have $\delta_{\ell} \leq a+1 \leq k-1$ for each $\ell \in[n]$. Then there exists a unique $\lambda_{a+1} \in[n]$ such that

$$
\begin{equation*}
\sum_{\ell=1}^{\lambda_{a+1}-1} \delta_{\ell}<\sum_{\ell=a+1}^{k-1} \ell \leq \sum_{\ell=1}^{\lambda_{a+1}} \delta_{\ell} \tag{15}
\end{equation*}
$$

According to Algorithm 1, we have

$$
\begin{equation*}
\left|S_{j}\right|=\delta_{j}, \forall j \in\left\{1,2, \cdots, \lambda_{a+1}\right\} \tag{16}
\end{equation*}
$$

and each of $S_{1}, S_{2}, \cdots, S_{\lambda_{a+1}}$ is a subset of $[k]$. Moreover, since $\delta_{\lambda_{a+1}} \leq a+1$, then from (15), we obtain

$$
\sum_{\ell=1}^{\lambda_{a+1}}\left|S_{\ell}\right|=\sum_{\ell=1}^{\lambda_{a+1}} \delta_{\ell}=\left(\sum_{\ell=a+1}^{k-1} \ell\right)+t_{0}
$$

for some $t_{0} \in\{0,1, \cdots, a\}$. We need to consider the following two cases.

Case 1. $t_{0} \in\{1,2, \cdots, a\}$.
Then according to Algorithm 1, we have

- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq a-t_{0}$,

$$
\begin{align*}
S_{j} & =\left\{t_{0}+1, t_{0}+2, \cdots, a-\ell+1,1,2, \cdots, t_{0}\right\} \\
& =\{1,2, \cdots, a-\ell+1\} \tag{17}
\end{align*}
$$

- For $j=\lambda_{a+1}+\ell$ and $a-t_{0}+1 \leq \ell \leq a-1$,

$$
\begin{equation*}
S_{j}=\{1,2, \cdots, a-\ell\} \tag{18}
\end{equation*}
$$

Moreover, $\lambda_{1}:=\lambda_{a+1}+a-1$ is the value of $j$ at the end of the while loop of Algorithm 1 and

$$
\begin{align*}
& \left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{\lambda_{1}}\right|\right)= \\
& \left(\delta_{1}, \cdots, \delta_{\lambda_{a+1}}, a, a-1, \cdots, t_{0}+1, t_{0}-1, \cdots, 2,1\right) \tag{19}
\end{align*}
$$

Case 2. $t_{0}=0$.
Then according to Algorithm 1, we have

- For $j=\lambda_{a+1}+\ell$ and $\ell \in[a]$,

$$
\begin{equation*}
S_{j}=\{1,2, \cdots, a-\ell+1\} \tag{20}
\end{equation*}
$$

Moreover, $\lambda_{1}:=\lambda_{a+1}+a$ is the value of $j$ at the end of the while loop of Algorithm 1 and

$$
\begin{align*}
& \left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{\lambda_{1}}\right|\right)= \\
& \left(\delta_{1}, \cdots, \delta_{\lambda_{a+1}}, a, a-1, \cdots, 2,1\right) \tag{21}
\end{align*}
$$

In both cases, clearly, each of $S_{\lambda_{a+1}+1}, \cdots, S_{\lambda_{1}}$ is a subset of $[k]$. Moreover, we have $\lambda_{1} \leq n$, which can be proved by contradiction as follows. Suppose $\lambda_{1}>n$. Then we have

$$
\sum_{j=1}^{\lambda_{1}}\left|S_{j}\right|+\sum_{\ell=1}^{a} \ell>\sum_{j=1}^{n}\left|S_{j}\right|+\sum_{\ell=1}^{a} \ell .
$$

Moreover, since $\delta_{j} \leq a+1$ for all $j \in[n]$ (see (3)), then from (19) and (21), we have

$$
\sum_{j=\lambda_{a+1}+1}^{n}\left|S_{j}\right|+\sum_{\ell=1}^{a} \ell \geq \sum_{j=\lambda_{a+1}+1}^{n} \delta_{j}
$$

From the above two inequalities, we have

$$
\begin{equation*}
\sum_{j=1}^{\lambda_{1}}\left|S_{j}\right|+\sum_{\ell=1}^{a} \ell>\sum_{j=1}^{n} \delta_{j}=k(k-1) \tag{22}
\end{equation*}
$$

where the last equation comes from (6). However, combining the facts $\sum_{j=1}^{\lambda_{1}}\left|S_{j}\right|=K=\frac{k(k-1)}{2}$ and $a<k-1$, we have

$$
\sum_{j=1}^{\lambda_{1}}\left|S_{j}\right|+\sum_{\ell=1}^{a} \ell<\frac{k(k-1)}{2}+\sum_{\ell=1}^{k-1} \ell=k(k-1)
$$

which contradicts to (22). Hence we proved that $\lambda_{1} \leq n$.
Further, according to Algorithm 1, we have

$$
\begin{equation*}
S_{\lambda_{1}+1}=\cdots=S_{n}=\emptyset \tag{23}
\end{equation*}
$$

So in Case 1, we have

$$
\begin{align*}
& \left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right)= \\
& (\delta_{1}, \cdots, \delta_{\lambda_{a+1}}, a, a-1 \cdots, t_{0}+1, t_{0}-1, \cdots, 2,1, \overbrace{0, \cdots, 0}^{n-\lambda_{1} \text { zeros }}) \tag{24}
\end{align*}
$$

where $t_{0} \in\{1,2, \cdots, a\}$ and $\lambda_{1}=\lambda_{a+1}+a-1$; in Case 2, we have

$$
\begin{align*}
& \left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right)= \\
& (\delta_{1}, \cdots, \delta_{\lambda_{a+1}}, a, a-1 \cdots, 2,1, \overbrace{0, \cdots, 0}^{n-\lambda_{1} \text { zeros }}) \tag{25}
\end{align*}
$$

where $\lambda_{1}=\lambda_{a+1}+a-1$. In both cases, each $S_{j}$ is a subset of $[k]$ and, as multisets, $\sqcup_{j=1}^{n} S_{j}=\sqcup_{j=1}^{\lambda_{1}} S_{j}=S$.

In Example 1, we have $a+1=5$. From (13), we can obtain $\lambda_{6}=2, \lambda_{5}=3$ and $t_{0}=3$. So this example falls into Case 1 and $\lambda_{1}=\lambda_{a+1}+a-1=6$.

In Example 2, we have $a+1=4$. From (14), we can obtain $\lambda_{6}=2, \lambda_{5}=3, \lambda_{4}=4$ and $t_{0}=0$. So this example falls into Case 2 and $\lambda_{1}=\lambda_{a+1}+a=7$.

## Appendix B <br> Proof of Claim 2

To prove Claim 2, we continue considering the two cases discussed in Appendix A.

First, consider Case 1. We need to divide this case into the following four subcases according to the value of $r$.

Case 1.1: $r \leq \lambda_{a+1}$.
Then by (3), (10) and (24), we have

$$
\begin{aligned}
\boldsymbol{\theta}= & \boldsymbol{\delta}-\left(\left|S_{1}\right|,\left|S_{2}\right|, \cdots,\left|S_{n}\right|\right) \\
= & (\overbrace{0, \cdots, 0}^{\lambda_{a+1} \text { zeros }}, 0,1, \cdots, a-t_{0}-1, a-t_{0}+1, \\
& \left.\cdots, a-2, a-1, \delta_{\lambda_{1}+1}, \cdots, \delta_{n}\right)
\end{aligned}
$$

where $\delta_{\lambda_{1}+1}=\cdots=\delta_{n}=a$. That is,
$\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} ; \\ \ell-1, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq a-t_{0} ; \\ \ell, & j=\lambda_{a+1}+\ell \text { and } a-t_{0}+1 \leq \ell \leq a-1 ; \\ \delta_{j}=a, & \lambda_{1}+1 \leq j \leq n .\end{cases}$

According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}+1, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $2 \leq \ell \leq a-t_{0}$,

$$
\begin{align*}
T_{j} & =\{k-(\ell-1)+1, k-(\ell-1)+2, \cdots, k\} \\
& =\{k-\ell+2, k-\ell+3, \cdots, k\} \tag{27}
\end{align*}
$$

- For $j=\lambda_{a+1}+\ell$ and $a-t_{0}+1 \leq \ell \leq a-1$,

$$
\begin{align*}
T_{j} & =\left\{k-\left(a-t_{0}\right)+1, \cdots, k, k-\ell+1, \cdots, k-\left(a-t_{0}\right)\right\} \\
& =\{k-\ell+1, k-\ell+2, \cdots, k\} \tag{28}
\end{align*}
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-\left(a-t_{0}\right)+1, \cdots, k}, \underbrace{k-a+1, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is then formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$. Clearly, each $T_{j}$ is a subset of $[k]$. Moreover, since

$$
\sum_{j=1}^{n} \theta_{j}=\sum_{j=1}^{n} \delta_{j}-\sum_{j=1}^{n}\left|S_{j}\right|=\frac{k(k-1)}{2}=K
$$

which is equal to the length of $\bar{T}$. So Algorithm 2 always divides $\bar{T}$ into $T_{1}, T_{2}, \cdots, T_{n}$ of size $\theta_{1}, \theta_{2}, \cdots, \theta_{n}$, respectively. Hence, as multisets, we have $\sqcup_{j=1}^{n} T_{j}=T$.

Note that $T_{j}=\emptyset$ for $1 \leq j \leq \lambda_{a+1}+1$, and $S_{j}=\emptyset$ for $\lambda_{1}+1 \leq j \leq n$. So $S_{j} \cap T_{j}=\emptyset$ for $j \in\left\{1, \cdots, \lambda_{a+1}+1\right\} \cup$ $\left\{\lambda_{1}+1, \cdots, n\right\}$. Moreover, for $\lambda_{a+1}+2 \leq j \leq \lambda_{1}$, by 17, (18), (20), (27) and (28), we have

$$
\max \left(S_{j}\right) \leq a-\ell+1<k-\ell+1 \leq \min \left(T_{j}\right)
$$

So $S_{j} \cap T_{j}=\emptyset$ for $j \in\left\{\lambda_{a+1}+2, \cdots, \lambda_{1}\right\}$. Hence, we have $S_{j} \cap T_{j}=\emptyset$ for all $j \in[n]$.

Case 1.2: $\lambda_{a+1}<r \leq \lambda_{a+1}+a-t_{0}$.
Then $r=\lambda_{a+1}+t_{1}$, where $1 \leq t_{1} \leq a-t_{0}$, and
$\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} ; \\ \ell, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq t_{1} ; \\ \ell-1, & j=\lambda_{a+1}+\ell \text { and } t_{1}+1 \leq \ell \leq a-t_{0} ; \\ \ell, & j=\lambda_{a+1}+\ell \text { and } a-t_{0}+1 \leq \ell \leq a-1 ; \\ \delta_{j}=a, & \lambda_{1}+1 \leq j \leq n .\end{cases}$
According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq t_{1}$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\}
$$

- For $j=\lambda_{a+1}+\ell$ and $t_{1}+1 \leq \ell \leq a-t_{0}$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\} \backslash\left\{k-l+t_{1}+1\right\} ;
$$

- For $j=\lambda_{a+1}+\ell$ and $a-t_{0}+1 \leq \ell \leq a-1$

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\}
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-\left(a-t_{0}-t_{1}\right)+1, \cdots, k}, \underbrace{k-a+1, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
Case 1.3: $\lambda_{a+1}+a-t_{0}<r \leq \lambda_{1}$.
Then $r=\lambda_{a+1}+t_{2}$, where $a-t_{0}+1 \leq t_{2} \leq a-1$, and

$$
\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} \\ \ell, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq a-t_{0} \\ \ell+1, & j=\lambda_{a+1}+\ell \text { and } a-t_{0}+1 \leq \ell \leq t_{2} \\ \ell, & j=\lambda_{a+1}+\ell \text { and } t_{2}+1 \leq \ell \leq a-1 \\ \delta_{j}=a, & \lambda_{1}+1 \leq j \leq n\end{cases}
$$

According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq a-t_{0}$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\} ;
$$

- For $j=\lambda_{a+1}+\ell$ and $a-t_{0}+1 \leq \ell \leq t_{2}$,

$$
T_{j}=\{k-\ell, k-\ell+1, \cdots, k\} ;
$$

- For $j=\lambda_{a+1}+\ell$ and $t_{2}+1 \leq \ell \leq a-1$

$$
T_{j}=\{k-\ell, k-\ell+1, \cdots, k\} \backslash\left\{k-a+t_{0}-\ell+t_{2}\right\} ;
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-2 a+t_{0}+t_{2}+1, \cdots, k}, \underbrace{k-a, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
Case 1.4: $\lambda_{1}<r<n$.
Then by (3), (10) and (24), we have
$\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} ; \\ \ell, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq a-t_{0} ; \\ \ell+1, & j=\lambda_{a+1}+\ell \text { and } a-t_{0}+1 \leq \ell \leq a-1 ; \\ \delta_{j} \leq a+1, & \lambda_{1}+1 \leq j \leq n .\end{cases}$
According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq a-t_{0}$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\} ;
$$

- For $j=\lambda_{a+1}+\ell$ and $a-t_{0}+1 \leq \ell \leq a-1$

$$
T_{j}=\{k-\ell, k-\ell+1, \cdots, k\} ;
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-\left(a-t_{0}\right), \cdots, k}, \underbrace{k-a, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
For all of these subcases, similar to Case 1.1, it can be verified that for each $j \in[n], T_{j}$ is a subset of $[k], S_{j} \cap T_{j}=\emptyset$ and, when viewed as multisets, we have $\sqcup_{j=1}^{n} T_{j}=T$.

Next, consider Case 2. We need to divide this case into the following three subcases according to the value of $r$.

Case 2.1: $r \leq \lambda_{a+1}$.
Then by (3), (10) and (25), we have

$$
\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} \\ \ell-1, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq a \\ \delta_{j}=a, & \lambda_{1}+1 \leq j \leq n\end{cases}
$$

According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}+1, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $2 \leq \ell \leq a$,

$$
T_{j}=\{k-\ell+2, k-\ell+3, \cdots, k\} ;
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-a+1, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
Case 2.2: $\lambda_{a+1}<r \leq \lambda_{1}$.
Then $r=\lambda_{a+1}+t_{1}$, where $1 \leq t_{1} \leq a$ and

$$
\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} \\ \ell, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq t_{1} \\ \ell-1, & j=\lambda_{a+1}+\ell \text { and } t_{1}+1 \leq \ell \leq a \\ \delta_{j}=a, & \lambda_{1}+1 \leq j \leq n\end{cases}
$$

According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq t_{1}$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\}
$$

- For $j=\lambda_{a+1}+\ell$ and $t_{1}+1 \leq \ell \leq a$,

$$
T_{j}=\{k-\ell+2, k-\ell+3, \cdots, k\} \backslash\left\{k-\ell+t_{1}+1\right\}
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-a+t_{1}+1, \cdots, k}, \underbrace{k-a+1, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
Case 2.3: $\lambda_{1}<r<n$.
Then by (3), (10) and (25), we have

$$
\theta_{j}= \begin{cases}0, & 1 \leq j \leq \lambda_{a+1} \\ \ell, & j=\lambda_{a+1}+\ell \text { and } 1 \leq \ell \leq a \\ \delta_{j} \leq a+1, & \lambda_{1}+1 \leq j \leq n\end{cases}
$$

According to Algorithm 2, we have

- For $1 \leq j \leq \lambda_{a+1}, T_{j}=\emptyset$;
- For $j=\lambda_{a+1}+\ell$ and $1 \leq \ell \leq a$,

$$
T_{j}=\{k-\ell+1, k-\ell+2, \cdots, k\}
$$

- Finally, $T_{\lambda_{1}+1}, T_{\lambda_{1}+2}, \cdots, T_{n}$ are obtained by dividing the sequence

$$
\underbrace{k-a, \cdots, k}, \underbrace{k-a-1, \cdots, k}, \cdots, \underbrace{2, \cdots, k}
$$

into $n-\lambda_{1}$ segments of length $a$, and $T_{\lambda_{1}+j}$ is formed by the elements of the $j$ th segment, $1 \leq j \leq n-\lambda_{1}$.
For all of these subcases, similar to Case 1.1, it can be verified that for each $j \in[n], T_{j}$ is a subset of $[k], S_{j} \cap T_{j}=\emptyset$ and, when viewed as multisets, we have $\sqcup_{j=1}^{n} T_{j}=T$.

Combining all of the above discussions, we proved that each $T_{j}$ is a subset of $[k], S_{j} \cap T_{j}=\emptyset$ and, when viewed as multisets, $\sqcup_{j=1}^{n} T_{j}=T$.

## Appendix C Proof of Claim 3

We again consider all the cases and subcases discussed in Appendices A and B.
We use the notations $\lambda_{1}$ and $\lambda_{a+1}$ with the same meaning as in Appendices A and B. We further define $\lambda_{j}$ for all $j \in$ $\{2, \cdots, a\} \cup\{a+2, \cdots, k-1\}$ as follows.

For Case 1, let

$$
\lambda_{j}= \begin{cases}\lambda_{a+1}+a-j+1, & \text { if } t_{0}+1 \leq j \leq a  \tag{29}\\ \lambda_{a+1}+a-j, & \text { if } 2 \leq j \leq t_{0}\end{cases}
$$

And for Case 2, let

$$
\begin{equation*}
\lambda_{j}=\lambda_{a+1}+a-j+1, \forall 2 \leq j \leq a \tag{30}
\end{equation*}
$$

For each $j \in\{a+2, \cdots, k-1\}$, let $\lambda_{j} \in[n]$ be such that

$$
\begin{equation*}
\sum_{\ell=1}^{\lambda_{j}-1} \delta_{\ell}<\sum_{\ell=j}^{k-1} \ell \leq \sum_{\ell=1}^{\lambda_{j}} \delta_{\ell} \tag{31}
\end{equation*}
$$

Note that by (3) and (5), we have $\delta_{j} \leq a+1 \leq k-1$ for each $j \in[n]$. Then for each $j \in\{a+2, \cdots, k-1\}$, it is easy to see that $\lambda_{j}$ is a uniquely determined value.

As an illustration, consider again Example 1 Note that $k=$ 7 and $a=4$, and in Appendix A, we have obtained $\lambda_{5}=3$ and $\lambda_{1}=6$. Now we can further obtain $\lambda_{6}=2, \lambda_{3}=\lambda_{4}=4$ and $\lambda_{2}=5$. In general, for Case 1, by (29) and (31), we always have

$$
\lambda_{k-1}<\cdots<\lambda_{t_{0}-1}<\lambda_{t_{0}}=\lambda_{t_{0}+1}<\cdots<\lambda_{1}
$$

For Example 2 note that $k=7$ and $a=3$, and in Appendix A, we have obtained $\lambda_{4}=4$ and $\lambda_{1}=7$. We can further obtain $\lambda_{6}=2, \lambda_{5}=3, \lambda_{3}=5$ and $\lambda_{2}=6$. In general, for Case 2, by (30) and (31), we always have

$$
\lambda_{k-1}<\lambda_{k-2}<\cdots<\lambda_{2}<\lambda_{1}
$$

Now let $X^{*}=\left\{\left(1,\left|S_{1}\right|\right),\left(2,\left|S_{2}\right|\right), \cdots,\left(\lambda_{1},\left|S_{\lambda_{1}}\right|\right)\right\}$ and suppose $X^{*}$ is represented as $X^{*}=X_{1}^{*} \sqcup X_{2}^{*} \sqcup \cdots \sqcup X_{k}^{*}$ for some $\sigma^{*} \in \mathscr{S}_{k}$ and some $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right) \in \mathcal{X}_{\sigma^{*}}$. Then for all subcases as discussed in the proof of Claim 2, it is a mechanical (but somewhat tedious) work to check, just as in Example 1 that

$$
X_{k}^{*}=\emptyset
$$

and

$$
X_{i}^{*}=Z_{i} \cap\left\{1,2, \cdots, \lambda_{i}\right\}, \forall i \in\{1,2, \cdots, k-1\}
$$

Hence, $\sigma^{*}: i \mapsto k-i+1, \forall i \in[k]$, is the unique permutation in $\mathscr{S}_{k}$ and $\left(X_{1}^{*}, X_{2}^{*}, \cdots, X_{k}^{*}\right)$ is the unique choice in $\mathcal{X}_{\sigma^{*}}$.

## REFERENCES

[1] S. H. Dau, W. Song, Z. Dong and C. Yuen, "Balanced Sparsest generator matrices for MDS codes," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2013, pp. 1889-1893.
[2] W. Halbawi, T. Ho, H. Yao, and I. Duursma, "Distributed Reed-Solomon codes for simple multiple access networks," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2013, pp. 651-655.
[3] M. Yan and A. Sprintson, "Algorithms for weakly secure data exchange," in Proc. Int. Symp. on Network Coding (NetCod), 2013, pp. 1-6.
[4] M. Yan, A. Sprintson, and I. Zelenko, "Weakly secure data exchange with generalized Reed-Solomon codes," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2014, pp. 1366-1370.
[5] S. H. Dau, W. Song and C. Yuen, "On the Existence of MDS Codes Over Small Fields With Constrained Generator Matrices," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2014, pp. 1787-1791.
[6] -_, "On simple multiple access networks," IEEE J. Sel. Areas Commun., vol. 33, no. 2, pp. 236-249, Feb 2015.
[7] W. Halbawi, M. Thill, and B. Hassibi, "Coding with Constraints: Minimum Distance Bounds and Systematic Constructions," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2015, pp. 1302-1306
[8] W. Halbawi, Z. Liu, and B. Hassibi, "Balanced Reed-Solomon codes," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2016, pp. 935-939.
[9] _-, "Balanced Reed-Solomon codes for all parameters," in Proc. IEEE Information Theory Workshop (ITW), 2016, pages 409-413.
[10] A. Heidarzadeh, and A. Sprintson "An Algebraic-Combinatorial Proof Technique for the GM-MDS Conjecture," 2017, available online at https://arxiv.org/abs/1702.01734.
[11] S. Li, and M. Gastpar, "Cooperative Data Exchange based on MDS Codes," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2017, pp. 14111415.
[12] W. Halbawi, I. Duursma, S. H. Dau, and B. Hassibi, "Balanced and Sparse Tamo-Barg Codes," in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2017, pp. 1018-1022.
[13] W. C. Huffman and V. Pless, Fundamentals of Error-Correcting Codes. Cambridge University Press, New York, 2003.


[^0]:    ${ }^{1}$ Another conjecture which is equivalent to the GM-MDS Conjecture was proposed in [4]

