Variable-Length Resolvability for Mixed Sources and its Application to Variable-Length Source Coding

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Abstract—In the problem of variable-length δ -channel resolvability, the channel output is approximated by encoding a variable-length uniform random number under the constraint that the variational distance between the target and approximated distributions should be within a given constant δ asymptotically. In this paper, we assume that the given channel input is a mixed source whose components may be general sources. To analyze the minimum achievable length rate of the uniform random number, called the δ -resolvability, we introduce a variant problem of the variable-length δ -channel resolvability. A general formula for the δ -resolvability in this variant problem is established for a general channel. When the channel is an identity mapping, it is shown that the δ -resolvability in the original and variant problems coincide. This relation leads to a direct derivation of a single-letter formula for the δ -resolvability when the given source is a mixed memoryless source. We extend the result to the second-order case. As a byproduct, we obtain the first-order and second-order formulas for fixed-to-variable length source coding allowing error probability up to δ .

I. INTRODUCTION

In the problem of *variable-length* δ -channel resolvability, the channel output is approximated by encoding a variablelength uniform random number under the constraint that the distance (e.g. variational distance) between the target and approximated distribution should be within a given constant δ asymptotically. This problem, introduced by Yagi and Han [12], is a generalized form of the *fixed-length* δ -channel resolvability [3], [4] in which the fixed-length uniform random number is used as a coin distribution. The minimum achievable length rate of the uniform random number, referred to as the δ -resolvability, is the subject of analysis. In [12], a general formula for the δ -resolvability has been established for any given source and channel. Recently, a single-letter formula for the δ -resolvability has been given in [13] when the source and the channel are stationary and memoryless. An interesting next step may be a mixed memoryless sources and/or a mixed memoryless channel [2], which are stationary but non-ergodic stochastic processes.

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In this paper, we assume that the given channel input is a mixed source with components which may be general sources. To establish a general formula of the δ -resolvability for a general channel, we introduce a variant problem of the variable-length δ -channel resolvability. When the channel is an identity mapping, it is shown that the δ -resolvability in the original and variant problems coincide. This relationship is of use to derive a single-letter formula for the δ -resolvability when the given source is a mixed memoryless source. We also extend the result to the second-order case. It is known that the δ -resolvability coincides with the minimum achievable coding rate of the weak fixed-to-variable length (FV) source coding allowing error probability up to δ . As a byproduct, we obtain the first-order and second-order formulas for this minimum achievable coding rate.

II. PROBLEM OF VARIABLE-LENGTH CHANNEL RESOLVABILITY

In this section, we review the problem of channel resolvability in the variable-length setting.

Let \mathcal{X} and \mathcal{Y} be finite or countably infinite alphabets. Let $\mathbf{W} = \{W^n\}_{n=1}^{\infty}$ be a general channel, where $W^n : \mathcal{X}^n \to \mathcal{Y}^n$ denotes a stochastic mapping. We denote by $\mathbf{Y} = \{Y^n\}_{n=1}^{\infty}$ the output process via \mathbf{W} due to the input process $\mathbf{X} = \{X^n\}_{n=1}^{\infty}$, where X^n and Y^n take values in \mathcal{X}^n and \mathcal{Y}^n , respectively. The probability distributions of X^n and Y^n are denoted by P_{X^n} and P_{Y^n} , respectively, and these symbols are used interchangeably.

Consider the problem of variable-length channel resolvability. Let \mathcal{U}^* denote the set of all sequences $\boldsymbol{u} \in \mathcal{U}^m$ over $m = 0, 1, 2, \cdots$, where $\mathcal{U}^0 = \{\lambda\}$ (λ is the null string). Let L_n denote a random variable which takes values in $\{0, 1, 2, \ldots\}$. We define the variable-length uniform random number $U^{(L_n)}$ so that $U^{(m)}$ is uniformly distributed over \mathcal{U}^m given $L_n = m$. In other words, for $\boldsymbol{u} \in \mathcal{U}^m$

$$P_{U^{(L_n)}}(\boldsymbol{u}, m) := \Pr\{U^{(L_n)} = \boldsymbol{u}, L_n = m\} = \frac{\Pr\{L_n = m\}}{K^m},$$
(1)

where $K = |\mathcal{U}|$. It should be noticed that variable-length sequences $\boldsymbol{u} \in \mathcal{U}^m$ are generated with joint probability $P_{U^{(L_n)}}(\boldsymbol{u},m)$. Consider the problem of approximating the target output distribution P_{Y^n} via W^n due to X^n by using another input $\tilde{X}^n = \varphi_n(U^{(L_n)})$ with a deterministic mapping (encoder) $\varphi_n : \mathcal{U}^* \to \mathcal{X}^n$. Let $d(P_{Y^n}, P_{\tilde{Y}^n}) := \frac{1}{2} \sum_{\boldsymbol{y}} |P_{Y^n}(\boldsymbol{y}) - P_{\tilde{Y}^n}(\boldsymbol{y})|$ be the variational distance between P_{Y^n} and $P_{\tilde{Y}^n}$.

Definition 1: Let $\delta \in [0, 1)$ be fixed arbitrarily. A resolution rate $R \geq 0$ is said to be δ -variable-length achievable or simply $v(\delta)$ -achievable for X (under the variational distance) if there exists a variable-length uniform random number $U^{(L_n)}$ and a deterministic mapping $\varphi_n : \mathcal{U}^* \to \mathcal{X}^n$ satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[L_n] \le R,\tag{2}$$

$$\limsup_{n \to \infty} d(P_{Y^n}, P_{\tilde{Y}^n}) \le \delta, \tag{3}$$

where $\mathbb{E}[\cdot]$ denotes the expected value and \tilde{Y}^n denotes the output via W^n due to the input $\tilde{X}^n = \varphi_n(U^{(L_n)})$. The infimum of all $v(\delta)$ -achievable rates for X:

$$S_{\mathbf{v}}(\delta | \boldsymbol{X}, \boldsymbol{W}) := \inf\{R : R \text{ is } \mathbf{v}(\delta) \text{-achievable for } \boldsymbol{X}\}$$
 (4)

is called the δ -variable-length channel resolvability or simply $v(\delta)$ -channel resolvability for X.

When the channel W^n is an identity mapping, the addressed problem reduces to that of *source resolvability*.

Definition 2: Assume that the channel W^n is an identity mapping. The infimum of all $v(\delta)$ -achievable rates for X:

$$S_{\mathbf{v}}(\delta|\mathbf{X}) := \inf\{R: R \text{ is } \mathbf{v}(\delta)\text{-achievable for } \mathbf{X}\}$$
(5)

is called the δ -variable-length source resolvability or simply $v(\delta)$ -source resolvability for X.

Let $\mathcal{P}(\mathcal{X}^n)$ denote the set of all probability distributions on \mathcal{X}^n . For $\delta \in [0, 1]$, defining the δ -ball using the variational distance as

$$B_{\delta}(X^n) = \{P_{V^n} \in \mathcal{P}(\mathcal{X}^n) : d(P_{X^n}, P_{V^n}) \le \delta\}, \quad (6)$$

we introduce the smooth entropy:

$$H_{[\delta]}(X^n) := \inf_{P_{V^n} \in B_{\delta}(X^n)} H(V^n), \tag{7}$$

where $H(V^n)$ denotes the Shannon entropy of P_{V^n} . The $H_{[\delta]}(X^n)$ is a nonincreasing monotone function of δ . Based on this quantity for a general source $X = \{X^n\}_{n=1}^{\infty}$, we define

$$H_{[\delta]}(\boldsymbol{X}) = \limsup_{n \to \infty} \frac{1}{n} H_{[\delta]}(X^n).$$
(8)

(9)

The following theorem indicates that the $v(\delta)$ -resolvability $S_v(\delta|\mathbf{X})$ can be characterized by the smooth entropy for \mathbf{X} . Theorem 1 ([12]): For any general target source \mathbf{X} ,

$$S_{\mathbf{v}}(\delta | \mathbf{X}) = \lim_{\gamma \downarrow 0} H_{[\delta + \gamma]}(\mathbf{X}) \quad (\delta \in [0, 1)).$$

III. RESOLVABILITY FOR MIXED SOURCES AND NON-MIXED CHANNELS

A. Definitions

In this section, the source $X = \{X^n\}_{n=1}^{\infty}$ is a mixed source with general component sources. Let $\Theta := \{1, 2, \dots\}$ be the index set of component sources $X_i = \{X_i^n\}_{n=1}^{\infty}, i \in \Theta$, which may be a finite or countably infinite set. The probability distribution of mixed source X^n is given by

$$P_{X^n}(\boldsymbol{x}) = \sum_{i \in \Theta} \alpha_i P_{X_i^n}(\boldsymbol{x}) \quad (\forall n = 1, 2, \cdots; \forall \boldsymbol{x} \in \mathcal{X}^n),$$
(10)

where $\alpha_i \ge 0$ with $\sum_{i\in\Theta} \alpha_i = 1$. Let $\boldsymbol{Y} = \{Y^n\}_{n=1}^{\infty}$ be the channel output via \boldsymbol{W} due to input \boldsymbol{X} . It is easily verified that the output distribution is given as a mixture of output distributions:

$$P_{Y^n}(\boldsymbol{y}) = \sum_{i \in \Theta} \alpha_i P_{Y_i^n}(\boldsymbol{y}) \qquad (\forall \boldsymbol{y} \in \mathcal{Y}^n), \qquad (11)$$

where Y_i^n denotes the output via W^n due to input X_i^n . The mixed source is formally denoted by $\{(X_i, \alpha_i)\}_{i \in \Theta}$. Hereafter, the mixing ratio $\{\alpha_i\}_{i \in \Theta}$ is omitted if it is clear from the context, and we occasionally denote the mixed source simply by $\{X_i\}$.

In this section, we consider a variant of the channel resolvability problems for mixed sources. Let $L_n^{(i)}$ denote a variablelength uniform random number for $i \in \Theta$. Let the random variable of length L_n be specified by

$$\Pr\{L_n = m\} = \sum_{i \in \Theta} \alpha_i \Pr\{L_n^{(i)} = m\} \quad (\forall m = 0, 1, 2, \cdots).$$
(12)

In other words, the length of a variable-length uniform random number $U^{(L_n)}$ obeys a mixture of the probability distributions for the lengths of component uniform random numbers $U^{(L_n^{(i)})}$. The average length of the uniform random number $U^{(L_n)}$ is given by

$$\mathbb{E}[L_n] = \sum_{i \in \Theta} \alpha_i \mathbb{E}[L_n^{(i)}].$$
(13)

In the following problem, there are component encoders $\varphi_n^{(i)}$: $\mathcal{U}^* \to \mathcal{X}^n$, each of which approximates the channel output Y_i^n via W^n due to the *i*-th component source X_i^n .

Definition 3: Let $\delta \in [0, 1)$ be fixed arbitrarily. A resolution rate $R \ge 0$ is said to be δ -variable-length achievable or simply $v(\delta)$ -achievable for mixed source $\{(X_i, \alpha_i)\}_{i \in \Theta}$ (under the variational distance) if there exists a set of variable-length uniform random number $U^{(L_n^{(i)})}$ and a deterministic mapping

¹More generally, all results provided in this section hold for any mixed source with a *general mixture*. Any stationary process can be characterized as a mixed source with general mixture whose components are ergodic processes.

 $\varphi_n^{(i)}: \mathcal{U}^* \to \mathcal{X}^n$ satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[L_n] \le R, \tag{14}$$

$$\limsup_{n \to \infty} \sum_{i \in \Theta} \alpha_i d(P_{Y_i^n}, P_{\tilde{Y}_i^n}) \le \delta, \tag{15}$$

where \tilde{Y}_i^n denotes the output via W^n due to the input $\tilde{X}_i^n = \varphi_n^{(i)}(U^{(L_n^{(i)})})$. The infimum of all $v(\delta)$ -achievable rates for $\{(X_i, \alpha_i)\}_{i \in \Theta}$:

$$S_{\mathbf{v}}^{\dagger}(\delta | \{ \boldsymbol{X}_{i} \}, \boldsymbol{W})$$

:= inf { R : R is v(\delta)-achievable for { \mathbf{X}_{i} } } (16)

is called the δ -variable-length channel resolvability or simply $v(\delta)$ -channel resolvability for $\{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$.

Remark 1: In this problem, the condition for the approximation measure (15) is changed from (3). It is well-known that the variational distance is *jointly convex* in its arguments, and in general it holds that

$$d(P_{Y^n}, P_{\tilde{Y}^n}) \le \sum_i \alpha_i d(P_{Y^n_i}, P_{\tilde{Y}^n_i}), \tag{17}$$

where

$$P_{\tilde{Y}^n}(\boldsymbol{y}) = \sum_{i \in \Theta} \alpha_i P_{\tilde{Y}^n_i}(\boldsymbol{y}) \quad (\forall \boldsymbol{y} \in \mathcal{Y}^n).$$
(18)

Equation (15) imposes a more stringent condition than the one in (3). Since $S_v(\delta | \mathbf{X}, \mathbf{W})$ coincides with the δ -mean channel resolvability [11], for which the coin distribution may be any general source, in general we have

$$S_{\mathbf{v}}(\delta|\boldsymbol{X}, \boldsymbol{W}) \le S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_{i}\}, \boldsymbol{W}).$$
(19)

When the channel W^n is an identity mapping, the addressed problem reduces to that of *source resolvability* for $\{X_i\}$.

Definition 4: Assume that the channel W^n is an identity mapping. The infimum of all $v(\delta)$ -achievable rates for $\{(X_i, \alpha_i)\}_{i \in \Theta}$:

$$S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_{i}\}) := \inf\{R: R \text{ is } \mathbf{v}(\delta)\text{-achievable for } \{\boldsymbol{X}_{i}\}\}$$
(20)

is called the δ -variable-length source resolvability or simply $v(\delta)$ -source resolvability for $\{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$.

B. Theorems

To characterize $S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_i\}, \boldsymbol{W})$, we define

$$H^{\dagger}_{[\delta],W^n}(\{X^n_i\}) := \inf_{\{P_{V^n_i}\} \in B^{\dagger}_{\delta}(\{X^n_i\},W^n)} \sum_{i \in \Theta} \alpha_i H(V^n_i).$$
(21)

where

$$B^{\uparrow}_{\delta}(\{X_i^n\}, W^n) = \Big\{\{P_{V_i^n}\}_{i \in \Theta} \subset \mathcal{P}(\mathcal{X}^n) : \sum_{i \in \Theta} \alpha_i d(P_{Y_i^n}, P_{Z_i^n}) \le \delta\Big\},$$
(22)

where Z_i^n denotes the output random variable via W^n due to the input V_i^n . In addition, we also define the asymptotic version:

$$H^{\dagger}_{[\delta],\boldsymbol{W}}(\{\boldsymbol{X}_i\}) := \limsup_{n \to \infty} \frac{1}{n} H^{\dagger}_{[\delta],W^n}(\{X_i^n\}).$$
(23)

Both $H_{[\delta],W^n}^{\dagger}(\{X_i^n\})$ and $H_{[\delta],\boldsymbol{W}}^{\dagger}(\{\boldsymbol{X}_i\})$ are nonincreasing monotone functions in δ . When the channel \boldsymbol{W} is an identity mapping, $H_{[\delta],W^n}^{\dagger}(\{X_i^n\})$ and $H_{[\delta],\boldsymbol{W}}^{\dagger}(\{\boldsymbol{X}_i\})$ are denoted simply by $H_{[\delta]}^{\dagger}(\{X_i^n\})$ and $H_{[\delta]}^{\dagger}(\{\boldsymbol{X}_i\})$, respectively. We establish the following theorem:

Theorem 2: For any mixed source $X = \{(X_i, \alpha_i)\}_{i \in \Theta}$, it holds that

$$S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_{i}\},\boldsymbol{W}) = \lim_{\gamma \downarrow 0} H_{[\delta+\gamma],\boldsymbol{W}}^{\dagger}(\{\boldsymbol{X}_{i}\}) \quad (\forall \delta \in [0,1)).$$
(24)

(Proof) The proof is described in Sect. IV-A.

When W is an identity mapping, we have the following corollary.

Corollary 1: For any mixed source $\mathbf{X} = \{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$, it holds that

$$S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_{i}\}) = \lim_{\gamma \downarrow 0} H_{[\delta+\gamma]}^{\dagger}(\{\boldsymbol{X}_{i}\}) \quad (\forall \delta \in [0,1)).$$
(25)

As is noted in Remark 1, we have (19) in general. It is not clear if $S_v^{\dagger}(\delta | \{ X_i \}, W)$ is equal to $S_v(\delta | X, W)$. The following theorem provides an interesting relationship between the two $v(\delta)$ -source resolvability problems for mixed sources.

Theorem 3: For any mixed source $X = \{(X_i, \alpha_i)\}_{i \in \Theta}$, it holds that

$$S_{\mathbf{v}}(\delta|\mathbf{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\mathbf{X}_i\}) \quad (\forall \delta \in [0,1)).$$
(26)

(Proof) The proof is described in Sect. IV-B. *Remark 2:* The v(δ)-source resolvability $S_v(\delta | \mathbf{X})$ is equal to the minimum rate of the FV source coding achieving the decoding error probability asymptotically not greater than $\delta \in$ [0, 1) [12]. We denote by $R_v^*(\delta | \mathbf{X})$ this minimum rate, and

$$R_{\mathbf{v}}^*(\delta|\boldsymbol{X}) = S_{\mathbf{v}}(\delta|\boldsymbol{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_i\}) \quad (\forall \delta \in [0,1)).$$
(27)

for any mixed source $\mathbf{X} = \{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$. To characterize $R_v^*(\delta | \mathbf{X})$, it suffices to analyze $S_v^{\dagger}(\delta | \{\mathbf{X}_i\})$, which may be easier for some mixed sources. In the succeeding sections, we demonstrate this claim for mixed memoryless sources. \Box

IV. PROOF OF THEOREMS 2 AND 3

A. Proof of Theorem 2

then from Theorem 3, we obtain

1) Converse Part: Let R be $v(\delta)$ -achievable for $\{X_i\}$. Then, there exists $U^{(L_n^{(i)})}$ and $\varphi_n^{(i)}$ satisfying (14) and

$$\limsup_{n \to \infty} \delta_n \le \delta, \tag{28}$$

where we define

$$\delta_n = \sum_{i \in \Theta} \alpha_i d(P_{Y_i^n}, P_{\tilde{Y}_i^n}) \tag{29}$$

and \tilde{Y}_i^n is the output via W^n due to the input $\tilde{X}_i^n = \varphi_n^{(i)}(U^{(L_n^{(i)})})$. Equation (28) implies that for any given $\gamma > 0$, $\delta_n \leq \delta + \gamma$ for all $n \geq n_0$ with some $n_0 > 0$, and therefore

$$H^{\dagger}_{[\delta+\gamma],W^{n}}(\{X_{i}^{n}\}) \leq H^{\dagger}_{[\delta_{n}],W^{n}}(\{X_{i}^{n}\}) \quad (\forall n \geq n_{0}) \quad (30)$$

because $H^{\dagger}_{[\delta],W^n}(\{X_i^n\})$ is a nonincreasing monotone function of δ . Since $\{P_{\tilde{X}_i^n}\} \subset B^{\dagger}_{\delta_n}(\{X_i^n\}, W^n)$, we have

$$H_{[\delta_n],W^n}^{\dagger}(\{X_i^n\}) \le \sum_{i \in \Theta} \alpha_i H(\tilde{X}_i^n).$$
(31)

On the other hand, it follows that

$$\sum_{i \in \Theta} \alpha_i H(\tilde{X}_i^n) \le \sum_{i \in \Theta} \alpha_i H(U^{(L_n^{(i)})})$$
$$= \sum_{i \in \Theta} \alpha_i \mathbb{E}[L_n^{(i)}] + \sum_{i \in \Theta} \alpha_i H(L_n^{(i)}), \quad (32)$$

where the inequality is due to the fact that $\varphi_n^{(i)}$ is a deterministic mapping and $\tilde{X}_i^n = \varphi_n^{(i)}(U^{(L_n^{(i)})})$. By invoking the well-known relation (cf. [1, Corollary 3.12]) it holds that

$$H(L_n^{(i)}) \le \log(e \cdot \mathbb{E}[L_n^{(i)}]). \tag{33}$$

In view of (14), (33) leads to

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i \in \Theta} \alpha_i H(L_n^{(i)})$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i \in \Theta} \alpha_i \log(e \cdot \mathbb{E}[L_n^{(i)}])$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log\left(e \cdot \sum_{i \in \Theta} \alpha_i \mathbb{E}[L_n^{(i)}]\right) = 0.$$
(34)

Combining (30)-(32) yields

$$\begin{split} H^{\dagger}_{[\delta+\gamma],\boldsymbol{W}}(\{\boldsymbol{X}_i\}) \\ &= \limsup_{n \to \infty} \frac{1}{n} H^{\dagger}_{[\delta+\gamma],W^n}(\{\boldsymbol{X}_i^n\}) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[L_n] + \limsup_{n \to \infty} \frac{1}{n} \sum_{i \in \Theta} \alpha_i H(L_n^{(i)}) \leq R, \end{split}$$

where we have used (13) for the first inequality and (14) and (34) for the second inequality. Since $\gamma > 0$ is arbitrary, we obtain

$$\lim_{\gamma \downarrow 0} H^{\dagger}_{[\delta + \gamma], \boldsymbol{W}}(\{\boldsymbol{X}_i\}) \le R.$$
(35)

2) Direct Part: By the analogous argument to the proof of the direct part of Theorem 1 [12], we can show that the rate $R := H^* + 3\gamma$ is $v(\delta)$ -achievable for $\{\mathbf{X}_i\}$, where $H^* = \lim_{\gamma' \downarrow 0} H^{\dagger}_{[\delta+\gamma'], \mathbf{W}}(\{\mathbf{X}_i\})$ and $\gamma > 0$ is an arbitrarily small constant. The proof sketch is as follows:

(i) We choose some $\{P_{V_i^n}\} \subset B_{\delta+\gamma}^{\dagger}(\{X_i^n\}, W^n)$ satisfying

$$\sum_{i} \alpha_{i} H(V_{i}^{n}) \leq H_{[\delta+\gamma],W^{n}}^{\dagger}(\{X_{i}^{n}\}) + \gamma.$$
(36)

By definition, we have

$$\sum_{i\in\Theta} \alpha_i d(P_{Y_i^n}, P_{Z_i^n}) \le \delta + \gamma, \tag{37}$$

where Z_i^n denotes the output via W^n due to the input V_i^n .

(ii) Define

$$S_n^{(i)}(m) := \left\{ \boldsymbol{x} \in \mathcal{X}^n : \left\lceil \log \frac{1}{P_{V_i^n}(\boldsymbol{x})} + n\gamma \right\rceil = m \right\}.$$
(38)

For each $i \in \Theta$, we set

$$\Pr[L_n^{(i)} = m] := \Pr[V_i^n \in S_n^{(i)}(m)].$$
(39)

In the same way as in the proof of Theorem 1 [12], we arrange an encoder $\varphi_n^{(i)}$ to generate $\tilde{X}_i^n = \varphi_n^{(i)}(U^{(L_n^{(i)})})$. (iii) The average length rate can be evaluated as

$$\mathbb{E}[L_n^{(i)}] \le \left(1 + \frac{1}{K^{n\gamma}}\right) \left(H(V_i^n) + n\gamma + 1\right), \quad (40)$$

whereas the variational distance satisfies

$$d(P_{Z_{i}^{n}}, P_{\tilde{Y}_{i}^{n}}) \leq d(P_{V_{i}^{n}}, P_{\tilde{X}_{i}^{n}}) \leq \frac{1}{2}K^{-n\gamma} + \gamma.$$
(41)

From (40) and (41), we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[L_n] = \limsup_{n \to \infty} \frac{1}{n} \sum_{i \in \Theta} \alpha_i \mathbb{E}[L_n^{(i)}]$$
$$\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i \in \Theta} \alpha_i H(V_i^n) + 2\gamma$$
$$\leq H^* + 3\gamma = R \tag{42}$$

and

$$\begin{split} &\limsup_{n \to \infty} \sum_{i \in \Theta} \alpha_i d(P_{Y_i^n}, P_{\tilde{Y}_i^n}) \\ &\leq \limsup_{n \to \infty} \sum_{i \in \Theta} \alpha_i (d(P_{Y_i^n}, P_{Z_i^n}) + d(P_{Z_i^n}, P_{\tilde{Y}_i^n})) \\ &\leq \limsup_{n \to \infty} \sum_{i \in \Theta} \alpha_i d(P_{Y_i^n}, P_{Z_i^n}) + \gamma \leq \delta + 2\gamma, \end{split}$$

where the first inequality is due to the triangle inequality and the third inequality follows from (37). Since $\gamma > 0$ is an arbitrary small constant, we conclude that R is $v(\delta)$ -achievable for $\{X_i\}$.

B. Proof of Theorem 3

Assume, without loss of generality, that the elements of \mathcal{X}^n are indexed as $x_1, x_2, \dots \in \mathcal{X}^n$ so that

$$P_{X^n}(\boldsymbol{x}_j) \ge P_{X^n}(\boldsymbol{x}_{j+1}) \quad (\forall j = 1, 2, \cdots).$$
(43)

For a given $\delta \in [0, 1)$, let j^* denote the integer satisfying

$$\sum_{j=1}^{j^*-1} P_{X^n}(\boldsymbol{x}_j) < 1 - \delta, \qquad \sum_{j=1}^{j^*} P_{X^n}(\boldsymbol{x}_j) \ge 1 - \delta. \quad (44)$$

Let V_{δ}^{n} be a random variable taking values in \mathcal{X}^{n} whose probability distribution is given by

$$P_{V_{\delta}^{n}}(\boldsymbol{x}_{j}) = \begin{cases} P_{X^{n}}(\boldsymbol{x}_{j}) + \delta & \text{for } j = 1\\ P_{X^{n}}(\boldsymbol{x}_{j}) & \text{for } j = 2, 3, \cdots, j^{*} - 1\\ P_{X^{n}}(\boldsymbol{x}_{j}) - \varepsilon & \text{for } j = j^{*}\\ 0 & \text{otherwise,} \end{cases}$$

$$(45)$$

where we define $\varepsilon = \delta - \sum_{j \ge j^*+1} P_{X^n}(\boldsymbol{x}_j)$. It is easily checked that $0 \le \varepsilon \le P_{X^n}(\boldsymbol{x}_{j^*})$ and the probability distribution $P_{V_{\delta}^n}$ majorizes² any $P_{V^n} \in B_{\delta}(X^n)$ [5]. Since the Shannon entropy is a *Schur concave* function³ [9], we immediately obtain the following lemma, which provides a characterization of $H_{[\delta]}(X^n)$.

Lemma 1 ([5]):

$$H_{[\delta]}(X^n) = H(V^n_{\delta}) \quad (\forall \delta \in [0,1)).$$

$$(46)$$

Let j^* be the integer satisfying (44). Let V^n be a random variable taking values in \mathcal{X}^n whose probability distribution is given by

$$P_{V^n}(\boldsymbol{x}_j) = \begin{cases} P_{X^n}(\boldsymbol{x}_j) & \text{for } j = 1, 2, \cdots, j^* - 1\\ \eta & \text{for } j = j^* \\ 0 & \text{otherwise,} \end{cases}$$
(47)

where we define $\eta = \sum_{j \ge j^*} P_{X^n}(\boldsymbol{x}_j)$. To prove Theorem 3, the following lemma is of use.

Lemma 2: Let $X^n = \{(X_i^n, \alpha_i)\}_{i \in \Theta}$ be a mixed source. Then,

$$H(V^n) \le H_{[\delta]}(X^n) + \frac{2\log e}{e} \quad (\forall \delta \in [0,1)).$$

$$(48)$$

(*Proof*) Let V_{δ}^{n} be defined as in (45). From Lemma 1, we have

$$H(V^{n}) - H_{[\delta]}(X^{n}) = H(V^{n}) - H(V_{\delta}^{n}) \leq P_{V^{n}}(\boldsymbol{x}_{1}) \log \frac{1}{P_{V^{n}}(\boldsymbol{x}_{1})} + P_{V^{n}}(\boldsymbol{x}_{j^{*}}) \log \frac{1}{P_{V^{n}}(\boldsymbol{x}_{j^{*}})} \leq \frac{2 \log e}{e},$$
(49)

where the last inequality is due to $x \log x \ge -\frac{\log e}{e}$ for all x > 0.

For every $i \in \Theta$, let $P_{V_i^n}$ be the probability distribution satisfying

$$P_{V_i^n}(\boldsymbol{x}_j) = \begin{cases} P_{X_i^n}(\boldsymbol{x}_j) & \text{for } j = 1, 2, \cdots, j^* - 1\\ \eta_i & \text{for } j = j^* \\ 0 & \text{otherwise,} \end{cases}$$
(50)

²For a sequence $\boldsymbol{u} = (u_1, u_2, \cdots, u_L)$ of length L, we denote by $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_L)$ the permuted version of \boldsymbol{u} satisfying $\tilde{u}_i \geq \tilde{u}_{i+1}$ for all $i = 1, 2, \cdots, L$, where ties are arbitrarily broken. We say $\boldsymbol{u} = (u_1, u_2, \cdots, u_L)$ majorizes $\boldsymbol{v} = (v_1, v_2, \cdots, v_L)$ if $\sum_{i=1}^j \tilde{u}_i \geq \sum_{i=1}^j \tilde{v}_i$ for all $j = 1, 2, \cdots, L$.

³A function f(u) is said to be *Schur concave* if $f(u) \leq f(v)$ for any pair (u, v) such that v is majorized by u.

where we define $\eta_i = \sum_{j \ge j^*} P_{X_i^n}(\boldsymbol{x}_j)$. Then, we can easily verify that

$$P_{V^n}(\boldsymbol{x}) = \sum_{i \in \Theta} \alpha_i P_{V_i^n}(\boldsymbol{x}) \quad (\forall \boldsymbol{x} \in \mathcal{X}^n).$$
(51)

That is, $\{(V_i^n, \alpha_i)\}_{i \in \Theta}$ is a mixed source. Defining

$$D_n^{(i)} = \left\{ \boldsymbol{x} \in \mathcal{X}^n : P_{V_i^n}(\boldsymbol{x}) > P_{X_i^n}(\boldsymbol{x}) \right\},$$
(52)

the average variational distance can be evaluated as

$$\sum_{i \in \Theta} \alpha_i d(P_{X_i^n}, P_{V_i^n}) = \sum_{i \in \Theta} \alpha_i \sum_{\boldsymbol{x} \in D_n^{(i)}} (P_{V_i^n}(\boldsymbol{x}) - P_{X_i^n}(\boldsymbol{x}))$$
$$= \sum_{i \in \Theta} \alpha_i (P_{V_i^n}(\boldsymbol{x}_{j^*}) - P_{X_i^n}(\boldsymbol{x}_{j^*}))$$
$$= \sum_{i \in \Theta} \alpha_i (\eta_i - P_{X_i^n}(\boldsymbol{x}_{j^*}))$$
$$= \sum_{i \in \Theta} \alpha_i \sum_{j > j^*} P_{X_i^n}(\boldsymbol{x}_j)$$
$$= \sum_{j > j^*} P_{X^n}(\boldsymbol{x}_j) \le \delta,$$
(53)

where the inequality is due to (44). Since the Shannon entropy is a concave function, (51) and (53) imply that

$$H(V^n) \ge \sum_{i \in \Theta} \alpha_i H(V_i^n) \ge H^{\dagger}_{[\delta]}(\{X_i^n\}).$$
 (54)

Combining (48) and (54) with Theorem 1 and Corollary 1, we obtain

$$S_{\mathbf{v}}(\delta|\mathbf{X}, \mathbf{W}) \ge S_{\mathbf{v}}^{\dagger}(\delta|\{\mathbf{X}_i\}, \mathbf{W}).$$
(55)

The reverse inequality obviously holds (cf. (19)), and hence we obtain the claim.

Remark 3: As is seen from the above proof arguments, Theorems 2 and 3 hold even with general probability space Θ .

V. RESOLVABILITY FOR MIXED MEMORYLESS SOURCES

In this section, we assume that the source $X = \{X^n\}_{n=1}^{\infty}$ is a *mixed memoryless source* and the channel W is an identity mapping. Each component source $X_i = \{X_i^n\}_{n=1}^{\infty}, i \in \Theta$ is stationary and memoryless, which is specified by a source X_i over \mathcal{X} as

$$P_{X_{i}^{n}}(\boldsymbol{x}) = \prod_{j=1}^{n} P_{X_{i}}(x_{j}) \quad (\forall \boldsymbol{x} = (x_{1}, x_{2}, \dots, x_{n}) \in \mathcal{X}^{n}).$$
(56)

Without loss of essential generality, we assume that

$$+\infty > H(X_1) \ge H(X_2) \ge \cdots,$$
(57)

where the component sources $\{X_i\}_{i\in\Theta}$ are indexed in the decreasing order of $H(X_i)$.

For given $\delta \in [0,1)$, we define the positive integer i^* satisfying

$$\sum_{i < i^*} \alpha_i \le \delta, \qquad A_{i^*} := \sum_{i \le i^*} \alpha_i > \delta.$$
(58)

We demonstrate an application of the general relationship (26) between the two variable-length resolvability problems to establish a single-letter formula for the $v(\delta)$ -source resolvability.

Theorem 4: For any mixed memoryless source $X = \{(X_i, \alpha_i)\}_{i \in \Theta}$, it holds that

$$S_{\mathbf{v}}(\delta|\mathbf{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\mathbf{X}_{i}\})$$
$$= (A_{i^{*}} - \delta)H(X_{i^{*}}) + \sum_{i>i^{*}} \alpha_{i}H(X_{i}).$$
(59)

for all $\delta \in [0, 1)$.

Remark 4: As was mentioned in Remark 2, we have $S_{v}(\delta|\mathbf{X}) = R_{v}^{*}(\delta|\mathbf{X})$ for all $\delta \in [0, 1)$ for any general source \mathbf{X} , where $R_{v}^{*}(\delta|\mathbf{X})$ denotes the minimum rate of the FV source coding achieving the decoding error probability asymptotically not greater than $\delta \in [0, 1)$. For mixed *memoryless* source $\mathbf{X} = \{\mathbf{X}_{i}\}$, Koga and Yamamoto [6] (for Θ with $|\Theta| = 2$) and Kuzuoka [8] (for any finite Θ) have shown that $R_{v}^{*}(\delta|\mathbf{X})$ is characterized as

$$R_{v}^{*}(\delta|\mathbf{X}) = (A_{i^{*}} - \delta)H(X_{i^{*}}) + \sum_{i > i^{*}} \alpha_{i}H(X_{i})$$
(60)

for all $\delta \in [0, 1)$ if the source alphabet \mathcal{X} is *finite*. Since formula (59) holds for any countably infinite Θ and \mathcal{X} , the relation $S_{v}(\delta | \mathbf{X}) = R_{v}^{*}(\delta | \mathbf{X})$ implies that formula (60) actually holds for a wider class of mixed memoryless sources.

Since any stationary memoryless source is a mixed source with a singleton set Θ , we immediately obtain the following corollary.

Corollary 2 ([6], [12]): Let X be a stationary memoryless source X. Then, it holds that

$$S_{\mathbf{v}}(\delta|\mathbf{X}) = (1-\delta)H(X) \tag{61}$$

for all $\delta \in [0, 1)$.

(Proof of Theorem 4)

The following argument demonstrates the usefulness of the general relationship (26) between the two variable-length resolvability problems. Since it holds that

$$S_{\mathbf{v}}(\delta|\boldsymbol{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\boldsymbol{X}_i\}) = \lim_{\gamma \downarrow 0} H_{[\delta+\gamma]}^{\dagger}(\{\boldsymbol{X}_i\})$$
(62)

as is shown in Corollary 1, we first focus on the quantity $H^{\dagger}_{[\delta]}(\{X^n_i\})$. The δ -ball $B^{\dagger}_{\delta}(\{X^n_i\})$, which is defined as $B^{\dagger}_{\delta}(\{X^n_i\}, W^n)$ with an identity mapping W^n , can be written as

$$B_{\delta}^{\dagger}(\{X_{i}^{n}\})$$

$$=\{\{P_{V_{i}^{n}}\} \subset \mathcal{P}(\mathcal{X}^{n}) : \exists\{\delta_{i} \geq 0\} \text{ s.t. } \sum_{i} \alpha_{i}\delta_{i} = \delta,$$

$$d(P_{X_{i}^{n}}, P_{V_{i}^{n}}) \leq \delta_{i}, \forall i \in \Theta\}$$

$$= \bigcup_{\{\delta_{i} \geq 0: \sum_{i} \alpha_{i}\delta_{i} = \delta\}} \bigcup_{i \in \Theta} B_{\delta_{i}}(X_{i}^{n}).$$
(63)

Then, it obviously holds that

$$H_{[\delta]}^{!}({X_{i}^{n}}) = \inf_{\{P_{V_{i}^{n}}\}\in B_{\delta}^{\dagger}({X_{i}^{n}})} \sum_{i\in\Theta} \alpha_{i}H(V_{i}^{n})$$

$$= \inf_{\{\delta_{i}\geq0:\sum_{i}\alpha_{i}\delta_{i}=\delta\}} \inf_{P_{V_{i}^{n}}\in B_{\delta_{i}}(X_{i}^{n})} \sum_{i\in\Theta} \alpha_{i}H(V_{i}^{n})$$

$$\geq \inf_{\{\delta_{i}>0:\sum_{i}\alpha_{i}\delta_{i}=\delta\}} \sum_{P_{i}\alpha_{i}} \alpha_{i} \inf_{B_{i}\in CB_{\delta}(X_{i}^{n})} H(V_{i}^{n})$$
(64)

$$= \inf_{\{\delta_i \ge 0: \sum_i \alpha_i \delta_i = \delta\}} \sum_{i \in \Theta} \alpha_i H_{[\delta_i]}(X_i^n).$$
(65)

It is known (cf. [14]) that

$$\liminf_{n \to \infty} \frac{1}{n} H_{[\delta]}(X_i^n) = (1 - \delta) H(X_i) \quad (\forall \delta \in [0, 1))$$
 (66)

for any stationary memoryless source $X_i = \{X_i^n\}_{n=1}^{\infty}$, and thus

$$H_{[\delta]}^{\dagger}(\{\boldsymbol{X}_{i}\}) = \limsup_{n \to \infty} \frac{1}{n} H_{[\delta]}^{\dagger}(\{\boldsymbol{X}_{i}^{n}\})$$

$$\geq \liminf_{n \to \infty} \frac{1}{n} H_{[\delta]}^{\dagger}(\{\boldsymbol{X}_{i}^{n}\})$$

$$\geq \inf_{\{\delta_{i} \geq 0: \sum_{i} \alpha_{i} \delta_{i} = \delta\}} \sum_{i \in \Theta} \alpha_{i} \liminf_{n \to \infty} \frac{1}{n} H_{[\delta_{i}]}(\boldsymbol{X}_{i}^{n})$$

$$= \inf_{\{\delta_{i} \geq 0: \sum_{i} \alpha_{i} \delta_{i} = \delta\}} \sum_{i \in \Theta} \alpha_{i}(1 - \delta_{i}) H(\boldsymbol{X}_{i})$$

$$= \inf_{\{\alpha_{i} \geq \varepsilon_{i} \geq 0: \sum_{i} \varepsilon_{i} = \delta\}} \sum_{i \in \Theta} (\alpha_{i} - \varepsilon_{i}) H(\boldsymbol{X}_{i}), \quad (67)$$

where the second inequality is due to Fatou's lemma. Noticing that the inf in (67) is a linear program and in view of (57), we find that the solution is given by

$$\varepsilon_i = \begin{cases} \alpha_i & \text{for } i < i^* \\ \delta - \sum_{i < i^*} \alpha_i & \text{for } i = i^* \\ 0 & \text{for } i > i^*, \end{cases}$$
(68)

yielding

$$H_{[\delta]}^{\dagger}(\{\boldsymbol{X}_{i}\}) \ge (A_{i^{*}} - \delta)H(X_{i^{*}}) + \sum_{i > i^{*}} \alpha_{i}H(X_{i}).$$
(69)

The right-hand side is right-continuous in $\delta \ge 0$, and thus it follows from Corollary 1 that

$$S_{\mathbf{v}}(\delta|\mathbf{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\mathbf{X}_{i}\})$$

$$\geq (A_{i^{*}} - \delta)H(X_{i^{*}}) + \sum_{i > i^{*}} \alpha_{i}H(X_{i}), \qquad (70)$$

where it should be noted that $i^* = i^*(\delta)$ is right-continuous in δ .

To show the reverse inequality, we start with the characterization (64). We choose

$$\delta_i = \begin{cases} 1 & \text{for } i < i^* \\ \frac{\delta - \sum_{i < i^*} \alpha_i}{\alpha_i} & \text{for } i = i^* \\ 0 & \text{for } i > i^*. \end{cases}$$
(71)

We also set probability distributions $\{P_{V_i^n}\}$ on \mathcal{X}^n by

$$P_{V_i^n}(\boldsymbol{x}) = \begin{cases} \Delta(\boldsymbol{x}) & \text{for } i < i^* \\ (1 - \delta_i) P_{X_i^n}(\boldsymbol{x}) + \delta_i \Delta(\boldsymbol{x}) & \text{for } i = i^* \\ P_{X_i^n}(\boldsymbol{x}) & \text{for } i > i^*, \end{cases}$$
(72)

where $\Delta(\boldsymbol{x}) = \mathbf{1}\{\boldsymbol{x} = \boldsymbol{x}_0\}$ is the delta distribution with some specific $\boldsymbol{x}_0 \in \mathcal{X}^n$. Then, it is easily verified that

$$\delta_i \ge 0 \ (\forall i \in \Theta) \quad \text{s.t.} \quad \sum_{i \in \Theta} \alpha_i \delta_i = \delta,$$
 (73)

$$d(P_{X_i^n}, P_{V_i^n}) \le \delta_i \quad (\forall i \in \Theta),$$
(74)

meaning $P_{V_i^n} \in B_{\delta_i}(X_i^n)$ for all $i \in \Theta$. Also, $H(V_{i^*}^n)$ can be evaluated as

$$H(V_{i^{*}}^{n}) = \sum_{\boldsymbol{x}\in\mathcal{X}^{n}\setminus\{\boldsymbol{x}_{0}\}} P_{V_{i^{*}}^{n}}(\boldsymbol{x}) \log \frac{1}{(1-\delta_{i^{*}})P_{X_{i^{*}}^{n}}(\boldsymbol{x})} + P_{V_{i^{*}}^{n}}(\boldsymbol{x}_{0}) \log \frac{1}{P_{V_{i^{*}}^{n}}(\boldsymbol{x}_{0})} \leq \sum_{\boldsymbol{x}\in\mathcal{X}^{n}} (1-\delta_{i^{*}})P_{X_{i^{*}}^{n}}(\boldsymbol{x}) \log \frac{1}{(1-\delta_{i^{*}})P_{X_{i^{*}}^{n}}(\boldsymbol{x})} + \frac{\log e}{e} \leq (1-\delta_{i^{*}})H(X_{i^{*}}^{n}) + \frac{2\log e}{e},$$
(75)

where the inequalities are due to $x \log x \ge -\frac{\log e}{e}$ for all $x \ge 0$. With these choices of $\{\delta_i\}$ and $\{P_{V_i^n}\}$ satisfying (72)–(74), it follows from (64) that

$$\frac{1}{n}H_{[\delta]}^{\dagger}(\{X_{i}^{n}\}) \leq \frac{1}{n}\sum_{i\in\Theta}\alpha_{i}H(V_{i}^{n})$$

$$= \frac{1}{n}\sum_{i\geq i^{*}}\alpha_{i}H(V_{i}^{n})$$

$$= \frac{\alpha_{i^{*}}}{n}H(V_{i^{*}}^{n}) + \sum_{i>i^{*}}\alpha_{i}H(X_{i}).$$
(76)

Taking the limit superior in n on both sides, we obtain

$$\limsup_{n \to \infty} \frac{1}{n} H^{\dagger}_{[\delta]}(\{X_i^n\})$$

$$\leq \limsup_{n \to \infty} \frac{\alpha_{i^*}}{n} H(V_{i^*}^n) + \sum_{i > i^*} \alpha_i H(X_i)$$

$$\cdot \leq \alpha_{i^*}(1 - \delta_{i^*}) H(X_{i^*}) + \sum_{i > i^*} \alpha_i H(X_i)$$

$$= (A_{i^*} - \delta) H(X_{i^*}) + \sum_{i > i^*} \alpha_i H(X_i), \quad (77)$$

where the second inequality follows from (75). Again, the right-hand side is right-continuous in $\delta \ge 0$, and Corollary 1 indicates that

$$S_{\mathbf{v}}(\delta|\mathbf{X}) = S_{\mathbf{v}}^{\dagger}(\delta|\{\mathbf{X}_{i}\})$$

$$\leq (A_{i^{*}} - \delta)H(X_{i^{*}}) + \sum_{i>i^{*}} \alpha_{i}H(X_{i}).$$
(78)

We complete the proof.

VI. SECOND-ORDER RESOLVABILITY FOR MIXED SOURCES

A. Definitions

In this section, we generalize the addressed problems to the second order case. The first definition corresponds to Definition 1 in the first order [12].

Definition 5: A second-order rate $L \in (-\infty, +\infty)$ is said to be $v(\delta, R)$ -achievable (under the variational distance) for X with $\delta \in [0, 1)$ and $R \ge 0$ if there exists a variable-length uniform random number $U^{(L_n)}$ and a deterministic mapping $\varphi_n : \mathcal{U}^* \to \mathcal{X}^n$ satisfying

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left(\mathbb{E}[L_n] - nR \right) \le L, \tag{79}$$

$$\limsup_{n \to \infty} d(P_{X^n}, P_{\tilde{X}^n}) \le \delta, \tag{80}$$

where $\tilde{X}^n = \varphi_n(U^{(L_n)})$ and $\mathbb{E}[L_n]$ is specified as in (13). The infimum of all $v(\delta, R)$ -achievable rates for X is denoted by

$$T_{\mathbf{v}}(\delta, R | \mathbf{X}) := \inf\{L : L \text{ is } \mathbf{v}(\delta, R) \text{-achievable for } \mathbf{X}\}.$$
(81)

We also consider a variant problem for mixed sources $X = \{X_i\}$.

Definition 6: A second-order rate $L \in (-\infty, +\infty)$ is said to be $v(\delta, R)$ -achievable (under the variational distance) for mixed source $\{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$ with $\delta \in [0, 1)$ and $R \geq 0$ if there exists a set of variable-length uniform random number $U^{(L_n^{(i)})}$ and a deterministic mapping $\varphi_n^{(i)} : \mathcal{U}^* \to \mathcal{X}^n$ satisfying

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left(\mathbb{E}[L_n] - nR \right) \le R, \tag{82}$$

$$\limsup_{n \to \infty} \sum_{i \in \Theta} \alpha_i d(P_{X_i^n}, P_{\tilde{X}_i^n}) \le \delta,$$
(83)

where $\tilde{X}_i^n = \varphi_n^{(i)}(U^{(L_n^{(i)})})$. The infimum of all $v(\delta, R)$ -achievable rates for $\{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$ is denote by:

$$T_{\mathbf{v}}^{\dagger}(\delta, R | \{ \boldsymbol{X}_i \}) := \inf \{ L : L \text{ is } \mathbf{v}(\delta, R) \text{-achievable}$$

for $\{ \boldsymbol{X}_i \} \}.$ (84)

Remark 5: It is easily verified that

$$T_{\rm v}(\delta, R | \mathbf{X}) = \begin{cases} +\infty & \text{for } R < S_{\rm v}(\delta | \mathbf{X}) \\ -\infty & \text{for } R > S_{\rm v}(\delta | \mathbf{X}). \end{cases}$$
(85)

Hence, only the case $R = S_v(\delta | \mathbf{X})$ is of our interest. The same remark also applies to $T_v^{\dagger}(\delta, R | \{\mathbf{X}_i\})$.

B. Theorems

 \square

The following theorems indicate that $T_v(\delta, R | \mathbf{X})$ and $T_v^{\dagger}(\delta, R | \{\mathbf{X}_i\})$ can also be characterized by the smooth entropies.

Theorem 5: For any mixed source $\{(X_i, \alpha_i)\}_{i \in \Theta}$,

$$T_{\mathbf{v}}(\delta, R | \boldsymbol{X}) = \lim_{\gamma \downarrow 0} \limsup_{n \to \infty} \frac{1}{\sqrt{n}} (H_{[\delta + \gamma]}(X^n) - nR), \quad (86)$$

$$T_{\mathbf{v}}^{\dagger}(\delta, R | \{ \boldsymbol{X}_i \}) = \lim_{\gamma \downarrow 0} \limsup_{n \to \infty} \frac{1}{\sqrt{n}} (H_{[\delta + \gamma]}^{\dagger}(\{ X_i^n \}) - nR)$$
(87)

for all $\delta \in [0, 1)$ and $R \ge 0$.

(*Proof*) For the proof of (86), see [12]. Formula (87) can be proven in a parallel way to Theorem 2. \Box

As in the first order case, we have the equivalence between $T_{v}(\delta, R|\mathbf{X})$ and $T_{v}^{\dagger}(\delta, R|\{\mathbf{X}_{i}\})$ for any mixed source $\{\mathbf{X}_{i}\}$. *Theorem 6:* For any mixed source $\{(\mathbf{X}_{i}, \alpha_{i})\}_{i \in \Theta}$,

$$T_{\mathbf{v}}(\delta, R | \boldsymbol{X}) = T_{\mathbf{v}}^{\dagger}(\delta, R | \{ \boldsymbol{X}_i \}) \quad (\delta \in [0, 1), R \ge 0).$$
(88)

(Proof) This theorem can be proven in a parallel way to Theorem 3.

We now turn to analyzing the $v(\delta, R)$ -source resolvability for mixed *memoryless* sources. We assume the following properties:

- (i) The index set Θ is finite.
- (ii) Each component source X_i has the finite third absolute moment of $\log \frac{1}{P_{Y_i}(X_i)}$.
- moment of $\log \frac{1}{P_{X_i}(X_i)}$. (iii) Component sources $\{X_i\}$ satisfy

$$+\infty > H(X_1) > H(X_2) > \cdots .$$
(89)

The following lemma is useful to establish a single-letter formula of the $v(\delta, R)$ -source resolvability.

Lemma 3 ([7]): Assume that a stationary memoryless source X^n has a finite absolute moment of $\log \frac{1}{P_X(X)}$. Then, it holds that

$$H_{[\delta]}(X^n) = (1 - \delta)nH(X) - \sqrt{\frac{nV(X)}{2\pi}}e^{-\frac{(Q^{-1}(\delta))^2}{2}} + O(1),$$
(90)

where V(X) denotes the variance of $\log \frac{1}{P_X(X)}$ (varentropy) and Q^{-1} denotes the inverse of the complementary cumulative distribution function of the standard Gaussian distribution. \Box

Theorem 7: Let $\mathbf{X} = \{(\mathbf{X}_i, \alpha_i)\}_{i \in \Theta}$ be a mixed memoryless source satisfying (i)–(iii). For $R = S_v(\delta | \mathbf{X}) = S_v^{\dagger}(\delta | \{\mathbf{X}_i\})$ given by (59), it holds that

$$T_{v}(\delta, R | \mathbf{X}) = T_{v}^{\dagger}(\delta, R | \{\mathbf{X}_{i}\})$$

= $-\alpha_{i^{*}} \sqrt{\frac{V(X_{i^{*}})}{2\pi}} e^{-\frac{(Q^{-1}(\delta_{i^{*}}))^{2}}{2}},$ (91)

where i^* is the integer satisfying (58) and δ_{i^*} is defined as in (71).

(*Proof*) The direct part is comparatively easy and we omit the proof due to the space limitation.

To prove the converse part, we define $D(\delta) := \{\{\delta_i\} : \delta_i \ge 0, \sum_{i \in \Theta} \alpha_i \delta_i = \delta\}$. Using (65) and Lemma 3, we obtain

$$\frac{1}{\sqrt{n}} H^{\dagger}_{[\delta]}(\{X_{i}^{n}\}) \geq \inf_{\{\tilde{\delta}_{i}\}\in D(\delta)} \sum_{i\in\Theta} \alpha_{i} \Big\{ (1-\tilde{\delta}_{i})\sqrt{n} H(X_{i}) - \sqrt{\frac{V(X_{i})}{2\pi}} e^{-\frac{(Q^{-1}(\tilde{\delta}_{i}))^{2}}{2}} + o(1) \Big\}, \quad (92)$$

and thus for all $n > n_0$ with some $n_0 > 0$ the minimizer $\{\tilde{\delta}_i\} \in D(\delta)$ on the right-hand side is $\{\delta_i\}$ given in (71): i.e.,

$$\frac{1}{\sqrt{n}} H^{\dagger}_{[\delta]}(\{X_{i}^{n}\}) \geq \sum_{i \geq i^{*}} \alpha_{i}(1-\delta_{i})\sqrt{n} H(X_{i}) - \alpha_{i^{*}} \sqrt{\frac{V(X_{i^{*}})}{2\pi}} e^{-\frac{(Q^{-1}(\delta_{i^{*}}))^{2}}{2}} + o(1) \quad (93)$$

for all $n > n_0$, where we have used the fact that $e^{-\frac{(Q^{-1}(0))^2}{2}} = 0$. Since $R = S_v(\delta|\mathbf{X})$ is given by

$$R = \sum_{i \ge i^*} \alpha_i (1 - \delta_i) H(X_i) \tag{94}$$

due to Theorem 4, it follows that

$$\lim_{\gamma \downarrow 0} \limsup_{n \to \infty} \frac{1}{\sqrt{n}} (H^{\dagger}_{[\delta + \gamma]}(\{X_i^n\}) - nR) \\ \geq -\alpha_{i^*} \sqrt{\frac{V(X_{i^*})}{2\pi}} e^{-\frac{(Q^{-1}(\delta_{i^*}))^2}{2}}.$$
(95)

In view of Theorems 5 and 6, we complete the proof of the converse part. $\hfill \Box$

Remark 6: As in the first order case, $T_v(\delta, R|\mathbf{X})$ is equal to $R_v^*(\delta, R|\mathbf{X})$, which denotes the minimum achievable rate of the FV δ -source coding [14]. Theorem 7 also indicates that

$$R_{\rm v}^*(\delta, R | \mathbf{X}) = -\alpha_{i^*} \sqrt{\frac{V(X_{i^*})}{2\pi}} e^{-\frac{(Q^{-1}(\delta_{i^*}))^2}{2}}, \qquad (96)$$

for a mixed memoryless source $X = \{(X_i, \alpha_i)\}_{i \in \Theta}$ satisfying (i)–(iii).

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