

# Achievable Rate Region for Iterative Multi-User Detection via Low-cost Gaussian Approximation

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**Abstract**—We establish a multi-user extrinsic information transfer (EXIT) chart area theorem for the interleave-division multiple-access (IDMA) scheme, a special form of superposition coding, in multiple access channels (MACs). A low-cost multi-user detection (MUD) based on the Gaussian approximation (GA) is assumed. The evolution of mean-square errors (MSE) of the GA-based MUD during iterative processing is studied. We show that the  $K$ -dimensional tuples formed by the MSEs of  $K$  users constitute a conservative vector field. The achievable rate is a potential function of this conservative field, so it is the integral along any path in the field with value of the integral solely determined by the two path terminals. Optimized codes can be found given the integration paths in the MSE fields by matching EXIT type functions. The above findings imply that i) low-cost GA-based MUD can provide near capacity performance; ii) the sum-rate capacity (region) can be achieved independently of the integration path in the MSE fields; and iii) the integration path can be an extra degree of freedom for code design.

## I. INTRODUCTION

Theoretically, successive interference cancellation (SIC) together with time-sharing or rate-splitting can achieve the entire capacity region [1]. SIC involves subtraction of successfully detected signals. If practical forward error control (FEC) codes are used, each subtraction incurs an extra overhead in terms of either power or rate relative to an ideal capacity-achieving code [2, Fig. 13.3]. Such overheads accumulate during SIC steps, moving it away from the capacity limit particularly when the number of users is large. Also, both time-sharing and rate-splitting involve segmenting a data frame of a user into several sub-frames. The reduced sub-frame length implies reduced coding gain for a practical turbo or low-density parity-check (LDPC) type code, which further worsens the accumulation of losses.

Iterative detection can alleviate the loss accumulation problem using soft cancellation instead of hard subtraction. A turbo or LDPC code involving iterative detection can be optimized by matching the so-called extrinsic information transfer (EXIT) functions of two local processors [3], [4]. In a single-user point-to-point channel,

such matching can offer near capacity performance, as shown the area properties [5], [6].

Interleave-division multiple-access (IDMA) is a low-cost transmission scheme for MACs [7]. A Gaussian approximation (GA) of cross-user interference is key to a low-cost IDMA detector [8]. For comparison, consider a common *a posteriori* probability (APP) multi-user detector (MUD) [7] and let  $K$  be the number of users. The per-user complexity of a GA-based MUD remains roughly the same for all  $K$ , while that of an APP-based MUD is exponential in  $K$ .

A question naturally arises: At such low cost, what is the achievable performance of IDMA under GA-based MUD? Some partial answers to this question are available. It is shown that IDMA is capacity approaching when all users see the same channel [9]. It is also known that the GA-based MUD can achieve some points in the capacity region for multiple-input multiple-output (MIMO) MACs [10].

This paper provides a comprehensive analysis of the achievable performance of IDMA under GA-based MUD. We approach the problem based on multi-dimensional curve matching of EXIT type functions. Let  $v_k$  be the mean-square error (MSE) (i.e., the variance) for the GA-based MUD for user  $k$ , with  $v_k = 0$  indicating perfect decoding. Using the mutual information (MI) and minimum MSE (MMSE) theorem [11], [6], we show that the achievable sum-rate can be evaluated using a line integral along a valid path in the  $K$ -dimensional vector field  $\mathbf{v} = [v_1, v_2, \dots, v_K]^T$ . Furthermore, the integral is path-independent and its value is solely determined by the two terminations. The path independence property greatly simplifies the code optimization problem. We gain some interesting insights from the discussions in this paper.

- A low-cost GA-based MUD can provide near optimal performance. In particular, it is capacity-achieving for Gaussian signaling.
- FEC codes optimized for single-user channels may

not be good choices for MACs. The FEC codes should be carefully designed to match the GA-based MUD, which facilitates iterative detection. We will provide examples for the related code design.

- A multi-user area theorem of EXIT chart is established for the code design.
- The sum-rate capacity is a potential function in the field formed by  $\mathbf{v}$ , which leads to the path independence property.
- All points of the MAC capacity region are achievable using only one FEC code per user. This avoids the loss related to the frame segmentation of SIC as aforementioned.
- The above results can be extended to MIMO MAC channels straightforwardly.

We will provide simulation results to show that properly designed IDMA can approach the sum-rate MAC capacity for various decoding paths in the MSE field within 1 dB.

## II. ITERATIVE IDMA RECEIVER

Consider a general  $K$ -user MAC system, which is described by

$$y = \sum_{i=1}^K \sqrt{P_i} h_i x_i + n \quad (1)$$

where  $P_i$  denotes the received signal strength of the  $i$ th user's signal,  $h_i$  denotes the fading coefficients of the user,  $x_i$  is the  $i$ th transmit complex-valued signal and  $n$  is the additive (circularly symmetric complex) white Gaussian noise (AWGN) with zero mean and unit variance, i.e.,  $\mathcal{CN}(0, \sigma^2 = 1)$ .

The iterative receiver is depicted in Fig. 1. The

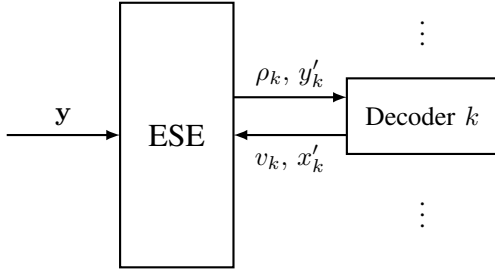


Fig. 1. The iterative multi-user detection and decoding model.

elementary signal estimator (ESE) module has access to the channel observation  $y$  and feedbacks  $x'_k$  from all the users' decoders. It performs the so called soft interference cancellation (SoIC) and provides each decoder a "clean observation", i.e., a signal with reduced interference. The decoders perform its decoding based on the signals  $y'_k$  from ESE.

To guarantee that the signals can be perfectly recovered, properly designed channel codes shall be applied. This can be done by, e.g., EXIT chart based design (see [5]).

### A. ESE functions

The decoder feedbacks  $x'_k$  are characterized by the MSE of its estimates, denoted by  $v_i$ . The "decoder observation" after SoIC is characterized by the signal-to-noise ratio (SNR), denoted by  $\rho_i$ , assuming that the interference is Gaussian-distributed<sup>1</sup>, i.e.,

$$y'_k = \sqrt{\rho_k} x_k + z \quad (2)$$

where  $z$ , comprised of AWGN and multi-user interference, is assumed to be  $\mathcal{CN}(0, 1)$ . This assumption greatly simplifies the multi-user detection, henceforth it is referred to as GA-based MUD. Therefore, the ESE transfer function for the  $i$ th user is given by

$$\rho_k = \frac{P_k |h_k|^2}{\sum_{j=1, j \neq i}^K P_j |h_j|^2 v_j + \sigma^2}, \quad \forall k = 1, 2, \dots, K. \quad (3a)$$

We can also express (3a) in a vector form as

$$\boldsymbol{\rho} = \boldsymbol{\phi}(\mathbf{v}) \quad (3b)$$

where  $\boldsymbol{\rho} = [\rho_1, \rho_2, \dots, \rho_K]^T$  and  $\mathbf{v} = [v_1, v_2, \dots, v_K]^T$ . Due to the fact that the MSE is bounded by  $0 \leq v_i \leq \mathbb{E}[|x_i|^2] = 1$ , we obtain that the SNR is also bounded by

$$\rho_{k, \min} = \frac{P_k |h_k|^2}{\sum_{j=1, j \neq i}^K P_j |h_j|^2 + \sigma^2} \leq \rho_k \leq \frac{P_k |h_k|^2}{\sigma^2} = \rho_{k, \max}. \quad (3c)$$

This bound implies that the single user decoder shall be able to decode its signal before its ESE input reaches the maximum, i.e.,  $\exists \rho'_k \leq \rho_{k, \max}, v_k(\rho'_k) = 0$ .

As the consequence of the iterative processing, we can write the SNR and MSE vector as a function depending on a "time" or "iteration" variable  $t$ , i.e.,  $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ ,  $\mathbf{v} = \mathbf{v}(t)$  and  $\boldsymbol{\rho}(t) = \boldsymbol{\phi}(\mathbf{v}(t))$ .

### B. DEC functions

The decoder (DEC) transfer functions can be characterized by

$$v_k = \psi_k(\rho_k), \quad 0 \leq v_k \leq 1, \quad \forall k = 1, 2, \dots, K. \quad (4a)$$

Similar to (3b), we can write (4a) in a vector form as

$$\mathbf{v} = \boldsymbol{\psi}(\boldsymbol{\rho}). \quad (4b)$$

Note that the boundaries for the uncooperative MAC<sup>2</sup> are given by

$$\mathbf{v} = \boldsymbol{\psi}(\boldsymbol{\rho}(t=0)) = \mathbf{1} \quad (4c)$$

$$\mathbf{v} = \boldsymbol{\psi}(\boldsymbol{\rho}(t=\infty)) = \mathbf{0} \quad (4d)$$

<sup>1</sup>The Gaussian assumption is valid for a large number of users with arbitrary uncorrelated transmit symbols  $x_i$  as the consequence of the central limit theorem, or if the transmit signals  $x_i$  are Gaussian by themselves.

<sup>2</sup>For MAC with cooperative encoders, a more general constraint can be  $\mathbf{1}^T \mathbf{v}(t=0) = K$  and  $v_k \geq 0, \forall k = 1, 2, \dots, K$ .

where  $t$  is a variable which addresses the evolution of the SNR  $\rho$  or MSE  $\mathbf{v}$  through iterative processing. (4c) indicates that no a priori information is present to the ESE at the beginning of iterations. (4d) ensures error-free decoding at the end. The decoders are typically APP decoders, so that the MSEs  $\mathbf{v}$  are also the conditional MMSE, i.e.,  $v_k = \mathbb{E}[|x_k - \mathbb{E}[x_k|x'_k]|^2]$ . Moreover, it is commonly assumed that the symbol estimates after APP decoding can be modeled as an observation from the AWGN channel, i.e.,

$$x'_k = \sqrt{\rho'_k} x_k + w \quad (5)$$

where  $w$  follows  $\mathcal{CN}(0, 1)$ .

### C. Matching condition

The matching codes which allow error-free decoding yet with highest code rate (will be shown in Sec. III-A) shall satisfy

$$\psi(\rho(t)) = \phi^{-1}(\rho(t)). \quad (6)$$

In other words, it is sufficient to match the code components along a  $K$ -dimensional line, which is given by  $\rho(t)$ . It is noteworthy to mention that it is not necessary to match the functions in the entire  $K$ -dimensional space, i.e., requiring  $\psi(\rho) = \phi^{-1}(\rho)$ ,  $\forall \rho$ . Matching along the path given by  $\rho(t)$  is much easier and achieves the MAC capacity (see Sec. III-A).

## III. ACHIEVABLE RATES

The achievable rates under the GAs in (2) and (5) can be written as [11], [6], [12]

$$R_k = \int_0^\infty f(\rho_k + f^{-1}(v_k)) d\rho_k, \quad \forall k = 1, 2, \dots, K. \quad (7a)$$

where  $f(\rho_k) = v_k$  denotes the MMSE.

### A. Gaussian alphabets

We consider Gaussian signals, i.e.,  $x_i$  are Gaussian-distributed which can be achieved by using, e.g., superposition coded modulation (SCM) [13]. Therefore, the MMSE is given by  $f_G(\rho) = \frac{1}{1+\rho}$ , and the achievable rates are

$$R_k = \int_0^\infty \frac{1}{\rho_k + v_k^{-1}} d\rho_k = - \int_{v_k=1}^{v_k=0} \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} dv_k, \quad \forall k = 1, 2, \dots, K \quad (7b)$$

where  $\mathbf{g}^T = [P_1 |h_1|^2, P_2 |h_2|^2, \dots, P_K |h_K|^2]^T$  contains the powers of all users,  $\mathbf{v} = [v_1, v_2, \dots, v_K]^T$  and  $g_k = P_k |h_k|^2$  denotes the  $k$ th element of vector  $\mathbf{g}$ . The derivation is shown in Appendix A.

The sum-rate of all users is

$$R_{\text{sum}} = \sum_{k=1}^K R_k = - \int_{L(t)} \frac{\mathbf{g}}{\mathbf{g}^T \mathbf{v}(t) + \sigma^2} \cdot d\mathbf{v}(t) \quad (8a)$$

where (8a) denotes a line integral defined by  $L = \mathbf{v}(t)$ ,  $t \in [0, \infty]$ . It can be further shown that the integrands constitute a gradient of a scalar field (a.k.a., potential function), i.e.,  $\nabla_{\mathbf{v}} \log(\sigma^2 + \mathbf{g}^T \mathbf{v}) = \frac{\mathbf{g}}{\mathbf{g}^T \mathbf{v} + \sigma^2}$ . Thus, the achievable sum-rate can be written as

$$R_{\text{sum}} = - \int_{L=\mathbf{v}(t)} [\nabla \log(\sigma^2 + \mathbf{g}^T \mathbf{v})] \mathbf{v}'(t) dt \quad (8b)$$

$$\stackrel{(4c),(4d)}{=} \log \left( 1 + \frac{\sum_{k=1}^K P_k |h_k|^2}{\sigma^2} \right)$$

which is independent of the path taken for code matching. In other words, any path with matched DEC functions can achieve the sum-rate capacity. The matching condition given in (6) is thus also proved, since it can be easily verified that  $R_k < - \int_{v_k=1}^{v_k=0} \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} dv_k$  and thus  $R_{\text{sum}} < \log \left( 1 + \frac{\sum_{k=1}^K P_k |h_k|^2}{\sigma^2} \right)$ , if  $\psi(\rho(t)) < \phi^{-1}(\rho(t))$ . On the contrary, if  $\psi(\rho(t)) > \phi^{-1}(\rho(t))$ , error-free decoding is not possible.

This leads to the following theorem.

*Assumptions<sup>3</sup>:*

- 1) Exchanged messages of the *extrinsic* and *a priori* channel are observations from AWGN channels, given in (2) and (5).
- 2) The channel decoder satisfies the matching condition in (6) and has MAP (i.e., APP) performance.

**Theorem 1.** *Under the above assumptions, the achievable sum-rate in IDMA with GA-based MUD for any path  $L(t) : \mathbf{v}_s = \mathbf{1} \rightarrow \mathbf{v}_e = \mathbf{0}$  (starting from  $\mathbf{v}_s = \mathbf{1}$  to  $\mathbf{v}_e = \mathbf{0}$ ) is given by*

$$R_{\text{sum}} = - \int_{L(t)} f_G(\rho(t) + f_G^{-1}(\mathbf{v}(t))) \cdot d\rho(t) \\ = \log \left( 1 + \frac{\sum_{k=1}^K P_k |h_k|^2}{\sigma^2} \right).$$

*Proof:* see above. ■

### B. Finite alphabets

If the symbols  $x_i \in \mathcal{S}_i$  are taken from finite alphabets  $|\mathcal{S}_i| < \infty$ , the capacity formula, in general, can not be expressed in closed-form. However, eq. (7a) is still valid and can be used to evaluate, by numerical integrals, the achievable rates.

We show in [14, Fig. 2] that the loss to Gaussian capacity due to finite modulation can be approached by imposing a larger number of users or data layers, depending on the target sum-rate. There, we also provide numerical results showing that near-capacity performance can be achieved with quadrature phase shift keying (QPSK). Due to space limitation, we refer interested readers to [14, Sec. III-B] for further discussions.

<sup>3</sup>These assumptions have been widely used for turbo-type iterative receivers. It is generally accepted that these assumptions are sufficiently accurate for practical systems.

### C. Example: path vs rate tuples

Consider a simple two-user case, i.e.,  $K = 2$ . Fig. 2 illustrates some special paths and their corresponding achievable rate pairs. The simplest path is a straight line

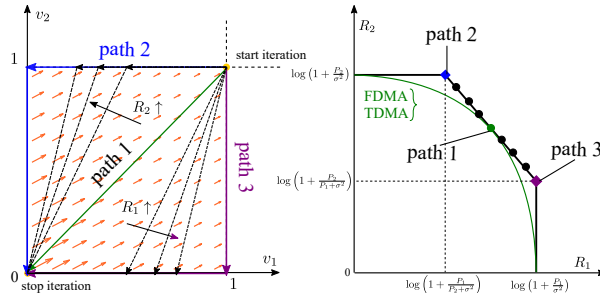


Fig. 2. Illustration (exemplary for two users) of different integration paths achieving different rate pairs  $(R_1, R_2)$ ; the arrows in the left figure illustrate the two-dimensional MSE vector field; the achieved rate pairs are marked in the right figure for the corresponding paths.

between the starting point  $\mathbf{v}(t=0) = \mathbf{1}$  and the stop point  $\mathbf{v}(t=\infty) = \mathbf{0}$ , denoted by *path 1*. It is straightforward to obtain  $R_k = \frac{g_k}{\mathbf{g}^T \mathbf{1}} \log \left( 1 + \frac{\mathbf{g}^T \mathbf{1}}{\sigma^2} \right)$ ,  $\forall k$ . In this case, the achievable rate of each user is proportional to the received signal power strength  $g_k$ . For the two-user case, the rate tuple coincides with the point where TDMA/FDMA achieves the sum-rate capacity. In *path 1*, it satisfies  $v_i(t) = v_j(t)$ ,  $\forall i, j, t$ . The matching code for the  $k$ th user shall have the following MSE characteristic function

$$v_k = \begin{cases} 1 & \rho_k \leq \rho_{k,\min} \\ \frac{1}{\mathbf{g}^T \mathbf{1} - g_k} \cdot \left( \frac{1}{\rho_k} - \sigma^2 \right) & \rho_{k,\min} \leq \rho_k \leq \rho_{k,\max} \\ 0 & \rho_k \geq \rho_{k,\max} \end{cases}.$$

*Path 2* and *path 3* are comprised of  $K$  segments and each segment satisfies  $\frac{dv_l}{dt} \neq 0$  and  $\frac{dv_k}{dt} = 0, \forall k \neq l$ . There exist  $K!$  such paths, which constitute the  $K!$  SIC points. The decoding functions are step functions with sharp transitions at threshold SNRs  $\rho_{k,\text{SIC}}$ . This type of decoding functions may pose difficulties for practical code designs, compared to that with smooth transitions.

#### D. Achievable rate region

To achieve other points in the MAC capacity region, i.e., with maximum sum-rate but different individual user rates, other paths shall be used. In the following theorem, we show that the entire MAC capacity region can be achieved by showing the existence of paths. Examples for constructing a dedicated path achieving a feasible rate tuple are provided in [14, Sec. V-A, case 2].

**Theorem 2.** IDMA with GA-based MUD and the assumptions in Theorem 1 achieves every rate tuple in the  $K$ -user MAC capacity region  $\mathcal{C}(K)$ . Given a feasible target rate tuple  $\mathbf{R} = [R_1, R_2, \dots, R_K] \in \mathcal{C}(K)$ , there

exists at least one path defined by  $L(t) = \mathbf{v}(t) : \mathbf{v}_s = \mathbf{1} \rightarrow \mathbf{v}_e = \mathbf{0}$  which achieves  $\mathbf{R}$ .

*Proof:* See Appendix B.

*Remark:* It is easy to prove that there exists a unique path for each of the  $K!$  SIC corner points and the decoding functions shall be step functions. For other rate tuples, it can be verified that there exist many different paths achieving that rate tuple. The choice of the integration path poses varying degrees of difficulty for the design of matching codes. Thus, the design of an appropriate integration path could be an extra degree of freedom for code design.

*Numerical Results:* We designed matching (binary) LDPC codes for  $K = 3$  with unequal-power distribution and QPSK for different integration paths and rate tuples at sum-rate one. Bit error rate (BER) simulations and density evolution results show that the gap to Gaussian capacity is below 1 dB for all cases [14, Sec. V].

#### IV. MU-MIMO CHANNEL

Assume that each transmitter has  $N_{t,i}$  antennas and the receiver has  $N_R$  antennas respectively; then, the received signal can be written as

$$\mathbf{y} = \sum_{k=1}^K \sqrt{P_k} \mathbf{H}_k \mathbf{x}_k + \mathbf{n} \quad (9)$$

where  $\mathbf{H}_k$  is the channel of the  $k$ th user,  $\mathbf{n}$  denotes the uncorrelated noise  $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \sigma^2\mathbf{I}$ . In this case, the ESE module is replaced by a linear MMSE (LMMSE) receiver [12]. Under the LMMSE-based ESE, the SNR of user  $k$  can be written as [15]

$$\rho_k = \frac{\sum_{i=1}^{N_{t,k}} \mathbf{h}_{k,i}^H \mathbf{R}^{-1} \mathbf{h}_{k,i}}{1 - v_k \sum_{i=1}^{N_{t,k}} \mathbf{h}_{k,i}^H \mathbf{R}^{-1} \mathbf{h}_{k,i}} \quad (10)$$

where  $\mathbf{h}_{k,i}$  denotes the  $i$ th column of the  $k$ th user's channel matrix  $\mathbf{H}_k$  and

$$\mathbf{R} = \sigma^2 \mathbf{I} + \mathbf{H} \mathbf{V} \mathbf{H}^H$$

with  $\mathbf{V} = \text{diag}(P_1 v_1, P_2 v_2, \dots, P_K v_K)$  and  $\mathbf{H}$  being the concatenated channels of all users. Following a similar approach in Appendix A, the sum-rate can be obtained as

$$\begin{aligned} R_{\text{sum}} &= \sum_{i=1}^K R_i = - \int_{\mathbf{v}=1}^{\mathbf{v}=\mathbf{0}} \nabla \log \det [\mathbf{R}] d\mathbf{v} \\ &= \log \det \left[ \mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{H}^H \mathbf{P} \mathbf{H} \right] \end{aligned} \quad (11)$$

where  $\mathbf{P} = \text{diag}(P_1, P_2, \dots, P_K)$ . Path independence follows from the condition

$$\frac{\partial}{\partial v_k} \log \det [\mathbf{R}] = \text{trace} [\mathbf{R}^{-1} \mathbf{H}_k \mathbf{H}_k^H] = \sum_{i=1}^{N_{t,k}} \mathbf{h}_{k,i}^H \mathbf{R}^{-1} \mathbf{h}_{k,i}$$

with Jacobi's formula.

## V. CONCLUSION

It is proved that the simple interleave-division multiple-access (IDMA), relying on a low-cost Gaussian approximation (GA) based multi-user detector (MUD), is capacity-achieving for general Gaussian multiple access channels (GMAC) with arbitrary number of users, power distribution and with single or multiple antennas. We show that IDMA with matching codes is capacity-achieving for arbitrary decoding path in the mean-square error (MSE) vector field. This property is further used to prove that IDMA achieves not only the sum-rate capacity, but the entire GMAC capacity region. The construction of capacity-achieving codes is also provided by establishing the area theorem for multi-user extrinsic information transfer (EXIT) chart.

## APPENDIX A PROOF OF (7b)

Let  $\rho_{k,\max}$  and  $\rho_{k,\min}$  as defined in (3c). The achievable rates can be thus expressed as

$$\begin{aligned}
R_k &= \int_{\rho_{k,\min}}^{\rho_{k,\max}} \frac{1}{\rho_k + v_k^{-1}} d\rho_k + \int_0^{\rho_{k,\min}} \frac{1}{\rho_k + 1} d\rho_k \\
&\stackrel{\rho'_k = \frac{d\rho_k}{dv_k}}{=} \int_{v_k=1}^{v_k=0} \frac{\rho'_k}{\rho_k + v_k^{-1}} dv_k + \int_0^{\rho_{k,\min}} \frac{1}{\rho_k + 1} d\rho_k \\
&= \int_1^0 \frac{\rho'_k - v_k^{-2} + v_k^{-2}}{\rho_k + v_k^{-1}} dv_k + \underbrace{\log(1 + \rho_{k,\min})}_{=w_0} \\
&= \left[ \log(\rho_k + v_k^{-1}) + \int_1^0 \frac{v_k^{-2}}{\rho_k + v_k^{-1}} dv_k \right]_{v_k=1}^{v_k=0} + w_0 \\
&\stackrel{(1a)}{=} \left[ \log(\rho_k + v_k^{-1}) + \int \left( v_k^{-1} - \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} \right) dv_k \right]_{v_k=1}^{v_k=0} \\
&\quad + w_0 \\
&= \left[ \log(\rho_k v_k + 1) - \int \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} dv_k \right]_{v_k=1}^{v_k=0} + w_0 \\
&= - \int_1^0 \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} dv_k
\end{aligned}$$

where  $g_k = P_k |h_k|^2$  is the  $k$ th element of the vector  $\mathbf{g}$ .

## APPENDIX B PROOF OF THEOREM 2

The user rate  $R_k = - \int_{v_k=1}^{v_k=0} \frac{g_k}{\mathbf{g}^T \mathbf{v} + \sigma^2} dv_k$  is obviously a continuous and monotone decreasing function of  $v$ . If  $v_k$  are unbounded, then  $R_k$  are unbounded with the single sum-rate constraint  $\sum R_k \leq \log\left(\frac{\mathbf{g}^T \mathbf{1} + \sigma^2}{\sigma^2}\right)$ . However, the value range of  $R_k$  is constrained by the fact that  $0 \leq v_l \leq 1, \forall l$ . Therefore, it is bounded by

$$R_k \leq - \int_{v_k=1}^{v_k=0} \frac{g_k}{g_k v_k + \sigma^2} dv_k = \log\left(\frac{g_k + \sigma^2}{\sigma^2}\right)$$

and similarly  $R_k \geq \log\left(\frac{g_k + \sigma^2}{\sum_{l \neq k} g_l + \sigma^2}\right)$ . Further, the constraints on  $v_l$  leads to

$$\begin{aligned}
R_k + R_l &\leq \log\left(\frac{g_k + g_l + \sigma^2}{\sigma^2}\right), \forall k \neq l \\
R_k + R_l + R_m &\leq \log\left(\frac{g_k + g_l + g_m + \sigma^2}{\sigma^2}\right), \forall k \neq l \neq m \\
&\vdots \\
\sum R_k &\leq \log\left(\frac{\mathbf{g}^T \mathbf{1} + \sigma^2}{\sigma^2}\right)
\end{aligned}$$

and these constraints constitute the capacity region.

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