Vector Gaussian Successive Refinement With Degraded Side Information

Yinfei Xu, Member, IEEE, Xuan Guang, Member, IEEE, Jian Lu, Member, IEEE, and Jun Chen, Senior Member, IEEE

Abstract

We investigate the problem of the successive refinement for Wyner-Ziv coding with degraded side information and obtain a complete characterization of the rate region for the quadratic vector Gaussian case. The achievability part is based on the evaluation of the Tian-Diggavi inner bound that involves Gaussian auxiliary random vectors. For the converse part, a matching outer bound is obtained with the aid of a new extremal inequality. Herein, the proof of this extremal inequality depends on the integration of the monotone path argument and the doubling trick as well as information-estimation relations.

Index Terms

Extremal inequality, lossy source coding, mean squared error, rate region, side information, successive refinement, vector Gaussian source, Wyner-Ziv problem.

I. INTRODUCTION

The research on network source coding can be traced back to the seminal work by Slepian and Wolf [1], where they considered, among other things, the problem of lossless source coding with side information at the decoder. Wyner and Ziv [2] studied the lossy source coding version of this problem (which later bears their names) and characterized its information-theoretic limit. Subsequently, the Wyner-Ziv problem was extended in various ways. One particular extension, known as successive refinement for Wyner-Ziv coding with degraded side information, is as follows: A source is encoded and decoded, in a successive manner, to meet different distortion constraints with the aid of progressively enhanced decoder side information. This extended Wyner-Ziv problem was tackled by Steinberg and Merhav [3] for the two-stage case and by Tian and Diggavi [4] for the multi-stage case. Specifically, the computable characterizations of rate regions in the discrete memoryless setting (with a general distortion measure) and in the scalar Gaussian setting (with the quadratic error distortion measure) were obtained accordingly.

In this paper, we consider the same extended Wyner-Ziv problem with a particular attention paid to the vector Gaussian setting (under covariance distortion constraints). The heart of the present paper is a new inequality regarding the optimality of the Gaussian solution to a certain extremal problem. It is well known that extremal inequalities play an important role in characterizing the fundamental limits of Gaussian network source and channel coding problems is well known. Indeed, they are indispensable to the converse argument for the Gaussian broadcast channel coding problem [5]-[14], the Gaussian interference channel coding problem [15]–[17], the Gaussian multi-terminal source coding problem [18]–[22], the secret key generation problem [23], the Gaussian multiple description problem [24]–[27], and others [28], [29]. Basic extremal inequalities that rely on the differential-entropy-maximizing property of the Gaussian distribution can only handle simple situations where the objective functional can be greedily optimized. When there are two or more conflicting terms, Shannon's entropy power inequality is often used to resolve the tension. However, the proportionality condition on the relevant covariance matrices needed for the tightness of the entropy power inequality is quite restrictive, typically only satisfied in scalar source and channel coding problems. As a consequence, more sophisticated extremal inequalities are needed to deal with vector Gaussian sources and channels. The proofs of such extremal inequalities, as well as the proof of the entropy power inequality, are often proved by invoking the monotone path argument or its variants. The conventional monotone path argument nevertheless appears to have its own limitations. For example, it fails to yield a tight outer bound on the capacity region of the two-user vector Gaussian broadcast channel with private and common messages. The desired result is eventually obtained by Geng and Nair [30] through a different approach involving so-called doubling trick. On the other hand, this approach obscures some useful information regarding the optimal Gaussian solution. Fortunately, this problem can be remedied via a systematic integration of the monotone path argument and the doubling trick, as shown by Wang and Chen [31] in their new proof of Courtade's extremal inequality [32]. In this work, we make use of this integrated strategy, together with the properties of the minimum mean square error (MMSE) and the Fisher information, to establish a new extremal inequality, which is further leveraged to characterize the rate region of the aforementioned extended Wyner-Ziv problem in the vector Gaussian source setting. It will

Y. Xu and J. Lu are with the School of Information Science and Engineering, Southeast University, Nanjing, 210096, China (email: {yinfeixu, lujian1980}@seu.edu.cn).

X. Guang is with the School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, China (email: xguang@nankai.edu.cn).

J. Chen is with the Department of Electrical and Computer Engineering, McMaster University, ON, L8S 4K1, Canada (email: junchen@ece.mcmaster.ca). This paper was presented in part at the 2019 IEEE International Symposium on Information Theory.

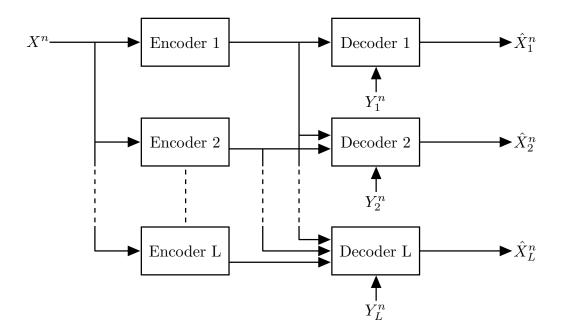


Fig. 1. Successive refinement for Wyner-Ziv coding with degraded side information.

be seen that the new extremal inequality avoids the comparison of distortion matrices, and thus is particularly handy when dealing with a large number of covariance distortion constraints.

The rest of this paper is organized as follows. We present the problem formulation and the main result in Section II. Section III is devoted to proving a new extremal inequality, which constitutes the main technical part of this paper. The main result is proved in Section IV. We conclude the paper in Section V.

II. PROBLEM STATEMENT AND MAIN RESULT

Let X be a $p \times 1$ -dimensional random vector with mean zero and covariance matrix $K_0 \succ 0$. Moreover, let

$$Y_i = X + N_i, \qquad i \in [1:L], \tag{1}$$

where N_i is a $p \times 1$ -dimensional random vector with mean zero and covariance matrix $K_i \succ 0$, $i \in [1 : L]$. It is assumed that

$$\boldsymbol{K}_1 \succ \ldots \succ \boldsymbol{K}_{L-1} \succ \boldsymbol{K}_L, \tag{2}$$

and X, $N_i - N_{i+1}$, $i \in [1:L]$, are mutually independent and jointly Gaussian¹. This assumption implies that

$$X \to Y_L \to Y_{L-1} \to \dots \to Y_1 \tag{3}$$

forms a Markov chain. Let $(X(t), Y_i(t), i \in [1 : L])_{t=1}^{\infty}$ be i.i.d. copies of $(X, Y_i, i \in [1 : L])$. The system model can be described as follows (see also Fig. 1).

• L encoding functions $(\phi_i^{(n)}, i \in [1:L])$:

$$\phi_i^{(n)}: \mathcal{X}^n \mapsto \mathcal{M}_i^{(n)}, \qquad i \in [1:L], \tag{4}$$

where $\phi_i^{(n)}$ maps the source sequence X^n to the codeword $M_i(X^n)$, $i \in [1:L]$. • L decoding functions $(\varphi_i^{(n)}, i \in [1:L])$:

$$\varphi_i^{(n)} : \prod_{j \in [1:i]} \mathcal{M}_j^{(n)} \times \mathcal{Y}_i^n \mapsto \hat{\mathcal{X}}^n, \qquad i \in [1:L],$$
(5)

where $\varphi_i^{(n)}$ produces the source reconstruction $\hat{X}_i^n(M_j, j \in [1:i], Y_i^n)$ by using codewords $(M_j, j \in [1:i])$ and side information Y_i^n . In particular, under covariance distortion contraints, there is no loss of optimality in assuming that $\phi_i^{(n)}$ performs MMSE estimation, i.e., $\hat{X}_i^n(M_j, j \in [1:i], Y_i^n) = \mathbb{E}[X^n|M_j, j \in [1:i], Y_i^n]$, $i \in [1:L]$.

¹Here N_{L+1} is a null random vector with covariance matrix $K_{L+1} = 0$.

Definition 1: A rate tuple $(R_i, i \in [1 : L])$ is said to be achievable subject to covariance distortion constraints $(\mathbf{D}_i \succ \mathbf{0}, i \in [1 : L])$ if there exists a sequence of encoding functions $(\phi_i^{(n)}, i \in [1 : L])$ and decoding functions $(\varphi_i^{(n)}, i \in [1 : L])$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \left| \mathcal{M}_i^{(n)} \right| \le R_i, \qquad i \in [1:L], \tag{6}$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[\left(X(t) - \hat{X}_i(t) \right) \left(X(t) - \hat{X}_i(t) \right)^T \right] \preceq \boldsymbol{D}_i, \qquad i \in [1:L].$$
(7)

The rate region $\mathcal{R}^*(\mathbf{D}_i, i \in [1:L])$ is defined as the set of all such achievable rate tuples.

The following theorem states a computable characterization of $\mathcal{R}^*(\mathbf{D}_i, i \in [1 : L])$, which is the main result of this paper. Theorem 1: $\mathcal{R}^*(\mathbf{D}_i, i \in [1 : L]) = \mathcal{R}(\mathbf{D}_i, i \in [1 : L])$, where $\mathcal{R}(\mathbf{D}_i, i \in [1 : L])$ is the convex hull of the set of $(R_i, i \in [1 : L])$ such that

$$R_1 \ge \frac{1}{2} \log \frac{|\mathbf{K}_0^{-1} + \mathbf{K}_1^{-1} + \mathbf{B}_1|}{|\mathbf{K}_0^{-1} + \mathbf{K}_1^{-1}|},\tag{8}$$

$$\sum_{j=1}^{i} R_j \ge \frac{1}{2} \log \frac{|\boldsymbol{K}_0^{-1} + \boldsymbol{K}_1^{-1} + \boldsymbol{B}_1|}{|\boldsymbol{K}_0^{-1} + \boldsymbol{K}_1^{-1}|} + \sum_{j=2}^{i} \frac{1}{2} \log \frac{|\boldsymbol{K}_0^{-1} + \boldsymbol{K}_j^{-1} + \sum_{k=1}^{j} \boldsymbol{B}_k|}{|\boldsymbol{K}_0^{-1} + \boldsymbol{K}_j^{-1} + \sum_{k=1}^{j-1} \boldsymbol{B}_k|}, \qquad i \in [2:L],$$
(9)

for some $(\boldsymbol{B}_i, i \in [1:L])$ satisfying

$$\boldsymbol{B}_i \succeq \boldsymbol{0}, \qquad i \in [1:L], \tag{10}$$

$$\sum_{j=1}^{i} \boldsymbol{B}_{j} \succeq \boldsymbol{D}_{i}^{-1} - \boldsymbol{K}_{0}^{-1} - \boldsymbol{K}_{i}^{-1}, \qquad i \in [1:L].$$
(11)

The proof of Theorem 1 can be found in Section IV, and it relies critically on the extremal inequality established in Section III.

III. AN EXTREMAL INEQUALITY

Theorem 2: Given $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_L \ge 0$, let $(B_i^*, i \in [1:L])$ be any positive semi-definite matrices such that

$$\sum_{j=1}^{i} \boldsymbol{B}_{j}^{*} \succeq \boldsymbol{D}_{i}^{-1} - \boldsymbol{K}_{0}^{-1} - \boldsymbol{K}_{i}^{-1}, \qquad i \in [1:L],$$
(12)

and

$$\frac{\mu_i}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_i^{-1} + \sum_{j=1}^i \boldsymbol{B}_j^* \right)^{-1} - \frac{\mu_{i+1}}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_{i+1}^{-1} + \sum_{j=1}^i \boldsymbol{B}_j^* \right)^{-1} = \boldsymbol{\Psi}_i - \boldsymbol{\Psi}_{i+1} + \boldsymbol{\Lambda}_i, \qquad i \in [1:L-1], \quad (13)$$

$$\frac{V_L}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_L^{-1} + \sum_{j=1}^L \boldsymbol{B}_j^* \right)^{-1} = \boldsymbol{\Psi}_L + \boldsymbol{\Lambda}_L,$$
(14)

$$\boldsymbol{B}_{i}^{*}\boldsymbol{\Psi}_{i} = \boldsymbol{0}, \qquad i \in [1:L],$$

$$(15)$$

$$\left(\boldsymbol{K}_{0}^{-1} + \boldsymbol{K}_{i}^{-1} + \sum_{j=1}^{i} \boldsymbol{B}_{j}^{*} - \boldsymbol{D}_{i}^{-1}\right) \boldsymbol{\Lambda}_{i} = \boldsymbol{0}, \qquad i \in [1:L],$$
(16)

for some positive semi-definite matrices $(\Psi_i, i \in [1 : L])$ and $(\Lambda_i, i \in [1 : L])$. For any random objects $(W_i, i \in [1 : L])$ satisfying the Markov chain contraint

$$(W_i, i \in [1:L]) \to X \to Y_L \to Y_{L-1} \to \dots \to Y_1$$
(17)

and the covariance distortion constraints

$$\operatorname{cov}(X|Y_i, W_j, j \in [1:i]) \preceq \boldsymbol{D}_i, \qquad i \in [1:L],$$
(18)

the following extremal inequality holds:

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$$\sum_{i=1}^{L-1} (\mu_i h(Y_i|W_j, j \in [1:i]) - \mu_{i+1} h(Y_{i+1}|W_j, j \in [1:i]) - (\mu_i - \mu_{i+1}) h(X|W_j, j \in [1:i])) + \mu_L h(Y_L|W_j, j \in [1:L]) - \mu_L h(X|W_j, j \in [1:L])$$

$$\geq \sum_{i=1}^{L-1} \left(-\frac{\mu_{i+1}}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_0^{-1} + \mathbf{K}_{i+1}^{-1} + \sum_{j=1}^{i} \mathbf{B}_j^* \right) \right| + \frac{\mu_i}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_0^{-1} + \mathbf{K}_i^{-1} + \sum_{j=1}^{i} \mathbf{B}_j^* \right) \right| \right) + \frac{\mu_L}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_0^{-1} + \mathbf{K}_L^{-1} + \sum_{j=1}^{L} \mathbf{B}_j^* \right) \right|.$$
(19)

Remark 1: For the special case L = 2, $\Lambda_1 = 0$, and $\mu_1 = \mu_2 = 1$, the extremal inequality (19) can be regarded as a variant of [11, Theorem 5], the original proof of which relies on the enhancement argument developed in [33]. However, when L > 2, the enhancement argument appears to be inadequate for resolving the difficulty caused by the introduction of $(\Psi_i, i \in [1 : L])$ and $(\Lambda_i, i \in [1 : L])$. We shall overcome this difficulty via a judicious application of the monotone path argument and the doubling trick.

For notational simplicity, we define

$$\boldsymbol{\Delta}_{i}^{-1} \triangleq \boldsymbol{K}_{0}^{-1} + \sum_{j=1}^{i} \boldsymbol{B}_{j}^{*}, \qquad i \in [1:L].$$
⁽²⁰⁾

The proof of Theorem 2 is divided into four steps.

A. Constructing the Monotone Path

We first construct 3L zero-mean Gaussian random vectors

 $X_1^G, \dots, X_L^G, \ Y_1^G, \dots, Y_L^G, \ \tilde{Y}_2^G, \dots, \tilde{Y}_{L+1}^G,$

which are independent of $(X_i, Y_i, W_i, i \in [1 : L])$. Specifically, they are defined as follows.

1) : Let $X_L^{\hat{G}}, W_i^{\hat{G}}, i \in [2:L]$, be mutually independent Gaussian random vectors with covariance matrices $\Delta_L, \Delta_{i-1} - \Delta_i$, $i \in [2:L]$, respectively. We define

$$X_i^G = X_{i+1}^G + W_{i+1}^G, \qquad i \in [1:L-1].$$
(21)

It is easy to see that

$$X_i^G \sim \mathcal{N}\left(\mathbf{0}, \mathbf{\Delta}_i\right), \qquad i \in [1:L].$$
(22)

2) : Let $N_i^G - N_{i+1}^G$, $i \in [1:L]$, be mutually independent Gaussian random vectors with covariance matrices $K_i - K_{i+1}$, $i \in [1:L]$, respectively. We assume that $(N_i^G, i \in [1:L+1])$ is independent of $(X_i^G, i \in [1:L])$. Define

$$Y_i^G = X_i^G + N_i^G, \qquad i \in [1:L],$$
(23)

$$\tilde{Y}_i^G = X_{i-1}^G + N_i^G, \quad i \in [2:L+1].$$
(24)

It is clear that

$$Y_i^G \sim \mathcal{N}\left(\mathbf{0}, \mathbf{\Delta}_i + \mathbf{K}_i\right), \qquad i \in [1:L], \tag{25}$$

$$\tilde{Y}_i^G \sim \mathcal{N}\left(\mathbf{0}, \mathbf{\Delta}_{i-1} + \mathbf{K}_i\right), \qquad i \in [2:L+1].$$
(26)

Using the covariance preserved transform (see, e.g., [34]), we define

$$X_{i,\gamma} = \sqrt{1 - \gamma} X + \sqrt{\gamma} X_i^G, \qquad i \in [1:L], \tag{27}$$

$$Y_{i,\gamma} = \sqrt{1 - \gamma} Y_i + \sqrt{\gamma} Y_i^G, \qquad i \in [1:L],$$
(28)

$$\tilde{Y}_{i,\gamma} = \sqrt{1-\gamma}Y_i + \sqrt{\gamma}\tilde{Y}_i^G, \qquad i \in [2:L+1]^2,$$
(29)

$$Y_{i,\gamma}^* = \sqrt{\gamma}Y_i - \sqrt{1-\gamma}Y_i^G, \qquad i \in [1:L], \tag{30}$$

for any $\gamma \in (0,1)$. Consider the following function:

$$g(\gamma) = \sum_{i=1}^{L-1} \left(\mu_i h(Y_{i,\gamma} | Y_{i,\gamma}^*, W_j, j \in [1:i]) - \mu_{i+1} h(\tilde{Y}_{i+1,\gamma} | Y_{i,\gamma}^*, W_j, j \in [1:i]) - (\mu_i - \mu_{i+1}) h(X_{i,\gamma} | Y_{i,\gamma}^*, W_j, j \in [1:i]) \right) + \mu_L h(Y_{L,\gamma} | Y_{L,\gamma}^*, W_j, j \in [1:L]) - \mu_L h(X_{L,\gamma} | Y_{L,\gamma}^*, W_j, j \in [1:L]).$$

$$(31)$$

²Here $Y_{L+1} = X$ and $\tilde{Y}_{L+1,\gamma} = X_{L,\gamma}$.

Notice that g(0) coincides with the left-hand side of (19) while g(1) coincides with the right-hand side of (19). Therefore, it suffices to show that $g(\gamma)$ decreases monotonically along the path parameterized by γ , i.e.,

$$\frac{d}{d\gamma}g(\gamma) \le 0, \qquad \gamma \in (0,1).$$
(32)

Remark 2: The construction of $(Y_{i,\gamma}^*, i \in [1 : L])$ is inspired by the doubling trick introduced in [30]. A similar construction can be found in [31].

B. Derivative of $g(\gamma)$

In this step, we utilize a vector generalization of I-MMSE relationship from [35]. First rewrite (31) as

$$g(\gamma) = \sum_{i=1}^{L-1} \left(\mu_i h(Y_{i,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]) - \mu_{i+1} h(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]) - (\mu_i - \mu_{i+1}) h(X_{i,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]) \right) + \mu_L h(Y_{L,\gamma}, Y_{L,\gamma}^* | W_j, j \in [1:L]) - \mu_L h(X_{L,\gamma}, Y_{L,\gamma}^* | W_j, j \in [1:L]).$$

$$(33)$$

In view of (28) and (30), it can be verified that

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$$h\left(Y_{i,\gamma}, Y_{i,\gamma}^{*} \middle| W_{j}, j \in [1:i]\right)$$

$$= h\left(\sqrt{1-\gamma}Y_{i} + \sqrt{\gamma}Y_{i}^{G}, \sqrt{\gamma}Y_{i} - \sqrt{1-\gamma}Y_{i}^{G} \middle| W_{j}, j \in [1:i]\right)$$
(34)

$$= h\left(Y_{i}, Y_{i}^{G} \middle| W_{j}, j \in [1:i]\right), \qquad i \in [1:L].$$
(35)

Since Y_i and Y_i^G do not depend on $\gamma,$ it follows that

$$\frac{d}{d\gamma}h\left(Y_i, Y_i^G \middle| W_j, j \in [1:i]\right) = 0, \qquad i \in [1:L].$$
(36)

Moreover, as shown in Appendices B and C,

$$\frac{d}{d\gamma}h\left(X_{i,\gamma},Y_{i,\gamma}^{*}\middle|W_{j},j\in[1:i]\right) = \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\Delta_{i}^{-1}+K_{i}^{-1}\right)^{-1}\left(J\left(X_{i,\gamma}\middle|Y_{i,\gamma}^{*},W_{j},j\in[1:i]\right)-\Delta_{i}^{-1}\right)\right\}, \quad i\in[1:L].$$

$$\frac{d}{d\gamma}h\left(\tilde{Y}_{i+1,\gamma},Y_{i,\gamma}^{*}\middle|W_{j},j\in[1:i]\right) = \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\left(\Delta_{i}^{-1}+K_{i}^{-1}\right)^{-1}-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}\right)-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\right\} + \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\left(\Delta_{i}^{-1}+K_{i}^{-1}\right)^{-1}-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}\right)-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\right\} + \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\Delta_{i}^{-1}+K_{i}^{-1}\right)^{-1}-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}\right)-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\right\} + \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}-\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}\right\}\right\}$$

$$J\left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^*, W_j, j \in [1:i]\right) \boldsymbol{K}_{i+1}\left(\boldsymbol{\Delta}_i^{-1} + \boldsymbol{K}_{i+1}^{-1}\right) - \boldsymbol{\Delta}_i^{-1}\left(\boldsymbol{\Delta}_i + \boldsymbol{K}_{i+1}\right) \boldsymbol{\Delta}_i^{-1}\right) \bigg\}, \qquad i \in [1:L-1].$$
(38)

Combining (35), (36), (37), and (38) gives

$$-2(1-\gamma)\frac{d}{d\gamma}g(\gamma)$$

$$=\sum_{i=1}^{L-1} \operatorname{tr}\left\{\left(\mu_{i+1}\left(\Delta_{i}^{-1}+K_{i}^{-1}\right)^{-1}-\mu_{i+1}\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)^{-1}\right)\left(\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}\right)\right\}$$

$$-J\left(\tilde{Y}_{i+1,\gamma}\Big|Y_{i,\gamma}^{*},W_{j},j\in[1:i]\right)K_{i+1}\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)-\Delta_{i}^{-1}\left(\Delta_{i}+K_{i+1}\right)\Delta_{i}^{-1}\right)\right\}$$

$$+\sum_{i=1}^{L-1}\operatorname{tr}\left\{(\mu_{i}-\mu_{i+1})(\Delta_{i}^{-1}+K_{i}^{-1})^{-1}\left(J\left(X_{i,\gamma}\Big|Y_{i,\gamma}^{*},W_{j},j\in[1:i]\right)-\Delta_{i}^{-1}\right)\right\}$$

$$+\operatorname{tr}\left\{\mu_{L}(\Delta_{L}^{-1}+K_{L}^{-1})\left(J\left(X_{L,\gamma}\Big|Y_{L,\gamma}^{*},W_{j},j\in[1:L]\right)-\Delta_{L}^{-1}\right)\right\},\qquad\gamma\in(0,1).$$
(39)

Hence, for the purpose of proving (32), it suffices to show that (39) is greater than or equal to 0.

C. Lower Bound of (39)

is given by

In this step, we establish a lower bound of (39) with the Karush-Kuhn-Tucker (KKT) conditions in (13) and (14) properly incorporated. First notice that the covariance matrix of random vector

$$\begin{pmatrix} \sqrt{1-\gamma}N_{i+1} + \sqrt{\gamma}N_{i+1}^{G} \\ \sqrt{\gamma}N_{i} - \sqrt{1-\gamma}N_{i}^{G} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{K}_{i+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{i} \end{pmatrix}.$$
(40)

So $\sqrt{1-\gamma}N_{i+1} + \sqrt{\gamma}N_{i+1}^G$ is independent of $\sqrt{\gamma}N_i - \sqrt{1-\gamma}N_i^G$, which, together with (30), implies that $\sqrt{1-\gamma}N_{i+1} + \sqrt{\gamma}N_{i+1}^G$ is independent of $Y_{i,\gamma}^*$ as well. For $i \in [1:L-1]$, we have

$$\tilde{Y}_{i+1,\gamma} = X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^G.$$
(41)

In view of the fact that $\sqrt{1-\gamma}N_{i+1} + \sqrt{\gamma}N_{i+1}^G$ is independent of $X_{i,\gamma}$, the Fisher information inequality (see Lemma 5 in Appendix A) can be invoked to show

$$\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) - \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1}$$

$$= \left(\boldsymbol{I} + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \left(\boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{I} \right)$$

$$- \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} - \boldsymbol{\Delta}_{i}^{-1}$$

$$= \left(I + \boldsymbol{V}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \left(\boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{I} \right)$$

$$= \left(I + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \left(\boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{I} \right)$$

$$= \left(I + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \left(\boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{I} \right)$$

$$= \left(I + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \left(\boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{I} \right)$$

$$= \left(I + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \right)$$

$$= \left(I + \boldsymbol{\Delta}_{i}^{-1} \boldsymbol{K}_{i+1} \right) J \left(X_{i,\gamma} + \sqrt{1 - \gamma} N_{i+1} + \sqrt{\gamma} N_{i+1}^{G} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \right)$$

$$\leq J\left(X_{i,\gamma}\middle|Y_{i,\gamma}^{*},W_{j},j\in[1:i]\right) + \boldsymbol{\Delta}_{i}^{-1}\boldsymbol{K}_{i+1}J\left(\sqrt{1-\gamma}N_{i+1}+\sqrt{\gamma}N_{i+1}^{G}\right)\boldsymbol{K}_{i+1}\boldsymbol{\Delta}_{i}^{-1} - \boldsymbol{\Delta}_{i}^{-1}\boldsymbol{K}_{i+1}\boldsymbol{\Delta}_{i}^{-1} - \boldsymbol{\Delta}_{i}^{-1}$$

$$(43)$$

$$=J\left(X_{i,\gamma}\Big|Y_{i,\gamma}^*,W_j,j\in[1:i]\right)-\boldsymbol{\Delta}_i^{-1}.$$
(44)

Since $K_i \succ K_{i+1}$, it follows that

$$\left(\Delta_{i}^{-1} + K_{i}^{-1}\right)^{-1} - \left(\Delta_{i}^{-1} + K_{i+1}^{-1}\right)^{-1} \succ \mathbf{0}.$$
(45)

Therefore,

$$- 2(1-\gamma)\frac{d}{d\gamma}g(\gamma)$$

$$\geq \sum_{i=1}^{L-1} \operatorname{tr} \left\{ \left(\mu_{i} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right)^{-1} - \mu_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right)^{-1} \right) \left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} \right) \right) \right\}$$

$$- \operatorname{tr} \left\{ \mu_{L} \left(\boldsymbol{\Delta}_{L}^{-1} + \boldsymbol{K}_{L}^{-1} \right) \left(J \left(\boldsymbol{X}_{L,\gamma} \middle| \boldsymbol{Y}_{L,\gamma}^{*}, \boldsymbol{W}_{j}, j \in [1:L] \right) - \boldsymbol{\Delta}_{L}^{-1} \right) \right\}$$

$$= \sum_{i=1}^{L-1} \operatorname{tr} \left\{ \left(\boldsymbol{\Psi}_{i} - \boldsymbol{\Psi}_{i+1} + \boldsymbol{\Lambda}_{i} \right) \left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} \right) \right) \right\}$$

$$+ \operatorname{tr} \left\{ \left(\boldsymbol{\Psi}_{L} + \boldsymbol{\Lambda}_{L} \right) \left(J \left(\boldsymbol{\tilde{Y}}_{L+1,\gamma} \middle| \boldsymbol{Y}_{L,\gamma}^{*}, \boldsymbol{W}_{j}, j \in [1:L] \right) - \boldsymbol{\Delta}_{L}^{-1} \right) \right\},$$

$$(47)$$

where (47) is due to the KKT properties in (13) and (14). Via suitable rearrangement, this lower bound can be written in the following equivalent form:

$$- 2(1-\gamma)\frac{d}{d\gamma}g(\gamma)$$

$$\geq \operatorname{tr}\left\{\Psi_{1}\left(\left(\Delta_{1}^{-1}+K_{2}^{-1}\right)K_{2}J\left(\tilde{Y}_{2,\gamma}\middle|Y_{1,\gamma}^{*},W_{1}\right)K_{2}\left(\Delta_{1}^{-1}+K_{2}^{-1}\right)-\Delta_{1}^{-1}\left(\Delta_{1}+K_{2}\right)\Delta_{1}^{-1}\right)\right\}$$

$$+ \sum_{i=2}^{L}\operatorname{tr}\left\{\Psi_{i}\left(\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)K_{i+1}J\left(\tilde{Y}_{i+1,\gamma}\middle|Y_{i,\gamma}^{*},W_{j},j\in[1:i]\right)K_{i+1}\left(\Delta_{i}^{-1}+K_{i+1}^{-1}\right)\right)\right\}$$

$$- \left(\Delta_{i-1}^{-1}+K_{i}^{-1}\right)K_{i}J\left(\tilde{Y}_{i,\gamma}\middle|Y_{i-1,\gamma}^{*},W_{j},j\in[1:i-1]\right)K_{i}\left(\Delta_{i-1}^{-1}+K_{i}^{-1}\right)\right)$$

$$- \Delta_{i}^{-1}\left(\Delta_{i}+K_{i+1}\right)\Delta_{i}^{-1}+\Delta_{i-1}^{-1}\left(\Delta_{i-1}+K_{i}\right)\Delta_{i-1}^{-1}\right)\right\}$$

$$(48a)$$

$$(48b)$$

$$+\sum_{i=1}^{L} \operatorname{tr} \left\{ \Lambda_{i} \left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) - \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1} \right) \right\}.$$
(48c)

Now it suffices to show that (48a)-(48c) are all lower bounded by 0.

D. Lower Bound of (48a)

From (190) in Appendix C,

$$\left(\boldsymbol{\Delta}_{1}^{-1} + \boldsymbol{K}_{2}^{-1} \right) \boldsymbol{K}_{2} J \left(\tilde{Y}_{2,\gamma} \middle| Y_{1,\gamma}^{*}, W_{1} \right) \boldsymbol{K}_{2} \left(\boldsymbol{\Delta}_{1}^{-1} + \boldsymbol{K}_{2}^{-1} \right) - \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \boldsymbol{\Delta}_{1}^{-1}$$

$$= \frac{1 - \gamma}{\gamma} \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right) \left(\left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right)^{-1}$$

$$- \frac{1}{\gamma} \operatorname{cov} \left(Y_{2} \middle| \tilde{Y}_{2,\gamma}, Y_{1,\gamma}^{*}, W_{1} \right) \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right) \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \boldsymbol{\Delta}_{1}^{-1}.$$

$$(49)$$

Combining the data processing inequality for MMSE (see Lemma 8 in Appendix A) and (181) gives

$$\cos\left(Y_{2}\Big|\tilde{Y}_{2,\gamma},Y_{1,\gamma}^{*},W_{1}\right) \leq \cos\left(Y_{2}\Big|\tilde{Y}_{2,\gamma},Y_{1,\gamma}^{*}\right) \\
= \left(\left(\boldsymbol{K}_{0}+\boldsymbol{K}_{2}\right)^{-1}+\frac{1-\gamma}{\gamma}\left(\boldsymbol{\Delta}_{1}+\boldsymbol{K}_{2}\right)^{-1}+\frac{1}{\gamma}\left(\boldsymbol{K}_{1}-\boldsymbol{K}_{2}\right)^{-1}\right)^{-1}.$$
(50)

Substituting (50) into (49) yields the following lower bound:

$$\left(\boldsymbol{\Delta}_{1}^{-1} + \boldsymbol{K}_{2}^{-1} \right) \boldsymbol{K}_{2} J \left(\tilde{Y}_{2,\gamma} \middle| Y_{1,\gamma}^{*}, W_{1} \right) \boldsymbol{K}_{2} \left(\boldsymbol{\Delta}_{1}^{-1} + \boldsymbol{K}_{2}^{-1} \right) - \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \boldsymbol{\Delta}_{1}^{-1} \succeq \frac{(1-\gamma)^{2}}{\gamma^{2}} \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right) \left(\frac{\gamma}{1-\gamma} \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right)^{-1} \right) - \frac{1}{1-\gamma} \left(\left(\boldsymbol{K}_{0} + \boldsymbol{K}_{2} \right)^{-1} + \frac{1-\gamma}{\gamma} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \frac{1}{\gamma} \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right)^{-1} \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right) \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \boldsymbol{\Delta}_{1}^{-1}$$

$$= \frac{1-\gamma}{\gamma} \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right)$$
(51)

$$\gamma = \left(\left(\mathbf{K}_{0} + \mathbf{K}_{2} \right)^{-1} + \frac{1 - \gamma}{\gamma} \left(\mathbf{\Delta}_{1} + \mathbf{K}_{2} \right)^{-1} + \frac{1}{\gamma} \left(\mathbf{K}_{1} - \mathbf{K}_{2} \right)^{-1} \right)^{-1} \left(\left(\mathbf{K}_{0} + \mathbf{K}_{2} \right)^{-1} - \left(\mathbf{\Delta}_{1} + \mathbf{K}_{2} \right)^{-1} \right) \left(\mathbf{\Delta}_{1} + \mathbf{K}_{2} \right) \mathbf{\Delta}_{1}^{-1}$$
(52)

$$= \frac{1-\gamma}{\gamma} \Delta_{1}^{-1} (\Delta_{1} + K_{2}) \left((\Delta_{1} + K_{2})^{-1} + (K_{1} - K_{2})^{-1} \right) \\ \left((K_{0} + K_{2})^{-1} + \frac{1-\gamma}{\gamma} (\Delta_{1} + K_{2})^{-1} + \frac{1}{\gamma} (K_{1} - K_{2})^{-1} \right)^{-1} (K_{0} + K_{2})^{-1} (\Delta_{1} - K_{0}) \Delta_{1}^{-1}$$

$$= \frac{1-\gamma}{\gamma} \Delta_{1}^{-1} (\Delta_{1} + K_{2}) \left((\Delta_{1} + K_{2})^{-1} + (K_{1} - K_{2})^{-1} \right)$$
(53)

$$\left((\boldsymbol{K}_{0} + \boldsymbol{K}_{2})^{-1} + \frac{1 - \gamma}{\gamma} (\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2})^{-1} + \frac{1}{\gamma} (\boldsymbol{K}_{1} - \boldsymbol{K}_{2})^{-1} \right)^{-1} (\boldsymbol{K}_{0} + \boldsymbol{K}_{2})^{-1} \boldsymbol{K}_{0}^{-1} (\boldsymbol{K}_{0}^{-1} - \boldsymbol{\Delta}_{1}^{-1}).$$
(54)

From the complementary slackness condition in (15), i.e.,

$$\boldsymbol{B}_{1}^{*}\boldsymbol{\Psi}_{1} = \left(\boldsymbol{K}_{0}^{-1} - \boldsymbol{\Delta}_{1}^{-1}\right)\boldsymbol{\Psi}_{1} = \boldsymbol{0},$$
(55)

we have

$$\operatorname{tr}\left\{\Psi_{1}\left(\left(\boldsymbol{\Delta}_{1}^{-1}+\boldsymbol{K}_{2}^{-1}\right)\boldsymbol{K}_{2}J\left(\tilde{Y}_{2,\gamma}\middle|Y_{1,\gamma}^{*},W_{1}\right)\boldsymbol{K}_{2}\left(\boldsymbol{\Delta}_{1}^{-1}+\boldsymbol{K}_{2}^{-1}\right)-\boldsymbol{\Delta}_{1}^{-1}\left(\boldsymbol{\Delta}_{1}+\boldsymbol{K}_{2}\right)\boldsymbol{\Delta}_{1}^{-1}\right)\right\}$$
(56)

$$\geq \operatorname{tr} \left\{ \frac{\gamma}{\gamma} \boldsymbol{\Delta}_{1}^{-1} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right) \left(\left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right) \right. \\ \left(\left(\boldsymbol{K}_{0} + \boldsymbol{K}_{2} \right)^{-1} + \frac{1 - \gamma}{\gamma} \left(\boldsymbol{\Delta}_{1} + \boldsymbol{K}_{2} \right)^{-1} + \frac{1}{\gamma} \left(\boldsymbol{K}_{1} - \boldsymbol{K}_{2} \right)^{-1} \right)^{-1} \left(\boldsymbol{K}_{0} + \boldsymbol{K}_{2} \right)^{-1} \boldsymbol{K}_{0}^{-1} \left(\boldsymbol{K}_{0}^{-1} - \boldsymbol{\Delta}_{1}^{-1} \right) \boldsymbol{\Psi}_{1} \right\} \\ = 0.$$
(57)

This proves that (48a) is lower bounded by 0.

E. Lower Bound of (48b)

To the end of showing that (48b) is lower bounded by 0, we introduce

$$N'_{i+1} \triangleq \sqrt{1-\gamma} \left(N_i - N_{i+1} \right) + \sqrt{\gamma} \left(N_i^G - N_{i+1}^G \right), \qquad i \in [1:L].$$
(58)

Note that N'_{i+1} is a Gaussian random vector with covariance matrix $K_i - K_{i+1}$ and is independent of $(\tilde{Y}_{i+1,\gamma}, Y^*_{i,\gamma})$. Moreover,

$$\tilde{Y}_{i,\gamma} = \tilde{Y}_{i+1,\gamma} + N'_{i+1}, \quad i \in [2:L].$$
(59)

In view of the fact that N'_{i+1} is independent of $Y^*_{i,\gamma}$, we can invoke the Fisher information inequality (see Lemma 5 in Appendix A) to show

$$(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i}^{-1}) \, \boldsymbol{K}_{i} J \left(\tilde{Y}_{i,\gamma} \middle| Y_{i-1,\gamma}^{*}, W_{j}, j \in [1:i-1] \right) \, \boldsymbol{K}_{i} \left(\boldsymbol{K}_{i}^{-1} + \boldsymbol{\Delta}_{i-1}^{-1} \right) = \left(\boldsymbol{\Delta}_{i-1}^{-1} \boldsymbol{K}_{i} + \boldsymbol{I} \right) J \left(\tilde{Y}_{i+1,\gamma} + N_{i+1}' \middle| Y_{i-1,\gamma}^{*}, W_{j}, j \in [1:i-1] \right) \left(\boldsymbol{I} + \boldsymbol{K}_{i} \boldsymbol{\Delta}_{i-1}^{-1} \right)$$

$$(60)$$

$$\leq \left(\boldsymbol{\Delta}_{i-1}^{-1} \boldsymbol{K}_{i+1} + \boldsymbol{I} \right) J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i-1,\gamma}^*, W_j, j \in [1:i-1] \right) \left(\boldsymbol{I} + \boldsymbol{K}_{i+1} \boldsymbol{\Delta}_{i-1}^{-1} \right) + \boldsymbol{\Delta}_{i-1}^{-1} \left(\boldsymbol{K}_i - \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i-1}^{-1}$$
(61)

$$\leq \left(\Delta_{i-1}^{-1} K_{i+1} + I \right) J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^*, W_j, j \in [1:i] \right) \left(I + K_{i+1} \Delta_{i-1}^{-1} \right) + \Delta_{i-1}^{-1} \left(K_i - K_{i+1} \right) \Delta_{i-1}^{-1}$$
(62)

$$= \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^*, W_j, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{K}_{i+1}^{-1} + \boldsymbol{\Delta}_{i-1}^{-1} \right) + \boldsymbol{\Delta}_{i-1}^{-1} \left(\boldsymbol{K}_i - \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i-1}^{-1}, \tag{63}$$

where (62) follows by the Markov chain contraint $(Y_{i-1,\gamma}^*, W_j, j \in [1:i-1]) \rightarrow (Y_{i,\gamma}^*, W_j, j \in [1:i]) \rightarrow \tilde{Y}_{i+1,\gamma}$ and the data processing inequality for Fisher information (see Lemma 7 in Appendix A). Meanwhile, due to the complementary slackness condition in (15), i.e.,

$$\boldsymbol{B}_{i}^{*}\boldsymbol{\Psi}_{i} = \left(\boldsymbol{\Delta}_{i}^{-1} - \boldsymbol{\Delta}_{i-1}^{-1}\right)\boldsymbol{\Psi}_{i} = \boldsymbol{0}, \qquad i \in [2:L],$$

$$(64)$$

we have

$$\operatorname{tr} \left\{ \Psi_{i} \left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \right. \\ \left. - \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \boldsymbol{K}_{i} J \left(\tilde{Y}_{i,\gamma} \middle| Y_{i-1,\gamma}^{*}, W_{j}, j \in [1:i-1] \right) \boldsymbol{K}_{i} \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \right. \\ \left. - \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{\Delta}_{i-1}^{-1} \left(\boldsymbol{\Delta}_{i-1} + \boldsymbol{K}_{i} \right) \boldsymbol{\Delta}_{i-1}^{-1} \right) \right\}$$

$$= \operatorname{tr} \left\{ \Psi_{i} \left(\left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \right. \\ \left. - \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \boldsymbol{K}_{i} J \left(\tilde{Y}_{i,\gamma} \middle| Y_{i-1,\gamma}^{*}, W_{j}, j \in [1:i-1] \right) \boldsymbol{K}_{i} \left(\boldsymbol{\Delta}_{i-1}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \right. \\ \left. + \boldsymbol{\Delta}_{i-1}^{-1} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i-1}^{-1} \right) \right\}$$

$$\geq 0, \qquad i \in [2:L].$$

$$(65)$$

This proves that (48b) is lower bounded by 0.

F. Lower Bound of (48c)

To the end of showing that (48c) is lower bounded by 0, we introduce

$$N''_{i+1} \triangleq \sqrt{\gamma} \left(N_i - N_{i+1} \right) - \sqrt{1 - \gamma} \left(N_i^G - N_{i+1}^G \right), \qquad i \in [1:L].$$
(66)

Note that N''_{i+1} is a Gaussian random vector with covariance matrix $K_i - K_{i+1}$ and is independent of $(Y_{i+1}, \tilde{Y}^G_{i+1})$. It can be verified that

$$cov\left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right)
= cov\left(Y_{i+1} \middle| \sqrt{1 - \gamma} \tilde{Y}_{i+1,\gamma} + \sqrt{\gamma} Y_{i,\gamma}^{*}, \tilde{Y}_{i+1,\gamma}, W_{j}, j \in [1:i]\right)$$
(67)

$$= \cos\left(Y_{i+1}\Big|(1-\gamma)Y_{i+1} + \sqrt{\gamma(1-\gamma)}\tilde{Y}_{i+1}^{G} + \gamma Y_{i} - \sqrt{\gamma(1-\gamma)}Y_{i}^{G}, \tilde{Y}_{i+1,\gamma}, W_{j}, j \in [1:i]\right)$$
(68)

$$= \cos\left(Y_{i+1}\Big|Y_{i+1} + \sqrt{\gamma}N_{i+1}'', Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}}\tilde{Y}_{i+1}^G, W_j, j \in [1:i]\right)$$
(69)

$$\leq \operatorname{cov}\left(Y_{i+1} \middle| \left(\frac{1-\gamma}{\gamma} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + \frac{1}{\gamma} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right) Y_{i+1} + \sqrt{\frac{1-\gamma}{\gamma}} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} \tilde{Y}_{i+1}^{G} + \sqrt{\frac{1}{\gamma}} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1} N_{i+1}'', W_{j}, j \in [1:i] \right).$$

$$(70)$$

where (70) is due to the data processing inequality for MMSE (see Lemma 8 in Appendix A). Let

$$\boldsymbol{P}_{i+1} \triangleq \left(\frac{1-\gamma}{\gamma} \left(\boldsymbol{\Delta}_i + \boldsymbol{K}_{i+1}\right)^{-1} + \frac{1}{\gamma} \left(\boldsymbol{K}_i - \boldsymbol{K}_{i+1}\right)^{-1}\right)^{-1},\tag{71}$$

$$S_{i+1}^{G} \triangleq \boldsymbol{P}_{i+1} \left(\sqrt{\frac{1-\gamma}{\gamma}} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} \tilde{Y}_{i+1}^{G} + \sqrt{\frac{1}{\gamma}} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} N_{i+1}'' \right).$$
(72)

It follows by the theory of linear MMSE estimtion that

$$N_i - N_{i+1} = S_{i+1}^G + T_{i+1}^G, (73)$$

where T_{i+1}^G is a Gaussian random vector with covariance matrix $K_i - K_{i+1} - P_{i+1}$, and is independent of S_{i+1}^G . Due to the Markov chain

$$(W_j, j \in [1:i]) \to Y_{i+1} \to Y_{i+1} + S^G_{i+1} \to Y_{i+1} + S^G_{i+1} + T^G_{i+1},$$
(74)

we can invoke Lemma 6 in Appendix A to show that

$$\operatorname{cov}\left(Y_{i+1}\Big|\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^*, W_j, j \in [1:i]\right)^{-1}$$
(75)

$$\succeq \operatorname{cov}\left(Y_{i+1} \middle| Y_{i+1} + S_{i+1}^G, W_j, j \in [1:i]\right)^{-1}$$
(76)

$$\succeq \operatorname{cov}\left(Y_{i+1} \middle| Y_{i+1} + S_{i+1}^G + T_{i+1}^G, W_j, j \in [1:i]\right)^{-1} - (\boldsymbol{K}_i - \boldsymbol{K}_{i+1})^{-1} + \boldsymbol{P}_{i+1}^{-1}$$
(77)

$$= \operatorname{cov}\left(Y_{i+1} \middle| Y_i, W_j, j \in [1:i]\right)^{-1} + \frac{1-\gamma}{\gamma} \left(\left(\boldsymbol{\Delta}_i + \boldsymbol{K}_{i+1}\right)^{-1} + \left(\boldsymbol{K}_i - \boldsymbol{K}_{i+1}\right)^{-1} \right).$$
(78)

1) : Note that the following Markov chain condition holds:

$$(W_j, j \in [1:i]) \to X \to Y_{i+1} \to Y_i.$$

$$(79)$$

Since X, Y_i , and Y_{i+1}) are jointly Gaussian, it follows that

$$\mathbb{E}\left[Y_{i+1}|X,Y_i\right] \tag{80}$$

$$= \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right) \boldsymbol{K}_{i}^{-1} \boldsymbol{X} + \boldsymbol{K}_{i+1} \boldsymbol{K}_{i}^{-1} \boldsymbol{Y}_{i}.$$
(81)

Furthermore, we have

$$Y_{i+1} = (\mathbf{K}_i - \mathbf{K}_{i+1}) \, \mathbf{K}_i^{-1} \left(X + \tilde{N}_{i+1} \right) + \mathbf{K}_{i+1} \mathbf{K}_i^{-1} Y_i, \tag{82}$$

where \tilde{N}_{i+1} is a zero-mean Gaussian random vector with covariance matrix

$$\tilde{K}_{i+1} = \left(K_{i+1}^{-1} - K_i^{-1}\right)^{-1} \succ \mathbf{0},$$
(83)

and is independent of (X, Y_i) . Therefore,

$$\operatorname{cov}\left(Y_{i+1}\middle|Y_{i}, W_{j}, j \in [1:i]\right)$$
$$= \operatorname{cov}\left(\left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right) \boldsymbol{K}_{i}^{-1}\left(X + \tilde{N}_{i+1}\right)\middle|Y_{i}, W_{j}, j \in [1:i]\right)$$
(84)

$$\leq (\mathbf{K}_{i} - \mathbf{K}_{i+1}) \mathbf{K}_{i}^{-1} \left(\mathbf{D}_{i} + \left(\mathbf{K}_{i+1}^{-1} - \mathbf{K}_{i}^{-1} \right)^{-1} \right) \mathbf{K}_{i}^{-1} \left(\mathbf{K}_{i} - \mathbf{K}_{i+1} \right),$$
(85)

where (85) is because of covariance distortion constraint $cov(X|Y_i, W_j, j \in [1:i]) \leq D_i$ in (18).

2) : It can be verified that

$$\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1}$$

$$= \left(\boldsymbol{K}_{i+1}^{-1} - \left(\boldsymbol{K}_{i+1} - \boldsymbol{K}_{i} \right)^{-1} - \boldsymbol{K}_{i+1}^{-1} + \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1}$$

$$(86)$$

$$= \boldsymbol{K}_{i+1} \left(\left(\boldsymbol{K}_{i+1}^{-1} - \boldsymbol{K}_{i}^{-1} \right)^{-1} - \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right)^{-1} \right)^{-1} \boldsymbol{K}_{i+1}$$
(87)

$$= K_{i+1} \left(K_{i+1}^{-1} - K_{i}^{-1} \right) \left(\left(\Delta_{i}^{-1} + K_{i}^{-1} \right)^{-1} + \left(K_{i+1}^{-1} - K_{i}^{-1} \right)^{-1} \right) \left(K_{i+1}^{-1} - K_{i}^{-1} \right) K_{i+1}$$
(88)

 $\leq (K_{i} - K_{i+1}) K_{i}^{-1} \left(D_{i} + \left(K_{i+1}^{-1} - K_{i}^{-1} \right)^{-1} \right) K_{i}^{-1} \left(K_{i} - K_{i+1} \right),$ (89)

where (89) is because of $\boldsymbol{\Delta}_i^{-1} + \boldsymbol{K}_i^{-1} \succeq \boldsymbol{D}_i^{-1}$.

Combing (85) and (89) yields

$$cov\left(Y_{i+1}\Big|\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right) \\
\leq \gamma\left(K_{i} - K_{i+1}\right)K_{i}^{-1}\left(D_{i} + \left(K_{i+1}^{-1} - K_{i}^{-1}\right)^{-1}\right)K_{i}^{-1}\left(K_{i} - K_{i+1}\right).$$
(90)

In view of (190), we have

$$\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) - \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1}$$

$$= \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \left(J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) - \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} \right) \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1}$$

$$= \frac{1 - \gamma}{4} \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)$$

$$(91)$$

$$\begin{pmatrix} \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1} - \frac{1}{\gamma} \operatorname{cov} \left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \right) \\ \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right) \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1} \tag{92}$$

$$\succeq \frac{1-\gamma}{\gamma} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right)^{-1} - \boldsymbol{D}_{i} \right) \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right)$$
(93)

$$=\frac{1-\gamma}{\gamma}\left(\boldsymbol{\Delta}_{i}^{-1}+\boldsymbol{K}_{i}^{-1}\right)\boldsymbol{D}_{i}\left(\boldsymbol{D}_{i}^{-1}-\boldsymbol{\Delta}_{i}^{-1}-\boldsymbol{K}_{i}^{-1}\right).$$
(94)

From the complementary slackness condition in (16), i.e.,

$$\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} - \boldsymbol{D}_{i}^{-1}\right)\boldsymbol{\Lambda}_{i} = \boldsymbol{0}, \qquad i \in [1:L],$$
(95)

we have

$$\operatorname{tr}\left\{ \Lambda_{i}\left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) \boldsymbol{K}_{i+1} J\left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \boldsymbol{K}_{i+1} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1} \right) - \frac{1}{\gamma} \boldsymbol{\Delta}_{i}^{-1} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \boldsymbol{\Delta}_{i}^{-1} \right) \right\}$$
(96)
$$\geq \operatorname{tr}\left\{ \frac{1-\gamma}{\gamma} \Lambda_{i} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right) \boldsymbol{D}_{i} \left(\boldsymbol{D}_{i}^{-1} - \boldsymbol{\Delta}_{i}^{-1} - \boldsymbol{K}_{i}^{-1} \right) \right\} = 0, \qquad i \in [1:L].$$
(97)

This proves that (48c) is lower bounded by 0.

IV. PROOF OF THEOREM 1

The proof of Theorem 1 is divided into three steps. We first adapt the argument in [3], [4] to show that every rate tuple in $\mathcal{R}(\mathbf{D}_i, i \in [1 : L])$ is achievable, i.e., $\mathcal{R}(\mathbf{D}_i, i \in [1 : L]) \subseteq \mathcal{R}^*(\mathbf{D}_i, i \in [1 : L])$. We then study the supporting hyperplanes of $\mathcal{R}(\mathbf{D}_i, i \in [1 : L])$ and characterize the optimal solution of the relevant minimization problem via KKT analysis. Finally we derive a matching converse by leveraging the extremal inequality in Theorem 2.

A. Achievability

It is easy to adapt the achevability argument in [3], [4] to prove the following result.

Lemma 1: $(R_i, i \in [1 : L]) \in \mathcal{R}^*(\mathcal{D}_i, i \in [1 : L])$ if there exist auxiliary random vectors $(W_i, i \in [1 : L])$ jointly Gaussian with $(X, Y_i, i \in [1 : L])$ satisfying

• the Markov chain constraint

$$(W_i, i \in [1:L]) \to X \to Y_L \to Y_{L-1} \to \dots \to Y_1, \tag{98}$$

• the rate constraints

$$R_1 \ge I(X; W_1 | Y_1),$$
(99)

$$\sum_{j=1}^{i} R_j \ge I(X; W_1 | Y_1) + \sum_{j=2}^{i} I(X; W_j | W_{j-1}, \dots, W_1, Y_j), \qquad i \in [2:L],$$
(100)

• the covariance distortion constraints

$$\operatorname{cov}(X|Y_i, W_j, j \in [1:i]) \preceq \boldsymbol{D}_i, \qquad i \in [1:L].$$

$$(101)$$

Equipped with Lemma 1, we proceed to show that every rate tuple in $\mathcal{R}(D_i, i \in [1 : L])$ is achievable. First choose auxiliary Gaussian random vectors $(W_i, i \in [1 : L])$ such that

$$\operatorname{cov}(X|W_j, j \in [1:i]) = \left(\boldsymbol{K}_0^{-1} + \sum_{j=1}^i \boldsymbol{B}_j\right)^{-1}, \qquad i \in [1:L].$$
(102)

It can be verified that

$$h(X|Y_{i}, W_{j}, j \in [1:i]) = \frac{1}{2} \log \left| (2\pi e) \left(\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j} \right)^{-1} \right|, \qquad i \in [1:L],$$

$$h(X|Y_{i+1}|W_{i}| i \in [1:i])$$

$$(103)$$

$$= \frac{1}{2} \log \left| (2\pi e) \left(\mathbf{K}_0^{-1} + \mathbf{K}_{i+1}^{-1} + \sum_{j=1}^i \mathbf{B}_j \right)^{-1} \right|, \quad i \in [1:L-1].$$
(104)

Moreover, we have

$$h(X|Y_i) = h(X|X + N_i) = \frac{1}{2} \log \left| (2\pi e) \left(\mathbf{K}_0^{-1} + \mathbf{K}_i^{-1} \right)^{-1} \right|, \qquad i \in [1:L],$$
(105)

$$\operatorname{cov}\left(X|Y_{i}, W_{j}, j \in [1:i]\right) = \left(\boldsymbol{K}_{0}^{-1} + \boldsymbol{K}_{i}^{-1} + \sum_{j=1}^{i} \boldsymbol{B}_{j}\right)^{-1}, \qquad i \in [1:L].$$

$$(106)$$

Now one can readily prove $\mathcal{R}(\mathbf{D}_i, i \in [1:L]) \subseteq \mathcal{R}^*(\mathbf{D}_i, i \in [1:L])$ by invoking Lemma 1 and a timesharing argument.

B. Supporting Hyperplane Characterization

Since $\mathcal{R}(\mathbf{D}_i, i \in [1 : L])$ is convex, it is completely specified by its supporting hyperplanes. The characterization of the supporting hyperplanes boils down to solving the following optimization problem

$$R^* \triangleq \inf_{(R_1,...,R_L) \in \mathcal{R}(\mathbf{D}_i, i \in [1:L])} \sum_{i=1}^{L} \mu_i R_i,$$
(107)

where $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_L \ge 0$. It is clear that

$$R^{*} = \min_{(\mathbf{B}_{i}, i \in [1:L])} \frac{\mu_{1}}{2} \log \frac{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{1}^{-1} + \mathbf{B}_{1}|}{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{1}^{-1}|} + \sum_{i=2}^{L} \frac{\mu_{i}}{2} \log \frac{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j}|}{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i-1} \mathbf{B}_{j}|}$$
(108)
subject to $\mathbf{B}_{i} \succeq \mathbf{0}, \qquad i \in [1; L],$
$$\sum_{j=1}^{i} \mathbf{B}_{j} \succeq \mathbf{D}_{i}^{-1} - \mathbf{K}_{0}^{-1} - \mathbf{K}_{i}^{-1}, \qquad i \in [1; L].$$

Theorem 3: The minimizer $(\mathbf{B}_i^*, i \in [1:L])$ of (108) must satisfy

$$\frac{\mu_i}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_i^{-1} + \sum_{j=1}^i \boldsymbol{B}_j^* \right)^{-1} - \frac{\mu_{i+1}}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_{i+1}^{-1} + \sum_{j=1}^i \boldsymbol{B}_j^* \right)^{-1} = \boldsymbol{\Psi}_i - \boldsymbol{\Psi}_{i+1} + \boldsymbol{\Lambda}_i, \qquad i \in [1:L-1], \quad (109)$$
$$\frac{\mu_L}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_1^{-1} + \sum_{j=1}^L \boldsymbol{B}_j^* \right)^{-1} = \boldsymbol{\Psi}_I + \boldsymbol{\Lambda}_I, \qquad (110)$$

$$\frac{\iota_L}{2} \left(\boldsymbol{K}_0^{-1} + \boldsymbol{K}_L^{-1} + \sum_{j=1}^L \boldsymbol{B}_j^* \right) = \boldsymbol{\Psi}_L + \boldsymbol{\Lambda}_L, \qquad (110)$$

for some positive semi-definite matrices $(\Psi_i, i \in [1:L])$ and $(\Lambda_i, i \in [1:L])$ such that

 $\boldsymbol{B}_{i}^{*}\boldsymbol{\Psi}_{i} = \boldsymbol{0}, \qquad i \in [1:L], \tag{111}$

$$\left(\boldsymbol{K}_{0}^{-1} + \boldsymbol{K}_{i}^{-1} + \sum_{j=1}^{i} \boldsymbol{B}_{j}^{*} - \boldsymbol{D}_{i}^{-1}\right) \boldsymbol{\Lambda}_{i} = \boldsymbol{0}, \qquad i \in [1:L].$$
(112)

Proof: The Lagrangian of (108) is given by

$$\frac{\mu_{1}}{2} \log \frac{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{1}^{-1} + \mathbf{B}_{1}|}{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{1}^{-1}|} + \sum_{i=2}^{L} \frac{\mu_{i}}{2} \log \frac{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j}|}{|\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i-1} \mathbf{B}_{j}|} - \sum_{i=1}^{L} \operatorname{tr}\{\mathbf{B}_{i} \Psi_{i} + (\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} - \mathbf{D}_{i}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j}) \Lambda_{i}\},$$
(113)

where positive semi-definite matrices (Ψ_i , $i \in [1 : L]$) and (Λ_i , $i \in [1 : L]$) serve as Lagrange multipliers. Note that (109)-(112) follow directly form the KKT conditions. The proof is complete by verifying a set of constraint qualifications in [36, Sections 4-5].

Remark 3: It is worth noting that (109)-(112) in Theorem 3 correspond exactly to (13)-(16) in Theorem 2.

C. Converse

It is easy to adapt the converse argument in [3], [4] to prove the following result.

Lemma 2: For any $(R_i, i \in [1 : L]) \in \mathcal{R}^*(\mathcal{D}_i, i \in [1 : L])$ and any $\epsilon > 0$, there exist auxiliary random objects jointly distributed with $(X, Y_i, i \in [1 : L])$ satisfying

• the Markov chain constraint

$$(W_i, i \in [1:L]) \to X \to Y_L \to Y_{L-1} \to \dots \to Y_1, \tag{114}$$

• the rate constraints

$$R_1 + \epsilon \ge I(X; W_1 | Y_1), \tag{115}$$

$$\sum_{j=1}^{i} (R_j + \epsilon) \ge I(X; W_1 | Y_1) + \sum_{j=2}^{i} I(X; W_j | W_{j-1}, \dots, W_1, Y_j), \qquad i \in [2:L],$$
(116)

• the covariance distortion constraints

$$\operatorname{cov}(X|Y_i, W_j, j \in [1:i]) \preceq \boldsymbol{D}_i + \epsilon \mathbf{I}, \qquad i \in [1:L].$$
(117)

Now we proceed to show that $\mathcal{R}^*(\mathbf{D}_i, i \in [1:L]) \subseteq \mathcal{R}(\mathbf{D}_i, i \in [1:L])$. For any $(R_1, \ldots, R_L) \in \mathcal{R}^*(\mathbf{D}_i, i \in [1:L])$ and any $\epsilon > 0$, it follows by Lemma 2, Theorem 3, and Theorem 2 that

$$\sum_{i=1}^{L} \mu_{i}(R_{i} + \epsilon)$$

$$\geq \mu_{1}I(X; W_{1}|Y_{1}) + \sum_{j=2}^{i} \mu_{i}I(X; W_{j}|W_{j-1}, \dots, W_{1}, Y_{j})$$

$$= \mu_{1}h(X|Y_{1}) + \sum_{i=1}^{L-1} (\mu_{i}h(Y_{i}|W_{j}, j \in [1:i]) - \mu_{i+1}h(Y_{i+1}|W_{j}, j \in [1:i]) - (\mu_{i} - \mu_{i+1})h(X|W_{j}, j \in [1:i]))$$

$$+ \mu_{L}h(Y_{L}|W_{j}, j \in [1:L]) - \mu_{L}h(X|W_{j}, j \in [1:L])$$

$$\geq -\frac{\mu_{1}}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} \right) \right| + \sum_{i=1}^{L-1} \left(-\frac{\mu_{i+1}}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_{0}^{-1} + \mathbf{K}_{i+1}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j}^{*}(\epsilon) \right) \right|$$

$$+ \frac{\mu_{i}}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} + \sum_{j=1}^{i} \mathbf{B}_{j}^{*}(\epsilon) \right) \right| \right) + \frac{\mu_{L}}{2} \log \left| (2\pi e)^{-1} \left(\mathbf{K}_{0}^{-1} + \mathbf{K}_{L}^{-1} + \sum_{j=1}^{L} \mathbf{B}_{j}^{*}(\epsilon) \right) \right|, \quad (120)$$

where $(\mathbf{B}_i^*(\epsilon), i \in [1:L])$ denotes the minimizer of (108) with $(\mathbf{D}_i, i \in [1:L])$ replaced by $(\mathbf{D}_i + \epsilon \mathbf{I}, i \in [1:L])$. Now one can readily show

$$\sum_{i=1}^{L} \mu_i R_i \ge R^* \tag{121}$$

via a simple limiting argument. This completes the proof of Theorem 1.

V. CONCLUSION

We have studied the problem of successive refinement for Wyner-Ziv coding with degraded side information and obtained a computable characterization of the rate region in the quadratic vector Gaussian setting. From the technical perspective, our main contribution is a new extremal inequality, which is established via a refined monotone path argument inspired by the doubling trick in [30]. In a recent paper [37], Unal and Wagner considered the vector Gaussian Heegard-Berger/Kaspi problem with no degradedness assumption on side information and obtained several conclusive results through careful comparisons of the relevant covariance distortions. In contrast, our proof technique does not require such comparisons and thus is potentially better suited to the non-degraded side information case. It is of considerable interest to investigate whether this technique can yield new results beyond those in [37].

APPENDIX A

PRELIMINARIES ON FISHER INFORMATION AND MMSE

Here is a summary of some basic properties of Fisher information and MMSE, which will be used extensively in the proof of extremal inequality (19).

We begin with the definition of conditional Fisher information matrix and MMSE matrix.

Definition 2: Let (X, U) be a pair of jointly distributed random vectors with differentiable conditional probability density function:

$$f(\boldsymbol{x}|u) \triangleq f(x_i, i \in [1:m]|u). \tag{122}$$

The vector-valued score function is defined as

$$\nabla \log f(\boldsymbol{x}|u) = \left[\frac{\partial \log f(\boldsymbol{x}|u)}{\partial x_1}, \cdots, \frac{\partial \log f(\boldsymbol{x}|u)}{\partial x_m}\right]^T.$$
(123)

The conditional Fisher information of X respect to U is given by

$$J(X|U) = \mathbb{E}\left[\left(\nabla \log f(\boldsymbol{x}|u)\right) \cdot \left(\nabla \log f(\boldsymbol{x}|u)\right)^{T}\right].$$
(124)

Definition 3: Let (X, Y, U) be a set of jointly distributed random vectors. The conditional covariance matrix of X given (Y, U) is defined as

$$\operatorname{cov}(X|Y,U) = \mathbb{E}\left[\left(X - \mathbb{E}[X|Y,U]\right) \cdot \left(X - \mathbb{E}[X|Y,U]\right)^{T}\right].$$
(125)

Lemma 3 (Matrix Version of de Bruijn's Identity): Let (X, U) be a pair of jointly distributed random vectors, and $N \sim N(\mathbf{0}, \mathbf{\Sigma})$ be a Gaussian random vector independent of (X, U). Then

$$\nabla_{\Sigma} h(X+N|U) = \frac{1}{2}J(X+N|U).$$
(126)

Lemma 3 is a conditional version of [38, Theorem 1], which provides a link between differential entropy and Fisher information.

Lemma 4: Let (X,U) be a pair of jointly distributed random vectors, and $N \sim \mathcal{N}(\mathbf{0}, \Sigma)$ be a Gaussian random vector independent of (X,U). Then

$$J(X+N|U) + \boldsymbol{\Sigma}^{-1} \operatorname{cov}(X|X+N,U)\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1}.$$
(127)

The complementary identity in Lemma 4 provides a link between Fisher information and MMSE, and its proof can be found in [38, Corollary 1].

Lemma 5: Let (X, Y, U) be a set of jointly distributed random vectors. Assume that X and Y are conditionally independent given U. Then for any square matrix A and B,

$$(\boldsymbol{A} + \boldsymbol{B})J(X + Y|U)(\boldsymbol{A} + \boldsymbol{B})^T \leq \boldsymbol{A}J(X|U)\boldsymbol{A}^T + \boldsymbol{B}J(Y|U)\boldsymbol{B}^T.$$
(128)

Proof: From the conditional version of matrix Fisher information inequality in [39, Appendix II], we have

$$J(X+Y|U) \preceq \mathbf{K}J(X|U)\mathbf{K}^{T} + (\mathbf{I}-\mathbf{K})J(Y|U)(\mathbf{I}-\mathbf{K})^{T},$$
(129)

for any square matrix K. Setting

$$\boldsymbol{K} = (\boldsymbol{A} + \boldsymbol{B})^{-1} \boldsymbol{A} \tag{130}$$

proves (128).

Lemma 6: Let X be a Gaussian random vector and U be an arbitrary random vector. Let N_1 and N_2 be two zero-mean Gaussian random vectors, independent of (X, U), with covariance matrices Σ_1 and Σ_2 , respectively. If

$$\Sigma_2 \succ \Sigma_1 \succ \mathbf{0},\tag{131}$$

then

$$\cos(X|X+N_1,U)^{-1} - \Sigma_1^{-1} \succeq \cos(X|X+N_2,U)^{-1} - \Sigma_2^{-1}.$$
(132)

Lemma 6 can be proved by combining the Cramér-Rao inequality and the complementary identity in Lemma 4. See [31, Lemma 4] for details.

Lemma 7 (Data Processing Inequality for Fisher Information): Let (X, U, V) be a set of jointly distributed random vectors. Assume that $U \to V \to X$ form a Markov chain. Then

$$J(X|U) \preceq J(X|V). \tag{133}$$

Lemma 7 is analogous to [40, Lemma 3], and can be easily proved using the chain rule of Fisher information matrix [40, Lemma 1].

Lemma 8 (Data Processing Inequality for MMSE): Let (X, U, V) be a set of jointly distributed random vectors. Assume $U \rightarrow V \rightarrow X$ form a Markov chain. Then

$$\operatorname{cov}(X|U) \succeq \operatorname{cov}(X|V).$$
 (134)

See [41, Proposition 5] for a detailed proof of Lemma 8.

APPENDIX B

DERIVATIVE OF THE BIVARIATE DIFFERENTIAL ENTROPY $h(X_{i,\gamma}, Y_{i,\gamma}^*|W_j, j \in [1:i])$ In view of (27) and (30), we have

$$h\left(X_{i,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]\right)$$

$$=h\left(\sqrt{1-\gamma}X+\sqrt{\gamma}X_{i}^{G},\sqrt{\gamma}Y_{i}-\sqrt{1-\gamma}Y_{i}^{G}|W_{j},j\in[1:i]\right)$$
(136)

$$=h\left(X+\sqrt{\frac{\gamma}{1-\gamma}}X_i^G, Y_i-\sqrt{\frac{1-\gamma}{\gamma}}Y_i^G\middle|W_j, j\in[1:i]\right)+\frac{n}{2}\log\gamma+\frac{n}{2}\log(1-\gamma).$$
(137)

Recall from (23) that

$$Y_i^G = X_i^G + N_i^G. aga{138}$$

The covariance matrix of

$$\left(\begin{array}{c} \sqrt{\gamma/(1-\gamma)}X_i^G\\ -\sqrt{(1-\gamma)/\gamma}Y_i^G \end{array}\right)$$

is given by

$$\boldsymbol{\Sigma}_{i,*} \triangleq \begin{pmatrix} \frac{\gamma}{1-\gamma} \boldsymbol{\Delta}_i & -\boldsymbol{\Delta}_i \\ -\boldsymbol{\Delta}_i & \frac{1-\gamma}{\gamma} (\boldsymbol{\Delta}_i + \boldsymbol{K}_i) \end{pmatrix}.$$
(139)

It is easy to verify that

$$\boldsymbol{\Sigma}_{i,*}^{-1} = \begin{pmatrix} \frac{1-\gamma}{\gamma} (\boldsymbol{\Delta}_i^{-1} + \boldsymbol{K}_i^{-1}) & \boldsymbol{K}_i^{-1} \\ \boldsymbol{K}_i^{-1} & \frac{\gamma}{1-\gamma} \boldsymbol{K}_i^{-1} \end{pmatrix}$$
(140)

and

$$\nabla_{\gamma} \boldsymbol{\Sigma}_{i,*} = \begin{pmatrix} \frac{1}{(1-\gamma)^2} \boldsymbol{\Delta}_i & \mathbf{0} \\ \mathbf{0} & -\frac{1}{\gamma^2} (\boldsymbol{\Delta}_i + \boldsymbol{K}_i) \end{pmatrix}.$$
 (141)

Combining (140) and (141) gives

$$\operatorname{tr}\left\{\left(\nabla_{\gamma}\boldsymbol{\Sigma}_{i,*}\right)\boldsymbol{\Sigma}_{i,*}^{-1}\right\} = 0,\tag{142}$$

$$\Sigma_{i,*}^{-1} \left(\nabla_{\gamma} \Sigma_{i,*} \right) \Sigma_{i,*}^{-1} = \begin{pmatrix} -\frac{1}{\gamma^2} (\Delta_i^{-1} + K_i^{-1}) & \mathbf{0} \\ \mathbf{0} & \frac{1}{(1-\gamma)^2} K_i^{-1} \end{pmatrix}.$$
 (143)

By invoking the chain rule of matrix calculus and Lemma 3 in Appendix A, we have

$$\frac{d}{d\gamma}h\left(X_{i,\gamma}, Y_{i,\gamma}^{*}|W_{j}, j \in [1:i]\right) = \frac{d}{d\gamma}\left\{h\left(X + \sqrt{\frac{\gamma}{1-\gamma}}X_{i}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}}Y_{i}^{G}\middle|W_{j}, j \in [1:i]\right) + \frac{n}{2}\log\gamma + \frac{n}{2}\log(1-\gamma)\right\}$$
(144)

$$= \frac{1}{2} \operatorname{tr} \left\{ \left(\nabla_{\gamma} \boldsymbol{\Sigma}_{i,*} \right) J \left(\left(\sqrt{\frac{1}{1-\gamma}} X_{i,\gamma}^T \quad \sqrt{\frac{1}{\gamma}} Y_{i,\gamma}^{*T} \right)^T \middle| W_j, j \in [1:i] \right) \right\} + \frac{n}{2} \left(\frac{1}{\gamma} - \frac{1}{1-\gamma} \right).$$

$$(145)$$

(135)

It can be verified

$$\operatorname{tr}\left\{ \left(\nabla_{\gamma} \boldsymbol{\Sigma}_{i,*}\right) J\left(\left(\sqrt{\frac{1}{1-\gamma}} X_{i,\gamma}^{T} \quad \sqrt{\frac{1}{\gamma}} Y_{i,\gamma}^{*T} \right)^{T} \middle| W_{j}, j \in [1:i] \right) \right\}$$

$$= \operatorname{tr}\left\{ \left(\nabla_{\gamma} \boldsymbol{\Sigma}_{i,*}\right) \boldsymbol{\Sigma}_{i,*}^{-1} - \boldsymbol{\Sigma}_{i,*}^{-1} \left(\nabla_{\gamma} \boldsymbol{\Sigma}_{i,*}\right) \boldsymbol{\Sigma}_{i,*}^{-1} \right.$$

$$\operatorname{cov}\left(\left(X^{T} \quad Y_{i}^{T} \right)^{T} \middle| X + \sqrt{\frac{\gamma}{1-\gamma}} X_{i}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}} Y_{i}^{G}, W_{j}, j \in [1:i] \right) \right\}$$

$$= \operatorname{tr}\left\{ \left(-\frac{1}{\gamma^{2}} (\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}) \quad \mathbf{0} \right) \right\}$$

$$(146)$$

$$= \operatorname{tr} \left\{ \begin{pmatrix} \gamma^{2} \left(\boldsymbol{\Delta}_{i}^{T} + \boldsymbol{K}_{i}^{T} \right)^{T} \boldsymbol{\nabla}_{i}^{T} \\ \mathbf{0} & \frac{1}{(1-\gamma)^{2}} \boldsymbol{K}_{i}^{-1} \end{pmatrix} \right. \\ \left. \operatorname{cov} \left(\begin{pmatrix} \left(\boldsymbol{X}^{T} + \boldsymbol{Y}_{i}^{T} \right)^{T} \middle| \boldsymbol{X} + \sqrt{\frac{\gamma}{1-\gamma}} \boldsymbol{X}_{i}^{G}, \boldsymbol{Y}_{i} - \sqrt{\frac{1-\gamma}{\gamma}} \boldsymbol{Y}_{i}^{G}, \boldsymbol{W}_{j}, j \in [1:i] \end{pmatrix} \right\},$$
(147)

where (146) follows by Lemma 4 in Appendix A, and (147) is due to (142) and (143). Notice that

$$\operatorname{cov}\left(\begin{pmatrix} X^{T} & Y_{i}^{T} \end{pmatrix}^{T} \middle| X + \sqrt{\frac{\gamma}{1-\gamma}} X_{i}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}} Y_{i}^{G} \right)$$
$$= \left(\begin{pmatrix} \mathbf{K}_{0} & \mathbf{K}_{0} \\ \mathbf{K}_{0} & \mathbf{K}_{0} + \mathbf{K}_{i} \end{pmatrix}^{-1} + \mathbf{\Sigma}_{i,*}^{-1} \end{pmatrix}^{-1}$$
(148)

$$= \left(\begin{pmatrix} \mathbf{K}_{0}^{-1} + \mathbf{K}_{i}^{-1} & -\mathbf{K}_{i}^{-1} \\ -\mathbf{K}_{i}^{-1} & \mathbf{K}_{i}^{-1} \end{pmatrix} + \begin{pmatrix} \frac{1-\gamma}{\gamma} (\mathbf{\Delta}_{i}^{-1} + \mathbf{K}_{i}^{-1}) & \mathbf{K}_{i}^{-1} \\ \mathbf{K}_{i}^{-1} & \frac{\gamma}{1-\gamma} \mathbf{K}_{i}^{-1} \end{pmatrix} \right)^{-1}$$
(149)

$$= \begin{pmatrix} \left(\boldsymbol{K}_{0}^{-1} + \frac{1-\gamma}{\gamma}\boldsymbol{\Delta}_{i}^{-1} + \frac{1}{\gamma}\boldsymbol{K}_{i}^{-1}\right)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & (1-\gamma)\boldsymbol{K}_{i} \end{pmatrix}.$$
(150)

Thus, we have the Markov chain

$$(W_j, j \in [1:i]) \to X \to \left(X + \sqrt{\frac{\gamma}{1-\gamma}} X_i^G, Y_i - \sqrt{\frac{1-\gamma}{\gamma}} Y_i^G\right) \to Y_i.$$
(151)

As a consequence,

$$\operatorname{cov}\left(\begin{pmatrix} X^{T} & Y_{i}^{T} \end{pmatrix}^{T} \middle| X_{i,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right)$$
$$= \begin{pmatrix} \operatorname{cov}\left(X \middle| X_{i,\gamma}, Y_{i,\gamma}, W_{j}, j \in [1:i] \right) & \mathbf{0} \\ \mathbf{0} & (1-\gamma)\mathbf{K}_{i} \end{pmatrix}.$$
(152)

By combining (145), (147) and (152), we obtain

$$\frac{d}{d\gamma}h\left(X_{i,\gamma}, Y_{i,\gamma}^{*}\middle|W_{j}, j\in[1:i]\right) = \frac{1}{2}\operatorname{tr}\left\{\begin{pmatrix} -\frac{1}{\gamma^{2}}(\boldsymbol{\Delta}_{i}^{-1}+\boldsymbol{K}_{i}^{-1}) & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{(1-\gamma)^{2}}\boldsymbol{K}_{i}^{-1} \end{pmatrix}\begin{pmatrix}\operatorname{cov}\left(X\middle|X_{i,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j\in[1:i]\right) & \boldsymbol{0} \\ \boldsymbol{0} & (1-\gamma)\boldsymbol{K}_{i} \end{pmatrix}\right\} + \frac{n}{2}\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right) \qquad (153)$$

$$= -\frac{1}{2\gamma} \operatorname{tr} \left\{ \frac{1}{\gamma} (\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}) \operatorname{cov} \left(X \middle| X_{i,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) - \boldsymbol{I} \right\}$$
(154)

$$= -\frac{1}{2\gamma} \operatorname{tr} \left\{ (\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}) \left(\frac{1}{\gamma} \operatorname{cov} \left(X \Big| X_{i,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) - (\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1})^{-1} \right) \right\}.$$
(155)

On the other hand, it follows by the theory of linear MMSE estimation that

$$\sqrt{\gamma}X_i^G = -\sqrt{\gamma(1-\gamma)} \left(\boldsymbol{\Delta}_i^{-1} + (1-\gamma)\boldsymbol{K}_i^{-1}\right)^{-1} \boldsymbol{K}_i^{-1} \left(\sqrt{\gamma}N_i - \sqrt{1-\gamma}Y_i^G\right) + \sqrt{\gamma}\hat{X}_i^G,$$
(156)

where $\hat{X}_{i,\gamma}$ is a Gaussian random vector with mean zero and covariance matrix $(\boldsymbol{\Delta}_{i}^{-1} + (1-\gamma)\boldsymbol{K}_{i}^{-1})^{-1}$, and is independent of $\sqrt{\gamma}N_{i} - \sqrt{1-\gamma}Y_{i}^{G}$. Thus, we have

$$X_{i,\gamma} = \sqrt{1 - \gamma} X + \sqrt{\gamma} X_i^G \tag{157}$$

$$=\sqrt{1-\gamma}X - \sqrt{\gamma(1-\gamma)}\left(\boldsymbol{\Delta}_{i}^{-1} + (1-\gamma)\boldsymbol{K}_{i}^{-1}\right)^{-1}\boldsymbol{K}_{i}^{-1}\left(\sqrt{\gamma}N_{i} - \sqrt{1-\gamma}Y_{i}^{G}\right) + \sqrt{\gamma}\hat{X}_{i}^{G}$$
(158)

$$= \sqrt{1 - \gamma} \left(\mathbf{\Delta}_{i}^{-1} + (1 - \gamma) \mathbf{K}_{i}^{-1} \right)^{-1} \left(\mathbf{\Delta}_{i}^{-1} + \mathbf{K}_{i}^{-1} \right) X + \sqrt{\gamma} \hat{X}_{i}^{G} - \sqrt{\gamma(1 - \gamma)} \left(\mathbf{\Delta}_{i}^{-1} + (1 - \gamma) \mathbf{K}_{i}^{-1} \right)^{-1} \mathbf{K}_{i}^{-1} Y_{i,\gamma}^{*}.$$
(159)

The complementary Fisher information representation of $\cos\left(X \middle| X_{i,\gamma}, Y_{i,\gamma}^*, W_j, j \in [1:i]\right)$ can thereby be expressed as

$$\operatorname{cov}\left(X\Big|X_{i,\gamma}, Y_{i,\gamma}^*, W_j, j \in [1:i]\right)$$
(160)

$$= \cos\left(X \left| \sqrt{1 - \gamma} \left(\Delta_{i}^{-1} + (1 - \gamma) \, \boldsymbol{K}_{i}^{-1} \right)^{-1} \left(\Delta_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right) X + \sqrt{\gamma} \hat{X}_{i}^{G}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right)$$
(161)

$$= \frac{\gamma}{1-\gamma} \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right)^{-1} \left(\boldsymbol{\Delta}_{i}^{-1} + (1-\gamma) \, \boldsymbol{K}_{i}^{-1} - \gamma J \left(X_{i,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \right) \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1} \right)^{-1}.$$
(162)

Equivalently, we can write

$$\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}\right) \left(\frac{1}{\gamma} \operatorname{cov}\left(\boldsymbol{X} \middle| \boldsymbol{X}_{i,\gamma}, \tilde{\boldsymbol{Y}}_{i,\gamma}^{*}, \boldsymbol{W}_{j}, j \in [1:L]\right) - \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}\right)^{-1}\right) \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}\right)$$
(163)

$$= \frac{\gamma}{1-\gamma} \boldsymbol{\Delta}_{i}^{-1} - \frac{\gamma}{1-\gamma} J\left(X_{i,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right).$$
(164)

Finally, substituting (164) into (155) gives

$$\frac{d}{d\gamma}h\left(X_{i,\gamma}, Y_{i,\gamma}^{*}\middle|W_{j}, j \in [1:i]\right) = \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}\right)^{-1}\left(J\left(X_{i,\gamma}\middle|Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right) - \boldsymbol{\Delta}_{i}^{-1}\right)\right\}.$$
(165)

APPENDIX C

DERIVATIVE OF THE BIVARIATE DIFFERENTIAL ENTROPY $h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]\right)$

In view of (29) and (30),

$$h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^* | W_j, j \in [1:i]\right) \tag{166}$$

$$=h\left(\sqrt{1-\gamma}Y_{i+1}+\sqrt{\gamma}\tilde{Y}_{i+1}^G,\sqrt{\gamma}Y_i-\sqrt{1-\gamma}Y_i^G\middle|W_j,j\in[1:i]\right)$$
(167)

$$= h\left(Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}}\tilde{Y}_{i+1}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}}Y_{i}^{G}\middle|W_{j}, j \in [1:i]\right) + \frac{n}{2}\log\gamma + \frac{n}{2}\log(1-\gamma).$$
(168)

By the definition of Y_i^G and \tilde{Y}_{i+1}^G in (23) and (24) as well as the construction of $(N_i^G, i \in [1:L])$, we can write

$$Y_i^G = \tilde{Y}_{i+1}^G + \left(N_i^G - N_{i+1}^G\right), \tag{169}$$

where $N_i^G - N_{i+1}^G$ is a Gaussian random vector with covariance matrix $K_i - K_{i+1}$, and is independent of \tilde{Y}_{i+1}^G . Therefore, the covariance matrix of

$$\left(\begin{array}{c} \sqrt{\gamma/(1-\gamma)}\tilde{Y}_{i+1}^{G} \\ -\sqrt{(1-\gamma)/\gamma}Y_{i}^{G} \end{array}\right)$$

is given by

$$\tilde{\Sigma}_{i} \triangleq \begin{pmatrix} \frac{\gamma}{1-\gamma} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i} \right) & -\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i} \right) \\ -\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i} \right) & \frac{1-\gamma}{\gamma} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) \end{pmatrix}.$$
(170)

It can be verified that

$$\tilde{\boldsymbol{\Sigma}}_{i}^{-1} = \begin{pmatrix} \frac{1-\gamma}{\gamma} \left((\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1})^{-1} + (\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1})^{-1} \right) & (\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1})^{-1} \\ (\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1})^{-1} & \frac{\gamma}{1-\gamma} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \end{pmatrix}$$
(171)

and

$$\nabla_{\gamma} \tilde{\boldsymbol{\Sigma}}_{i} = \begin{pmatrix} \frac{1}{(1-\gamma)^{2}} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right) & \boldsymbol{0} \\ \boldsymbol{0} & -\frac{1}{\gamma^{2}} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i} \right) \end{pmatrix}.$$
(172)

Combining (171) and (172) gives

$$\operatorname{tr}\left\{\left(\nabla_{\gamma}\tilde{\boldsymbol{\Sigma}}_{i}\right)\tilde{\boldsymbol{\Sigma}}_{i}^{-1}\right\}=0,$$
(173)

$$\tilde{\Sigma}_{i}^{-1} \left(\nabla_{\gamma} \tilde{\Sigma}_{i} \right) \tilde{\Sigma}_{i}^{-1} = \begin{pmatrix} -\frac{1}{\gamma^{2}} \left(\left(\Delta_{i} + K_{i+1} \right)^{-1} + \left(K_{i} - K_{i+1} \right)^{-1} \right) & \mathbf{0} \\ \mathbf{0} & \frac{1}{(1-\gamma)^{2}} \left(K_{i} - K_{i+1} \right)^{-1} \end{pmatrix}.$$
(174)

By invoking the chain rule of matrix calculus and Lemma 3 in Appendix A, we have

$$\frac{d}{d\gamma}h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}|W_{j}, j \in [1:i]\right)$$

$$= \frac{d}{d\gamma}\left\{h\left(Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}}\tilde{Y}_{i+1}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}}Y_{i}^{G}\middle|W_{j}, j \in [1:i]\right) + \frac{n}{2}\log\gamma + \frac{n}{2}\log(1-\gamma)\right\}$$

$$= \frac{1}{2}\operatorname{tr}\left\{\left(\nabla_{\gamma}\tilde{\Sigma}_{i}\right)J\left(\left(\sqrt{\frac{1}{1-\gamma}}\tilde{Y}_{i+1,\gamma}^{T} \quad \sqrt{\frac{1}{\gamma}}Y_{i,\gamma}^{*}\right)^{T}\middle|W_{j}, j \in [1:i]\right)\right\} + \frac{n}{2}\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)$$
(176)

It can be verified that

$$\operatorname{tr}\left\{\left(\nabla_{\gamma}\tilde{\boldsymbol{\Sigma}}_{i}\right)J\left(\left(\sqrt{\frac{1}{1-\gamma}}\tilde{Y}_{i+1,\gamma}^{T} \quad \sqrt{\frac{1}{\gamma}}{Y}_{i,\gamma}^{*}^{T}\right)^{T} \middle| W_{j}, j \in [1:i]\right)\right\}$$

$$=\operatorname{tr}\left\{\left(\nabla_{\gamma}\tilde{\boldsymbol{\Sigma}}_{i}\right)\tilde{\boldsymbol{\Sigma}}_{i}^{-1} - \tilde{\boldsymbol{\Sigma}}_{i}^{-1}\left(\nabla_{\gamma}\tilde{\boldsymbol{\Sigma}}_{i}\right)\tilde{\boldsymbol{\Sigma}}_{i}^{-1}$$

$$\operatorname{cov}\left(\left(Y_{i+1}^{T} \quad Y_{i}^{T}\right)^{T} \middle| Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}}\tilde{Y}_{i+1}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}}Y_{i}^{G}, W_{j}, j \in [1:i]\right)\right\}$$

$$=\operatorname{tr}\left\{\left(\begin{array}{c}-\frac{1}{\gamma^{2}}\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right)\right) \mathbf{0}$$

$$\frac{1}{(1-\gamma)^{2}}\left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right)$$

$$(177)$$

$$\cos\left(\begin{pmatrix} Y_{i+1}^{T} & Y_{i}^{T} \end{pmatrix}^{T} \middle| Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}} \tilde{Y}_{i+1}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}} Y_{i}^{G}, W_{j}, j \in [1:i] \end{pmatrix} \right\},$$
(178)

where (177) follows by Lemma 4 in Appendix A, and (178) is due to (173) and (174). Notice that

$$cov \left(\begin{pmatrix} Y_{i+1}^{T} & Y_{i}^{T} \end{pmatrix}^{T} \middle| Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}} \tilde{Y}_{i+1}^{G}, Y_{i} - \sqrt{\frac{1-\gamma}{\gamma}} Y_{i}^{G} \end{pmatrix} \\
= \left(\begin{pmatrix} K_{0} + K_{i+1} & K_{0} + K_{i+1} \\ K_{0} + K_{i+1} & K_{0} + K_{i} \end{pmatrix}^{-1} + \tilde{\Sigma}_{i}^{-1} \right)^{-1} \\
\left(\begin{pmatrix} (K_{0} + K_{i+1})^{-1} + (K_{i} - K_{i+1})^{-1} & -(K_{i} - K_{i+1})^{-1} \\ -(K_{i} - K_{i+1})^{-1} \end{pmatrix} \right)$$
(179)

$$= \left(\begin{pmatrix} (\mathbf{K}_{0} + \mathbf{K}_{i+1})^{-1} + (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} & -(\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} \\ -(\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} & (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} \end{pmatrix} + \begin{pmatrix} \frac{1-\gamma}{\gamma} \left((\mathbf{\Delta}_{i} + \mathbf{K}_{i+1})^{-1} + (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} \right) & (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} \\ (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} & \frac{\gamma}{1-\gamma} \left(\mathbf{K}_{i} - \mathbf{K}_{i+1} \right)^{-1} \end{pmatrix} \right)^{-1}$$
(180)

$$= \left(\begin{pmatrix} (\mathbf{K}_{0} + \mathbf{K}_{i+1})^{-1} + \frac{1-\gamma}{\gamma} (\mathbf{\Delta}_{i} + \mathbf{K}_{i+1})^{-1} + \frac{1}{\gamma} (\mathbf{K}_{i} - \mathbf{K}_{i+1})^{-1} \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ (1-\gamma) (\mathbf{K}_{i} - \mathbf{K}_{i+1}) \end{pmatrix} \right).$$
(181)

Thus, we have the Markov chain

$$(W_j, j \in [1:i]) \to Y_{i+1} \to \left(Y_{i+1} + \sqrt{\frac{\gamma}{1-\gamma}} \tilde{Y}_{i+1}^G, Y_i - \sqrt{\frac{1-\gamma}{\gamma}} Y_i^G\right) \to Y_i.$$

$$(182)$$

As a consequence,

$$cov\left(\begin{pmatrix} Y_{i+1}^{T} & Y_{i}^{T} \end{pmatrix}^{T} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) \\
= \begin{pmatrix} cov\left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) & \mathbf{0} \\ \mathbf{0} & (1-\gamma)(\mathbf{K}_{i} - \mathbf{K}_{i+1}) \end{pmatrix}.$$
(183)

Combining (176), (178) and (183), we obtain

$$\frac{d}{d\gamma}h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}\middle|W_{j}, j \in [1:i]\right) = \frac{1}{2}\operatorname{tr}\left\{\begin{pmatrix} -\frac{1}{\gamma^{2}}\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right) & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{(1-\gamma)^{2}}\left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right) \\ \begin{pmatrix} \operatorname{cov}\left(Y_{i+1}\middle|\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right) & \boldsymbol{0} \\ \boldsymbol{0} & (1-\gamma)(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}) \end{pmatrix}\right\} + \frac{n}{2}\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right) \quad (184)$$

$$= -\frac{1}{2\gamma} \operatorname{tr} \left\{ \frac{1}{\gamma} \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right) \operatorname{cov} \left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) - \boldsymbol{I} \right\}.$$
(185)

On the other hand, it follows by the theory of linear MMSE estimation that

$$\sqrt{\gamma} \tilde{Y}_{i+1}^{G} = -\sqrt{\gamma(1-\gamma)} \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + (1-\gamma) \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \left(\sqrt{\gamma} N_{i} - \sqrt{\gamma} N_{i+1} - \sqrt{1-\gamma} Y_{i}^{G} \right) + \sqrt{\gamma} \hat{Y}_{i+1}^{G},$$
(186)

where $\hat{Y}_{i+1,\gamma}$ is a Gaussian random vector with mean zero and covariance matrix $\left((\Delta_i + K_{i+1})^{-1} + (1-\gamma) (K_i - K_{i+1})^{-1} \right)^{-1}$, and is independent of $\sqrt{\gamma} (N_i - N_{i+1}) - \sqrt{1-\gamma} Y_i^G$. Thus, we have $\tilde{Y}_{i+1} = \sqrt{1-\gamma} Y_{i+1} + \sqrt{\gamma} \tilde{Y}_{i+1}^G$ $= \sqrt{1-\gamma} Y_{i+1} - \sqrt{\gamma(1-\gamma)} \left((\Delta_i + K_{i+1})^{-1} + (1-\gamma) (K_i - K_{i+1})^{-1} \right)^{-1} (K_i - K_{i+1})^{-1}$ $\left(\sqrt{\gamma} N_i - \sqrt{\gamma} N_{i+1} - \sqrt{1-\gamma} Y_i^G \right) + \sqrt{\gamma} \hat{Y}_{i+1}^G$ $= \sqrt{1-\gamma} \left((\Delta_i + K_{i+1})^{-1} + (1-\gamma) (K_i - K_{i+1})^{-1} \right)^{-1} \left((\Delta_i + K_{i+1})^{-1} + (K_i - K_{i+1})^{-1} \right) Y_{i+1}$ $+ \sqrt{\gamma} \hat{Y}_{i+1}^G - \sqrt{\gamma(1-\gamma)} \left((\Delta_i + K_{i+1})^{-1} + (1-\gamma) (K_i - K_{i+1})^{-1} \right)^{-1} (K_i - K_{i+1})^{-1} Y_{i,\gamma}^*.$ (188)

The complementary Fisher information representation of $\operatorname{cov}\left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^*, W_j, j \in [1:i]\right)$ can be thereby expressed as

$$cov\left(Y_{i+1}\Big|\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right) = \frac{\gamma}{1-\gamma}\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right)^{-1}\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + (1-\gamma)\left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1} - \gamma J\left(\tilde{Y}_{i+1,\gamma}\Big|Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right)\right)\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1}\right)^{-1}\right)^{-1}.$$
(189)

Equivalently, we can write

$$\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right) \left(\frac{1}{\gamma} \operatorname{cov} \left(Y_{i+1} \middle| \tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right) - \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1} \right) \left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right) = \frac{\gamma}{1 - \gamma} \left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} - \frac{\gamma}{1 - \gamma} J \left(\tilde{Y}_{i+1,\gamma} \middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i] \right).$$
(190)

Substituting (190) into (185) gives

$$\frac{d}{d\gamma}h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*}\middle|W_{j}, j\in[1:i]\right) = \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\left(\boldsymbol{\Delta}_{i}+\boldsymbol{K}_{i+1}\right)^{-1}+\left(\boldsymbol{K}_{i}-\boldsymbol{K}_{i+1}\right)^{-1}\right)^{-1}\left(J\left(\tilde{Y}_{i+1,\gamma}\middle|Y_{i,\gamma}^{*}, W_{j}, j\in[1:i]\right)-\left(\boldsymbol{\Delta}_{i}+\boldsymbol{K}_{i+1}\right)^{-1}\right)\right\}. \quad (191)$$

Furthermore, it follows by the Woodbury matrix inversion lemma that

$$\left(\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1} \right)^{-1} + \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \right)^{-1} - \boldsymbol{K}_{i+1} \left(\boldsymbol{K}_{i} - \boldsymbol{K}_{i+1} \right)^{-1} \boldsymbol{K}_{i+1} - \boldsymbol{K}_{i+1} \left(\boldsymbol{\Lambda}_{i+1} + \boldsymbol{K}_{i+1} \right)^{-1} \boldsymbol{K}_{i+1} \right)^{-1} \boldsymbol{K}_{i+1}$$
(102)

$$= K_{i+1} \left(K_{i+1} - K_{i+1} \left(K_{i+1} - K_{i} \right)^{-1} K_{i+1} - K_{i+1} + K_{i+1} \left(\Delta_{i} + K_{i+1} \right)^{-1} K_{i+1} \right) \quad K_{i+1}$$
(192)

$$= K_{i+1} \left(\left(K_{i+1}^{-1} - K_{i}^{-1} \right)^{-1} - \left(\Delta_{i}^{-1} + K_{i+1}^{-1} \right)^{-1} \right)^{-1} K_{i+1}$$
(193)

$$= K_{i+1} \left(\Delta_i^{-1} + K_{i+1}^{-1} \right) \left(\left(\Delta_i^{-1} + K_i^{-1} \right)^{-1} - \left(\Delta_i^{-1} + K_{i+1}^{-1} \right)^{-1} \right) \left(\Delta_i^{-1} + K_{i+1}^{-1} \right) K_{i+1}.$$
(194)

So we can rewrite (191) as

$$\frac{d}{d\gamma}h\left(\tilde{Y}_{i+1,\gamma}, Y_{i,\gamma}^{*} \middle| W_{j}, j \in [1:i]\right)
= \frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i}^{-1}\right)^{-1} - \left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1}\right)^{-1}\right)\left(\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1}\right)\boldsymbol{K}_{i+1}\right)
J\left(\tilde{Y}_{i+1,\gamma}\middle| Y_{i,\gamma}^{*}, W_{j}, j \in [1:i]\right)\boldsymbol{K}_{i+1}\left(\boldsymbol{\Delta}_{i}^{-1} + \boldsymbol{K}_{i+1}^{-1}\right) - \boldsymbol{\Delta}_{i}^{-1}\left(\boldsymbol{\Delta}_{i} + \boldsymbol{K}_{i+1}\right)\boldsymbol{\Delta}_{i}^{-1}\right)\right\}.$$
(195)

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