A Unified Framework of State Evolution for Message-Passing Algorithms

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Abstract—This paper presents a unified framework to understand the dynamics of message-passing algorithms in compressed sensing. State evolution is rigorously analyzed for a general error model that contains the error model of approximate messagepassing (AMP), as well as that of orthogonal AMP. As a byproduct, AMP is proved to converge asymptotically if the sensing matrix is orthogonally invariant and if the moment sequence of its asymptotic singular-value distribution coincide with that of the Marčhenko-Pastur distribution up to the order that is at most twice as large as the maximum number of iterations.

I. INTRODUCTION

Consider the recovery of an unknown N-dimensional signal vector $\boldsymbol{x} \in \mathbb{R}^N$ from an M-dimensional linear measurement vector $\boldsymbol{y} \in \mathbb{R}^M$, given by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}. \tag{1}$$

In (1), the sensing matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ is known, while the noise vector $\mathbf{w} \in \mathbb{R}^{M}$ is unknown. The purpose of this paper is to present a unified framework for analyzing the asymptotic performance of signal recovery via message-passing (MP).

An important example of MP is approximate messagepassing (AMP) [1]. Bayes-optimal AMP can be regarded as an *exact* approximation of belief propagation [2] in the large-system limit—both M and N tend to infinity while the compression rate $\delta = M/N$ is kept $\mathcal{O}(1)$. Bayati *et al.* [3], [4] analyzed the rigorous dynamics of AMP in the large system limit via state evolution (SE) when the sensing matrix Ahas independent and identically distributed (i.i.d.), zero-mean, and sub-Gaussian elements. Their result implies that, in spite of its low complexity, AMP can achieve the Bayes-optimal performance in a range of the compression rate δ . However, AMP fails to converge when the sensing matrix is non-zero mean [5] or ill-conditioned [6].

Another important example of MP is orthogonal AMP (OAMP) [7]. OAMP is also called vector AMP (VAMP) [8] and was originally proposed by Opper and Winther [9, Appendix D]. Bayes-optimal OAMP can be regarded as an large-system approximation of expectation propagation (EP) [10], [11]. The rigorous SE of OAMP was presented in the same conference when the sensing matrix is orthogonally invariant on the real field [8] or unitarily invariant on the complex field [11]. These rigorous results imply that OAMP converges for a wider class of sensing matrices than AMP because the class of orthogonally invariant matrices contains matrices with dependent elements. One disadvantage of OAMP is high

complexity due to the requirement of one matrix inversion¹ per iteration. See [12] for a complexity reduction of OAMP.

This paper proposes an SE framework for understanding both AMP and OAMP from a unified point of view. The proposed framework is based on a general recursive model of errors that contains the error models of both AMP and OAMP. The main point of the model is that the current errors depend on the whole history of errors in the preceding iterations, while the current errors in OAMP are determined only by the errors in the latest iteration. Under the assumption of orthogonally invariant sensing matrices, we present a rigorous SE analysis of the general error model in the large-system limit.

The main contributions of this paper are twofold: One is the rigorous SE of the general error model that contains those of both AMP and OAMP. The result provides a framework for designing new MP algorithms that have the advantages of both AMP and OAMP [13]: low complexity and the convergence property for orthogonally invariant sensing matrices.

The other contribution is a detailed convergence analysis of AMP. AMP with the maximum number T of iterations is proved to converge for orthogonally invariant sensing matrices if the moment sequence of the asymptotic eigenvalue (EV) distribution of $A^T A$ coincides with that of the Marčhenko-Pastur distribution [14] up to order 2T at most. When Ahas i.i.d. zero-mean elements, the asymptotic EV distribution coincides with the Marčhenko-Pastur distribution perfectly. Thus, the i.i.d. assumption of A is too strong in guaranteeing the convergence of AMP, as long as a finite number of iterations are assumed.

II. PRELIMINARIES

A. General Error Model

Consider the singular-value decomposition (SVD) $A = U\Sigma V^{T}$ of the sensing matrix, in which U and V are $M \times M$ and $N \times N$ orthogonal matrices, respectively. We consider the following general error model in iteration t:

$$\boldsymbol{b}_{t} = \boldsymbol{V}^{\mathrm{T}} \tilde{\boldsymbol{q}}_{t}, \quad \tilde{\boldsymbol{q}}_{t} = \boldsymbol{q}_{t} - \sum_{t'=0}^{t-1} \langle \partial_{t'} \boldsymbol{\psi}_{t-1} \rangle \boldsymbol{h}_{t'}, \qquad (2)$$

$$\boldsymbol{m}_t = \boldsymbol{\phi}_t(\boldsymbol{b}_0, \dots, \boldsymbol{b}_t, \tilde{\boldsymbol{w}}), \tag{3}$$

¹ The singular-value decomposition (SVD) of A allows us to circumvent this requirement [8]. However, the SVD itself is high complexity, unless the sensing matrix has some special structure.

$$\boldsymbol{h}_t = \boldsymbol{V} \tilde{\boldsymbol{m}}_t, \quad \tilde{\boldsymbol{m}}_t = \boldsymbol{m}_t - \sum_{t'=0}^{t} \langle \partial_{t'} \boldsymbol{\phi}_t \rangle \boldsymbol{b}_{t'}, \qquad (4)$$

$$\boldsymbol{q}_{t+1} = \boldsymbol{\psi}_t(\boldsymbol{h}_0, \dots, \boldsymbol{h}_t, \boldsymbol{x}), \tag{5}$$

with $\tilde{\boldsymbol{w}} = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{w}$ and the initial conditions $\boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0 = -\boldsymbol{x}$.

In the general error model, the notation $\langle \boldsymbol{v} \rangle$ denotes the arithmetic mean $\langle \boldsymbol{v} \rangle = N^{-1} \sum_{n=1}^{N} [\boldsymbol{v}]_n$ for $\boldsymbol{v} \in \mathbb{R}^N$. The functions $\phi_t : \mathbb{R}^{N \times (t+1)} \times \mathbb{R}^M \to \mathbb{R}^N$ and $\psi_t : \mathbb{R}^{N \times (t+2)} \to \mathbb{R}^N$ are the element-wise mapping of input vectors, i.e.

$$[\boldsymbol{\phi}_t(\boldsymbol{b}_0,\ldots,\boldsymbol{b}_t,\tilde{\boldsymbol{w}})]_n = \phi_{t,n}([\boldsymbol{b}_0]_n,\ldots,[\boldsymbol{b}_t]_n,[\tilde{\boldsymbol{w}}]_n), \quad (6)$$

$$[\boldsymbol{\psi}_t(\boldsymbol{h}_0,\ldots,\boldsymbol{h}_t,\boldsymbol{x})]_n = \psi_{t,n}([\boldsymbol{h}_0]_n,\ldots,[\boldsymbol{h}_t]_n,[\boldsymbol{x}]_n) \quad (7)$$

for some functions $\phi_{t,n} : \mathbb{R}^{t+2} \to \mathbb{R}$ and $\psi_{t,n} : \mathbb{R}^{t+2} \to \mathbb{R}$. Finally, the notations $\partial_{t'}\phi_t$ and $\partial_{t'}\psi_t$ represent N-dimensional vectors of which the *n*th elements $[\partial_{t'}\phi_t]_n$ and $[\partial_{t'}\phi_t]_n$ are given by the partial derivatives of $\phi_{t,n}$ and $\psi_{t,n}$ with respect to the *t*'th variable, respectively.

The functions ϕ_t and ψ_t may depend on the singular-values of the sensing matrix. Since the support of the asymptotic singular-value distribution of A is assumed to be compact in this paper, we do not write the dependencies of Σ explicitly.

The general error model is composed of two systems with respect to (b_t, m_t) and (h_t, q_{t+1}) , respectively. We refer to the former and latter systems as modules A and B, respectively.

Remark 1: Suppose that the functions ϕ_t and ψ_t depend only on the latest variables, i.e. $m_t = \phi_t(b_t, \tilde{w})$ and $q_{t+1} = \psi_t(h_t, x)$. Then, the general error model reduces to that of OAMP [11]. The functions ϕ_t and ψ_t characterize the types of the linear filter and the thresholding function used in OAMP. Furthermore, the normalized squared norm $N^{-1} ||q_{t+1}||^2$ corresponds to the mean-square error (MSE) for the OAMP estimation of x in iteration t.

B. AMP

We formulate an AMP error model similar to the general error model. Let x_t denote the AMP estimator of x in iteration t. The update rules of AMP [1] are given by

$$\boldsymbol{x}_{t+1} = \boldsymbol{\theta}_t (\boldsymbol{x}_t + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{z}_t), \qquad (8)$$

$$\boldsymbol{z}_{t} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_{t} + \frac{\xi_{t-1}}{\delta}\boldsymbol{z}_{t-1}, \quad \xi_{t} = \left\langle \boldsymbol{\theta}_{t}'(\boldsymbol{x}_{t} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{z}_{t}) \right\rangle,$$
(9)

with $\boldsymbol{z}_{-1} = \boldsymbol{0}$ and $\boldsymbol{x}_0 = \boldsymbol{0}$. In (8), the thresholding function satisfies the separation condition $[\boldsymbol{\theta}_t(\boldsymbol{v})]_n = \theta_t([\boldsymbol{v}]_n)$ for $\boldsymbol{v} \in \mathbb{R}^N$ with a common scalar function $\theta_t : \mathbb{R} \to \mathbb{R}$.

Let $h_t = x_t + A^T z_t - x$ and $q_{t+1} = x_{t+1} - x$ denote the estimation errors before and after thresholding, respectively. From the definition (5), we find

$$\boldsymbol{\psi}_t(\boldsymbol{h}_t, \boldsymbol{x}) = \boldsymbol{\theta}_t(\boldsymbol{x} + \boldsymbol{h}_t) - \boldsymbol{x}. \tag{10}$$

Then, the extrinsic vector \tilde{q}_t in (2) for t > 0 is given by

$$\tilde{\boldsymbol{q}}_t = \boldsymbol{q}_t - \langle \boldsymbol{\theta}'_{t-1}(\boldsymbol{x} + \boldsymbol{h}_{t-1}) \rangle \boldsymbol{h}_{t-1} = \boldsymbol{q}_t - \xi_{t-1} \boldsymbol{h}_{t-1}.$$
(11)

To define the function ϕ_t in (3), we let

$$\boldsymbol{m}_t = \boldsymbol{V}^{\mathrm{T}} \boldsymbol{h}_t. \tag{12}$$

Substituting the definition of h_t yields

$$\boldsymbol{m}_{t} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{q}_{t} + \boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{z}_{t} = \boldsymbol{b}_{t} + \xi_{t-1}\boldsymbol{m}_{t-1} + \boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{z}_{t}, \quad (13)$$

with $b_t = V^{\mathrm{T}} \tilde{q}_t$ and $m_{-1} = 0$, where the second equality follows from (11) and (12). Left-multiplying (9) by $\Sigma^{\mathrm{T}} U^{\mathrm{T}}$ and using (1), we obtain

$$\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{z}_{t} = -\boldsymbol{\Lambda}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{q}_{t} + \boldsymbol{\Sigma}^{\mathrm{T}}\tilde{\boldsymbol{w}} + \frac{\xi_{t-1}}{\delta}\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{z}_{t-1}, \quad (14)$$

with $\Lambda = \Sigma^T \Sigma$. Applying (11), (12) and (13) to (14), we arrive at

$$\boldsymbol{m}_{t} = (\boldsymbol{I}_{N} - \boldsymbol{\Lambda})\boldsymbol{b}_{t} - \frac{\xi_{t-1}}{\delta}\boldsymbol{b}_{t-1} + \boldsymbol{\Sigma}^{\mathrm{T}}\tilde{\boldsymbol{w}} \\ + \xi_{t-1} \left\{ \left(1 + \frac{1}{\delta} \right) \boldsymbol{I}_{N} - \boldsymbol{\Lambda} \right\} \boldsymbol{m}_{t-1} - \frac{\xi_{t-1}\xi_{t-2}}{\delta} \boldsymbol{m}_{t-2}, (15)$$

with $b_t = 0$ and $m_t = 0$ for t < 0. The right-hand side (RHS) of (15) defines the function ϕ_t recursively. Note that m_t depends on all vectors $\{b_0, \ldots, b_t\}$.

The only difference between the general and AMP error models is in (4) and (12). Instead of \tilde{m}_t , the vector m_t is used to define h_t in the AMP. We will prove $\langle \partial_{t'} \phi_t \rangle \stackrel{\text{a.s.}}{=} 0$ in the second main theorem.

C. Assumptions

We follow [3] to postulate Lipschitz-continuous functions as ϕ_t and ψ_t in the general error model.

Assumption 1: $\phi_{t,n}$ and $\psi_{t,n}$ are Lipschitz-continuous. Furthermore, $\phi_t(\mathbf{b}_0, \ldots, \mathbf{b}_t, \tilde{\mathbf{w}})$ and $\psi_t(\mathbf{h}_0, \ldots, \mathbf{h}_t, \mathbf{x})$ are not a linear combination of the first t+1 vectors plus some function of the last vector.

The latter assumption implies that \tilde{q}_t and \tilde{m}_t in (2) and (4) depend on (h_0, \ldots, h_t) and (b_0, \ldots, b_t) , respectively.

We assume the following moment conditions on x and w to guarantee the existence of the second moments of the variables in the general error model.

Assumption 2: The signal vector \boldsymbol{x} has independent elements with bounded $(4 + \epsilon)$ th moments for some $\epsilon > 0$.

Assumption 3: The noise vector \boldsymbol{w} has bounded $(4 + \epsilon)$ th moments for some $\epsilon > 0$ and satisfies $M^{-1} \|\boldsymbol{w}\|^2 \xrightarrow{\text{a.s.}} \sigma^2$ as $M \to \infty$.

We follow [8], [11] to postulate orthogonally invariant sensing matrices.

Assumption 4: The sensing matrix A is orthogonally invariant. More precisely, the orthogonal matrices U and V in the SVD $A = U\Sigma V^{T}$ is independent of the other random variables and Haar-distributed [14]. The empirical EV distribution of $A^{T}A$ converges almost surely (a.s.) to an asymptotic distribution with a compact support in the large-system limit.

D. Marčhenko-Pastur Distribution

We review the Marčhenko-Pastur distribution. Assume that the sensing matrix $A \in \mathbb{R}^{M \times N}$ has independent zero-mean Gaussian elements with variance 1/M. The *k*th moment $M^{-1}\text{Tr}\{(AA^{T})^{k}\}$ of the empirical EV distribution of AA^{T} converges a.s. to that of the Marčhenko-Pastur distribution in the large-system limit. Instead of presenting the Marčhenko-Pastur distribution explicitly, we characterize it via the η transform $\eta: [0,\infty) \to (0,1]$, defined as

$$\eta(x) = \lim_{M = \delta N \to \infty} \frac{1}{M} \operatorname{Tr} \left\{ (\boldsymbol{I}_M + x \boldsymbol{A} \boldsymbol{A}^{\mathrm{T}})^{-1} \right\}.$$
(16)

As shown in [14, Eq. (2.120)], the η -transform of the Marčhenko-Pastur distribution is the positive solution to

$$\eta = 1 - \frac{1}{\delta} + \frac{1}{\delta(1 + x\eta)}.$$
(17)

The η -transform defines the Marčhenko-Pastur distribution uniquely because the distribution is uniquely determined by the Stieltjes transform, which is given via analytic continuation of the *n*-transform [14].

We need the asymptotic EV distribution of $A^{T}A$, rather than AA^{T} . Define the η -transform of $A^{\mathrm{T}}A$ as

$$\tilde{\eta}(x) = \lim_{M = \delta N \to \infty} \frac{1}{N} \operatorname{Tr} \left\{ (\boldsymbol{I}_N + x \boldsymbol{A}^{\mathrm{T}} \boldsymbol{A})^{-1} \right\}.$$
(18)

Since AA^{T} and $A^{T}A$ have identical positive eigenvalues, we find the relationship

$$\tilde{\eta}(x) = \delta \eta(x) + 1 - \delta. \tag{19}$$

Substituting this into (17) yields

$$\tilde{\eta} = \frac{\delta}{\delta + x(\tilde{\eta} + \delta - 1)}.$$
(20)

It is possible to calculate the moment sequence of the asymptotic EV distribution of $A^{\mathrm{T}}A$ via the η -transform $\tilde{\eta}$. Since the η -transform is uniformly bounded for all $x \in [0, \infty)$, (A-d) There is some C > 0 such that the minimum eigenvalue we use the eigen-decomposition $A^{\mathrm{T}}A = V\Lambda V^{\mathrm{T}}$ and the definition (18) to obtain

$$\tilde{\eta}(x) = \sum_{k=0}^{\infty} (-x)^k \mu_k, \qquad (21)$$

$$\mu_k = \lim_{M = \delta N \to \infty} \frac{1}{N} \operatorname{Tr}(\mathbf{\Lambda}^k).$$
 (22)

This implies that the kth moment μ_k of the asymptotic EV distribution of $A^{T}A$ is given via the kth derivative of the η transform at the origin. Direct calculation of the derivatives based on (20) yields $\mu_0 = 1$, $\mu_1 = 1$, and $\mu_2 = 1 + \delta^{-1}$.

III. MAIN RESULTS

A. State Evolution

We analyze the dynamics of the general error model in the large-system limit. Let

$$\boldsymbol{B}_t = (\boldsymbol{b}_0, \dots, \boldsymbol{b}_{t-1}) \in \mathbb{R}^{N \times t}, \tag{23}$$

$$\tilde{\boldsymbol{M}}_t = (\tilde{\boldsymbol{m}}_0, \dots, \tilde{\boldsymbol{m}}_{t-1}) \in \mathbb{R}^{N \times t}, \tag{24}$$

$$\boldsymbol{H}_{t} = (\boldsymbol{h}_{0}, \dots, \boldsymbol{h}_{t-1}) \in \mathbb{R}^{N \times t}, \tag{25}$$

$$\tilde{\boldsymbol{Q}}_t = (\tilde{\boldsymbol{q}}_0, \dots, \tilde{\boldsymbol{q}}_{t-1}) \in \mathbb{R}^{N \times t}.$$
(26)

Define the set $\mathfrak{E}_{t,t'} = \{ \boldsymbol{B}_{t'}, \tilde{\boldsymbol{M}}_{t'}, \boldsymbol{H}_{t}, \tilde{\boldsymbol{Q}}_{t+1}, \boldsymbol{x}, \boldsymbol{w}, \boldsymbol{U}, \boldsymbol{\Sigma} \}.$ The set $\mathfrak{E}_{t,t}$ contains the whole history of the estimation errors just before evaluating (2) in iteration t, as well as all random variables with the only exception of V, while $\mathfrak{E}_{t,t+1}$ includes the whole history just before evaluating (4). We use the conditioning technique by Bolthausen [15] to obtain the following theorem:

Theorem 1: Postulate Assumptions 1–4. For all $\tau = 0, 1, ...$ and $\tau' = 0, \ldots, \tau$, the following properties hold for module A in the large-system limit.

(A-a) Let $\beta_t = (\tilde{Q}_t^T \tilde{Q}_t)^{-1} \tilde{Q}_t^T \tilde{q}_t$, and $\tilde{q}_t^{\perp} = P_{\tilde{Q}_t}^{\perp} \tilde{q}_t$, with $\boldsymbol{P}_{\tilde{\boldsymbol{Q}}_{t}}^{\perp} = \boldsymbol{I}_{N} - \tilde{\boldsymbol{Q}}_{t} (\tilde{\boldsymbol{Q}}_{t}^{\mathrm{T}} \tilde{\boldsymbol{Q}}_{t})^{-1} \tilde{\boldsymbol{Q}}_{t}^{\mathrm{T}}$. For $\tau > 0$, the vector \boldsymbol{b}_{τ} conditioned on $\mathfrak{E}_{\tau,\tau}$ is statistically equivalent to

$$b_{\tau}|_{\mathfrak{E}_{\tau,\tau}} \sim B_{\tau}\beta_{\tau} + B_{\tau}o(1) + \tilde{M}_{\tau}o(1) + \Phi^{\perp}_{(\tilde{M}_{\tau},B_{\tau})}\omega_{t}$$
(27)

In (27), the notation o(1) denotes a finite-dimensional vector of which all elements are o(1). For a matrix M, the notation Φ_M^\perp represents the matrix that is composed of all left-singular vectors of M associated with zero singular values. ω_t is independent of the other random variables, orthogonally invariant, and has bounded (4 + ϵ)th moments for some $\epsilon > 0$ satisfying $\|\boldsymbol{\omega}_t\|^2 = \|\tilde{\boldsymbol{q}}_t^{\perp}\|^2$.

$$\frac{1}{N}\boldsymbol{b}_{\tau'}^{\mathrm{T}}\boldsymbol{b}_{\tau} - \frac{1}{N}\tilde{\boldsymbol{q}}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{q}}_{\tau} \stackrel{\mathrm{a.s.}}{\to} 0.$$
(28)

(A-c) Suppose that $\tilde{\phi}_{\tau}(\boldsymbol{B}_{\tau+1}, \tilde{\boldsymbol{w}}) : \mathbb{R}^{N \times (\tau+1)} \times \mathbb{R}^M \to \mathbb{R}^N$ satisfies the separation condition like (6), and that each function $[\phi_{\tau}]_n$ is Lipschitz-continuous. Then,

$$\frac{1}{N}\boldsymbol{b}_{\tau'}^{\mathrm{T}}\left(\tilde{\boldsymbol{\phi}}_{\tau}-\sum_{t'=0}^{\tau}\left\langle\partial_{t'}\tilde{\boldsymbol{\phi}}_{\tau}\right\rangle\boldsymbol{b}_{t'}\right)\stackrel{\mathrm{a.s.}}{\to}0.$$
 (29)

of $N^{-1}(\tilde{\boldsymbol{M}}_{\tau+1}^{\mathrm{T}}\tilde{\boldsymbol{M}}_{\tau+1})^{-1}$ is a.s. larger than C.

For module B, on the other hand, the following properties hold in the large-system limit:

(B-a) Let $\boldsymbol{\alpha}_t = (\tilde{\boldsymbol{M}}_t^{\mathrm{T}} \tilde{\boldsymbol{M}}_t)^{-1} \tilde{\boldsymbol{M}}_t^{\mathrm{T}} \tilde{\boldsymbol{m}}_t$ and $\tilde{\boldsymbol{m}}_t^{\perp} = \boldsymbol{P}_{\tilde{\boldsymbol{M}}_t}^{\perp} \tilde{\boldsymbol{m}}_t$. Then, the vector \boldsymbol{h}_{τ} conditioned on $\mathfrak{E}_{\tau,\tau+1}$ is statistically equivalent to

$$\boldsymbol{h}_0|_{\mathfrak{E}_{0,1}} \sim o(1)\boldsymbol{q}_0 + \boldsymbol{\Phi}_{\boldsymbol{q}_0}^{\perp} \tilde{\boldsymbol{\omega}}_0 \tag{30}$$

for $\tau = 0$, otherwise

(A-b)

$$h_{\tau}|_{\mathfrak{E}_{\tau,\tau+1}} \sim H_{\tau} \alpha_{\tau} + H_{\tau} o(1) + Q_{\tau+1} o(1) + \Phi^{\perp}_{(\tilde{Q}_{\tau+1},H_{\tau})} \tilde{\boldsymbol{\omega}}_{t}.$$
(31)

In (30), $\tilde{\omega}_t$ is an independent and orthogonally invariant vector, and has bounded $(4 + \epsilon)$ th moments for some $\epsilon > 0$ satisfying $\|\tilde{\boldsymbol{\omega}}_0\|^2 = \|\tilde{\boldsymbol{m}}_0\|^2$ and $\|\tilde{\boldsymbol{\omega}}_t\|^2 = \|\tilde{\boldsymbol{m}}_t^{\perp}\|^2$ for t > 0.

$$\frac{1}{N}\boldsymbol{h}_{\tau'}^{\mathrm{T}}\boldsymbol{h}_{\tau} - \frac{1}{N}\tilde{\boldsymbol{m}}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{m}}_{\tau} \stackrel{\mathrm{a.s.}}{\to} 0.$$
(32)

(B-c) Suppose that $\tilde{\psi}_{\tau}(H_{\tau+1}, x) : \mathbb{R}^{N \times (\tau+2)} \to \mathbb{R}^N$ satisfies the separation condition like (7), and that each function $[\psi_{\tau}]_n$ is Lipschitz-continuous. Then,

$$\frac{1}{N}\boldsymbol{h}_{\tau'}^{\mathrm{T}}\left(\tilde{\boldsymbol{\psi}}_{\tau}-\sum_{t'=0}^{\tau}\left\langle\partial_{t'}\tilde{\boldsymbol{\psi}}_{\tau}\right\rangle\boldsymbol{h}_{t'}\right)\stackrel{\mathrm{a.s.}}{\to}0.$$
 (33)

(B-d) There is some C > 0 such that the minimum eigenvalue of $N^{-1} (\tilde{\boldsymbol{Q}}_{\tau+2}^{\mathrm{T}} \tilde{\boldsymbol{Q}}_{\tau+2})^{-1}$ are a.s. larger than C. *Proof:* See Appendix A.

Theorem 1 was proved in [8], [11] for the case of functions ϕ_t and ψ_t that depend only on b_t and h_t , respectively. Theorem 1 is a generalization of [8], [11] to the case of the general functions (6) and (7).

Properties (A-c) and (B-c) imply the orthogonality between b_{τ} and \tilde{m}_t and between h_{τ} and \tilde{q}_{t+1} in the general error model. Thus, we refer to MP algorithms as long-memory OAMP (LM-OAMP) if their error models are contained in the general error model.

If q_{t+1} corresponds to the estimation error of an MP algorithm in iteration t, we need to evaluate the MSE $N^{-1} ||q_{t+1}||^2$ in the large-system limit. While Theorem 1 allows us to analyze the MSE, this paper does not discuss any more analysis in the general error model. The MSE should be considered for each concrete MP algorithm.

Because of space limitation, we have focused on a performance measure, such as MSE, that requires the existence of the second moments of the variables in the general error model. As considered in [3], it is straightforward to extend Theorem 1 to the case of general performance measures in terms of pseudo-Lipschitz functions.

B. AMP

We next prove that the general error model contains the AMP error model under an assumption on the asymptotic EV distribution of $A^{T}A$.

Theorem 2: Consider the AMP error model, postulate Assumptions 1–4, and suppose that the moment sequence of the asymptotic EV distribution of $A^T A$ coincides with that of the Marchenko-Pastur distribution up to order T. Then, $\tilde{m}_t \stackrel{\text{a.s.}}{=} m_t + B_{t+1}o(1)$ holds for all t < T in the large-system limit.

Proof: See Section IV.

The only difference between the general and AMP error models is in (4) and (12). Thus, Theorem 2 implies that the general error model contains the AMP error model in the large-system limit. As long as the number of iterations is finite, it should be possible to construct orthogonally invariant sensing matrices satisfying two conditions: One is that the sensing matrices have dependent elements. The other condition is that the moment sequence of the asymptotic EV distribution of $A^{T}A$ is equal to that of the Marčhenko-Pastur distribution up to the required order. Thus, we conclude that Theorems 1 and 2 are the first rigorous result on the asymptotic dynamics of the AMP for non-independent sensing matrices.

Remark 2: Instead of evaluating $N^{-1}\boldsymbol{m}_t^{\mathrm{T}}\boldsymbol{m}_t$ directly, we present a sufficient condition for guaranteeing that the MSE $N^{-1}\|\boldsymbol{q}_{t+1}\|^2$ coincides with that for the case of zero-mean i.i.d. Gaussian sensing matrices [3]. From (15), $N^{-1}\|\boldsymbol{m}_t\|^2$ depends on the asymptotic moments $\{\mu_k\}$ up to order 2t + 2. Thus, the MSE $N^{-1}\|\boldsymbol{q}_{t+1}\|^2$ coincides with that in [3] for all t < T in the large-system limit if the moment sequence of the asymptotic EV distribution of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is equal to that of the

Marčhenko-Pastur distribution up to order 2T. A future work is to analyze what occurs between the orders T and 2T.

IV. PROOF OF THEOREM 2

Let $g_{\tau',\tau}^{(k)} = \langle \mathbf{\Lambda}^k \partial_{\tau'} \phi_{\tau} \rangle$ with ϕ_{τ} defined as the RHS of (15). The goal is to prove $g_{\tau',\tau}^{(0)} \xrightarrow{\text{a.s.}} 0$ for all $0 \leq \tau < T$ and $0 \leq \tau' \leq \tau$ in the large-system limit.

The proof is by induction with respect to τ . For $\tau = \tau'$, we use (15) to obtain

$$g_{\tau,\tau}^{(k)} \stackrel{\text{a.s.}}{=} \mu_k - \mu_{k+1} + o(1)$$
 (34)

in the large-system limit, where the *k*th moment μ_k is defined in (22). In particular, for $\tau = 0$ and $\tau = 1$ we use $\mu_0 = \mu_1 = 1$ to find $g_{\tau,\tau}^{(0)} \xrightarrow{\text{a.s.}} 0$ in the large-system limit.

Let $\tau = 1$. Since we have proved $g_{0,0}^{(0)} \xrightarrow{\text{a.s.}} 0$, we can use Property (B-a) for $\tau = 0$. Thus, ξ_0 converges a.s. to a constant independent of \boldsymbol{b}_0 in the large-system limit. Using (15) yields

$$\frac{g_{\tau-1,\tau}^{(k)}}{\xi_{\tau-1}} \stackrel{\text{a.s.}}{=} -\frac{\mu_k}{\delta} + \left(1 + \frac{1}{\delta}\right) g_{\tau-1,\tau-1}^{(k)} - g_{\tau-1,\tau-1}^{(k+1)} + o(1)$$

$$\stackrel{\text{a.s.}}{=} -\frac{\mu_{k+1}}{\delta} + g_{\tau-1,\tau-1}^{(k)} - g_{\tau-1,\tau-1}^{(k+1)} + o(1)$$
(35)

for $\tau = 1$, where the second equality follows from the identity $\mu_k \stackrel{\text{a.s.}}{=} g_{\tau-1,\tau-1}^{(k)} + \mu_{k+1} + o(1)$ obtained from (34). Thus, we find $g_{0,0}^{(0)}/\xi_0 \stackrel{\text{a.s.}}{\to} 0$ in the large-system limit.

Assume that there is some t < T such that $g_{\tau',\tau}^{(k)} \stackrel{\text{a.s.}}{\to} 0$ holds for all $0 \le \tau < t$ and $0 \le \tau' \le \tau$. We prove $g_{\tau',t}^{(0)} \stackrel{\text{a.s.}}{\to} 0$ for all $\tau' \le t$. The induction hypothesis allows us to use Property (B-a) for all $\tau < t$, so that, for all $\tau < t$, ξ_{τ} converges a.s. to a constant independent of $\{b_0, \ldots, b_{\tau}\}$ in the largesystem limit. This observation implies that (35) holds for all $\tau \le t$. Furthermore, we use (15) to obtain

$$\frac{g_{\tau',\tau}^{(k)}}{\xi_{\tau-1}} \stackrel{\text{a.s.}}{=} \left(1 + \frac{1}{\delta}\right) g_{\tau',\tau-1}^{(k)} - g_{\tau',\tau-1}^{(k+1)} - \frac{\xi_{\tau-2}}{\delta} g_{\tau',\tau-2}^{(k)} + o(1)$$
(36)

for all $\tau \leq t$ and $\tau' < \tau - 1$.

We simplify the recursive system (34), (35), and (36). Let $g_{\tau',\tau}^{(k)} = a_{\tau} \tilde{g}_{\tau',\tau}^{(k)} / a_{\tau'}$, with $a_0 = 1$ and $a_{\tau} = \xi_{\tau-1} a_{\tau-1}$ for all $1 \leq \tau \leq t$. Applying these definitions to (34), (35), and (36), we have

$$\tilde{g}_{\tau,\tau}^{(k)} \stackrel{\text{a.s.}}{=} \mu_k - \mu_{k+1} + o(1),$$
 (37)

$$\tilde{g}_{\tau-1,\tau}^{(k)} \stackrel{\text{a.s.}}{=} -\frac{\mu_{k+1}}{\delta} + \tilde{g}_{\tau-1,\tau-1}^{(k)} - \tilde{g}_{\tau-1,\tau-1}^{(k+1)} + o(1), \quad (38)$$

$$\tilde{g}_{\tau',\tau}^{(k)} \stackrel{\text{a.s.}}{=} \left(1 + \frac{1}{\delta}\right) \tilde{g}_{\tau',\tau-1}^{(k)} - \tilde{g}_{\tau',\tau-1}^{(k+1)} - \frac{\tilde{g}_{\tau',\tau-2}^{(k)}}{\delta} + o(1).$$
(39)

The simplified system (37)–(39) implies that $\tilde{g}_{\tau',\tau}^{(k)}$ is stationary with respect to τ' and τ . In other words, $\tilde{g}_{\tau',\tau}^{(k)}$ depends on τ and τ' only through the difference $\tau - \tau'$.

Let $g_{\tau}^{(k)} = \tilde{g}_{0,\tau}^{(k)}$ for $\tau \leq t$, which satisfies the recursive system (37), (38), and (39) with $\tilde{g}_{\tau',\tau}^{(k)}$ replaced by $g_{\tau-\tau'}^{(k)}$. It is sufficient to prove $g_t^{(0)} \stackrel{\text{a.s.}}{\to} 0$ in the large-system limit. By definition, $g_{\tau}^{(k)}$ is independent of the higher-order moments

 μ_j for all $j > \tau + k + 1$. As long as t < T is assumed, the sequence $\{g_0^{(0)}, \ldots, g_t^{(0)}\}$ is determined by the moments up to order T. Without loss of generality, we can assume that the asymptotic EV distribution of $A^T A$ coincides with the Marčhenko-Pastur distribution perfectly.

To prove $g_t^{(0)} \xrightarrow{\text{a.s.}} 0$, we define the generating function of $\{q_{\tau}^{(k)}\}$ as

$$G(x,y) = \sum_{\tau=0}^{\infty} G_{\tau}(x)y^{\tau}, \qquad (40)$$

with

$$G_{\tau}(x) = \sum_{k=0}^{\infty} g_{\tau}^{(k)} x^k - \frac{g_{\tau-1}^{(0)}}{x}, \quad g_{-1}^{(0)} = 0,$$
(41)

where $g_{\tau}^{(k)}$ satisfies the recursive system (37), (38), and (39) with $\tilde{g}_{\tau',\tau}^{(k)}$ replaced by $g_{\tau-\tau'}^{(k)}$. Note that we have extended the definition of $g_{\tau}^{(k)}$ with respect to τ from $\{0,\ldots,t\}$ to all nonnegative integers. From the induction hypothesis $g_{t-1}^{(0)} \xrightarrow{\text{a.s.}} 0$, it is sufficient to prove $G_t(0) \xrightarrow{\text{a.s.}} 0$.

We first derive an explicit formula of G(x, y). From (37), (38), and (39), we utilize the power-series representation (21) to obtain

$$G_0(x) \stackrel{\text{a.s.}}{=} \tilde{\eta}(-x) - \frac{\tilde{\eta}(-x) - 1}{x} + o(1),$$
 (42)

$$G_1(x) \stackrel{\text{a.s.}}{=} -\frac{\tilde{\eta}(-x) - 1}{\delta x} + \left(1 - \frac{1}{x}\right)G_0(x) + o(1), \quad (43)$$

$$G_{\tau}(x) \stackrel{\text{a.s.}}{=} \left(1 + \frac{1}{\delta} - \frac{1}{x}\right) G_{\tau-1}(x) - \frac{G_{\tau-2}(x)}{\delta} + o(1) \quad (44)$$

for all $\tau > 1$, where we have used $\mu_0 = 1$. From (44), we have

$$G(x,y) \stackrel{\text{a.s.}}{=} G_0(x) + yG_1(x) - \frac{y^2}{\delta}G(x,y) + \left(1 + \frac{1}{\delta} - \frac{1}{x}\right)y\{G(x,y) - G_0(x)\} + o(1).$$
(45)

Solving this equation with (42) and (43), we arrive at

$$G(x,y) = \frac{P(x,y)}{Q(x,y)} + o(1),$$
(46)

with

$$P(x,y) = (\delta x - \delta - xy)\tilde{\eta}(-x) + \delta, \qquad (47)$$

$$Q(x,y) = \delta y + (y-\delta)(y-1)x.$$
(48)

We next prove that the numerator P(x, y) is divisible by the denominator Q(x, y) for $y \in (0, \min\{1, \delta\})$. It is sufficient to prove that $P(-x^*, y) = 0$ holds for the zero $-x^*$ of Q(x, y), given by

$$x^* = \frac{\delta y}{(y - \delta)(y - 1)} > 0.$$
(49)

Calculating $P(-x^*, y)$ yields

$$P(-x^*, y) = \delta\left(1 - \frac{\tilde{\eta}(x^*)}{1 - y}\right).$$
(50)

Since the η -transform $\tilde{\eta}$ satisfies (20), we have

$$\left\{\tilde{\eta}(x^*) - \frac{y - \delta}{y}\right\} \{\tilde{\eta}(x^*) - (1 - y)\} = 0.$$
 (51)

The positivity of the η -transform implies that the correct solution is $\tilde{\eta}(x^*) = 1 - y$. Thus, we arrive at $P(-x^*, y) = 0$.

Finally, we prove $g_t^{(0)} \xrightarrow{\text{a.s.}} 0$. For $y \neq 0$, we use $\tilde{\eta}(0) = 1$ to find $\lim_{x\to 0} G(x, y) \xrightarrow{\text{a.s.}} 0$. Since we have proved that G(x, y) is a polynomial for all $y \in (0, \min\{1, \delta\})$, from (40) we can conclude $\lim_{x\to 0} G_{\tau}(x) \xrightarrow{\text{a.s.}} 0$ for all τ . In particular, we use (41) and the induction hypothesis $g_{t-1}^{(0)} \xrightarrow{\text{a.s.}} 0$ to arrive at $g_t^{(0)} \xrightarrow{\text{a.s.}} 0$. Thus, Theorem 2 holds.

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APPENDIX A Proof of Theorem 1

A. Properties of Pseudo-Lipschitz Functions

We present the definition and basic properties of pseudo-Lipschitz functions [3].

Definition 1: A function $f : \mathbb{R}^t \to \mathbb{R}$ is called pseudo-Lipschitz of order k if there are some constants L > 0 and $k \in \mathbb{N}$ such that, for all $x \in \mathbb{R}^t$ and $y \in \mathbb{R}^t$,

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le L(1 + \|\boldsymbol{x}\|^{k-1} + \|\boldsymbol{y}\|^{k-1})\|\boldsymbol{x} - \boldsymbol{y}\|.$$
 (52)

In proving the following propositions, we use the equivalence between norms on \mathbb{R}^t for finite $t \in \mathbb{R}$, i.e. $C_1 \| \cdot \|_q \leq \| \cdot \|_p \leq C_2 \| \cdot \|_q$ for some constants $C_1, C_2 > 0$. Note that $\| \cdot \|_2$ is abbreviated as $\| \cdot \|$.

Proposition 1: Let $f : \mathbb{R}^t \to \mathbb{R}$ denote any pseudo-Lipschitz function of order k. Then, there is some constant C > 0 such that $|f(\boldsymbol{x})| \leq L(1 + \|\boldsymbol{x}\|^k)$ for all $\boldsymbol{x} \in \mathbb{R}^t$.

Proof: Since f is pseudo-Lipschitz of order k, there is some constant L' > 0 such that $|f(\boldsymbol{x})| \leq |f(\boldsymbol{0})| + L'(1 + \|\boldsymbol{x}\|^{k-1})\|\boldsymbol{x}\|$ holds for all $\boldsymbol{x} \in \mathbb{R}^t$. For $\|\boldsymbol{x}\| < 1$, we have $|f(\boldsymbol{x})| \leq |f(\boldsymbol{0})| + 2L'$. Otherwise, $|f(\boldsymbol{x})| \leq |f(\boldsymbol{0})| + 2L'\|\boldsymbol{x}\|^k$. Thus, there is some constant L > 0 such that $|f(\boldsymbol{x})| \leq L(1 + \|\boldsymbol{x}\|^k)$ holds.

Proposition 1 implies that any pseudo-Lipschitz function $f(\mathbf{x})$ of order k is $\mathcal{O}(||\mathbf{x}||^k)$ as $||\mathbf{x}|| \to \infty$, while $f(\mathbf{x}) = \mathcal{O}(||\mathbf{x}||)$ holds for any Lipschitz-continuous function f.

Proposition 2: Let $x \in \mathbb{R}^t$ denote a random vector with bounded kth absolute moments for some $k \in \mathbb{N}$. Suppose that a function $f : \mathbb{R}^t \to \mathbb{R}$ is pseudo-Lipschitz of order k and almost everywhere (a.e.) differentiable. Then, we have $\mathbb{E}[|f(x)|] < \infty$ and $\mathbb{E}[|\partial_{t'}f(x)|] < \infty$.

Proof: Using Proposition 1, we obtain

$$\mathbb{E}[|f(\boldsymbol{x})|] \le C\left(1 + \mathbb{E}\left[\|\boldsymbol{x}\|^{k}\right]\right) < \infty, \tag{53}$$

where the boundedness follows from that of the kth absolute moments of x.

The boundedness $\mathbb{E}[\partial_{t'}f(x)] < \infty$ is also obtained by repeating the same argument, since (52) implies

$$\begin{aligned} |\partial_{t'} f(\boldsymbol{x})| &= \lim_{\Delta x \to 0} \left| \frac{f(\boldsymbol{x} + \Delta x \boldsymbol{e}_{t'}) - f(\boldsymbol{x})}{\Delta x} \right| \\ &\leq L \left(1 + 2 \|\boldsymbol{x}\|^{k-1} \right), \end{aligned}$$
(54)

where $e_{t'}$ denotes the t'th column of I_t . Thus, Proposition 2 holds.

Proposition 3: Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are pseudo-Lipschitz of orders k_1 and k_2 , respectively. Then, $h(\boldsymbol{x}) = f(x_1)g(x_2)$ is pseudo-Lipschitz of order $(k_1 + k_2)$.

Proof: From the pseudo-Lipschitz properties, there are some constants $L_{\rm f}, L_{\rm g} > 0$ such that

$$|f(x_1) - f(y_1)| \le L_f(1 + |x_1|^{k_1 - 1} + |y_1|^{k_1 - 1})|x_1 - y_1|,$$
 (55)

$$|g(x_2) - g(y_2)| \le L_{g}(1 + |x_2|^{k_2 - 1} + |y_2|^{k_2 - 1})|x_2 - y_2|.$$
 (56)

Without loss of generality, we assume $|x_1| \ge |y_1|$ and $|x_2| \ge |y_2|$. Using the triangle inequality yields

$$|f(x_1)g(x_2) - f(y_1)g(y_2)| \le |f(x_1) - f(y_1)||g(x_2)| + |f(y_1)||g(x_2) - g(y_2)|.$$
(57)

Applying the upper bounds $|f(y_1)| \leq C_f(1+|y_1|^{k_1})$ and $|g(x_2)| \leq C_g(1+|x_2|^{k_2})$ for some constants $C_f, C_g > 0$ obtained from Proposition 1, as well as (55) and (56), we obtain

$$|f(x_1)g(x_2) - f(y_1)g(y_2)|$$

$$\leq C(1 + |x_1|^{k_1-1})(1 + |x_2|^{k_2})|x_1 - y_1|$$

$$+ C(1 + |x_1|^{k_1})(1 + |x_2|^{k_2-1})|x_2 - y_2|$$
(58)

for some constant C > 0, where we have used $|x_1| \ge |y_1|$ and $|x_2| \ge |y_2|$. Since $|x_1|^k |x_2|^{k'} \le (|x_1| + |x_2|)^{k+k'}$ holds for all $k \ge 0$ and $k' \ge 0$, we have

$$|f(x_1)g(x_2) - f(y_1)g(y_2)| < C\{1 + \|\boldsymbol{x}\|_1^{k_1 + k_2 - 1}\}\|\boldsymbol{x} - \boldsymbol{y}\|_1.$$
(59)

Proposition 3 follows from the equivalence between the norms $\|\cdot\|_1$ and $\|\cdot\|$ on \mathbb{R}^2 .

B. Key Lemmas

We present three key lemmas used in proving Theorem 1.

Lemma 1 ([8]): Suppose that the $N \times N$ orthogonal matrix V is Haar-distributed. For 0 < t < N, consider a noiseless linear measurement $Y \in \mathbb{R}^{N \times t}$ of the unknown signal matrix V given by

$$Y = VX, (60)$$

where the known *sensing* matrix $X \in \mathbb{R}^{N \times t}$ is full rank. Then, the posterior distribution of V given X and Y is statistically equivalent to

$$V|_{\boldsymbol{X},\boldsymbol{Y}} \sim \boldsymbol{Y}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1}\boldsymbol{X}^{\mathrm{T}} + \boldsymbol{\Phi}_{\boldsymbol{Y}}^{\perp}\tilde{V}(\boldsymbol{\Phi}_{\boldsymbol{X}}^{\perp})^{\mathrm{T}},$$
 (61)

where the $(N-t) \times (N-t)$ orthogonal matrix \tilde{V} is Haardistributed.

Lemma 1 is the main lemma in the conditioning technique by Bolthausen [15]. The lemma is used to prove Properties (A-a) and (B-a).

Lemma 2: Let $\boldsymbol{z} = (z_1, \ldots, z_t)^{\mathrm{T}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$. For all $k \in \mathbb{N}$, any pseudo-Lipschitz of order k and a.e. differentiable function $f : \mathbb{R}^t \to \mathbb{R}$ satisfies

$$\mathbb{E}[z_1 f(\boldsymbol{z})] = \sum_{t'=1}^t \mathbb{E}[z_1 z_{t'}] \mathbb{E}\left[\partial_{t'} f(\boldsymbol{z})\right].$$
(62)

Proof: Proposition 2 implies that both sides in (62) are bounded. For the eigen-decomposition $\Sigma = U\Lambda U^{\mathrm{T}}$, we use the change of variables $\tilde{z} = U^{\mathrm{T}}z$ to obtain

$$\mathbb{E}[z_1 f(\boldsymbol{z})] = \sum_{\tau=1}^t U_{1\tau} \mathbb{E}[\tilde{z}_{\tau} f(\boldsymbol{U}\tilde{\boldsymbol{z}})].$$
(63)

Since \tilde{z} has independent elements, Stein's lemma implies

$$\mathbb{E}[z_1 f(\boldsymbol{z})] = \sum_{\tau=1}^{t} U_{1\tau} \mathbb{E}[\tilde{z}_{\tau}^2] \mathbb{E}\left[\frac{\partial f}{\partial \tilde{z}_{\tau}}(\boldsymbol{U}\tilde{\boldsymbol{z}})\right]$$
$$= \sum_{\tau=1}^{t} U_{1\tau}[\boldsymbol{\Lambda}]_{\tau\tau} \sum_{t'=1}^{t} U_{t'\tau} \mathbb{E}\left[\partial_{t'} f(\boldsymbol{z})\right]. \quad (64)$$

Using the definition $[\Sigma]_{1t'} = \sum_{\tau=1}^{t} U_{1\tau}[\Lambda]_{\tau\tau} U_{t'\tau}$, we arrive at Lemma 2.

Lemma 2 is used to prove Properties (A-c) and (B-c). The expression of the so-called Onsager terms—the second terms on \tilde{q}_t and \tilde{m}_t given in (2) and (4)—originates from Lemma 2.

Lemma 3: Suppose that scalar functions $\{f_n : \mathbb{R} \to \mathbb{R}\}$ are pseudo-Lipschitz of order k for some $k \in \mathbb{N}$, that $a \in \mathbb{R}^{N-t}$ is an orthogonally invariant vector with bounded $(2k + \epsilon)$ th moments for some $\epsilon > 0$, and that the limit $\lim_{N\to\infty} N^{-1} ||a||^2 \xrightarrow{\text{a.s.}} v > 0$ holds for fixed $t \ge 0$. Let $b \in \mathbb{R}^{N-t}$ denote a deterministic vector satisfying $N^{-1} ||b||^2 \to 0$ and $N^{-1} \sum_{n=1}^{N} b_n^{2k-2} < \infty$. Let $\tilde{a} = \Phi^{\perp} a$ for any $N \times (N - t)$ matrix Φ^{\perp} with orthonormal columns. Then, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left\{ f_n(\tilde{a}_n + b_n) - \mathbb{E} \left[f_n(\sqrt{v} z_n) \right] \right\} \stackrel{\text{a.s.}}{=} 0, \quad (65)$$

with $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$.

Proof: See Appendix B.

Lemma 3 is used to prove Properties (A-c) and (B-c). The moment conditions of x and w in Assumptions 2 and 3 are required for utilizing this lemma.

C. Properties in Module A for $\tau = 0$

The proof of Theorem 1 is by induction. We first prove the properties in Module A for $\tau = 0$. We need to prove Properties (A-b), (A-c), and (A-d) for $\tau = 0$. We only prove Property (A-c) for $\tau = 0$ since Properties (A-b) and (A-d) are trivial for $\tau = 0$. From the definition (2) and Assumption 2, b_0 is orthogonally invariant and has bounded $(4 + \epsilon)$ th moments for some $\epsilon > 0$. Furthermore, $\tilde{\boldsymbol{w}} = \boldsymbol{U}^T \boldsymbol{w}$ is orthogonally invariant and has bounded $(4 + \epsilon)$ th moments from Assumption 3. Note that $b\tilde{\phi}_{0,n}(b, \tilde{w}_n)$ is pseudo-Lipschitz of order 2 from Proposition 3. Thus, we can use Lemma 3.

Let $z_0 \sim \mathcal{N}(\mathbf{0}, v_0 \mathbf{I}_N)$ with $v_0 = \lim_{M = \delta N \to \infty} N^{-1} \| \mathbf{b}_0 \|^2$. Using Lemma 3 conditioned on $\tilde{\boldsymbol{w}}$ and then using the same lemma again, we obtain

$$\frac{1}{N}\boldsymbol{b}_{0}^{\mathrm{T}}\tilde{\boldsymbol{\phi}}_{0}(\boldsymbol{b}_{0},\tilde{\boldsymbol{w}}) \stackrel{\mathrm{a.s.}}{=} \frac{1}{N}\mathbb{E}\left[\boldsymbol{z}_{0}^{\mathrm{T}}\tilde{\boldsymbol{\phi}}_{0}(\boldsymbol{z}_{0},\tilde{\boldsymbol{w}})\right] + o(1)$$

$$= v_{0}\mathbb{E}\left[\left\langle\partial_{0}\tilde{\boldsymbol{\phi}}_{0}(\boldsymbol{z}_{0},\tilde{\boldsymbol{w}})\right\rangle\right] + o(1)$$

$$\stackrel{\mathrm{a.s.}}{=} \frac{1}{N}\boldsymbol{b}_{0}^{\mathrm{T}}\boldsymbol{b}_{0}\left\langle\partial_{0}\tilde{\boldsymbol{\phi}}_{0}(\boldsymbol{b}_{0},\tilde{\boldsymbol{w}})\right\rangle + o(1). \quad (66)$$

where the second equality follows from Lemma 2. For the last equality, we need a careful discussion: Lemma 3 implies that the empirical distribution of $(\boldsymbol{b}_0, \tilde{\boldsymbol{w}})$ converges weakly to the distribution of $(\boldsymbol{z}_0, \tilde{\boldsymbol{w}})$ in the large-system limit. We use [3, Lemma 5] to obtain the last equality. Thus, Property (A-c) holds for $\tau = 0$.

D. Properties in Module B for $\tau = 0$

Since Property (B-b) is trivial for $\tau = 0$, we only prove the other properties in Module B for $\tau = 0$. We first prove Property (B-a) for $\tau = 0$. Using Lemma 1 with $Y = q_t$ and $X = b_t$ for (4) yields

$$\boldsymbol{h}_{0}|_{\boldsymbol{\mathfrak{E}}_{0,1}} \sim \frac{\boldsymbol{b}_{0}^{\mathrm{T}} \tilde{\boldsymbol{m}}_{0}}{\|\boldsymbol{b}_{0}\|^{2}} \boldsymbol{q}_{0} + \boldsymbol{\Phi}_{\boldsymbol{q}_{0}}^{\perp} \tilde{\boldsymbol{\omega}}_{0} \stackrel{\mathrm{a.s.}}{=} o(1) \boldsymbol{q}_{0} + \boldsymbol{\Phi}_{\boldsymbol{q}_{0}}^{\perp} \tilde{\boldsymbol{\omega}}_{0}, \quad (67)$$

with $\tilde{\boldsymbol{\omega}}_0 = \tilde{\boldsymbol{V}}(\boldsymbol{\Phi}_{\boldsymbol{b}_0}^{\perp})^{\mathrm{T}}\tilde{\boldsymbol{m}}_0$, in which the last equality follows from (4) and (29) for $\tau = 0$. Note that $\tilde{\boldsymbol{\omega}}_0$ is an orthogonally invariant vector. Since \boldsymbol{b}_0 has bounded $(4 + \epsilon)$ th moments, from (3), (4), Assumption 1, and Assumption 3, $\tilde{\boldsymbol{m}}_0$ is so. Thus, $\tilde{\boldsymbol{\omega}}_0$ has bounded $(4 + \epsilon)$ th moments. Furthermore, we have $\|\tilde{\boldsymbol{\omega}}_0\|^2 = \tilde{\boldsymbol{m}}_0^{\mathrm{T}} \boldsymbol{P}_{\boldsymbol{b}_0}^{\perp} \tilde{\boldsymbol{m}}_0 = \|\tilde{\boldsymbol{m}}_0\|^2 + o(N)$, because of (29) for $\tau = 0$. Thus. Property (B-a) holds for $\tau = 0$.

We next prove Property (B-c) for $\tau = 0$. Let $\tilde{z}_0 \sim \mathcal{N}(\mathbf{0}, \tilde{v}_0 \boldsymbol{I}_N)$ with $\tilde{v}_0 = \lim_{M = \delta N \to \infty} N^{-1} \| \tilde{\boldsymbol{m}}_0 \|^2$. Using Property (B-a) and Lemma 3 yields

$$\frac{1}{N}\boldsymbol{h}_{0}^{\mathrm{T}}\tilde{\boldsymbol{\psi}}_{0}(\boldsymbol{h}_{0},\boldsymbol{x}) \stackrel{\mathrm{a.s.}}{=} \frac{1}{N}\mathbb{E}\left[\tilde{\boldsymbol{z}}_{0}^{\mathrm{T}}\tilde{\boldsymbol{\psi}}_{0}(\tilde{\boldsymbol{z}}_{0},\boldsymbol{x})\right] + o(1)$$

$$= \tilde{v}_{0}\mathbb{E}\left[\left\langle\partial_{0}\tilde{\boldsymbol{\psi}}_{0}(\tilde{\boldsymbol{z}}_{0},\boldsymbol{x})\right\rangle\right] + o(1)$$

$$\stackrel{\mathrm{a.s.}}{=} \frac{\|\boldsymbol{h}_{0}\|^{2}}{N}\left\langle\partial_{0}\tilde{\boldsymbol{\psi}}_{0}(\boldsymbol{h}_{0},\boldsymbol{x})\right\rangle + o(1), \quad (68)$$

where the second inequality is due to Lemma 2, and where the last inequality follows from the definition of \tilde{v}_0 , (4), and the same argument as in the derivation of (66). Thus, Property (B-c) holds for $\tau = 0$.

Finally, we prove Property (B-d) for $\tau = 0$. From [3, Lemmas 8 and 9], it is sufficient to prove $N^{-1} \|\tilde{q}_1^{\perp}\|^2$ converges a.s. to a strictly positive constant in the large-system limit. By definition, we have

$$\frac{\|\tilde{\boldsymbol{q}}_{1}^{\perp}\|^{2}}{N} = \frac{\|\tilde{\boldsymbol{q}}_{1}\|^{2}}{N} - \frac{(N^{-1}\tilde{\boldsymbol{q}}_{0}^{\mathrm{T}}\tilde{\boldsymbol{q}}_{1})^{2}}{N^{-1}\|\tilde{\boldsymbol{q}}_{0}\|^{2}}$$
$$\stackrel{\text{a.s.}}{=} \frac{\mathbb{E}[\|\tilde{\boldsymbol{q}}_{1}\|^{2}]}{N} - \frac{(N^{-1}\mathbb{E}[\tilde{\boldsymbol{q}}_{0}^{\mathrm{T}}\tilde{\boldsymbol{q}}_{1}])^{2}}{N^{-1}\mathbb{E}[\|\tilde{\boldsymbol{q}}_{0}\|^{2}]} + o(1), \quad (69)$$

with

$$\tilde{\boldsymbol{q}}_1 = \boldsymbol{\psi}_0(\tilde{\boldsymbol{z}}_0, \boldsymbol{x}) - \mathbb{E}\left[\langle \partial_0 \boldsymbol{\psi}_0(\tilde{\boldsymbol{z}}_0, \boldsymbol{x}) \rangle \right] \tilde{\boldsymbol{z}}_0, \tag{70}$$

where the second equality is obtained by repeating the same argument as in the derivation of (68).

In order to lower-bound (69), we use the Cauchy-Schwarz inequality twice,

$$(\mathbb{E}[\tilde{\boldsymbol{q}}_{0}^{\mathrm{T}}\tilde{\boldsymbol{q}}_{1}])^{2} = (\mathbb{E}\left\{\tilde{\boldsymbol{q}}_{0}^{\mathrm{T}}\mathbb{E}_{\tilde{\boldsymbol{z}}_{0}}[\tilde{\boldsymbol{q}}_{1}]\right\})^{2} \leq (\mathbb{E}\left\{\|\tilde{\boldsymbol{q}}_{0}\|\|\mathbb{E}_{\tilde{\boldsymbol{z}}_{0}}[\tilde{\boldsymbol{q}}_{1}]\|\right\})^{2} \\ \leq \mathbb{E}[\|\tilde{\boldsymbol{q}}_{0}\|^{2}]\mathbb{E}\{\|\mathbb{E}_{\tilde{\boldsymbol{z}}_{0}}[\tilde{\boldsymbol{q}}_{1}]\|^{2}\}.$$
(71)

Substituting this upper bound into (69) yields

$$\frac{\|\tilde{\boldsymbol{q}}_{1}^{\perp}\|^{2}}{N} \stackrel{\text{a.s.}}{\geq} \frac{1}{N} \mathbb{E}\left\{\mathbb{E}_{\tilde{\boldsymbol{z}}_{0}}[\|\tilde{\boldsymbol{q}}_{1}\|^{2}] - \|\mathbb{E}_{\tilde{\boldsymbol{z}}_{0}}[\tilde{\boldsymbol{q}}_{1}]\|^{2}\right\} + o(1), \quad (72)$$

which is strictly positive from Assumption 1. Thus, Property (B-d) holds for $\tau = 0$.

E. Properties in Module A by Induction

Suppose that Theorem 1 is correct for all $\tau < t$. We first prove Property (A-a) for $\tau = t$. The orthogonal matrix V^{T} conditioned on $\mathfrak{E}_{t,t}$ satisfies the constraint

$$(\tilde{\boldsymbol{M}}_t, \boldsymbol{B}_t) = \boldsymbol{V}^{\mathrm{T}}(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t).$$
(73)

We confirm that $(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t)$ is full rank. The induction hypothesis (B-c) for all $\tau < t$ implies the orthogonality $N^{-1}\boldsymbol{h}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{q}}_{\tau''} \stackrel{\mathrm{a.s.}}{=} 0$ for all $\tau'' \leq t$ and $\tau' < \tau''$. Thus, we have

$$(\boldsymbol{H}_{t}, \tilde{\boldsymbol{Q}}_{t})^{\mathrm{T}}(\boldsymbol{H}_{t}, \tilde{\boldsymbol{Q}}_{t}) \stackrel{\mathrm{a.s.}}{=} \begin{pmatrix} \tilde{\boldsymbol{M}}_{t}^{\mathrm{T}} \tilde{\boldsymbol{M}}_{t} & \boldsymbol{O} \\ \boldsymbol{O} & \tilde{\boldsymbol{Q}}_{t}^{\mathrm{T}} \tilde{\boldsymbol{Q}}_{t} \end{pmatrix} + o(N), (74)$$

where we have used the definition (4). The induction hypotheses (A-d) and (B-d) imply that $(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t)$ is full rank. Thus, we can use Lemma 1 to obtain

$$\boldsymbol{V}^{\mathrm{T}} \tilde{\boldsymbol{q}}_{t} |_{\mathfrak{E}_{t,t}} \sim (\tilde{\boldsymbol{M}}_{t}, \boldsymbol{B}_{t}) (\boldsymbol{H}_{t}, \tilde{\boldsymbol{Q}}_{t})^{\dagger} \tilde{\boldsymbol{q}}_{t} + \boldsymbol{\Phi}_{(\tilde{\boldsymbol{M}}_{t}, \boldsymbol{B}_{t})}^{\perp} \boldsymbol{\omega}_{t}, \quad (75)$$

with

$$\boldsymbol{\omega}_t = \tilde{\boldsymbol{V}}(\boldsymbol{\Phi}_{(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t)}^{\perp})^{\mathrm{T}} \tilde{\boldsymbol{q}}_t, \tag{76}$$

where \tilde{V} is an independent and Haar-distributed orthogonal matrix. Evaluating the pseudo-inverse matrix $(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t)^{\dagger}$, we have

$$V^{\mathrm{T}} \tilde{\boldsymbol{q}}_{t}|_{\mathfrak{E}_{t,t}} \sim \boldsymbol{B}_{t} \boldsymbol{\beta}_{t} + \boldsymbol{\Phi}_{(\tilde{\boldsymbol{M}}_{t},\boldsymbol{B}_{t})}^{\perp} \boldsymbol{\omega}_{t} \\ + \tilde{\boldsymbol{M}}_{t} \boldsymbol{o}(1) + \boldsymbol{B}_{t} \boldsymbol{o}(1), \tag{77}$$

with

$$\boldsymbol{\beta}_t = (\tilde{\boldsymbol{Q}}_t^{\mathrm{T}} \tilde{\boldsymbol{Q}}_t)^{-1} \tilde{\boldsymbol{Q}}_t^{\mathrm{T}} \tilde{\boldsymbol{q}}_t.$$
(78)

In order to complete the proof of Property (A-a) for $\tau = t$, we analyze the moment properties of ω_t . By definition, we have

$$\|\boldsymbol{\omega}_t\|^2 = \tilde{\boldsymbol{q}}_t^{\mathrm{T}} \boldsymbol{P}_{(\boldsymbol{H}_t, \tilde{\boldsymbol{Q}}_t)}^{\perp} \tilde{\boldsymbol{q}}_t \stackrel{\text{a.s.}}{=} \tilde{\boldsymbol{q}}_t^{\mathrm{T}} \boldsymbol{P}_{\tilde{\boldsymbol{Q}}_t}^{\perp} \tilde{\boldsymbol{q}}_t + o(N), \quad (79)$$

where the second equality follows from the orthogonality between $h_{\tau'}$ and $\tilde{q}_{\tau''}$. Furthermore, it is straightforward to confirm that ω_t has bounded $(4 + \epsilon)$ th moments. Thus, Property (A-a) holds for $\tau = t$.

See [11] for the proof of Properties (A-b) and (A-d) for $\tau = t$. Finally, we prove Property (A-c) for $\tau = t$. Let $\{\boldsymbol{z}_{\tau} \sim \mathcal{N}(\boldsymbol{0}, v_{\tau}\boldsymbol{I}_N)\}$ denote independent Gaussian random vectors with $v_0 = \lim_{M = \delta N \to \infty} N^{-1} \|\boldsymbol{q}_0\|^2$ and $v_{\tau} = \lim_{M = \delta N \to \infty} N^{-1} \|\boldsymbol{\tilde{q}}_{\tau}^{\perp}\|^2$ for $\tau > 0$, and define $\boldsymbol{\tilde{b}}_{\tau}$ recursively as

$$\tilde{\boldsymbol{b}}_{\tau} = \tilde{\boldsymbol{B}}_{\tau} \boldsymbol{\beta}_{\tau} + \boldsymbol{z}_{\tau}, \quad \tilde{\boldsymbol{B}}_{\tau} = (\tilde{\boldsymbol{b}}_0, \dots, \tilde{\boldsymbol{b}}_{\tau-1}), \quad (80)$$

conditioned on \tilde{Q}_{t+1} , with $\tilde{b}_0 = z_0$. Using Property (A-a) and Lemma 3 repeatedly yields

$$\frac{1}{N}\boldsymbol{b}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{\phi}}_{t} \stackrel{\mathrm{a.s.}}{=} \frac{1}{N}\mathbb{E}\left[\tilde{\boldsymbol{b}}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{\phi}}_{t}(\tilde{\boldsymbol{B}}_{t+1},\tilde{\boldsymbol{w}})\right] + o(1).$$
(81)

Since $\{\tilde{b}_{\tau}\}$ are jointly Gaussian conditioned on \tilde{Q}_{t+1} , we use Lemma 2 to obtain

$$\frac{1}{N}\boldsymbol{b}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{\phi}}_{t} \stackrel{\mathrm{a.s.}}{=} \frac{1}{N} \sum_{t'=0}^{t} \mathbb{E}\left[\tilde{\boldsymbol{b}}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{b}}_{t'}\right] \mathbb{E}\left[\left\langle\partial_{t'}\tilde{\boldsymbol{\phi}}_{t}(\tilde{\boldsymbol{B}}_{t+1},\tilde{\boldsymbol{w}})\right\rangle\right] + o(1)$$
$$\stackrel{\mathrm{a.s.}}{=} \frac{1}{N} \sum_{t'=0}^{t} \tilde{\boldsymbol{b}}_{\tau'}^{\mathrm{T}}\tilde{\boldsymbol{b}}_{t'}\left\langle\partial_{t'}\tilde{\boldsymbol{\phi}}_{t}(\boldsymbol{B}_{t+1},\tilde{\boldsymbol{w}})\right\rangle + o(1), \quad (82)$$

where the last equality follows from the repetition of the argument in (66). Thus, Property (A-c) holds for $\tau = t$.

F. Properties in Module B by Induction

Suppose that all properties in Modules A an B hold for all $\tau \leq t$ and $\tau < t$, respectively. It is possible to prove all properties in Module B for $\tau = t$, by repeating the proof in Appendix A-E. Thus, Theorem 1 is correct for all τ .

APPENDIX B Proof of Lemma 3

Since $\boldsymbol{a} \in \mathbb{R}^{N-t}$ is an orthogonally invariant vector, we can represent \boldsymbol{a} as $\boldsymbol{a} \sim \gamma \boldsymbol{u}_1$ with $\gamma = \|\boldsymbol{a}\| / \|\boldsymbol{u}_1\|$ and some standard Gaussian vector $\boldsymbol{u} = (\boldsymbol{u}_0^{\mathrm{T}}, \boldsymbol{u}_1^{\mathrm{T}})^{\mathrm{T}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$. For an $N \times N$ orthogonal matrix $\boldsymbol{\Phi} = (\boldsymbol{\Phi}^{\mathrm{H}}, \boldsymbol{\Phi}^{\mathrm{L}})$, let $\boldsymbol{z} = \boldsymbol{\Phi} \boldsymbol{u} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_N)$ and $\boldsymbol{\epsilon} = \boldsymbol{\Phi}^{\mathrm{H}} \boldsymbol{u}_0$. Then, we have

$$\tilde{\boldsymbol{a}} \sim \gamma(\boldsymbol{\Phi}\boldsymbol{u} - \boldsymbol{\Phi}^{\parallel}\boldsymbol{u}_0) = \gamma(\boldsymbol{z} - \boldsymbol{\epsilon}).$$
 (83)

Note that $\gamma^2 \stackrel{\text{a.s.}}{\to} v$ holds.

Let

$$S_N = \sum_{n=1}^N \tilde{f}_n(\tilde{a}_n), \tag{84}$$

$$\tilde{S}_N = \sum_{n=1}^N \tilde{f}_n(\gamma z_n), \tag{85}$$

$$\bar{S}_N = \sum_{n=1}^N \tilde{f}_n(\sqrt{v_N} z_n), \tag{86}$$

with $\tilde{f}_n(x) = f_n(x+b_n)$ and $v_N = ||\boldsymbol{a}||^2/(N-t)$. Since f_n is pseudo-Lipschitz of k, \tilde{f}_n is so. We first evaluate the difference $|\mathbb{E}[S_N] - \mathbb{E}[\bar{S}_N]|$. Using the triangle inequality yields

$$|\mathbb{E}[S_N] - \mathbb{E}[\bar{S}_N]| \le |\mathbb{E}[S_N] - \mathbb{E}[\tilde{S}_N]| + |\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]|.$$
(87)

We upper-bound the first term. From the pseudo-Lipschitz property of \tilde{f}_n , there is some constant L > 0 such that

$$\left| \mathbb{E}[S_N] - \mathbb{E}[\tilde{S}_N] \right| \leq L \sum_{n=1}^N \mathbb{E}\left[|\gamma \epsilon_n| \right] + L \sum_{n=1}^N \mathbb{E}\left[|\gamma z_n|^{k-1} |\gamma \epsilon_n| \right] \\ + L \sum_{n=1}^N \mathbb{E}\left[|\tilde{a}_n|^{k-1} |\gamma \epsilon_n| \right].$$
(88)

For the first term on the upper bound (88), we use the upper bound $\|\boldsymbol{\epsilon}\|_1 \leq \sqrt{N} \|\boldsymbol{\epsilon}\|$ to obtain

$$\sum_{n=1}^{N} \mathbb{E}\left[|\gamma \epsilon_{n}|\right] \leq \sqrt{N} \mathbb{E}\left[|\gamma| \|\boldsymbol{\epsilon}\|\right] \leq \sqrt{N} \left(\mathbb{E}[\gamma^{2}] \mathbb{E}[\|\boldsymbol{\epsilon}\|^{2}]\right)^{1/2}$$
$$= \sqrt{N} \left(\mathbb{E}[\gamma^{2}] \mathbb{E}[\|\boldsymbol{u}_{0}\|^{2}]\right)^{1/2} = \sqrt{N} \{\sqrt{vt} + o(1)\}, \tag{89}$$

where the second inequality follows from the Cauchy-Schwarz inequality.

For the second term on the upper bound (88), similarly we have

$$\sum_{n=1}^{N} \mathbb{E}\left[|\gamma z_{n}|^{k-1}|\gamma \epsilon_{n}|\right]$$

$$\leq \mathbb{E}\left[|\gamma|^{k} \left(\sum_{n=1}^{N} |z_{n}|^{2k-2}\right)^{1/2} \|\boldsymbol{\epsilon}\|\right]$$

$$\leq \left(\mathbb{E}[|\gamma|^{pk}]\right)^{1/p} \left\{\mathbb{E}\left[\left(\sum_{n=1}^{N} |z_{n}|^{2k-2}\right)^{q/2} \|\boldsymbol{\epsilon}\|^{q}\right]\right\}^{1/q}$$

$$\leq C \left\{\mathbb{E}\left[\left(\sum_{n=1}^{N} |z_{n}|^{2k-2}\right)^{q}\right] \mathbb{E}\left[\|\boldsymbol{\epsilon}\|^{2q}\right]\right\}^{(2q)^{-1}}, \quad (90)$$

for some constants C > 0 and q > 1. In these bounds, the first inequality follows from the Cauchy-Schwarz inequality. The second inequality is due to Hölder's inequality for all integers p > 1, q > 1, and $p + q \le 1$. The last inequality follows from the Cauchy-Schwarz inequality and $\mathbb{E}[|\gamma|^{pk}] < \infty$ for p > 1sufficiently close to 1. The expectation $\mathbb{E}[||\boldsymbol{\epsilon}||^{2q}] = \mathbb{E}[||\boldsymbol{u}_0||^{2q}]$ is bounded for fixed t. Furthermore, we use the boundedness of all moments of $|z_n|$ to have

$$\left\{ \mathbb{E}\left[\left(\sum_{n=1}^{N} |z_n|^{2k-2} \right)^q \right] \right\}^{(2q)^{-1}} < \left(C^{2q} N^q \right)^{(2q)^{-1}} = C\sqrt{N}$$
(91)

for some constant C > 0. Combining these observations, we arrive at

$$\sum_{n=1}^{N} \mathbb{E}\left[|\gamma z_n|^{k-1} |\gamma \epsilon_n| \right] < C\sqrt{N}$$
(92)

for some constant C > 0.

For the last term on the upper bound (88), we repeat the same argument to obtain

$$\sum_{n=1}^{N} \mathbb{E}\left[|\tilde{a}_n|^{k-1} |\gamma \epsilon_n| \right] < C\sqrt{N}$$
(93)

for some constant C > 0. Thus, we have proved

$$\frac{1}{N} |\mathbb{E}[S_N] - \mathbb{E}[\tilde{S}_N]| = \mathcal{O}(N^{-1/2}).$$
(94)

We upper-bound the difference $|\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]|$. Using the pseudo-Lipschitz property of \tilde{f}_n yields

$$|\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]| \le L \sum_{n=1}^N \mathbb{E}[|\gamma - \sqrt{v_N}||z_n|] + L \sum_{n=1}^N \mathbb{E}\left[|\gamma|^{k-1}|\gamma - \sqrt{v_N}||z_n|^k\right] + L \sum_{n=1}^N \mathbb{E}\left[v_N^{(k-1)/2}|\gamma - \sqrt{v_N}||z_n|^k\right]$$
(95)

for some constant L > 0. For the second term on the upper bound (95), we use the definitions $\gamma = \|\boldsymbol{a}\| / \|\boldsymbol{u}_1\|$ and $v_N = \|\boldsymbol{a}\|^2 / (N-t)$ to obtain

$$\sum_{n=1}^{N} \mathbb{E}[|\gamma|^{k-1}|\gamma - \sqrt{v_N}||z_n|^k] \\= \mathbb{E}\left[\frac{|\gamma|^{k-1}|N - t - \|\boldsymbol{u}_1\|^2|\|\boldsymbol{a}\|\|\boldsymbol{z}\|_k^k}{\sqrt{N - t}\|\boldsymbol{u}_1\|(\sqrt{N - t} + \|\boldsymbol{u}_1\|)}\right] \\< \mathbb{E}\left[|\gamma|^{k-1}|N - t - \|\boldsymbol{u}_1\|^2|\frac{\|\boldsymbol{a}\|\|\boldsymbol{z}\|_k^k}{(N - t)\|\boldsymbol{u}_1\|}\right] \\\leq C\left(\mathbb{E}\left[|N - t - \|\boldsymbol{u}_1\|^2|^q\left(\frac{\|\boldsymbol{a}\|\|\boldsymbol{z}\|_k^k}{(N - t)\|\boldsymbol{u}_1\|}\right)^q\right]\right)^{1/q} \\\leq C\left(\mathbb{E}\left[|N - t - \|\boldsymbol{u}_1\|^2|^2q\right]\mathbb{E}\left[\left(\frac{\|\boldsymbol{a}\|\|\boldsymbol{z}\|_k^k}{(N - t)\|\boldsymbol{u}_1\|}\right)^{2q}\right]\right)^{\frac{1}{2q}} (96)$$

for some constants C > 0 and q > 1. In these bounds, the second inequality follows from Hölder's inequality. The last inequality is due to the Cauchy-Schwarz inequality. Since $u_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{N-t})$ holds, it is straightforward to confirm

$$\mathbb{E}[(N-t-\|\boldsymbol{u}_1\|^2)^{2q}] = \mathbb{E}\left[\left\{\sum_{n=t+1}^N (u_n^2-1)\right\}^{2q}\right] = \mathcal{O}(N^q).$$
(97)

Furthermore, the latter expectation on the upper bound is bounded. Thus, we arrive at

$$\sum_{n=1}^{N} \mathbb{E}[|\gamma|^{k-1}|\gamma - \sqrt{v_N}||z_n|^k] = \mathcal{O}(\sqrt{N}).$$
(98)

Repeating the same argument for the first and last terms on the upper bound (95), we have

$$|\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]| = \mathcal{O}(\sqrt{N}).$$
(99)

Combining (94) and (99), we arrive at

$$\frac{1}{N} |\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]| = \mathcal{O}(N^{-1/2}).$$
(100)

We next prove that $(S_N - \mathbb{E}[S_N])/N$ converges a.s. to zero as $N \to \infty$. From a strong law of large numbers (SLLN) for dependent random variables [16, Corollary 1], it is sufficient to prove $\mathbb{V}[S_N] = \mathcal{O}(N^a)$ for some a < 2.

Let us prove $\mathbb{V}[S_N] = \mathbb{V}[\tilde{S}_N] + \mathcal{O}(N^a)$ for a < 2. By definition, we have

$$S_N^2 = \sum_{n,n'=1}^N \tilde{f}_n(\tilde{a}_n) \tilde{f}_{n'}(\tilde{a}_{n'}).$$
(101)

Proposition 3 implies that S_N^2 is the sum of the pseudo-Lipschitz functions $f(\tilde{a}_n, \tilde{a}_{n'}) = \tilde{f}_n(\tilde{a}_n)\tilde{f}_{n'}(\tilde{a}_{n'})$ of order 2k. Thus, we have

$$\left| \mathbb{E}[S_N^2] - \mathbb{E}[\tilde{S}_N^2] \right| < L \sum_{n,n'} \mathbb{E} \left[\gamma(\epsilon_n^2 + \epsilon_{n'}^2)^{1/2} \right] \\ + L \sum_{n,n'} \mathbb{E} \left[\gamma^{2k} (z_n^2 + z_{n'}^2)^{k-1/2} (\epsilon_n^2 + \epsilon_{n'}^2)^{1/2} \right] \\ + L \sum_{n,n'} \mathbb{E} \left[\gamma(\tilde{a}_n^2 + \tilde{a}_{n'}^2)^{k-1/2} (\epsilon_n^2 + \epsilon_{n'}^2)^{1/2} \right] (102)$$

for some constant L > 0.

For the first term on the upper bound (102), we use Hölder's inequality to obtain

$$\sum_{n,n'} \mathbb{E}\left[\gamma(\epsilon_n^2 + \epsilon_{n'}^2)^{1/2}\right] \le C \sum_{n,n'} \left\{ \mathbb{E}\left[(\epsilon_n^2 + \epsilon_{n'}^2)^{q/2}\right] \right\}^{1/q}$$
$$\le CN^2 \left\{ \mathbb{E}\left[\frac{1}{N^2} \sum_{n,n'} (\epsilon_n^2 + \epsilon_{n'}^2)^{q/2}\right] \right\}^{1/q}$$
(103)

for some constants C > 0 and $q \ge 2$, where the second inequality follows from Jensen's inequality. Applying the inequality $|x_1| + |x_2| \le 2^{1-2/q} (|x_1|^{q/2} + |x_2|^{q/2})^{2/q}$, we have

$$\sum_{n,n'} \mathbb{E}\left[\gamma(\epsilon_n^2 + \epsilon_{n'}^2)^{1/2}\right] \le CN^2 \left\{ \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^N |\epsilon_n|^q\right] \right\}^{1/q} (104)$$

for some constant C > 0. Since $\sum_{n=1}^{N} |\epsilon_n|^q \le ||\boldsymbol{\epsilon}||^q = ||\boldsymbol{u}_0||^q$ holds, we arrive at

$$\sum_{n,n'} \mathbb{E}\left[\gamma(\epsilon_n^2 + \epsilon_{n'}^2)^{1/2}\right] \le CN^{2-1/q}$$
(105)

for some constant C > 0.

For the second term on the upper bound (102), similarly we have

$$\sum_{n,n'} \mathbb{E} \left[\gamma^{2k} (z_n^2 + z_{n'}^2)^{k-1/2} (\epsilon_n^2 + \epsilon_{n'}^2)^{1/2} \right]$$

$$\leq CN^2 \left\{ \mathbb{E} \left[\frac{1}{N} \sum_{n=1}^N \epsilon_n^q \right] \right\}^{1/q} = \mathcal{O}(N^{2-1/q}) \quad (106)$$

for some constants C > 0 and $q \ge 2$, where we have used $\mathbb{E}[\gamma^{2pk}] < \infty$ and $\mathbb{E}[(z_n^2 + z_{n'}^2)^{q(2k-1)}] < \infty$ for some q and $p = (1 - q^{-1})^{-1}$. Repeating the same argument for the last term on the upper bound (102), we arrive at

$$\left| \mathbb{E}[S_N^2] - \mathbb{E}[\tilde{S}_N^2] \right| = \mathcal{O}(N^{2-1/q})$$
(107)

for some $q \ge 2$. Since (94) implies $(\mathbb{E}[S_N])^2 = (\mathbb{E}[\tilde{S}_N])^2 + \mathcal{O}(N^{3/2})$, we have $\mathbb{V}[S_N] = \mathbb{V}[\tilde{S}_N] + \mathcal{O}(N^{2-1/q})$ for some $q \ge 2$.

In order to prove the SLLN for $(S_N - \mathbb{E}[\bar{S}_N])/N$, we need to show $|\mathbb{E}[\tilde{S}_N^2] - \mathbb{E}[\bar{S}_N^2]| = \mathcal{O}(N^a)$ for some a < 2. This convergence and (99) imply that $N^{-1}(\mathbb{E}[\tilde{S}_N] - \mathbb{E}[\bar{S}_N]) \to 0$ and $\mathbb{V}[\tilde{S}_N] = \mathbb{V}[\bar{S}_N] + \mathcal{O}(N^{\max\{3/2, a\}})$. Furthermore, it is straightforward to confirm

$$\mathbb{V}\left[\bar{S}_{N}\right] = \sum_{n=1}^{N} \mathbb{E}\left\{\mathbb{V}\left[\left.\tilde{f}_{n}(\sqrt{v}_{N}z_{n})\right| \left\|\boldsymbol{a}\right\|\right]\right\} = \mathcal{O}(N).$$
(108)

Thus, we find the SLLN $(S_N - \mathbb{E}[\bar{S}_N])/N \stackrel{\text{a.s.}}{\to} 0.$

Let us prove $|\mathbb{E}[\tilde{S}_N^2] - \mathbb{E}[\bar{S}_N^2]| = \mathcal{O}(N^a)$ for some a < 2. Using the pseudo-Lipschitz property yields

$$\begin{aligned} & \left| \mathbb{E}[\tilde{S}_{N}^{2}] - \mathbb{E}[\bar{S}_{N}^{2}] \right| \\ \leq & L \sum_{n,n'} \mathbb{E}\left[|\gamma - \sqrt{v_{N}}| (z_{n}^{2} + z_{n'}^{2})^{1/2} \right] \\ & + L \sum_{n,n'} \mathbb{E}\left[\gamma^{2k-1} (z_{n}^{2} + z_{n'}^{2})^{k} |\gamma - \sqrt{v_{N}}| \right] \\ & + L \sum_{n,n'} \mathbb{E}\left[v_{N}^{k-1/2} (z_{n}^{2} + z_{n'}^{2})^{k} |\gamma - \sqrt{v_{N}}| \right] \end{aligned}$$
(109)

for some constant L > 0. For the second term on the upper bound (109), we use the definitions $\gamma = ||\mathbf{a}||/||\mathbf{u}_1||$ and $v_N = ||\mathbf{a}||^2/(N-t)$ to obtain

$$\sum_{n,n'} \mathbb{E} \left[\gamma^{2k-1} (z_n^2 + z_{n'}^2)^k | \gamma - \sqrt{v_N} | \right]$$

$$< \sum_{n,n'} \mathbb{E} \left[\gamma^{2k-1} (z_n^2 + z_{n'}^2)^k \frac{|N - t - \| \boldsymbol{u}_1 \|^2 | \| \boldsymbol{a} \|}{(N - t) \| \boldsymbol{u}_1 \|} \right]$$

$$\leq CN \left\{ \mathbb{E} \left[\left(|N - t - \| \boldsymbol{u}_1 \|^2 | \frac{\| \boldsymbol{a} \| \| \boldsymbol{z} \|_{2k}^2}{(N - t) \| \boldsymbol{u}_1 \|} \right)^q \right] \right\}^{1/q}$$

$$= \mathcal{O}(N^{3/2})$$
(110)

for some constants C > 0 and q > 1, where the second inequality follows from $z_n^2 + z_{n'}^2 \leq 2^{1-1/k}(z_n^{2k} + z_{n'}^{2k})^{1/k}$, Hölder's inequality, and from $\mathbb{E}[\gamma^{p(2k-1)}] < \infty$ for $p = (1 - q^{-1})^{-1}$. Repeating the same argument for the first and last terms on the upper bound (109), we arrive at

$$|\mathbb{E}[\tilde{S}_N^2] - \mathbb{E}[\bar{S}_N^2]| = \mathcal{O}(N^{3/2}).$$
(111)

Let $S_{0,N} = \sum_{n=1}^{N} f_n(\sqrt{v}z_n)$. In order to complete the proof of Lemma 3, we show $N^{-1}|\mathbb{E}[\bar{S}_N] - \mathbb{E}[S_{0,N}]| \to 0$. Define $\bar{S}_{0,N} = \sum_{n=1}^{N} \tilde{f}_n(\sqrt{v}z_n)$. Using the triangle inequality yields

$$|\mathbb{E}[S_N] - \mathbb{E}[S_{0,N}]| \le |\mathbb{E}[\bar{S}_N] - \mathbb{E}[\bar{S}_{0,N}]| + |\mathbb{E}[\bar{S}_{0,N}] - \mathbb{E}[S_{0,N}]|.$$
(112)

It is straightforward to prove $N^{-1}|\mathbb{E}[\bar{S}_N] - \mathbb{E}[\bar{S}_{0,N}]| \to 0$. Thus, we only evaluate the second term.

Using the pseudo-Lipschitz property yields

$$\frac{1}{N} \left| \mathbb{E}[\bar{S}_{0,N}] - \mathbb{E}[S_{0,N}] \right| \\
\leq \frac{L}{N} \sum_{n=1}^{N} |b_n| \mathbb{E}_{z_n} \left[1 + (b_n + \sqrt{v} z_n)^{k-1} + (\sqrt{v} z_n)^{k-1} \right] \\
\leq L \left(\frac{C_N}{N} \|b\|^2 \right)^{1/2}$$
(113)

for some constant L > 0, with

$$C_N = \frac{1}{N} \sum_{n=1}^{N} \left(\mathbb{E}_{z_n} \left[1 + (b_n + \sqrt{v} z_n)^{k-1} + (\sqrt{v} z_n)^{k-1} \right] \right)^2,$$
(114)

where the second inequality follows from the Cauchy-Schwarz inequality, Since $N^{-1} \| \boldsymbol{b} \|^2 \to 0$ and $N^{-1} \sum_{n=1}^N b_n^{2k-2} < \infty$

are assumed, we arrive at $N^{-1}|\mathbb{E}[\bar{S}_{0,N}] - \mathbb{E}[S_{0,N}]| \xrightarrow{\text{a.s.}} 0$. Thus, Lemma 3 holds.