Information Bottleneck on General Alphabets

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***Abstract*—We prove rigorously a source coding theorem that can probably be considered folklore, a generalization to arbitrary alphabets of a problem motivated by the Information Bottleneck method. For general random variables (Y, X), we show essentially that for some $n \in \mathbb{N}$, a function f with rate limit $\log|f| \leq nR$ and $\mathrm{I}(\mathsf{Y}^n; f(\mathsf{X}^n)) \geq nS$ exists if and only if there is a random variable U such that the Markov chain $\mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}$ holds, $\mathrm{I}(\mathsf{U}; \mathsf{X}) \leq R$ and $\mathrm{I}(\mathsf{U}; \mathsf{Y}) \geq S$. The proof relies on the well established discrete case and showcases a technique for lifting discrete coding theorems to arbitrary alphabets.**

I. Introduction

Since its inception [1], the *Information Bottleneck* (IB) method became a widely applied tool, especially in the context of machine learning problems. It has been successfully applied to various problems in machine learning [2], computer vision [3], and communications [5], [6], [7]. Furthermore, it is a valuable tool for channel output compression in a communication system [8], [9].

In the underlying information-theoretic problem, we define a pair $(S, R) \in \mathbb{R}^2$ to be *achievable* for the two arbitrary random sources (Y, X), if there exists a function f with rate limited range $\frac{1}{n}\log|f| \leq R$ and $\mathrm{I}(\mathbf{Y}; f(\mathbf{X})) \geq nS$, where $(\mathbf{Y}, \mathbf{X})$ are n independent and identically distributed (i.i.d.) copies of (Y, X).

While this Shannon-theoretic problem and variants thereof were also considered (e. g., [10], [11]), a large part of the literature is aimed at studying the IB function

$$S_{\mathrm{IB}}(R) = \sup_{\substack{\mathsf{U}\,:\,\mathrm{I}(\mathsf{U};\mathsf{X}) \leq R \\ \mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}}} \mathrm{I}(\mathsf{U}; \mathsf{Y}) \tag{1}$$

in different contexts. In particular, several works (e. g., [1], [2], [12], [13], [14]) intend to compute a probability distribution that achieves the supremum in (1). The resulting distribution is then used as a building block in numerical algorithms, e. g., for document clustering [2] or dimensionality reduction [12].

In the discrete case, $S_{\mathrm{IB}}(R)$ is equal to the maximum of all S such that (S, R) is in the *achievable region* (closure of the set of all achievable pairs). This statement has been re-proven many times in different contexts [15], [11], [16], [17]. In this note, we prove a theorem, which can probably be considered folklore, extending this result from discrete to arbitrary random variables. Formally speaking, using the definitions in [18], we prove that a pair (S, R) is in the achievable region of an arbitrary source (Y, X) if and only if, for every $\varepsilon > 0$, there exists a random variable U with $\mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}$, $\mathrm{I}(\mathsf{X}; \mathsf{U}) \leq R + \varepsilon$, and $\mathrm{I}(\mathsf{Y}; \mathsf{U}) \geq S - \varepsilon$. This provides a single-letter solution to the information-theoretic problem behind the information bottleneck method for arbitrary random sources and in particular it shows, that the information bottleneck for Gaussian random variables [12] is indeed the solution to a Shannon-theoretic problem.

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Funding by WWTF Grants MA16-053, ICT15-119, and NXT17-013.

The proof relies on the discrete case. Thus, the techniques employed could be useful for lifting other discrete coding theorems to the case of arbitrary alphabets.

II. Main Result

Let Y and X be random variables with arbitrary alphabets $\mathcal{S}_{\mathsf{Y}}$ and $\mathcal{S}_{\mathsf{X}}$, respectively. The bold-faced random vectors $\mathbf{Y}$ and $\mathbf{X}$ are n i.i.d. copies of Y and X, respectively. We then have the following definitions.

Definition 1. *A pair $(S, R) \in \mathbb{R}^2$ is* achievable *if for some $n \in \mathbb{N}$ there exists a measurable function $f\colon \mathcal{S}_{\mathsf{X}}^n \to \mathcal{M}$ for some finite set $\mathcal{M}$ with bounded cardinality $\frac{1}{n}\log|\mathcal{M}| \leq R$ and*

$$\frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) \geq S. \tag{2}$$

The set of all achievable pairs is denoted $\mathcal{R} \subseteq \mathbb{R}^2$.

Definition 2. *A pair $(S, R) \in \mathbb{R}^2$ is* IB-achievable *if there exists an additional random variable U with arbitrary alphabet $\mathcal{S}_{\mathsf{U}}$, satisfying $\mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}$ and*

$$R \geq \mathrm{I}(\mathsf{X}; \mathsf{U}), \tag{3}$$
$$S \leq \mathrm{I}(\mathsf{Y}; \mathsf{U}). \tag{4}$$

The set of all IB-achievable pairs is denoted $\mathcal{R}_{\mathrm{IB}} \subseteq \mathbb{R}^2$.

In what follows, we will prove the following theorem.

Theorem 3. *The equality $\overline{\mathcal{R}_{\mathrm{IB}}} = \overline{\mathcal{R}}$ holds.*

III. Preliminaries

When introducing a function, we implicitly assume it to be measurable w.r.t. the appropriate σ-algebras. The σ-algebra associated with a finite set is its power set and the σ-algebra associated with $\mathbb{R}$ is the Borel σ-algebra. The symbol $\varnothing$ is used for the empty set and for a constant random variable. When there is no possibility for confusion, we will not distinguish between a single-element set and its element, e. g., we write x instead of $\{x\}$ and $\mathbb{1}_x$ for the indicator function of $\{x\}$. We use $A \triangle B := (A \setminus B) \cup (B \setminus A)$ to denote the symmetric set difference.

Let (Ω, Σ, μ) be a probability space. A random variable $\mathsf{X}\colon \Omega \to \mathcal{S}_{\mathsf{X}}$ takes values in the measurable space $(\mathcal{S}_{\mathsf{X}}, \mathcal{A}_{\mathsf{X}})$. The push-forward probability measure $\mu_{\mathsf{X}}\colon \mathcal{A}_{\mathsf{X}} \to [0, 1]$ is defined by $\mu_{\mathsf{X}}(A) = \mu\big(\mathsf{X}^{-1}(A)\big)$ for all $A \in \mathcal{A}_{\mathsf{X}}$. We will state most results in terms of push-forward measures and usually ignore the background probability space. When multiple random variables are defined, we implicitly assume the push-forward measures to be consistent in the sense that, e. g., $\mu_{\mathsf{X}}(A) = \mu_{\mathsf{XY}}(A \times \mathcal{S}_{\mathsf{Y}})$ for all $A \in \mathcal{A}_{\mathsf{X}}$.

For $n \in \mathbb{N}$ let Ω^n denote the n-fold Cartesian product of (Ω, Σ, μ). A bold-faced random vector, e. g., $\mathbf{X}$, defined on Ω^n, is an n-fold copy of X, i. e., $\mathbf{X} = \mathsf{X}^n$. Accordingly, the corresponding push-forward measure, e. g., $\mu_{\mathbf{X}}$ is the n-fold product measure.

For a random variable X let a_{X}, b_{X}, and c_{X} denote arbitrary functions on $\mathcal{S}_{\mathsf{X}}$, each with finite range. We will use the symbol $\mathcal{M}_{\mathsf{X}}$ to denote the range of a_{X}, i. e., $a_{\mathsf{X}}\colon \mathcal{S}_{\mathsf{X}} \to \mathcal{M}_{\mathsf{X}}$.

Definition 4 ([19, Def. 8.11]). *The* conditional expectation *of a random variable* X *with* $\mathcal{S}_\mathsf{X} = \mathbb{R}$*, given a random variable* Y*, is a random variable* $\mathbb{E}[\mathsf{X}|\mathsf{Y}]$ *such that*

1) $\mathbb{E}[\mathsf{X}|\mathsf{Y}]$ *is* $\sigma(\mathsf{Y})$*-measurable, and*
2) *for all* $A \in \sigma(\mathsf{Y})$*, we have* $\mathbb{E}\big[\mathbb{1}_A \mathbb{E}[\mathsf{X}|\mathsf{Y}]\big] = \mathbb{E}[\mathbb{1}_A \mathsf{X}]$.

The conditional probability *of an event* $B \in \Sigma$ *given* Y *is defined as* $\mathrm{P}\{B|\mathsf{Y}\} := \mathbb{E}[\mathbb{1}_B|\mathsf{Y}]$.

The conditional expectation and therefore also the conditional probability exists and is unique up to equality almost surely by [19, Thm. 8.12]. Furthermore, if $(\mathcal{S}_\mathsf{X}, \mathcal{A}_\mathsf{X})$ is a standard space [18, Sec. 1.5], there even exists a *regular conditional distribution* of X given Y [19, Thm. 8.37].

Definition 5. *For two random variables* X *and* Y *a* regular conditional distribution *of* X *given* Y *is a function* $\kappa_{\mathsf{X}|\mathsf{Y}} \colon \Omega \times \mathcal{A}_\mathsf{X} \to [0,1]$ *such that*

1) *for every* $\omega \in \Omega$*, the set function* $\kappa_{\mathsf{X}|\mathsf{Y}}(\omega) := \kappa_{\mathsf{X}|\mathsf{Y}}(\omega; \,\cdot\,)$ *is a probability measure on* $(\mathcal{S}_\mathsf{X}, \mathcal{A}_\mathsf{X})$.
2) *for every set* $A \in \mathcal{A}_\mathsf{X}$*, the function* $\kappa_{\mathsf{X}|\mathsf{Y}}(\,\cdot\,; A)$ *is* $\sigma(\mathsf{Y})$*-measurable.*
3) *for* μ*-a. e.* $\omega \in \Omega$ *and all* $A \in \mathcal{A}_\mathsf{X}$*, we have* $\kappa_{\mathsf{X}|\mathsf{Y}}(\omega; A) = \mathrm{P}\big\{\mathsf{X}^{-1}(A)\big|\mathsf{Y}\big\}(\omega)$ *(cf. Def. 4).*

Note, in particular, that finite spaces are standard spaces.

Remark 1. If the random variable Y is discrete, then $\kappa_{\mathsf{X}|\mathsf{Y}}$ reduces to conditioning given events $\mathsf{Y} = y$ for $y \in \mathcal{S}_\mathsf{Y}$, i. e., $\kappa_{\mathsf{X}|\mathsf{Y}}(\omega; A) = \frac{\mu_{\mathsf{XY}}(A \times \mathsf{Y}(\omega))}{\mu_\mathsf{Y}(\mathsf{Y}(\omega))}$ (cf. [19, Lem. 8.10]).

We use the following definitions and results from [18], [19].

Definition 6. *For random variables* X *and* Y *with* $|\mathcal{S}_\mathsf{X}| < \infty$ *the* conditional entropy *is defined as [18, Sec. 5.5]*

$$\mathrm{H}(\mathsf{X}|\mathsf{Y}) := \int \mathrm{H}\big(\kappa_{\mathsf{X}|\mathsf{Y}}\big)\, d\mu, \tag{5}$$

where $\mathrm{H}(\,\cdot\,)$ *denotes discrete entropy on* $\mathcal{S}_\mathsf{X}$*. For arbitrary random variables* X*,* Y*, and* Z *the* conditional mutual information *is defined as [18, Lem. 5.5.7]*

$$\begin{aligned}\mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z}) &:= \sup_{a_\mathsf{X}, a_\mathsf{Y}} \int \mathrm{D}\big(\kappa_{a_\mathsf{X}(\mathsf{X})a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}} \big\| \kappa_{a_\mathsf{X}(\mathsf{X})|\mathsf{Z}} \times \kappa_{a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}}\big)\, d\mu & (6)\\ &= \sup_{a_\mathsf{X}, a_\mathsf{Y}} \big[\mathrm{H}(a_\mathsf{X}(\mathsf{X})|\mathsf{Z}) + \mathrm{H}(a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}) - \mathrm{H}(a_\mathsf{X}(\mathsf{X})a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z})\big], & (7)\end{aligned}$$

where $\mathrm{D}(\,\cdot\,\|\,\cdot\,)$ *denotes Kullback-Leibler divergence [18, Sec. 2.3] and the supremum is taken over all* a_X *and* a_Y *with finite range. The* mutual information *is given by [18, Lem. 5.5.1]* $\mathrm{I}(\mathsf{X};\mathsf{Y}) := \mathrm{I}(\mathsf{X};\mathsf{Y}|\varnothing)$.

Definition 7 ([19, Def. 12.20]). *For arbitrary random variables* X*,* Y*, and* Z*, the Markov chain* $\mathsf{X} \multimap \mathsf{Y} \multimap \mathsf{Z}$ *holds if, for any* $A \in \mathcal{A}_\mathsf{X}$*,* $B \in \mathcal{A}_\mathsf{Z}$*, the following holds* μ*-a. e.:*

$$\mathrm{P}\big\{\mathsf{X}^{-1}(A) \cap \mathsf{Z}^{-1}(B)\big|\mathsf{Y}\big\} = \mathrm{P}\big\{\mathsf{X}^{-1}(A)\big|\mathsf{Y}\big\}\mathrm{P}\big\{\mathsf{Z}^{-1}(B)\big|\mathsf{Y}\big\}. \tag{8}$$

In the following, we collect some properties of these definitions.

Lemma 8. *For random variables* X*,* Y*, and* Z *the following properties hold:*

(i) $\mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z}) \geq 0$ *with equality if and only if* $\mathsf{X} \multimap \mathsf{Z} \multimap \mathsf{Y}$.
(ii) *For discrete* X*, i. e.,* $|\mathcal{S}_\mathsf{X}| < \infty$*, we have* $\mathrm{I}(\mathsf{X};\mathsf{Y}) = \mathrm{H}(\mathsf{X}) - \mathrm{H}(\mathsf{X}|\mathsf{Y})$.
(iii) $\mathrm{I}(\mathsf{X};\mathsf{YZ}) = \mathrm{I}(\mathsf{X};\mathsf{Z}) + \mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z})$.
(iv) *If* $\mathsf{X} \multimap \mathsf{Y} \multimap \mathsf{Z}$*, then* $\mathrm{I}(\mathsf{X};\mathsf{Y}) \geq \mathrm{I}(\mathsf{X};\mathsf{Z})$.

Proof. (i): The claim $\mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z}) \geq 0$ follows directly from (6) and the non-negativity of divergence.

Assume that $\mathsf{X} \multimap \mathsf{Z} \multimap \mathsf{Y}$, i. e., $\mathrm{P}\big\{\mathsf{X}^{-1}(A) \cap \mathsf{Y}^{-1}(B)\big|\mathsf{Z}\big\} = \mathrm{P}\big\{\mathsf{X}^{-1}(A)\big|\mathsf{Z}\big\}\mathrm{P}\big\{\mathsf{Y}^{-1}(B)\big|\mathsf{Z}\big\}$ almost everywhere. Let $a_\mathsf{X} \colon \mathcal{S}_\mathsf{X} \to \mathcal{M}_\mathsf{X}$ and $a_\mathsf{Y} \colon \mathcal{S}_\mathsf{Y} \to \mathcal{M}_\mathsf{Y}$ be functions with finite range. Pick two arbitrary sets $A \subseteq \mathcal{M}_\mathsf{X}$, $B \subseteq \mathcal{M}_\mathsf{Y}$ and we obtain μ-a. e.

$$\begin{aligned}&\kappa_{a_\mathsf{X}(\mathsf{X})a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}}(\,\cdot\,; A \times B)\\ &\quad= \mathrm{P}\big\{\mathsf{X}^{-1}(a_\mathsf{X}^{-1}(A)) \cap \mathsf{Y}^{-1}(a_\mathsf{Y}^{-1}(B))\big|\mathsf{Z}\big\} & (9)\\ &\quad= \mathrm{P}\big\{\mathsf{X}^{-1}(a_\mathsf{X}^{-1}(A))\big|\mathsf{Z}\big\}\mathrm{P}\big\{\mathsf{Y}^{-1}(a_\mathsf{Y}^{-1}(B))\big|\mathsf{Z}\big\} & (10)\\ &\quad= \kappa_{a_\mathsf{X}(\mathsf{X})|\mathsf{Z}}(\,\cdot\,; A)\kappa_{a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}}(\,\cdot\,; B), & (11)\end{aligned}$$

where (9) and (11) follow from part 3 of Def. 5. This proves that μ-a. e. the equality of measures $\kappa_{a_\mathsf{X}(\mathsf{X})a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}} = \kappa_{a_\mathsf{X}(\mathsf{X})|\mathsf{Z}} \times \kappa_{a_\mathsf{Y}(\mathsf{Y})|\mathsf{Z}}$ holds. By the properties of Kullback-Leibler divergence [18, Thm. 2.3.1] we have $\mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z}) = 0$ due to (6).

On the other hand, assume $\mathrm{I}(\mathsf{X};\mathsf{Y}|\mathsf{Z}) = 0$ and choose arbitrary sets $A \in \mathcal{A}_\mathsf{X}$ and $B \in \mathcal{A}_\mathsf{Y}$. We define $a_\mathsf{X} := \mathbb{1}_A$, $a_\mathsf{Y} := \mathbb{1}_B$, $\hat{\mathsf{X}} := a_\mathsf{X}(\mathsf{X})$, and $\hat{\mathsf{Y}} := a_\mathsf{Y}(\mathsf{Y})$. By (6) we have $\mathrm{D}\big(\kappa_{\hat{\mathsf{X}}\hat{\mathsf{Y}}|\mathsf{Z}}(\omega)\big\|\kappa_{\hat{\mathsf{X}}|\mathsf{Z}}(\omega) \times \kappa_{\hat{\mathsf{Y}}|\mathsf{Z}}(\omega)\big) = 0$ for μ-a. e. $\omega \in \Omega$, which is equivalent to the equality μ-a. e. of the measures $\kappa_{\hat{\mathsf{X}}\hat{\mathsf{Y}}|\mathsf{Z}} = \kappa_{\hat{\mathsf{X}}|\mathsf{Z}} \times \kappa_{\hat{\mathsf{Y}}|\mathsf{Z}}$. We obtain μ-a. e.,

$$\begin{aligned}\mathrm{P}\big\{\mathsf{X}^{-1}(A) \cap \mathsf{Y}^{-1}(B)\big|\mathsf{Z}\big\} &= \kappa_{\hat{\mathsf{X}}\hat{\mathsf{Y}}|\mathsf{Z}}(\,\cdot\,; 1 \times 1) & (12)\\ &= \kappa_{\hat{\mathsf{X}}|\mathsf{Z}}(\,\cdot\,; 1)\kappa_{\hat{\mathsf{Y}}|\mathsf{Z}}(\,\cdot\,; 1) & (13)\\ &= \mathrm{P}\big\{\mathsf{X}^{-1}(A)\big|\mathsf{Z}\big\}\mathrm{P}\big\{\mathsf{Y}^{-1}(B)\big|\mathsf{Z}\big\}. & (14)\end{aligned}$$

(ii): See [18, Lem. 5.5.6].
(iii): See [18, Lem. 5.5.7].
(iv): Using Prop. (i) we have $\mathrm{I}(\mathsf{X};\mathsf{Z}|\mathsf{Y}) = 0$ and by Prop. (iii) it follows that

$$\begin{aligned}\mathrm{I}(\mathsf{X};\mathsf{Z}) &\leq \mathrm{I}(\mathsf{X};\mathsf{YZ}) & (15)\\ &= \mathrm{I}(\mathsf{X};\mathsf{Y}) + \mathrm{I}(\mathsf{X};\mathsf{Z}|\mathsf{Y}) = \mathrm{I}(\mathsf{X};\mathsf{Y}). & \blacksquare\end{aligned}$$

Occasionally we will interpret a probability measure on a finite space $\mathcal{M}$ as a vector in $[0,1]^\mathcal{M}$, equipped with the Borel σ-algebra. We will use the L_∞-distance on this space.

Definition 9. *For two probability measures* μ *and* ν *on a finite space* $\mathcal{M}$*, their distance is defined as the* L_∞*-distance* $\mathrm{d}(\mu,\nu) := \max_{m\in\mathcal{M}} |\mu(m) - \nu(m)|$*. The diameter of* $A \subseteq [0,1]^\mathcal{M}$ *is defined as* $\operatorname{diam}(A) = \sup_{\mu,\nu\in A} \mathrm{d}(\mu,\nu)$.

Lemma 10 ([20, Lem. 2.7]). *For two probability measures* μ *and* ν *on a finite space* $\mathcal{M}$ *with* $\mathrm{d}(\mu,\nu) \leq \varepsilon \leq \frac{1}{2}$ *the inequality* $|\mathrm{H}(\mu) - \mathrm{H}(\nu)| \leq -\varepsilon|\mathcal{M}|\log\varepsilon$ *holds.*

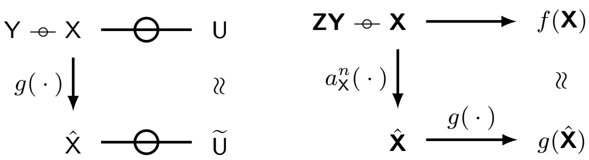
IV. Proof of $\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$

For finite spaces $\mathcal{S}_\mathsf{Y}$, $\mathcal{S}_\mathsf{X}$, and $\mathcal{S}_\mathsf{U}$, the statement $\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$ is well known, cf., [10, Sec. IV], [11, Sec. III.F]. We restate it in the form of the following lemma.

Lemma 11. *For random variables* Y*,* X*, and* U *with finite* $\mathcal{S}_\mathsf{Y}$*,* $\mathcal{S}_\mathsf{X}$*, and* $\mathcal{S}_\mathsf{U}$*, assume that* $\mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}$ *holds. Then, for any* $\varepsilon > 0$*, there exists* $n \in \mathbb{N}$ *and a function* $f \colon \mathcal{S}_\mathsf{X}^n \to \mathcal{M}$ *with* $\frac{1}{n}\log|\mathcal{M}| \leq \mathrm{I}(\mathsf{X};\mathsf{U}) + \varepsilon$ *such that* $\frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) \geq \mathrm{I}(\mathsf{Y};\mathsf{U}) - \varepsilon$.

In a first step, we will utilize Lem. 11 to show $\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$ for an arbitrary alphabet $\mathcal{S}_\mathsf{X}$, i. e., we wish to prove the following Proposition 12, lifting the restriction $|\mathcal{S}_\mathsf{X}| < \infty$.

Proposition 12. *For random variables* Y*,* X*, and* U *with finite* $\mathcal{S}_\mathsf{Y}$ *and* $\mathcal{S}_\mathsf{U}$*, assume that* $\mathsf{Y} \multimap \mathsf{X} \multimap \mathsf{U}$ *holds. Then, for any*



(a) $\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$. (b) $\mathcal{R} \subseteq \overline{\mathcal{R}_{\mathrm{IB}}}$.

Fig. 1: Illustrations.

$\varepsilon > 0$, there exists $n \in \mathbb{N}$ and a function $f\colon \mathcal{S}_{\mathsf{X}}^n \to \mathcal{M}$ with $\frac{1}{n}\log|\mathcal{M}| \le \mathrm{I}(\mathsf{X};\mathsf{U}) + \varepsilon$ such that

$$\frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) \ge \mathrm{I}(\mathsf{Y};\mathsf{U}) - \varepsilon. \tag{16}$$

Remark 2. Considering that both definitions of achievability (Defs. 1 and 2) only rely on the notion of mutual information, one may assume that Def. 6 can be used to directly infer Proposition 12 from Lem. 11. However, this is not the case. For an arbitrary discretization $a_{\mathsf{X}}(\mathsf{X})$ of X, we do have $\mathrm{I}(a_{\mathsf{X}}(\mathsf{X});\mathsf{U}) \le \mathrm{I}(\mathsf{X};\mathsf{U})$. However, the Markov chain $\mathsf{Y} -\!\circ\!- a_{\mathsf{X}}(\mathsf{X}) -\!\circ\!- \mathsf{U}$ does not hold in general. To circumvent this problem, we will use a discrete random variable $\hat{\mathsf{X}} = g(\mathsf{X})$ with an appropriate quantizer g and construct a new random variable $\widetilde{\mathsf{U}}$, satisfying the Markov chain $\mathsf{Y} -\!\circ\!- \hat{\mathsf{X}} -\!\circ\!- \widetilde{\mathsf{U}}$ such that $\mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}})$ is close to $\mathrm{I}(\mathsf{Y};\mathsf{U})$. Fig. 1a illustrates this strategy. We choose the quantizer g based on the conditional probability distribution of U given X, i. e., quantization based on $\kappa_{\mathsf{U}|\mathsf{X}}$ using L_∞-distance (cf. Def. 9). Subsequently, we will use that, by Lem. 10, a small L_∞-distance guarantees a small gap in terms of information measures.

Proof of Proposition 12. Let μ_{YXU} be a probability measure on $\Omega := \mathcal{S}_{\mathsf{Y}} \times \mathcal{S}_{\mathsf{X}} \times \mathcal{S}_{\mathsf{U}}$, such that $\mathsf{Y} -\!\circ\!- \mathsf{X} -\!\circ\!- \mathsf{U}$ holds. Fix $0 < \delta \le \frac{1}{2}$ and find a finite, measurable partition $(P_i)_{i\in\mathcal{I}}$ of the space of probability measures on $\mathcal{S}_{\mathsf{U}}$ such that for every $i \in \mathcal{I}$ we have $\operatorname{diam}(P_i) \le \delta$ and fix some $\nu_i \in P_i$ for every $i \in \mathcal{I}$. Define the random variable $\hat{\mathsf{X}}\colon \Omega \to \mathcal{I}$ as $\hat{\mathsf{X}} = i$ if $\kappa_{\mathsf{U}|\mathsf{X}} \in P_i$. The random variable $\hat{\mathsf{X}}$ is $\sigma(\mathsf{X})$-measurable (see Appendix A). We can therefore find a measurable function g such that $\hat{\mathsf{X}} = g(\mathsf{X})$ by the factorization lemma [19, Corollary 1.97]. Define the new probability space $\Omega \times \bigtimes_{i\in\mathcal{I}} \mathcal{S}_{\mathsf{U}}$, equipped with the probability measure $\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} := \mu_{\mathsf{YXU}} \times \bigtimes_{i\in\mathcal{I}} \nu_i$. Slightly abusing notation, we define the random variables $\mathsf{Y}, \mathsf{X}, \mathsf{U}$, and $\widetilde{\mathsf{U}}_i$ (for every $i \in \mathcal{I}$) as the according projections. We also use $\hat{\mathsf{X}} = g(\mathsf{X})$ and define the random variable $\widetilde{\mathsf{U}} = \widetilde{\mathsf{U}}_{\hat{\mathsf{X}}}$. From this construction we have $\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}}$-a. e. the equality of measures $\kappa_{\widetilde{\mathsf{U}}|\hat{\mathsf{X}}} = \kappa_{\widetilde{\mathsf{U}}|\mathsf{X}} = \nu_{\hat{\mathsf{X}}}$, as well as $\mathsf{Y} -\!\circ\!- \hat{\mathsf{X}} -\!\circ\!- \widetilde{\mathsf{U}}$ and $\mathsf{Y} -\!\circ\!- \mathsf{X} -\!\circ\!- \widetilde{\mathsf{U}}$ (see Appendix B). Therefore, we have $\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}}$-a. e.

$$\mathrm{d}(\kappa_{\widetilde{\mathsf{U}}|\hat{\mathsf{X}}}, \kappa_{\mathsf{U}|\mathsf{X}}) \le \delta, \text{ and } \qquad \mathrm{d}(\kappa_{\widetilde{\mathsf{U}}|\mathsf{X}}, \kappa_{\mathsf{U}|\mathsf{X}}) \le \delta, \tag{17}$$

by $\kappa_{\widetilde{\mathsf{U}}|\hat{\mathsf{X}}} = \kappa_{\widetilde{\mathsf{U}}|\mathsf{X}} = \nu_{\hat{\mathsf{X}}}$ and $\kappa_{\mathsf{U}|\mathsf{X}}, \nu_{\hat{\mathsf{X}}} \in P_{\hat{\mathsf{X}}}$. Thus, for any $u \in \mathcal{S}_{\mathsf{U}}$,

$$\begin{aligned}\mu_{\mathsf{U}}(u) &= \int \kappa_{\mathsf{U}|\mathsf{X}}(\,\cdot\,; u)\, d\mu_{\mathsf{YXU}} &(18)\\ &\le \int (\kappa_{\widetilde{\mathsf{U}}|\mathsf{X}}(\,\cdot\,; u) + \delta)\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} = \mu_{\widetilde{\mathsf{U}}}(u) + \delta &(19)\end{aligned}$$

and, by the same argument, $\mu_{\mathsf{U}}(u) \ge \mu_{\widetilde{\mathsf{U}}}(u) - \delta$, i. e., in total,

$$\mathrm{d}(\mu_{\mathsf{U}}, \mu_{\widetilde{\mathsf{U}}}) \le \delta. \tag{20}$$

Thus, we obtain

$$\begin{aligned}\mathrm{I}(\mathsf{X};\mathsf{U}) &= \mathrm{H}(\mu_{\mathsf{U}}) - \mathrm{H}(\mathsf{U}|\mathsf{X}) &(21)\\ &\overset{(20)}{\ge} \mathrm{H}\big(\mu_{\widetilde{\mathsf{U}}}\big) + \delta|\mathcal{S}_{\mathsf{U}}|\log\delta - \int \mathrm{H}\big(\kappa_{\mathsf{U}|\mathsf{X}}\big)\, d\mu_{\mathsf{YXU}} &(22)\\ &\overset{(17)}{\ge} \mathrm{H}\big(\mu_{\widetilde{\mathsf{U}}}\big) + 2\delta|\mathcal{S}_{\mathsf{U}}|\log\delta - \int \mathrm{H}\big(\kappa_{\widetilde{\mathsf{U}}|\hat{\mathsf{X}}}\big)\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} &(23)\\ &= \mathrm{I}(\hat{\mathsf{X}};\widetilde{\mathsf{U}}) + 2\delta|\mathcal{S}_{\mathsf{U}}|\log\delta, &(24)\end{aligned}$$

where (21) and (24) follow from Prop. (ii) of Lem. 8, and in both (22) and (23) we used Lem. 10. From $\mathsf{Y} -\!\circ\!- \mathsf{X} -\!\circ\!- \mathsf{U}$ and Prop. (i) of Lem. 8, we know that μ_{YXU}-a. e., we have the equality of measures $\kappa_{\mathsf{YU}|\mathsf{X}} = \kappa_{\mathsf{Y}|\mathsf{X}} \times \kappa_{\mathsf{U}|\mathsf{X}}$. Using this equality in (26) we obtain

$$\begin{aligned}\mu_{\mathsf{YU}}(y \times u) &= \int \kappa_{\mathsf{YU}|\mathsf{X}}(\,\cdot\,; y \times u)\, d\mu_{\mathsf{YXU}} &(25)\\ &= \int \kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,; y)\kappa_{\mathsf{U}|\mathsf{X}}(\,\cdot\,; u)\, d\mu_{\mathsf{YXU}} &(26)\\ &\overset{(17)}{\le} \int \kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,; y)(\kappa_{\widetilde{\mathsf{U}}|\mathsf{X}}(\,\cdot\,; u) + \delta)\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} &(27)\\ &\le \int \kappa_{\mathsf{Y}\widetilde{\mathsf{U}}|\mathsf{X}}(\,\cdot\,; y \times u)\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} + \delta &(28)\\ &= \mu_{\mathsf{Y}\widetilde{\mathsf{U}}}(y \times u) + \delta, &(29)\end{aligned}$$

where (25) and (29) follow from the defining property of conditional probability, part 2 of Def. 4, and (28) follows from $\mathsf{Y} -\!\circ\!- \mathsf{X} -\!\circ\!- \widetilde{\mathsf{U}}$ and Prop. (i) of Lem. 8. By the same argument, one can show that $\mu_{\mathsf{YU}}(y \times u) \ge \mu_{\mathsf{Y}\widetilde{\mathsf{U}}}(y \times u) - \delta$. Therefore, in total, $\mathrm{d}(\mu_{\mathsf{YU}}, \mu_{\mathsf{Y}\widetilde{\mathsf{U}}}) \le \delta$ and, by Lem. 10,

$$|\mathrm{H}(\mathsf{YU}) - \mathrm{H}(\mathsf{Y}\widetilde{\mathsf{U}})| \le -\delta|\mathcal{S}_{\mathsf{Y}}||\mathcal{S}_{\mathsf{U}}|\log\delta. \tag{30}$$

Thus, the mutual information can be bounded by

$$\begin{aligned}\mathrm{I}(\mathsf{Y};\mathsf{U}) &= \mathrm{H}(\mathsf{Y}) + \mathrm{H}(\mathsf{U}) - \mathrm{H}(\mathsf{YU}) &(31)\\ &\overset{(20)}{\le} \mathrm{H}(\mathsf{Y}) + \mathrm{H}(\widetilde{\mathsf{U}}) - \delta|\mathcal{S}_{\mathsf{U}}|\log\delta - \mathrm{H}(\mathsf{YU}) &(32)\\ &\overset{(30)}{\le} \mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}}) - \delta(|\mathcal{S}_{\mathsf{Y}}| + 1)|\mathcal{S}_{\mathsf{U}}|\log\delta &(33)\\ &\le \mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}}) - 2\delta|\mathcal{S}_{\mathsf{Y}}||\mathcal{S}_{\mathsf{U}}|\log\delta, &(34)\end{aligned}$$

where we applied Lem. 10 in (32) and (33). We apply Lem. 11 to the three random variables $\mathsf{Y}, \hat{\mathsf{X}}$, and $\widetilde{\mathsf{U}}$ and obtain a function $\hat{f}\colon \mathcal{I}^n \to \mathcal{M}$ with $\frac{1}{n}\mathrm{I}\big(\mathbf{Y}; \hat{f}(\hat{\mathbf{X}})\big) \ge \mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}}) - \delta$ and

$$\frac{1}{n}\log|\mathcal{M}| \le \mathrm{I}(\hat{\mathsf{X}};\widetilde{\mathsf{U}}) + \delta \overset{(24)}{\le} \mathrm{I}(\mathsf{X};\mathsf{U}) + \delta - 2\delta|\mathcal{S}_{\mathsf{U}}|\log\delta. \tag{35}$$

We have $\hat{\mathbf{X}} = g^n \circ \mathbf{X}$ and defining $f := \hat{f} \circ g^n$, we obtain

$$\begin{aligned}\frac{1}{n}\mathrm{I}(\mathbf{Y}; f(\mathbf{X})) = \frac{1}{n}\mathrm{I}(\mathbf{Y}; \hat{f}(\hat{\mathbf{X}})) &\ge \mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}}) - \delta &(36)\\ &\overset{(34)}{\ge} \mathrm{I}(\mathsf{Y};\mathsf{U}) + 2\delta|\mathcal{S}_{\mathsf{Y}}||\mathcal{S}_{\mathsf{U}}|\log\delta - \delta. &(37)\end{aligned}$$

Choosing δ such that $\varepsilon \ge -2\delta|\mathcal{S}_{\mathsf{Y}}||\mathcal{S}_{\mathsf{U}}|\log\delta + \delta$ completes the proof. ∎

We can now complete the proof by showing the following lemma.

Lemma 13. *$\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$.*

Proof. Assuming $(S,R) \in \mathcal{R}_{\mathrm{IB}}$, choose μ_{YXU} according to Def. 2. Clearly $\mathrm{I}(\mathsf{X};\mathsf{U}) < \infty$ to satisfy (3) and thus also $\mathrm{I}(\mathsf{Y};\mathsf{U}) < \infty$ by Prop. (iv) of Lem. 8 as $\mathsf{Y} -\!\circ\!- \mathsf{X} -\!\circ\!- \mathsf{U}$ holds. Pick $\varepsilon > 0$, select functions a_{X}, a_{U} such that $\mathrm{I}\big(a_{\mathsf{X}}(\mathsf{X}); a_{\mathsf{U}}(\mathsf{U})\big) \geq \mathrm{I}(\mathsf{X};\mathsf{U}) - \varepsilon$, and select functions b_{Y}, b_{U} such that $\mathrm{I}\big(b_{\mathsf{Y}}(\mathsf{Y}); b_{\mathsf{U}}(\mathsf{U})\big) \geq \mathrm{I}(\mathsf{Y};\mathsf{U}) - \varepsilon$ (cf. (7)). Using $\hat{\mathsf{U}} := \big(a_{\mathsf{U}}(\mathsf{U}), b_{\mathsf{U}}(\mathsf{U})\big)$ and $\hat{\mathsf{Y}} := b_{\mathsf{Y}}(\mathsf{Y})$, we have

$$0 = \mathrm{I}(\mathsf{Y};\mathsf{U}|\mathsf{X}) = \sup_{c_{\mathsf{Y}}, c_{\mathsf{U}}} \mathrm{I}(c_{\mathsf{Y}}(\mathsf{Y}); c_{\mathsf{U}}(\mathsf{U})|\mathsf{X}) \geq \mathrm{I}(\hat{\mathsf{Y}};\hat{\mathsf{U}}|\mathsf{X}) \geq 0 \quad (38)$$

as well as

$$\mathrm{I}(\mathsf{X};\mathsf{U}) = \sup_{c_{\mathsf{X}}, c_{\mathsf{U}}} \mathrm{I}\big(c_{\mathsf{X}}(\mathsf{X}); c_{\mathsf{U}}(\mathsf{U})\big) \quad (39)$$

$$\geq \sup_{c_{\mathsf{X}}} \mathrm{I}(c_{\mathsf{X}}(\mathsf{X}); \hat{\mathsf{U}}) = \mathrm{I}(\mathsf{X};\hat{\mathsf{U}}), \text{ and} \quad (40)$$

$$\mathrm{I}(\mathsf{Y};\mathsf{U}) - \varepsilon \leq \mathrm{I}\big(b_{\mathsf{Y}}(\mathsf{Y}); b_{\mathsf{U}}(\mathsf{U})\big) \leq \mathrm{I}(\hat{\mathsf{Y}};\hat{\mathsf{U}}). \quad (41)$$

We apply Proposition 12, substituting $\hat{\mathsf{U}} \to \mathsf{U}$ and $\hat{\mathsf{Y}} \to \mathsf{Y}$. Proposition 12 guarantees the existence of a function $f\colon \mathcal{S}_{\mathsf{X}}^n \to \mathcal{M}$ with $\frac{1}{n}\log|\mathcal{M}| \leq \mathrm{I}(\mathsf{X};\hat{\mathsf{U}}) + \varepsilon \overset{(40)}{\leq} \mathrm{I}(\mathsf{X};\mathsf{U}) + \varepsilon \overset{(3)}{\leq} R + \varepsilon$ and

$$\frac{1}{n}\mathrm{I}(\mathbf{Y}; f(\mathbf{X})) = \frac{1}{n}\sup_{c_{\mathbf{Y}}} \mathrm{I}(c_{\mathbf{Y}} \circ \mathbf{Y}; f(\mathbf{X})) \quad (42)$$

$$\geq \frac{1}{n}\mathrm{I}(b_{\mathsf{Y}}^n \circ \mathbf{Y}; f(\mathbf{X})) = \frac{1}{n}\mathrm{I}(\hat{\mathbf{Y}}; f(\mathbf{X})) \quad (43)$$

$$\overset{(16)}{\geq} \mathrm{I}(\hat{\mathsf{Y}};\hat{\mathsf{U}}) - \varepsilon \overset{(41)}{\geq} \mathrm{I}(\mathsf{Y};\mathsf{U}) - 2\varepsilon \overset{(4)}{\geq} S - 2\varepsilon. \quad (44)$$

Thus, $(S - 2\varepsilon, R - \varepsilon) \in \mathcal{R}$ and therefore $(S, R) \in \overline{\mathcal{R}}$. ∎

V. Proof of $\mathcal{R} \subseteq \overline{\mathcal{R}_{\mathrm{IB}}}$

We start with the well-known result $\mathcal{R}_{\mathrm{IB}} \subseteq \overline{\mathcal{R}}$ for finite spaces $\mathcal{S}_{\mathsf{Y}}$, $\mathcal{S}_{\mathsf{X}}$, and $\mathcal{S}_{\mathsf{U}}$, cf., [10, Sec. IV], [11, Sec. III.F]. The statement is rephrased in the following lemma.

Lemma 14. *Assume that the spaces $\mathcal{S}_{\mathsf{Y}}$ and $\mathcal{S}_{\mathsf{X}}$ are both finite and μ_{YX} is fixed. For some $n \in \mathbb{N}$, let $f\colon \mathcal{S}_{\mathsf{X}}^n \to \mathcal{M}$ be a function with $|\mathcal{M}| < \infty$. Then there exists a probability measure μ_{YXU}, extending μ_{YX}, such that $\mathcal{S}_{\mathsf{U}}$ is finite, $\mathsf{Y} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$, and*

$$\mathrm{I}(\mathsf{X};\mathsf{U}) \leq \frac{1}{n}\log|\mathcal{M}|, \quad (45)$$

$$\mathrm{I}(\mathsf{Y};\mathsf{U}) \geq \frac{1}{n}\mathrm{I}(\mathbf{Y}; f(\mathbf{X})). \quad (46)$$

We can slightly strengthen Lem. 14.

Corollary 15. *Assume that, in the setting of Lem. 14, we are given μ_{ZYX} on $\mathcal{S}_{\mathsf{Z}} \times \mathcal{S}_{\mathsf{Y}} \times \mathcal{S}_{\mathsf{X}}$, extending μ_{YX}, where $\mathcal{S}_{\mathsf{Z}}$ is arbitrary, not necessarily finite. Then there exists a probability measure μ_{ZYXU}, extending μ_{ZYX}, such that $\mathcal{S}_{\mathsf{U}}$ is finite and $\mathsf{ZY} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$, (45), and (46) hold.*

Proof. Apply Lem. 14 to obtain μ_{YXU} on $\mathcal{S}_{\mathsf{Y}} \times \mathcal{S}_{\mathsf{X}} \times \mathcal{S}_{\mathsf{U}}$ satisfying (45), (46), and $\mathsf{Y} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$. We define μ_{ZYXU} by

$$\mu_{\mathsf{ZYXU}}(A \times y \times x \times u) = \frac{\mu_{\mathsf{ZYX}}(A \times y \times x)}{\mu_{\mathsf{YX}}(y \times x)}\mu_{\mathsf{YXU}}(y \times x \times u) \quad (47)$$

for any $(y, x, u) \in \mathcal{S}_{\mathsf{Y}} \times \mathcal{S}_{\mathsf{X}} \times \mathcal{S}_{\mathsf{U}}$ and $A \in \mathcal{A}_{\mathsf{Z}}$. Pick arbitrary $A \in \mathcal{A}_{\mathsf{Z}}$, $y \in \mathcal{S}_{\mathsf{Y}}$, and $u \in \mathcal{S}_{\mathsf{U}}$. The Markov chain $\mathsf{ZY} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$ now follows as the events $\mathsf{Z}^{-1}(A) \cap \mathsf{Y}^{-1}(y)$ and $\mathsf{U}^{-1}(u)$ are independent given $\mathsf{X}^{-1}(x)$ for any $x \in \mathcal{S}_{\mathsf{X}}$ (cf. Rmk. 1). ∎

Again, we proceed by extending Cor. 15, lifting the restriction that $\mathcal{S}_{\mathsf{X}}$ is finite and obtain the following proposition.

Proposition 16. *Given a probability measure μ_{ZYX} as in Cor. 15, assume that $|\mathcal{S}_{\mathsf{Y}}| < \infty$. For some $n \in \mathbb{N}$, let $f\colon \mathcal{S}_{\mathsf{X}}^n \to \mathcal{M}$ be a function with $|\mathcal{M}| < \infty$. Then, for any $\varepsilon > 0$, there exists a probability measure μ_{ZYXU}, extending μ_{ZYX} with $\mathsf{ZY} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$ and*

$$\mathrm{I}(\mathsf{X};\mathsf{U}) \leq \frac{1}{n}\log|\mathcal{M}| \quad (48)$$

$$\mathrm{I}(\mathsf{Y};\mathsf{U}) \geq \frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) - \varepsilon. \quad (49)$$

Remark 3. In contrast to Proposition 12, Proposition 16 could be proved by the usual single-letterization + time-sharing strategy, by showing that the necessary Markov chains hold. However, we will rely on the discrete case (Lem. 14) and showcase a technique to lift it to general alphabets.

Remark 4. In the proof of Proposition 16, we face a similar problem as outlined in Rmk. 2. We need to construct a function $g(\hat{\mathbf{X}})$ of a "per-letter" quantization $\hat{\mathbf{X}} := a_{\mathsf{X}}^n(\mathbf{X})$, that is close to $f(\mathbf{X})$ in distribution. Fig. 1b provides a sketch.

Proof of Proposition 16. We can partition $\mathcal{S}_{\mathsf{X}}^n = \bigcup_{m \in \mathcal{M}} \mathcal{Q}_m$ into finitely many measurable, mutually disjoint sets $\mathcal{Q}_m := f^{-1}(m)$, $m \in \mathcal{M}$. We want to approximate the sets $\mathcal{Q}_m$ by a finite union of rectangles in the semiring [19, Def. 1.9] $\Xi := \big\{\mathcal{B} : \mathcal{B} = \bigtimes_{i=1}^n B_i \text{ with } B_i \in \mathcal{A}_{\mathsf{X}}\big\}$. We choose $\delta > 0$, which will be specified later. According to [19, Thm. 1.65(ii)], we obtain $\mathcal{B}^{(m)} := \bigcup_{k=1}^K \mathcal{B}_k^{(m)}$ for each $m \in \mathcal{M}$, where $\mathcal{B}_k^{(m)} \in \Xi$ are mutually disjoint sets, satisfying $\mu_{\mathbf{X}}(\mathcal{B}^{(m)} \triangle \mathcal{Q}_m) \leq \delta$. Since $\mathcal{B}_k^{(m)} \in \Xi$, we have $\mathcal{B}_k^{(m)} = \bigtimes_{i=1}^n B_{k,i}^{(m)}$ for some $B_{k,i}^{(m)} \in \mathcal{A}_{\mathsf{X}}$. We can construct functions a_{X} and g such that $g \circ a_{\mathsf{X}}^n(\boldsymbol{x}) = m$ whenever $\boldsymbol{x} \in \mathcal{B}^{(m)}$ and $\boldsymbol{x} \notin \mathcal{B}^{(\not m)}$ with $\mathcal{B}^{(\not m)} := \bigcup_{m' \neq m} \mathcal{B}^{(m')}$. Indeed, we obtain a_{X} by finding a measurable partition of $\mathcal{S}_{\mathsf{X}}$ that is finer than $(B_{k,i}^{(m)}, (B_{k,i}^{(m)})^{\mathrm{c}})$ for all i, k, m. For fixed $m \in \mathcal{M}$,

$$\mathcal{Q}_m \subseteq \mathcal{Q}_m \cup \big(\mathcal{B}^{(m)} \setminus \mathcal{B}^{(\not m)}\big) \quad (50)$$

$$\subseteq \big(\mathcal{B}^{(m)} \setminus \mathcal{B}^{(\not m)}\big) \cup \big(\mathcal{Q}_m \setminus \mathcal{B}^{(m)}\big) \cup \bigcup_{m' \neq m} \mathcal{Q}_m \cap \mathcal{B}^{(m')} \quad (51)$$

$$\subseteq \big(\mathcal{B}^{(m)} \setminus \mathcal{B}^{(\not m)}\big) \cup \big(\mathcal{Q}_m \triangle \mathcal{B}^{(m)}\big) \cup \bigcup_{m' \neq m} \mathcal{B}^{(m')} \setminus \mathcal{Q}_{m'} \quad (52)$$

$$\subseteq \big(\mathcal{B}^{(m)} \setminus \mathcal{B}^{(\not m)}\big) \cup \bigcup_{m'} \mathcal{B}^{(m')} \triangle \mathcal{Q}_{m'}, \quad (53)$$

where we used the fact that $\mathcal{Q}_m \cap \mathcal{Q}_{m'} = \varnothing$ for $m \neq m'$ in (52). Using $\hat{\mathsf{X}} := a_{\mathsf{X}}(\mathsf{X})$, we obtain for any $\boldsymbol{y} \in \mathcal{S}_{\mathsf{Y}}^n$

$$\mu_{\mathbf{Y}f(\mathbf{X})}(\boldsymbol{y} \times m) = \mu_{\mathbf{YX}}(\boldsymbol{y} \times \mathcal{Q}_m) \quad (54)$$

$$\overset{(53)}{\leq} \mu_{\mathbf{YX}}\big(\boldsymbol{y} \times (\mathcal{B}^{(m)} \setminus \mathcal{B}^{(\not m)})\big) + \sum_{m'} \mu_{\mathbf{X}}(\mathcal{B}^{(m')} \triangle \mathcal{Q}_{m'}) \quad (55)$$

$$\leq \mu_{\mathbf{Y}g(\hat{\mathbf{X}})}(\boldsymbol{y} \times m) + |\mathcal{M}|\delta. \quad (56)$$

On the other hand, we have

$$\mu_{\mathbf{Y}f(\mathbf{X})}(\boldsymbol{y} \times m) = \mu_{\mathbf{Y}}(\boldsymbol{y}) - \sum_{m' \neq m} \mu_{\mathbf{Y}f(\mathbf{X})}(\boldsymbol{y} \times m') \quad (57)$$

$$\overset{(56)}{\geq} \mu_{\mathbf{Y}}(\boldsymbol{y}) - \sum_{m' \neq m} \Big(\mu_{\mathbf{Y}g(\hat{\mathbf{X}})}(\boldsymbol{y} \times m') + |\mathcal{M}|\delta\Big) \quad (58)$$

$$\geq \mu_{\mathbf{Y}g(\hat{\mathbf{X}})}(\boldsymbol{y} \times m) - |\mathcal{M}|^2\delta. \quad (59)$$

We thus obtain $\mathrm{d}(\mu_{\mathbf{Y}f(\mathbf{X})}, \mu_{\mathbf{Y}g(\hat{\mathbf{X}})}) \leq |\mathcal{M}|^2\delta$. This also implies $\mathrm{d}(\mu_{f(\mathbf{X})}, \mu_{g(\hat{\mathbf{X}})}) \leq |\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^2\delta$. Assume $|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^2\delta \leq \frac{1}{2}$ and apply Cor. 15 substituting $\hat{\mathsf{X}} \to \mathsf{X}$, $\mathsf{XZ} \to \mathsf{Z}$, and the function

$g \to f$. This yields a random variable U with $\mathsf{XZY} \mathrel{-\!\!\circ\!\!-} \hat{\mathsf{X}} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$,

$$\mathrm{I}(\hat{\mathsf{X}};\mathsf{U}) \leq \frac{1}{n}\log|\mathcal{M}|, \text{ and } \quad \mathrm{I}(\mathsf{Y};\mathsf{U}) \geq \frac{1}{n}\mathrm{I}\big(\mathbf{Y}; g(\hat{\mathbf{X}})\big). \tag{60}$$

We also obtain $\mathsf{ZY} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$ due to

$$\begin{aligned} 0 &= \mathrm{I}(\mathsf{XZY};\mathsf{U}|\hat{\mathsf{X}}) && (61)\\ &= \mathrm{I}(\mathsf{XZY};\mathsf{U}) - \mathrm{I}(\mathsf{U};\hat{\mathsf{X}}) && (62)\\ &\geq \mathrm{I}(\mathsf{XZY};\mathsf{U}) - \mathrm{I}(\mathsf{U};\mathsf{X}) && (63)\\ &= \mathrm{I}(\mathsf{ZY};\mathsf{U}|\mathsf{X}) && (64)\\ &\geq 0, && (65) \end{aligned}$$

where (61) follows from $\mathsf{XZY} \mathrel{-\!\!\circ\!\!-} \hat{\mathsf{X}} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$ using Prop. (i) of Lem. 8, (62) and (64) follow from Prop. (iii) of Lem. 8, (63) is a consequence of Def. 6, and we used Prop. (i) of Lem. 8 in (65). This also immediately implies $0 = \mathrm{I}(\mathsf{X};\mathsf{U}|\hat{\mathsf{X}})$ and hence

$$\begin{aligned} \frac{1}{n}\log|\mathcal{M}| &\overset{(60)}{\geq} \mathrm{I}(\hat{\mathsf{X}};\mathsf{U}) = \mathrm{I}(\hat{\mathsf{X}};\mathsf{U}) + \mathrm{I}(\mathsf{X};\mathsf{U}|\hat{\mathsf{X}}) && (66)\\ &= \mathrm{I}(\mathsf{X}\hat{\mathsf{X}};\mathsf{U}) = \mathrm{I}(\mathsf{X};\mathsf{U}), && (67) \end{aligned}$$

where we used Prop. (iii) of Lem. 8 in (67). We also have

$$\begin{aligned} \mathrm{I}(\mathsf{Y};\mathsf{U}) &\overset{(60)}{\geq} \frac{1}{n}\mathrm{I}\big(\mathbf{Y}; g(\hat{\mathbf{X}})\big) && (68)\\ &= \frac{1}{n}\Big(\mathrm{H}(\mathbf{Y}) + \mathrm{H}(g(\hat{\mathbf{X}})) - \mathrm{H}(\mathbf{Y}g(\hat{\mathbf{X}}))\Big) && (69)\\ &\geq \frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) + \frac{1}{n}|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^3\delta\log(|\mathcal{M}|^2\delta)\\ &\qquad + \frac{1}{n}|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^3\delta\log(|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^2\delta) && (70)\\ &\geq \frac{1}{n}\mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) + \frac{2}{n}|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^3\delta\log(|\mathcal{M}|^2\delta) && (71) \end{aligned}$$

where we used Lem. 10 in (70). Select δ such that $\varepsilon \geq -\frac{2}{n}|\mathcal{S}_{\mathsf{Y}}|^n|\mathcal{M}|^3\delta\log(|\mathcal{M}|^2\delta)$. ∎

We can now finish the proof by showing the following lemma.

Lemma 17. *$\mathcal{R} \subseteq \overline{\mathcal{R}_{\mathrm{IB}}}$.*

Proof. Assume $(S,R) \in \mathcal{R}$ and choose $n \in \mathbb{N}$ and f, satisfying $\frac{1}{n}\log|\mathcal{M}| \leq R$ and (2). Choose any $\varepsilon > 0$ and find a_{Y} such that

$$\mathrm{I}\big(a_{\mathsf{Y}}^n(\mathbf{Y}); f(\mathbf{X})\big) \geq \mathrm{I}\big(\mathbf{Y}; f(\mathbf{X})\big) - \varepsilon \overset{(2)}{\geq} nS - \varepsilon. \tag{72}$$

This is possible by applying [18, Lem. 5.2.2] with the algebra that is generated by the rectangles (cf. the paragraph above [18, Lem. 5.5.1]). We apply Proposition 16, substituting $a_{\mathsf{Y}}(\mathsf{Y}) \to \mathsf{Y}$ and $\mathsf{Y} \to \mathsf{Z}$. For arbitrary $\varepsilon > 0$, Proposition 16 provides U with $\mathsf{Y}a_{\mathsf{Y}}(\mathsf{Y}) \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$ (i. e., $\mathsf{Y} \mathrel{-\!\!\circ\!\!-} \mathsf{X} \mathrel{-\!\!\circ\!\!-} \mathsf{U}$) and

$$\begin{aligned} \mathrm{I}(\mathsf{X};\mathsf{U}) &\leq \frac{1}{n}\log|\mathcal{M}| \leq R && (73)\\ \mathrm{I}(\mathsf{Y};\mathsf{U}) &\geq \mathrm{I}(a_{\mathsf{Y}}(\mathsf{Y});\mathsf{U}) && (74)\\ &\overset{(49)}{\geq} \frac{1}{n}\mathrm{I}\big(a_{\mathsf{Y}}^n(\mathbf{Y}); f(\mathbf{X})\big) - \varepsilon \overset{(72)}{\geq} S - 2\varepsilon. && (75) \end{aligned}$$

Hence, $(S - 2\varepsilon, R) \in \mathcal{R}_{\mathrm{IB}}$ and consequently $(S,R) \in \overline{\mathcal{R}_{\mathrm{IB}}}$. ∎

Appendix

A. $\hat{\mathsf{X}}$ is $\sigma(\mathsf{X})$-measurable

For $u \in \mathcal{S}_{\mathsf{U}}$ consider the $\sigma(\mathsf{X})$-measurable function $h_u := \kappa_{\mathsf{U}|\mathsf{X}}(\,\cdot\,;u)$ on $[0,1]$. We obtain the vector valued function $h := (h_u)_{u\in\mathcal{S}_{\mathsf{U}}}$ on $[0,1]^{|\mathcal{S}_{\mathsf{U}}|}$. This function h is $\sigma(\mathsf{X})$-measurable as every component is $\sigma(\mathsf{X})$-measurable. Thus, we have $\hat{\mathsf{X}}^{-1}(i) = h^{-1}(P_i) \in \sigma(\mathsf{X})$.

B. Distribution of $\widetilde{\mathsf{U}}$ and Conditional Independence

We will first show that $\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}}$-a. e.

$$\kappa_{\widetilde{\mathsf{U}}|\hat{\mathsf{X}}} = \kappa_{\widetilde{\mathsf{U}}|\mathsf{X}} = \nu_{\hat{\mathsf{X}}}. \tag{76}$$

Clearly, $\nu_{\hat{\mathsf{X}}}$ is a probability measure everywhere. Fixing $u \in \mathcal{S}_{\mathsf{U}}$, we need that $\nu_{\hat{\mathsf{X}}}(u)$ is $\sigma(\hat{\mathsf{X}})$-measurable, which is shown by the factorization lemma [19, Corollary 1.97], when writing $\nu_{\hat{\mathsf{X}}}(u) = \nu_{(\cdot)}(u) \circ \hat{\mathsf{X}}$. Also, this proves $\sigma(\mathsf{X})$-measurability as $\hat{\mathsf{X}}$ is $\sigma(\mathsf{X})$-measurable, i. e., $\sigma(\hat{\mathsf{X}}) \subseteq \sigma(\mathsf{X})$. It remains to show the defining property of conditional probability, part 2 of Def. 4. Choosing $B \in \sigma(\mathsf{X})$ and $u \in \mathcal{S}_{\mathsf{U}}$, we need to show that

$$\mathbb{E}[\mathbb{1}_B \nu_{\hat{\mathsf{X}}}(u)] = \mathbb{E}\Big[\mathbb{1}_B \mathbb{1}_{\{u\}}(\widetilde{\mathsf{U}})\Big]. \tag{77}$$

The statement for $B \in \sigma(\hat{\mathsf{X}})$ then follows by $\sigma(\hat{\mathsf{X}}) \subseteq \sigma(\mathsf{X})$, i. e., the $\sigma(\mathsf{X})$-measurability of $\hat{\mathsf{X}}$. We prove (77) by

$$\begin{aligned} \mathbb{E}[\mathbb{1}_B \nu_{\hat{\mathsf{X}}}(u)] &= \sum_{i\in\mathcal{I}} \mathbb{E}\big[\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_B \nu_i(u)\big] && (78)\\ &= \sum_{i\in\mathcal{I}} \nu_i(u)\mathbb{E}\big[\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_B\big] && (79)\\ &= \sum_{i\in\mathcal{I}} \mathbb{E}\Big[\mathbb{1}_u(\widetilde{\mathsf{U}}_i)\Big]\mathbb{E}\big[\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_B\big] && (80)\\ &= \sum_{i\in\mathcal{I}} \mathbb{E}\Big[\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}}_i)\Big] && (81)\\ &= \sum_{i\in\mathcal{I}} \mathbb{E}\Big[\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}})\Big] && (82)\\ &= \mathbb{E}\Big[\mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}})\Big], && (83) \end{aligned}$$

where we used Fubini's theorem [19, Thm. 14.16] in (81).

To prove $\mathrm{I}(\mathsf{Y};\widetilde{\mathsf{U}}|\mathsf{X}) = 0$, we need to show that for every $y \in \mathcal{S}_{\mathsf{Y}}$, $u \in \mathcal{S}_{\mathsf{U}}$, and $B \in \sigma(\mathsf{X})$, we have

$$\int \mathbb{1}_B \kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,;y)\nu_{\hat{\mathsf{X}}}(u)\, d\mu_{\mathsf{YXU}} = \int \mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}})\mathbb{1}_y(\mathsf{Y})\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} \tag{84}$$

and by integrating, we indeed obtain

$$\begin{aligned} &\int \mathbb{1}_B \kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,;y)\nu_{\hat{\mathsf{X}}}(u)\, d\mu_{\mathsf{YXU}} && (85)\\ &= \sum_{i\in\mathcal{I}} \int \mathbb{1}_B\mathbb{1}_i(\hat{\mathsf{X}})\kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,;y)\nu_i(u)\, d\mu_{\mathsf{YXU}} && (86)\\ &= \sum_{i\in\mathcal{I}} \nu_i(u) \int \mathbb{1}_B\mathbb{1}_i(\hat{\mathsf{X}})\kappa_{\mathsf{Y}|\mathsf{X}}(\,\cdot\,;y)\, d\mu_{\mathsf{YXU}} && (87)\\ &= \sum_{i\in\mathcal{I}} \int \mathbb{1}_u(\widetilde{\mathsf{U}}_i)\, d\mu_{\widetilde{\mathsf{U}}_{\mathcal{I}}} \int \mathbb{1}_B\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_y(\mathsf{Y})\, d\mu_{\mathsf{YXU}} && (88)\\ &= \sum_{i\in\mathcal{I}} \int \mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}}_i)\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_y(\mathsf{Y})\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} && (89)\\ &= \sum_{i\in\mathcal{I}} \int \mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}})\mathbb{1}_i(\hat{\mathsf{X}})\mathbb{1}_y(\mathsf{Y})\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}} && (90)\\ &= \int \mathbb{1}_B\mathbb{1}_u(\widetilde{\mathsf{U}})\mathbb{1}_y(\mathsf{Y})\, d\mu_{\mathsf{YXU}\widetilde{\mathsf{U}}_{\mathcal{I}}}, && (91) \end{aligned}$$

where we used part 2 of Def. 4 in (88) and Fubini's theorem [19, Thm. 14.16] in (89). By replacing $\kappa_{\mathsf{Y}|\mathsf{X}}$ with $\kappa_{\mathsf{Y}|\hat{\mathsf{X}}}$ and using $B \in \sigma(\hat{\mathsf{X}})$, the same argument can be used to show $\mathrm{I}(\mathsf{Y}; \widetilde{\mathsf{U}}|\hat{\mathsf{X}}) = 0$.

Acknowledgment

The authors would like to thank Michael Meidlinger for providing inspiration for this work.

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